Mixed finite element formulations for polyconvex anisotropic material formulations

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Motivation

- Dynamic simulations of fiber-reinforced materials in lightweight structures

- Keywords:
  - Anisotropic material behavior
  - Nearly-incompressible material behavior
  - Thermo-viscoelastic material behavior

- Solution strategy:
  1. Mixed finite elements to reduce locking effects
  2. Extension to a thermo-mechanical coupling
  3. Higher-order energy-momentum conserving time integrator for stable and accurate dynamic simulations
  4. Viscous internal variable as mixed field

Continuum mechanics and material formulation

- Strain energy function (thermo-viscoelastic matrix part Ψ_{M} and n_{F} thermoelastic fiber parts Ψ_{F_i})

  $$\Psi(\boldsymbol{C},\Theta)=\Psi_{\mathrm{M}}(\boldsymbol{C},\Theta,\boldsymbol{C}_v)+\sum_{i=1}^{n_{\mathrm{F}}}\Psi_{\mathrm{F}_i}(\boldsymbol{C},\Theta,\boldsymbol{M}_i)\qquad \boldsymbol{M}_i=(\boldsymbol{a}_i^0)^T\otimes\boldsymbol{a}_i^0$$

  $\boldsymbol{C}$ - right Cauchy-Green tensor, $\boldsymbol{C}_v$ - viscous right Cauchy-Green tensor, Θ - absolute temperature, $\boldsymbol{a}_i^0$ - fiber direction
- Components and specific dependencies ($J(\boldsymbol{C})=\sqrt{\det[\boldsymbol{C}]}$, $\boldsymbol{\Lambda}=\boldsymbol{C}\boldsymbol{C}_v^{-1}$)

  $$\begin{aligned}\Psi_{\mathrm{M}}(\boldsymbol{C},\Theta)&=\Psi_{\mathrm{M}}^{\mathrm{iso}}(\boldsymbol{C},\mathrm{cof}[\boldsymbol{C}],J)+\Psi_{\mathrm{M}}^{\mathrm{vol}}(J)+\Psi_{\mathrm{M}}^{\mathrm{cap}}(\Theta)+\Psi_{\mathrm{M}}^{\mathrm{coup}}(\Theta,J)+\Psi_{\mathrm{M}}^{\mathrm{vis}}(\boldsymbol{\Lambda})\\ \Psi_{\mathrm{F}_i}(\boldsymbol{C},\Theta,\boldsymbol{M}_i)&=\Psi_{\mathrm{F}_i}^{\mathrm{ela}}(\boldsymbol{C},\mathrm{cof}[\boldsymbol{C}],J,\boldsymbol{M}_i)+\Psi_{\mathrm{F}_i}^{\mathrm{cap}}(\Theta)+\Psi_{\mathrm{F}_i}^{\mathrm{coup}}(\Theta,\boldsymbol{C},\boldsymbol{M}_i)\end{aligned}$$
- Thermo-mechanical coupling [Groß 18]

  $$\Psi_{\mathrm{M}}^{\mathrm{coup}}=-2n_{\mathrm{dim}}\beta_{\mathrm{M}}(\Theta-\Theta_\infty)J\frac{\partial\Psi_{\mathrm{M}}^{\mathrm{vol}}(J)}{\partial J}\quad \Psi_{\mathrm{F}_i}^{\mathrm{coup}}=-2\beta_{\mathrm{F}_i}(\Theta-\Theta_\infty)\sqrt{I_4^i}\frac{\partial\Psi_{\mathrm{F}_i}^{\mathrm{ela}}(I_4^i\ldots)}{\partial I_4^i}$$

  Θ_∞ - ambient temperature, $I_4^i=\mathrm{tr}[\boldsymbol{C}\boldsymbol{M}_i]$
- Polyconvex material formulation for the hyperelastic parts

Total internal energy Π^{int} for the mixed principle of virtual power [Groß 18]

$$\Pi^{\text{int}} = \Pi_{\text{HW}} + \int_{\mathcal{B}_0} \frac{1}{2} \boldsymbol{S} : (\boldsymbol{C} - \tilde{\boldsymbol{C}}) \mathrm{d}V + \int_{\mathcal{B}_0} \tilde{\boldsymbol{S}} : \dot{\tilde{\boldsymbol{C}}} \mathrm{d}V + \int_{\mathcal{B}_0} \eta \, (\Theta - \tilde{\Theta}) \mathrm{d}V$$

- Independent mixed field $\tilde{\boldsymbol{C}}$ and corresponding Lagrangian multiplier $\boldsymbol{S}$
- Superimposed stress tensor $\tilde{\boldsymbol{S}}$ to derive energy–momentum scheme [Groß 18]
- Assumed temperature field $\tilde{\Theta}$ and the entropy density field η

Mixed elements based on Hu-Washizu functionals Π_{HW} [Simo 85] [Schr 11] [Schr 16]

D DP Co CoSKA CoCoA

$$\begin{aligned}\Pi_{\text{HW}} = & \int_{\mathcal{B}_0} \Psi(\dots) \mathrm{d}V + \int_{\mathcal{B}_0} p \, (J(\tilde{\boldsymbol{C}}) - \tilde{J}) \mathrm{d}V + \int_{\mathcal{B}_0} \tilde{p} \dot{\tilde{J}} \mathrm{d}V \\ & + \int_{\mathcal{B}_0} \boldsymbol{B} : (\operatorname{cof}[\tilde{\boldsymbol{C}}] - \tilde{\boldsymbol{H}}) \mathrm{d}V + \int_{\mathcal{B}_0} \tilde{\boldsymbol{B}} : \dot{\tilde{\boldsymbol{H}}} \mathrm{d}V + \int_{\mathcal{B}_0} \frac{1}{2} \boldsymbol{S}_A : (\tilde{\boldsymbol{C}} - \tilde{\boldsymbol{C}}_A) \mathrm{d}V + \int_{\mathcal{B}_0} \tilde{\boldsymbol{S}}_A : \dot{\tilde{\boldsymbol{C}}}_A \mathrm{d}V \\ & + \int_{\mathcal{B}_0} \boldsymbol{B}_A : (\operatorname{cof}[\tilde{\boldsymbol{C}}] - \tilde{\boldsymbol{H}}_A) \mathrm{d}V + \int_{\mathcal{B}_0} \tilde{\boldsymbol{B}}_A : \dot{\tilde{\boldsymbol{H}}}_A \mathrm{d}V + \int_{\mathcal{B}_0} p_A (J(\tilde{\boldsymbol{C}}) - \tilde{J}_A) \mathrm{d}V + \int_{\mathcal{B}_0} \tilde{p}_A \dot{\tilde{J}}_A \end{aligned}$$

$$\Psi_{\text{M}}(\dots) = \Psi_{\text{M}}^{\text{iso}}(\tilde{\boldsymbol{C}}, \tilde{\boldsymbol{H}}, \tilde{J}) + \Psi_{\text{M}}^{\text{vol}}(\tilde{J}) + \Psi_{\text{M}}^{\text{cap}}(\Theta) + \Psi_{\text{M}}^{\text{coup}}(\Theta, \tilde{J}) + \Psi_{\text{M}}^{\text{vis}}(\boldsymbol{\Lambda})$$

$$\Psi_{\text{F}_i}(\dots) = \Psi_{\text{F}_i}^{\text{ela}}(\tilde{\boldsymbol{C}}_A, \tilde{\boldsymbol{H}}_A, \tilde{J}_A, \boldsymbol{M}_i) + \Psi_{\text{F}_i}^{\text{cap}}(\Theta) + \Psi_{\text{F}_i}^{\text{coup}}(\Theta, \tilde{\boldsymbol{C}}_A, \boldsymbol{M}_i)$$

Superimposed fields

- Dependencies of the strain energy functions

$$\Psi_{\mathrm{M}}(\dots) = \Psi_{\mathrm{M}}^{\mathrm{iso}}(\tilde{\boldsymbol{C}}, \tilde{\boldsymbol{H}}, \tilde{J}) + \Psi_{\mathrm{M}}^{\mathrm{vol}}(\tilde{J}) + \Psi_{\mathrm{M}}^{\mathrm{cap}}(\Theta) + \Psi_{\mathrm{M}}^{\mathrm{coup}}(\Theta, \tilde{J}) + \Psi_{\mathrm{M}}^{\mathrm{vis}}(\boldsymbol{\Lambda})$$

$$\Psi_{\mathrm{F}_i}(\dots) = \Psi_{\mathrm{F}_i}^{\mathrm{ela}}(\tilde{\boldsymbol{C}}_A, \tilde{\boldsymbol{H}}_A, \tilde{J}_A, \boldsymbol{M}_i) + \Psi_{\mathrm{F}_i}^{\mathrm{cap}}(\Theta) + \Psi_{\mathrm{F}_i}^{\mathrm{coup}}(\Theta, \tilde{\boldsymbol{C}}_A, \boldsymbol{M}_i)$$

- Superimposed stress tensor $\tilde{\boldsymbol{S}}$ with $\tilde{\Psi} = \Psi_{\mathrm{M}}^{\mathrm{iso}}(\tilde{\boldsymbol{C}}) + \Psi_{\mathrm{M}}^{\mathrm{cap}}(\Theta) + \Psi_{\mathrm{M}}^{\mathrm{vis}}(\boldsymbol{\Lambda}) + \sum_{i=1}^{n_{\mathrm{F}}}[\Psi_{\mathrm{F}_i}^{\mathrm{cap}}(\Theta)]$

$$\tilde{\boldsymbol{S}} = \frac{\tilde{\Psi}(1) - \tilde{\Psi}(0) - \int \frac{\partial \tilde{\Psi}}{\partial \tilde{\boldsymbol{C}}} : \dot{\tilde{\boldsymbol{C}}} - \int \frac{\partial \tilde{\Psi}}{\partial \Theta} \dot{\Theta} - \int \frac{\partial \tilde{\Psi}}{\partial \boldsymbol{C}_v} : \dot{\boldsymbol{C}}_v}{\dot{\tilde{\boldsymbol{C}}} : \dot{\tilde{\boldsymbol{C}}}} \dot{\tilde{\boldsymbol{C}}}$$

- Superimposed pressure $\tilde{p}$ with $\tilde{\Psi} = \Psi_{\mathrm{M}}^{\mathrm{iso}}(\tilde{J}) + \Psi_{\mathrm{M}}^{\mathrm{vol}}(\tilde{J}) + \Psi_{\mathrm{M}}^{\mathrm{coup}}(\Theta, \tilde{J})$

$$\tilde{p} = \frac{\tilde{\Psi}(1) - \tilde{\Psi}(0) - \int \frac{\partial \tilde{\Psi}}{\partial \tilde{J}} \dot{\tilde{J}} - \int \frac{\partial \tilde{\Psi}}{\partial \Theta} \dot{\Theta}}{\dot{\tilde{J}}\dot{\tilde{J}}} \dot{\tilde{J}}$$

- Superimposed stress tensor $\tilde{\boldsymbol{S}}_A$ with $\tilde{\Psi} = \sum_{i=1}^{n_{\mathrm{F}}}[\Psi_{\mathrm{F}_i}^{\mathrm{ela}}(\tilde{\boldsymbol{C}}_A, \boldsymbol{M}_i) + \Psi_{\mathrm{F}_i}^{\mathrm{coup}}(\Theta, \tilde{\boldsymbol{C}}_A, \boldsymbol{M}_i)]$

$$\tilde{\boldsymbol{S}}_A = \frac{\tilde{\Psi}(1) - \tilde{\Psi}(0) - \int \frac{\partial \tilde{\Psi}}{\partial \tilde{\boldsymbol{C}}_A} : \dot{\tilde{\boldsymbol{C}}}_A - \int \frac{\partial \tilde{\Psi}}{\partial \Theta} \dot{\Theta}}{\dot{\tilde{\boldsymbol{C}}}_A : \dot{\tilde{\boldsymbol{C}}}_A} \dot{\tilde{\boldsymbol{C}}}_A$$

- The remaining superimposed fields be designed in the same manner

Mixed principle of virtual power of CoCoA Element $\dot{\mathcal{H}} = \dot{T} + \dot{\Pi}^{\mathrm{ext}} + \dot{\Pi}^{\mathrm{int}}$

- Kinetic power functional

$$\dot{T}(\dot{\boldsymbol{q}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}) = \int_{\mathcal{B}_0} (\rho_0 \boldsymbol{v} - \boldsymbol{p}) \cdot \dot{\boldsymbol{v}} \mathrm{d}V + \int_{\mathcal{B}_0} \dot{\boldsymbol{p}} \cdot (\dot{\boldsymbol{q}} - \boldsymbol{v}) \mathrm{d}V + \int_{\mathcal{B}_0} \boldsymbol{p} \cdot \ddot{\boldsymbol{q}} \mathrm{d}V$$

- External power functional

$$\dot{\Pi}^{\mathsf{ext}}(\dot{\boldsymbol{q}}, \boldsymbol{\lambda}, \tilde{\Theta}, \dot{\boldsymbol{C}}_v) = -\int_{\partial\mathcal{B}_0} \boldsymbol{t} \cdot \dot{\boldsymbol{q}} \mathrm{d}A - \int_{\partial\mathcal{B}_0} \boldsymbol{\lambda}_q \cdot (\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}^{\mathsf{ref}}) \mathrm{d}A - \int_{\partial\mathcal{B}_0} \boldsymbol{\lambda}_\Theta \cdot (\dot{\Theta} - \dot{\Theta}^{\mathsf{ref}}) \mathrm{d}A$$

$$+ \int_{\mathcal{B}_0} \nabla \left(\frac{\tilde{\Theta}}{\Theta} \right) \cdot \boldsymbol{Q} \mathrm{d}V + \int_{\mathcal{B}_0} \frac{\tilde{\Theta}}{\Theta} D^{\mathrm{int}} \mathrm{d}V + \int_{\mathcal{B}_0} \frac{1}{2} \dot{\boldsymbol{C}}_v : \mathbb{V}(\boldsymbol{C}_v) : \dot{\boldsymbol{C}}_v \mathrm{d}V$$

$$\boldsymbol{Q} = -\left[\sum_{i=1}^{n_{\mathrm{F}}} J \frac{k_{\mathrm{F}_i} - k_{\mathrm{M}}}{\tilde{\boldsymbol{C}}_A : \boldsymbol{M}_i} \boldsymbol{M}_i + kJ\tilde{\boldsymbol{C}}^{-1} \right] \nabla \Theta$$

- Viscous dissipation D^{int} and positive-definite viscosity tensor $\mathbb{V}$

$$D^{\mathrm{int}} = \dot{\boldsymbol{C}}_v : \mathbb{V}(\boldsymbol{C}_v) : \dot{\boldsymbol{C}}_v \quad \mathbb{V}(\boldsymbol{C}_v) = \frac{1}{4} \left(V_{\mathrm{vol}} - \frac{V_{\mathrm{dev}}}{n_{\mathrm{dim}}} \right) \boldsymbol{C}_v^{-1} \otimes \boldsymbol{C}_v^{-1} + \frac{V_{\mathrm{dev}}}{4} \mathbb{I}_{\mathsf{s}} : \boldsymbol{C}_v^{-1} \otimes \boldsymbol{C}_v^{-1}$$

- Variation with respect to the variables in the arguments of the total energy balance $\int_T \dot{\mathcal{H}} \mathrm{d}t = \int_T [\delta_* \dot{T} + \delta_* \dot{\Pi}^{\mathrm{ext}} + \delta_* \dot{\Pi}^{\mathrm{int}}] \mathrm{d}t = 0$ with the dependencies:
$\dot{\mathcal{H}}(\dot{\boldsymbol{q}}, \boldsymbol{\lambda}_q, \boldsymbol{\Theta}_q, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}, \tilde{\Theta}, \dot{\Theta}, \dot{\eta}, \dot{\boldsymbol{C}}_v, \dot{\tilde{\boldsymbol{C}}}, \dot{\tilde{\boldsymbol{C}}}_A, \dot{\tilde{\boldsymbol{H}}}, \dot{\tilde{\boldsymbol{H}}}_A, \dot{\tilde{J}}, \dot{\tilde{J}}_A, \boldsymbol{S}, \boldsymbol{S}_A, \boldsymbol{B}, \boldsymbol{B}_A, p, p_A)$

Weak formulation of CoCoA Element

$$\int_T \int_{\mathcal{B}_0} [\mathrm{Div}[\boldsymbol{F}\boldsymbol{S}] - \dot{\boldsymbol{p}}] \cdot \delta \dot{\boldsymbol{q}} \,\mathrm{d}V \mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} \left[\frac{1}{\rho_0}\boldsymbol{p} - \dot{\boldsymbol{q}}\right] \cdot \delta \dot{\boldsymbol{v}} \,\mathrm{d}V \mathrm{d}t = 0$$

$$\int_T \int_{\partial\mathcal{B}_0} \left[-\boldsymbol{t} - \boldsymbol{\lambda}_q\right] \cdot \delta \dot{\boldsymbol{q}} \,\mathrm{d}A \mathrm{d}t = 0$$

$$\int_T \int_{\partial\mathcal{B}_0} \left[\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}^{\mathsf{ref}}(t)\right] \cdot \delta \boldsymbol{\lambda}_q \,\mathrm{d}A \mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} \left[\Theta - \tilde{\Theta}\right] \delta \dot{\eta} \,\mathrm{d}V \mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} \left[\eta + \frac{\partial \Psi}{\partial \Theta}\right] \delta \dot{\Theta} \,\mathrm{d}V \mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} \left[\frac{\mathrm{Div}[\boldsymbol{Q}]}{\Theta} + \frac{D^{\mathrm{int}}}{\Theta} + \dot{\eta}\right] \delta \dot{\tilde{\Theta}} \,\mathrm{d}V \mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} \left[\frac{\partial \Psi}{\partial \boldsymbol{C}_v} + \dot{\boldsymbol{C}}_v : \mathbb{V}(\boldsymbol{C}_v)\right] : \delta \dot{\boldsymbol{C}}_v \,\mathrm{d}V \mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} \left[\frac{1}{2}\boldsymbol{S} - \left(\boxed{\frac{\partial \Psi}{\partial \tilde{\boldsymbol{C}}} + \tilde{\boldsymbol{S}} + \boldsymbol{B} : \mathbb{P} + \frac{p}{2J}\mathrm{cof}[\boldsymbol{C}] + \frac{1}{2}\boldsymbol{S}_A + \boldsymbol{B}_A : \mathbb{P} + \frac{p_A}{2J}\mathrm{cof}[\boldsymbol{C}]}\right)\right] : \delta \dot{\tilde{\boldsymbol{C}}} \,\mathrm{d}V \mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} \frac{1}{2}\left[\dot{\tilde{\boldsymbol{C}}}_A - \dot{\tilde{\boldsymbol{C}}}\right] : \delta \boldsymbol{S}_A \,\mathrm{d}V \mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} \left[\frac{1}{2}\boldsymbol{S}_A - \left[\frac{\partial \Psi}{\partial \tilde{\boldsymbol{C}}_A} + \tilde{\boldsymbol{S}}_A\right]\right] : \delta \dot{\tilde{\boldsymbol{C}}}_A \,\mathrm{d}V \mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} \left[\dot{\tilde{J}} - \dot{J}\right] \delta p \,\mathrm{d}V \mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} \left[p - \left[\frac{\partial \Psi}{\partial \tilde{J}} + \tilde{p}\right]\right] \delta \dot{\tilde{J}} \,\mathrm{d}V \mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} \frac{1}{2}\left[\dot{\tilde{\boldsymbol{C}}} - \dot{\boldsymbol{C}}\right] : \delta \boldsymbol{S} \,\mathrm{d}V \mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} \left[\dot{\tilde{\boldsymbol{H}}} - \mathrm{co}\dot{\mathrm{f}}[\boldsymbol{C}]\right] : \delta \boldsymbol{B} \,\mathrm{d}V \mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} \left[\boldsymbol{B} - \left[\frac{\partial \Psi}{\partial \tilde{\boldsymbol{H}}} + \tilde{\boldsymbol{B}}\right]\right] : \delta \dot{\tilde{\boldsymbol{H}}} \,\mathrm{d}V \mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} \left[\dot{\tilde{J}}_A - \dot{J}\right] \delta p_A \,\mathrm{d}V \mathrm{d}t = 0 = 0 \qquad \int_T \int_{\mathcal{B}_0} \left[p_A - \left[\frac{\partial \Psi}{\partial \tilde{J}_A} + \tilde{p}_A\right]\right] \delta \dot{\tilde{J}}_A \,\mathrm{d}V \mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} \left[\dot{\tilde{\boldsymbol{H}}}_A - \mathrm{co}\dot{\mathrm{f}}[\boldsymbol{C}]\right] : \delta \boldsymbol{B}_A \,\mathrm{d}V \mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} \left[\boldsymbol{B}_A - \left[\frac{\partial \Psi}{\partial \tilde{\boldsymbol{H}}_A} + \tilde{\boldsymbol{B}}_A\right]\right] : \delta \dot{\tilde{\boldsymbol{H}}}_A \,\mathrm{d}V \mathrm{d}t = 0 \qquad \mathbb{P} = \frac{\partial \mathrm{cof}[\boldsymbol{C}]}{\partial \boldsymbol{C}}$$

Approximation $k =$ Polynomial degree in time

- Discretization in space and time
- Lagrangian shape functions in space (N) [Wrig 08] [Bart 18]
  - Independent approximation of the different mixed fields
  - Lagrangian multiplier approximated equally as the corresponding mixed fields e.g. $\tilde{\boldsymbol{C}}_A$ & $\boldsymbol{S}_A$, $\tilde{J}$ & p or $\tilde{\boldsymbol{H}}$ & $\boldsymbol{B}$
- Lagrangian shape functions in time (M,M',$\tilde{M}$) [Bets 01] [Groß 18]

$$M_i(\alpha) = \prod_{\substack{j=1 \\ j\neq i}}^{k+1} \frac{\alpha - \alpha_j}{\alpha_i - \alpha_j},\ 1 \leq i \leq k+1 \qquad \tilde{M}_i(\alpha) = \prod_{\substack{j=1 \\ j\neq i}}^{k} \frac{\alpha - \alpha_j}{\alpha_i - \alpha_j}, 1 \leq i \leq k$$

- Time rate variables and mixed fields ($\boldsymbol{q}$,$\boldsymbol{v}$,$\boldsymbol{p}$,$\tilde{\Theta}$,Θ,η,$\boldsymbol{C}_v$,$\tilde{\boldsymbol{C}}$,$\tilde{\boldsymbol{C}}_A$,$\tilde{\boldsymbol{H}}$,$\tilde{\boldsymbol{H}}_A$,$\tilde{J}$,$\tilde{J}_A$)

$$(\bullet)^{e,h} = \sum_{I=1}^{k+1} \sum_{A=1}^{n_{no}} M_I(\alpha) \mathrm{N}^A(\boldsymbol{\xi}) (\bullet)_I^{eA} \qquad \left((\dot{\bullet})^{e,h} = \frac{1}{h_n} \sum_{I=1}^{k+1} \sum_{A=1}^{n_{no}} M_I'(\alpha) \mathrm{N}^A(\boldsymbol{\xi}) (\bullet)_I^{eA} \right)$$

- Lagrangian multiplier and variation fields ($\boldsymbol{\lambda}_q$,,$\boldsymbol{\lambda}_\Theta$,$\boldsymbol{S}$,$\boldsymbol{S}_A$,$\boldsymbol{B}$,$\boldsymbol{B}_A$,$p$,$p_A$,$\delta_*\bullet$)

$$(\bullet)^{e,h} = \sum_{I=1}^{k} \sum_{A=1}^{n_{no_{\mathrm{u}}}} \tilde{M}_I \mathrm{N}^A (\bullet)_I^{eA}$$

Implementation

- Staggered solution of mixed fields at element level, e.g.
  - 1. Solve Values of $\tilde{J}$ with equation $\int_T \int_{\mathcal{B}_0} \left[\dot{\tilde{J}} - \dot{J}\right] \delta p \mathrm{d}V \mathrm{d}t = 0$
  - 2. Solve Values of p with $\tilde{J}$ and equation $\int_T \int_{\mathcal{B}_0} \left[p - \left[\frac{\partial \Psi}{\partial \tilde{J}} + \tilde{p}\right]\right] \delta \dot{\tilde{J}} \mathrm{d}V \mathrm{d}t = 0$
  - Discontinuous at the boundaries of spatial elements
- Iterative solution of $\boldsymbol{C}_v$ is calculated on element level not on Gauss point level
- Eliminate $\mathbf{p}$ and η, e.g.

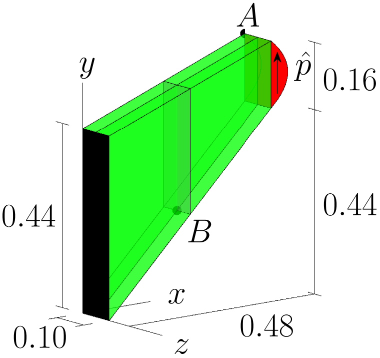
$$\int_T \int_{\mathcal{B}_0} [\mathrm{Div}[\boldsymbol{F}\boldsymbol{S}] - \dot{\boldsymbol{p}}] \cdot \delta \dot{\boldsymbol{q}} \mathrm{d}V \mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} \left[\frac{1}{\rho_0}\boldsymbol{p} - \dot{\boldsymbol{q}}\right] \cdot \delta \dot{\boldsymbol{v}} \mathrm{d}V \mathrm{d}t = 0$$

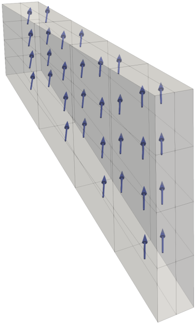
  - Rearrange first equation to $\boldsymbol{p}$ and insert in second equation
- Condensate at element level to pure displacement temperature form

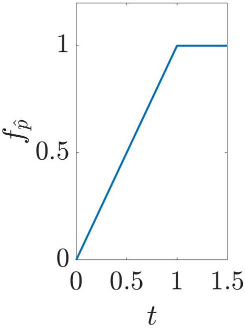
$$\begin{bmatrix} \mathbf{K}^e_{qq} & \mathbf{K}^e_{q\Theta} & \mathbf{K}^e_{q\lambda_q} & \mathbf{0} \\ \mathbf{K}^e_{\Theta q} & \mathbf{K}^e_{\Theta\Theta} & \mathbf{0} & \mathbf{K}^e_{\Theta\lambda_\Theta} \\ \mathbf{K}^e_{\lambda_q q} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}^e_{\lambda_\Theta\Theta} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{q} \\ \Delta\boldsymbol{\Theta} \\ \Delta\lambda_q \\ \Delta\lambda_\Theta \end{bmatrix} = \begin{bmatrix} \mathbf{r}^e_q \\ \mathbf{r}^e_\theta \\ \mathbf{r}^e_{\lambda_q} \\ \mathbf{r}^e_{\lambda_\Theta} \end{bmatrix}$$

- Schur complements: $\Rightarrow$ Except $\mathbf{K}_{\boldsymbol{C}_v \boldsymbol{C}_v}$, all inverted matrices are constant!

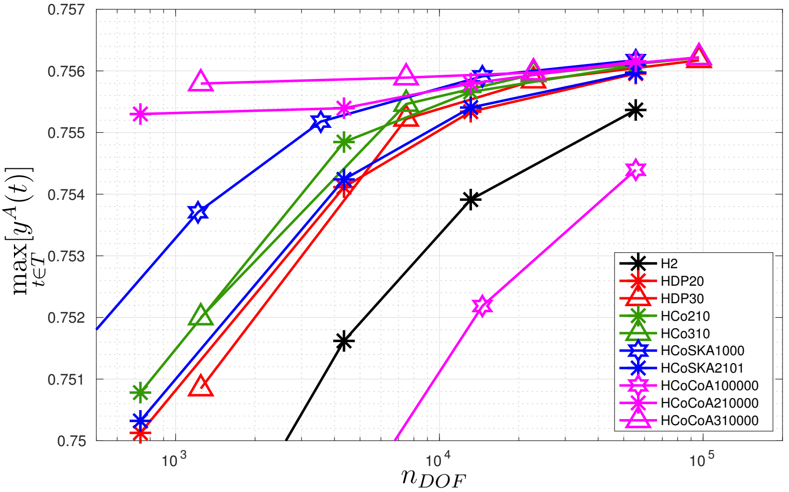
- High poisson ratio ($\nu \approx 0.4995$) & Stiff fiber ($\approx 10 \cdot 10^6$)
- High thermo-mechanical coupling, low heat Conduction ($k < 1$) & low viscosity (material formulation from [Schr 11])
- $T = 1.5$, $h_n = 0.001$, $k = 1$, $\hat{p} = 1.5e6 f_{\hat{p}}$
- Anisotropic direction $(\boldsymbol{a}_1^0)^T = [1\ 1\ 1]$





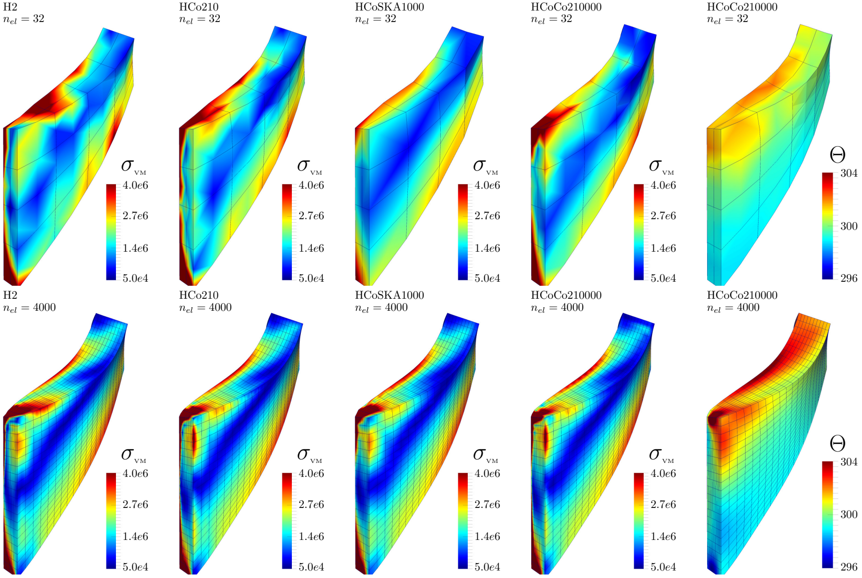


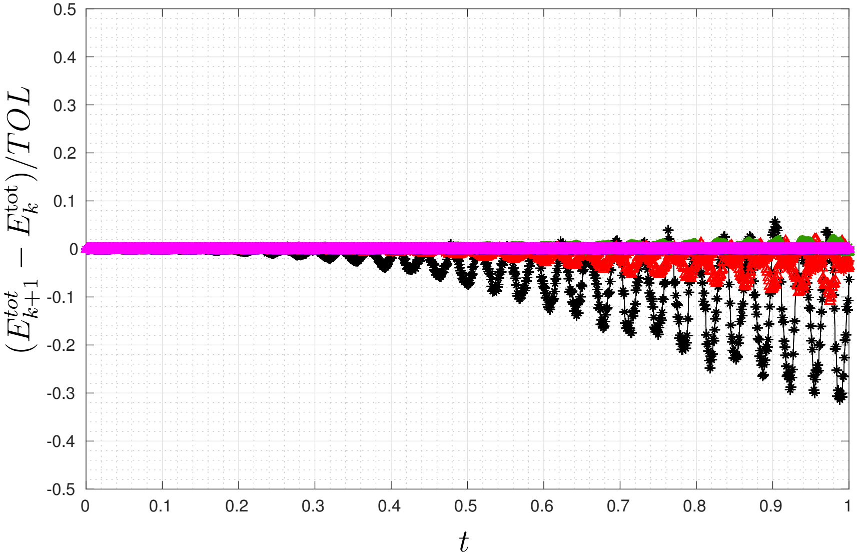
Convergence of the y-coordinate at Point A

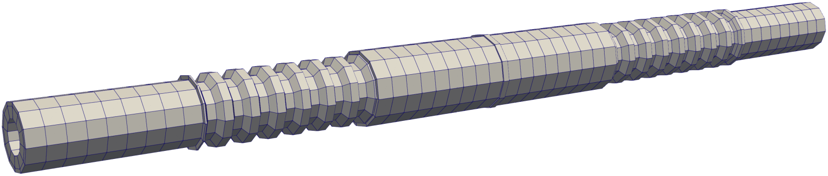


Digits (pol. degree of): H[$\boldsymbol{q}$] HDP[$\boldsymbol{q}$,$\tilde{J}$] HCo[$\boldsymbol{q}$,$\tilde{\boldsymbol{H}}$,$\tilde{J}$] HCoSKA/HCoCoA[$\boldsymbol{q}$,$\tilde{\boldsymbol{H}}$,$\tilde{J}$,$\tilde{\boldsymbol{C}}_A$,$\tilde{\boldsymbol{H}}_A$,$\tilde{J}_A$]

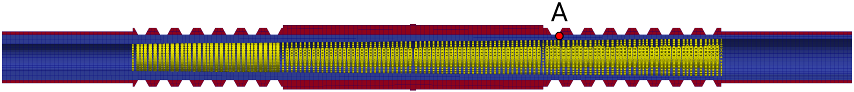
Numerical example - Cook cantilever beam
Deformed configuration, v. Mises equivalent stress and temperature distribution



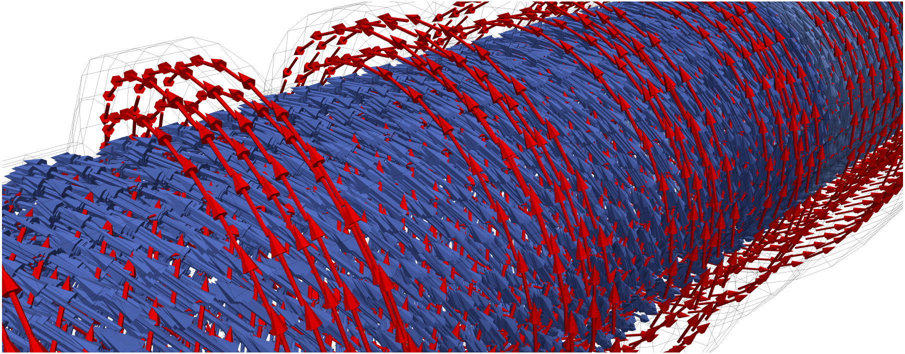

0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
$(E_{k+1}^{tot} - E_k^{\mathrm{tot}})/TOL$
t



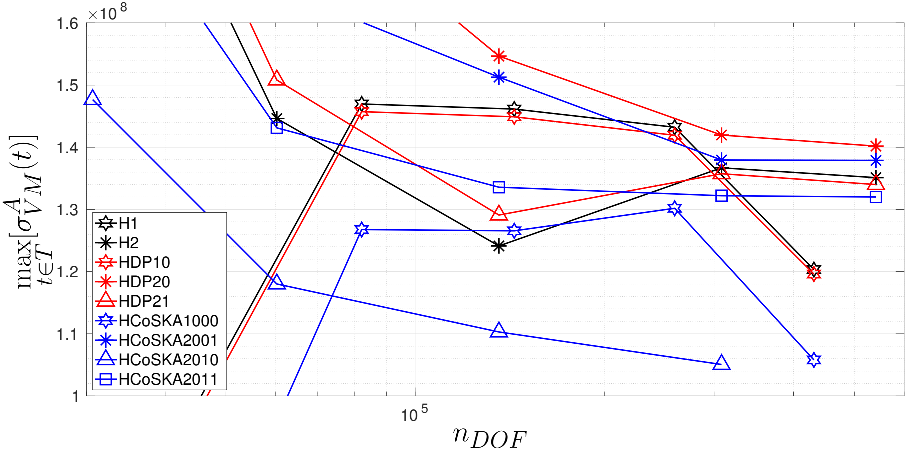
- Heatpipe (see [Quelle])with a rotating Dirichlet condition ($\Omega = 1 f_t$) on both sides and an internal pressure ($\hat{p} = 0.5e8 f_t$, yellow) with $f_t = \frac{1}{T} t$
- Elastic parameters and material functions from [Dal 17] (Fibre-reinforced plastic)
- $T = 1$, $h_n = 0.0001$, $k = 1$
- Low thermo-mechanical coupling, low heat conduction & high viscosity
- Tow material domains (blue & red) with different fibers



- Inner Domain (blue): Two fibers with an angle of 90 degrees to each other in circumferential direction
- Outer Domain (red): One fiber in circumferential direction



Numerical example - Rotating heatpipe
Convergence of σ_{VM} on Point A



Digits (pol. degree of): H[$\boldsymbol{q}$] HDP[$\boldsymbol{q}$,$\tilde{J}$] HCoSKA[$\boldsymbol{q}$,$\tilde{\boldsymbol{H}}$,$\tilde{J}$,$\tilde{\boldsymbol{C}}_A$]

Numerical example - Rotating heatpipe
Deformed configuration, v. Mises equivalent stress and temperature distribution

CoSKA2101 $n_{el} = 8184$ $T = 1$

CoSKA2101 $n_{el} = 8184$ $T = 1$

- Improved spatial convergence allows coarser meshe and reduce computing time
- Computing time for higher polynomial degrees in time is reduced significantly

H2	$k=1$		$k=2$	
n_{el}	n_{DOF}	t_{CPU}*	n_{DOF}	t_{CPU}*
1740	41184	3.6	82368	15.9
3536	81088	6.9	162176	33.5
8184	182400	15.9	364800	78.2
20274	411866	43.5	823732	230.8
CoSKA2101	$k=1$		$k=2$	
n_{el}	n_{DOF}	t_{CPU}*	n_{DOF}	t_{CPU}*
1740	41184	3.6	82368	17.5
3536	81088	7.9	162176	36.3
8184	182400	green 17.5	364800	86.1
20274	411866	46.3	823732	246.5

* Average calculation time of a time step with 2 Newton iterations

- Set $\Omega = 5f_t$
- $T = 1$, $h_n = 5 \cdot 10e - 5$, $k = 2$
- Apply an thermal dirichlet boundary condition on the left side

Conclusion

- Motivation:
  - Dynamic simulations of fiber-reinforced materials in light-weight structures
- Strategy:
  - Mixed finite elements to reduce locking effects
  - Extension to a thermo-mechanical coupling
  - Higher-order energy-momentum conserving time integrator for stable and accurate dynamic simulations
  - Viscous internal variable as mixed field
- Important results:
  - Excellent performance of the mixed elements is still preserved in a thermo-viscoelastic context.
  - Huge computing time reduction, especially in the context of the iterative calculation of the internal variable
  - Higher-order energy-momentum time integrators conserves energy
- Outlook:
  - Extend these formulations to higher-order gradients to capture the fibre-bending stiffness

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