Thermo-mechanical coupling in fiber-reinforced continua: Mixed finite element formulations and energy-momentum time integration

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Motivation

- Dynamic simulations of fiber-reinforced materials in lightweight structures

- Keywords:
  - Anisotropic material behavior
  - Nearly-incompressible material behavior
  - Thermo-mechanical material behavior [Thermal expansion and heat conduction]

- Solution strategy:
  1. Mixed finite elements to reduce locking effects
  2. Extension to a thermo-mechanical coupling
  3. Higher-order energy-momentum conserving time integrator for stable and accurate dynamic simulations [eg(k)]

Continuum mechanics and material formulation

- Strain energy function (matrix part Ψ_{M} and n_{F} fiber parts Ψ_{F_i})

$$\Psi(\boldsymbol{C},\Theta) = \Psi_{\mathrm{M}}(\boldsymbol{C},\Theta) + \sum_{i=1}^{n_{\mathrm{F}}} \Psi_{\mathrm{F}_i}(\boldsymbol{C},\Theta,\boldsymbol{M}_i) \qquad \boldsymbol{M}_i = (\boldsymbol{a}_i^0)^T \otimes \boldsymbol{a}_i^0$$

$\boldsymbol{C}$ - Right Cauchy-Green tensor, Θ - Absolute temperature, $\boldsymbol{a}_i^0$ - Fiber direction

- Components and specific dependencies ($J(\boldsymbol{C}) = \sqrt{\det[\boldsymbol{C}]}$)

$$\Psi_{\mathrm{M}}(\boldsymbol{C},\Theta) = \Psi_{\mathrm{M}}^{\mathrm{iso}}(\boldsymbol{C},\mathrm{cof}[\boldsymbol{C}],J) + \Psi_{\mathrm{M}}^{\mathrm{vol}}(J) + \Psi_{\mathrm{M}}^{\mathrm{cap}}(\Theta) + \Psi_{\mathrm{M}}^{\mathrm{coup}}(\Theta,J)$$

$$\Psi_{\mathrm{F}_i}(\boldsymbol{C},\Theta,\boldsymbol{M}_i) = \Psi_{\mathrm{F}_i}^{\mathrm{ela}}(\boldsymbol{C},\mathrm{cof}[\boldsymbol{C}],J,\boldsymbol{M}_i) + \Psi_{\mathrm{F}_i}^{\mathrm{cap}}(\Theta) + \Psi_{\mathrm{F}_i}^{\mathrm{coup}}(\Theta,\boldsymbol{C},\boldsymbol{M}_i)$$

- Polyconvex material formulation for the hyperelastic parts

$$\Psi_{\mathrm{M}}^{\mathrm{iso}}(\boldsymbol{C},\mathrm{cof}[\boldsymbol{C}],J) = \Psi_{\mathrm{M}}^{\mathrm{iso}}(\boldsymbol{C}) + \Psi_{\mathrm{M}}^{\mathrm{iso}}(\mathrm{cof}[\boldsymbol{C}]) + \Psi_{\mathrm{M}}^{\mathrm{iso}}(J)$$

$$\Psi_{\mathrm{F}_i}^{\mathrm{ela}}(\boldsymbol{C},\mathrm{cof}[\boldsymbol{C}],J,\boldsymbol{M}_i) = \Psi_{\mathrm{F}_i}^{\mathrm{ela}}(\boldsymbol{C},\boldsymbol{M}_i) + \Psi_{\mathrm{F}_i}^{\mathrm{ela}}(\mathrm{cof}[\boldsymbol{C}],\boldsymbol{M}_i) + \Psi_{\mathrm{F}_i}^{\mathrm{ela}}(J)$$

- Thermo-mechanical coupling [Groß 18]

$$\Psi_{\mathrm{M}}^{\mathrm{coup}} = -2n_{\mathrm{dim}}\beta_{\mathrm{M}}(\Theta-\Theta_\infty)J\frac{\partial\Psi_{\mathrm{M}}^{\mathrm{vol}}(J)}{\partial J} \qquad \Psi_{\mathrm{F}_i}^{\mathrm{coup}} = -2\beta_{\mathrm{F}_i}(\Theta-\Theta_\infty)\sqrt{I_4^i}\,\frac{\partial\Psi_{\mathrm{F}_i}^{\mathrm{ela}}(I_4^i\ldots)}{\partial I_4^i}$$

Θ_∞ - Ambient temperature, $I_4^i = \mathrm{tr}[\boldsymbol{C}\boldsymbol{M}_i]$

Total internal energy Π^{int}

- Total internal energy for the mixed principle of virtual power [Groß 18]

$$\Pi^{\text{int}} = \Pi_{\text{HW}} + \int_{\mathcal{B}_0} \frac{1}{2}\boldsymbol{S} : (\boldsymbol{C} - \tilde{\boldsymbol{C}})\mathrm{d}V + \int_{\mathcal{B}_0} \tilde{\boldsymbol{S}} : \tilde{\boldsymbol{C}}\mathrm{d}V + \int_{\mathcal{B}_0} \eta\,(\Theta - \tilde{\Theta})\mathrm{d}V$$

- Mixed elements based on Hu-Washizu functionals Π_{HW}
- Independent mixed field $\tilde{\boldsymbol{C}}$ and corrosponding Lagrangian multiplier $\boldsymbol{S}$
- Superimposed stress tensor $\tilde{\boldsymbol{S}}$ to derive energy–momentum scheme [Groß 18]
- Assumed temperature field $\tilde{\Theta}$ and the entropy density field η
- Displacement element $\Pi_{\text{HW}} = \int_{\mathcal{B}_0} \Psi(\tilde{\boldsymbol{C}}, \Theta)\mathrm{d}V$

Displacement-Pressure Element [Simo 85]

$$\Pi^{\text{DP}}_{\text{HW}} = \Pi^{\text{DP}}_{\text{HW}} + \int_{\mathcal{B}_0} p\,(J(\tilde{\boldsymbol{C}}) - \tilde{J})\mathrm{d}V + \int_{\mathcal{B}_0} \tilde{p}\tilde{J}\mathrm{d}V$$

$$\Psi_{\text{M}}(\dots) = \Psi^{\text{iso}}_{\text{M}}(\tilde{\boldsymbol{C}}, \operatorname{cof}[\tilde{\boldsymbol{C}}], \tilde{J}) + \Psi^{\text{vol}}_{\text{M}}(\tilde{J}) + \Psi^{\text{cap}}_{\text{M}}(\Theta) + \Psi^{\text{coup}}_{\text{M}}(\Theta, \tilde{J})$$

$$\Psi_{\text{F}_i}(\dots) = \Psi^{\text{ela}}_{\text{F}_i}(\tilde{\boldsymbol{C}}, \operatorname{cof}[\tilde{\boldsymbol{C}}], \tilde{J}, \boldsymbol{M}_i) + \Psi^{\text{cap}}_{\text{F}_i}(\Theta) + \Psi^{\text{coup}}_{\text{F}_i}(\Theta, \tilde{\boldsymbol{C}}, \boldsymbol{M}_i)$$

CoFEM Element [Schr 11]

$$\Pi_{\mathrm{HW}}^{\mathrm{CoFEM}} = \Pi_{\mathrm{HW}}^{\mathrm{DP}} + \int_{\mathcal{B}_0} \boldsymbol{B} : (\mathrm{cof}[\tilde{\boldsymbol{C}}] - \tilde{\boldsymbol{H}})\mathrm{d}V + \int_{\mathcal{B}_0} \tilde{\boldsymbol{B}} : \tilde{\boldsymbol{H}}\mathrm{d}V$$

$$\Psi_{\mathrm{M}}(\dots) = \Psi_{\mathrm{M}}^{\mathrm{iso}}(\tilde{\boldsymbol{C}}, \tilde{\boldsymbol{H}}, \tilde{J}) + \Psi_{\mathrm{M}}^{\mathrm{vol}}(\tilde{J}) + \Psi_{\mathrm{M}}^{\mathrm{cap}}(\Theta) + \Psi_{\mathrm{M}}^{\mathrm{coup}}(\Theta, \tilde{J})$$

$$\Psi_{\mathrm{F}_i}(\dots) = \Psi_{\mathrm{F}_i}^{\mathrm{ela}}(\tilde{\boldsymbol{C}}, \tilde{\boldsymbol{H}}, \tilde{J}, \boldsymbol{M}_i) + \Psi_{\mathrm{F}_i}^{\mathrm{cap}}(\Theta) + \Psi_{\mathrm{F}_i}^{\mathrm{coup}}(\Theta, \tilde{\boldsymbol{C}}, \boldsymbol{M}_i)$$

SKA Element [Schr 16]

$$\Pi_{\mathrm{HW}}^{\mathrm{CoSKA}} = \Pi_{\mathrm{HW}}^{\mathrm{CoFEM}} + \int_{\mathcal{B}_0} \frac{1}{2}\boldsymbol{S}_A : (\tilde{\boldsymbol{C}} - \tilde{\boldsymbol{C}}_A)\mathrm{d}V + \int_{\mathcal{B}_0} \tilde{\boldsymbol{S}}_A : \tilde{\boldsymbol{C}}_A\mathrm{d}V$$

$$\Psi_{\mathrm{M}}(\dots) = \Psi_{\mathrm{M}}^{\mathrm{iso}}(\tilde{\boldsymbol{C}}, \tilde{\boldsymbol{H}}, \tilde{J}) + \Psi_{\mathrm{M}}^{\mathrm{vol}}(\tilde{J}) + \Psi_{\mathrm{M}}^{\mathrm{cap}}(\Theta) + \Psi_{\mathrm{M}}^{\mathrm{coup}}(\Theta, \tilde{J})$$

$$\Psi_{\mathrm{F}_i}(\dots) = \Psi_{\mathrm{F}_i}^{\mathrm{ela}}(\tilde{\boldsymbol{C}}_A, \mathrm{cof}[\tilde{\boldsymbol{C}}_A], \sqrt{\det[\tilde{\boldsymbol{C}}_A]}, \boldsymbol{M}_i) + \Psi_{\mathrm{F}_i}^{\mathrm{cap}}(\Theta) + \Psi_{\mathrm{F}_i}^{\mathrm{coup}}(\Theta, \tilde{\boldsymbol{C}}_A, \boldsymbol{M}_i)$$

CoCoA Element

$$\Pi_{\mathrm{HW}}^{CoCoA} = \Pi_{\mathrm{HW}}^{\mathrm{CoSKA}} + \int_{\mathcal{B}_0} \boldsymbol{B}_A : (\mathrm{cof}[\tilde{\boldsymbol{C}}] - \tilde{\boldsymbol{H}}_A)\mathrm{d}V + \int_{\mathcal{B}_0} \tilde{\boldsymbol{B}}_A : \tilde{\boldsymbol{H}}_A\mathrm{d}V + \int_{\mathcal{B}_0} p_A(J(\tilde{\boldsymbol{C}}) - \tilde{J}_A)\mathrm{d}V + \int_{\mathcal{B}_0} \tilde{p}_A \tilde{J}_A \mathrm{d}V$$

$$\Psi_{\mathrm{M}}(\dots) = \Psi_{\mathrm{M}}^{\mathrm{iso}}(\tilde{\boldsymbol{C}}, \tilde{\boldsymbol{H}}, \tilde{J}) + \Psi_{\mathrm{M}}^{\mathrm{vol}}(\tilde{J}) + \Psi_{\mathrm{M}}^{\mathrm{cap}}(\Theta) + \Psi_{\mathrm{M}}^{\mathrm{coup}}(\Theta, \tilde{J})$$

$$\Psi_{\mathrm{F}_i}(\dots) = \Psi_{\mathrm{F}_i}^{\mathrm{ela}}(\tilde{\boldsymbol{C}}_A, \tilde{\boldsymbol{H}}_A, \tilde{J}_A, \boldsymbol{M}_i) + \Psi_{\mathrm{F}_i}^{\mathrm{cap}}(\Theta) + \Psi_{\mathrm{F}_i}^{\mathrm{coup}}(\Theta, \tilde{\boldsymbol{C}}_A, \boldsymbol{M}_i)$$

Superimposed fields for CoCoA Element

- Superimposed stress tensor $\tilde{\boldsymbol{S}}$ with $\tilde{\Psi} = \Psi_{\mathrm{M}}^{\mathrm{iso}}(\tilde{\boldsymbol{C}}) + \Psi_{\mathrm{M}}^{\mathrm{cap}}(\Theta) + \sum_{i=1}^{n_{\mathrm{F}}}[\Psi_{\mathrm{F}_i}^{\mathrm{cap}}(\Theta)]$
$$\tilde{\boldsymbol{S}} = \frac{\tilde{\Psi}(1) - \tilde{\Psi}(0) - \int \frac{\partial\tilde{\Psi}}{\partial\tilde{\boldsymbol{C}}} : \dot{\tilde{\boldsymbol{C}}} - \int \frac{\partial\tilde{\Psi}}{\partial\Theta}\dot{\Theta}}{\dot{\tilde{\boldsymbol{C}}} : \dot{\tilde{\boldsymbol{C}}}} \dot{\tilde{\boldsymbol{C}}}$$
- Superimposed pressure $\tilde{p}$ with $\tilde{\Psi} = \Psi_{\mathrm{M}}^{\mathrm{iso}}(\tilde{J}) + \Psi_{\mathrm{M}}^{\mathrm{vol}}(\tilde{J}) + \Psi_{\mathrm{M}}^{\mathrm{coup}}(\Theta, \tilde{J})$
$$\tilde{p} = \frac{\tilde{\Psi}(1) - \tilde{\Psi}(0) - \int \frac{\partial\tilde{\Psi}}{\partial\tilde{J}}\dot{\tilde{J}} - \int \frac{\partial\tilde{\Psi}}{\partial\Theta}\dot{\Theta}}{\dot{\tilde{J}}\dot{\tilde{J}}} \dot{\tilde{J}}$$
- Superimposed stress tensor $\tilde{\boldsymbol{S}}_A$ with $\tilde{\Psi} = \sum_{i=1}^{n_{\mathrm{F}}}[\Psi_{\mathrm{F}_i}^{\mathrm{ela}}(\tilde{\boldsymbol{C}}_A, \boldsymbol{M}_i) + \Psi_{\mathrm{F}_i}^{\mathrm{coup}}(\Theta, \tilde{\boldsymbol{C}}_A, \boldsymbol{M}_i)$
$$\tilde{\boldsymbol{S}}_A = \frac{\tilde{\Psi}(1) - \tilde{\Psi}(0) - \int \frac{\partial\tilde{\Psi}}{\partial\tilde{\boldsymbol{C}}_A} : \dot{\tilde{\boldsymbol{C}}}_A - \int \frac{\partial\tilde{\Psi}}{\partial\Theta}\dot{\Theta}}{\dot{\tilde{\boldsymbol{C}}}_A : \dot{\tilde{\boldsymbol{C}}}_A} \dot{\tilde{\boldsymbol{C}}}_A$$
- Superimposed conjugated stress t. $\tilde{\boldsymbol{B}}$ with $\tilde{\Psi} = \Psi_{\mathrm{M}}^{\mathrm{iso}}(\tilde{\boldsymbol{H}})$ and $\tilde{\boldsymbol{B}}_A$ with $\tilde{\Psi} = \sum_{i=1}^{n_{\mathrm{F}}} \Psi_{\mathrm{F}_i}^{\mathrm{ela}}(\tilde{\boldsymbol{H}}_A, \boldsymbol{M}_i)$
$$\tilde{\boldsymbol{B}} = \frac{\tilde{\Psi}(1) - \tilde{\Psi}(0) - \int \frac{\partial\tilde{\Psi}}{\partial\tilde{\boldsymbol{H}}} : \dot{\tilde{\boldsymbol{H}}}}{\dot{\tilde{\boldsymbol{H}}} : \dot{\tilde{\boldsymbol{H}}}} \dot{\tilde{\boldsymbol{H}}} \qquad \tilde{\boldsymbol{B}}_A = \frac{\tilde{\Psi}(1) - \tilde{\Psi}(0) - \int \frac{\partial\tilde{\Psi}}{\partial\tilde{\boldsymbol{H}}_A} : \dot{\tilde{\boldsymbol{H}}}_A}{\dot{\tilde{\boldsymbol{H}}}_A : \dot{\tilde{\boldsymbol{H}}}_A} \dot{\tilde{\boldsymbol{H}}}_A$$
- Superimposed pressure $\tilde{p}_A$ with $\tilde{\Psi} = \sum_{i=1}^{n_{\mathrm{F}}}[\Psi_{\mathrm{F}_i}^{\mathrm{ela}}(\tilde{J}_A)]$
$$\tilde{\boldsymbol{S}} = \frac{\tilde{\Psi}(1) - \tilde{\Psi}(0) - \int \frac{\partial\tilde{\Psi}}{\partial\tilde{J}_A}\dot{\tilde{J}}_A}{\dot{\tilde{J}}_A\dot{\tilde{J}}_A} \dot{\tilde{J}}_A$$

Mixed principle of virtual power of CoCoA Element

- Total energy balance law in functional form $\dot{\mathcal{H}} = \dot{T} + \dot{\Pi}^{\text{ext}} + \dot{\Pi}^{\text{int}}$

$$\dot{\mathcal{H}}(\dot{\boldsymbol{q}}, \boldsymbol{\lambda}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}, \tilde{\Theta}, \dot{\Theta}, \dot{\eta}, \dot{\tilde{\boldsymbol{C}}}, \dot{\tilde{\boldsymbol{C}}}_A, \dot{\tilde{\boldsymbol{H}}}, \dot{\tilde{\boldsymbol{H}}}_A, \dot{\tilde{J}}, \dot{\tilde{J}}_A, \boldsymbol{S}, \boldsymbol{S}_A, \boldsymbol{B}, \boldsymbol{B}_A, p, p_A) = 0$$

- Kinetic power functional

$$\dot{T}(\dot{\boldsymbol{q}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}) = \int_{\mathcal{B}_0} (\rho_0 \boldsymbol{v} - \boldsymbol{p}) \cdot \dot{\boldsymbol{v}} \mathrm{d}V + \int_{\mathcal{B}_0} \dot{\boldsymbol{p}} \cdot (\dot{\boldsymbol{q}} - \boldsymbol{v}) \mathrm{d}V + \int_{\mathcal{B}_0} \boldsymbol{p} \cdot \ddot{\boldsymbol{q}} \mathrm{d}V$$

- External power functional

$$\dot{\Pi}^{\mathsf{ext}}(\dot{\boldsymbol{q}}, \boldsymbol{\lambda}, \tilde{\Theta}) = -\int_{\partial\mathcal{B}_0} \boldsymbol{t} \cdot \dot{\boldsymbol{q}} \mathrm{d}A - \int_{\partial\mathcal{B}_0} \boldsymbol{\lambda} \cdot (\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}^{\mathsf{ref}}) \mathrm{d}A + \int_{\mathcal{B}_0} \nabla \left(\frac{\tilde{\Theta}}{\Theta} \right) \cdot \boldsymbol{Q} \mathrm{d}V.$$

- Piola heat flux vector ($J = J(\tilde{\boldsymbol{C}})$)

$$\boldsymbol{Q} = -\left[\sum_{i=1}^{n_{\mathrm{F}}} J \frac{k_{F_i} - k_M}{\tilde{\boldsymbol{C}}_A : \boldsymbol{M}_i} \boldsymbol{M}_i + k J \tilde{\boldsymbol{C}}^{-1} \right] \nabla \Theta$$

- Variation with respect to the variables in the arguments of the total energy balance

$$\int_T [\delta_* \dot{T} + \delta_* \dot{\Pi}^{\text{ext}} + \delta_* \dot{\Pi}^{\text{int}}] \mathrm{d}t = 0$$

Weak formulation of CoCoA Element

$$\int_T \int_{\mathcal{B}_0} [\mathrm{Div}[\boldsymbol{F}\boldsymbol{S}] - \dot{\boldsymbol{p}}] \cdot \delta\dot{\boldsymbol{q}} \,\mathrm{d}V\mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} \left[\frac{1}{\rho_0}\boldsymbol{p} - \dot{\boldsymbol{q}}\right] \cdot \delta\dot{\boldsymbol{v}} \,\mathrm{d}V\mathrm{d}t = 0 \qquad \int_T \int_{\partial\mathcal{B}_0} [-\boldsymbol{t} - \boldsymbol{\lambda}] \cdot \delta\dot{\boldsymbol{q}} \,\mathrm{d}A\mathrm{d}t = 0$$

$$\int_T \int_{\partial\mathcal{B}_0} \left[\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}^{\mathrm{ref}}(t)\right] \cdot \delta\boldsymbol{\lambda} \,\mathrm{d}A\mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} \left[\Theta - \tilde{\Theta}\right] \delta\dot{\eta} \,\mathrm{d}V\mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} \left[\eta + \frac{\partial\Psi}{\partial\Theta}\right] \delta\dot{\Theta} \,\mathrm{d}V\mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} \left[\frac{\mathrm{Div}[\boldsymbol{Q}]}{\Theta} + \dot{\eta}\right] \delta\tilde{\Theta} \,\mathrm{d}V\mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} \frac{1}{2}\left[\dot{\tilde{\boldsymbol{C}}} - \dot{\boldsymbol{C}}\right] : \delta\boldsymbol{S} \,\mathrm{d}V\mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} \left[\frac{1}{2}\boldsymbol{S} - \left(\boxed{\frac{\partial\Psi}{\partial\tilde{\boldsymbol{C}}} + \tilde{\boldsymbol{S}} + \boldsymbol{B} : \mathbb{P} + \frac{p}{2J}\mathrm{cof}[\boldsymbol{C}] + \frac{1}{2}\boldsymbol{S}_A + \boldsymbol{B}_A : \mathbb{P} + \frac{p_A}{2J}\mathrm{cof}[\boldsymbol{C}]}\right)\right] : \delta\tilde{\boldsymbol{C}} \,\mathrm{d}V\mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} \frac{1}{2}\left[\dot{\tilde{\boldsymbol{C}}}_A - \dot{\boldsymbol{C}}\right] : \delta\boldsymbol{S}_A \,\mathrm{d}V\mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} \left[\frac{1}{2}\boldsymbol{S}_A - \left[\frac{\partial\Psi}{\partial\tilde{\boldsymbol{C}}_A} + \tilde{\boldsymbol{S}}_A\right]\right] : \delta\dot{\tilde{\boldsymbol{C}}}_A \,\mathrm{d}V\mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} \left[\dot{\tilde{J}} - \dot{J}\right] \delta p \,\mathrm{d}V\mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} \left[p - \left[\frac{\partial\Psi}{\partial\tilde{J}} + \tilde{p}\right]\right] \delta\dot{\tilde{J}} \,\mathrm{d}V\mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} \left[\dot{\tilde{\boldsymbol{H}}} - \mathrm{cof}\dot{[}\boldsymbol{C}]\right] : \delta\boldsymbol{B} \,\mathrm{d}V\mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} \left[\boldsymbol{B} - \left[\frac{\partial\Psi}{\partial\tilde{\boldsymbol{H}}} + \tilde{\boldsymbol{B}}\right]\right] : \delta\dot{\tilde{\boldsymbol{H}}} \,\mathrm{d}V\mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} \left[\dot{\tilde{J}}_A - \dot{J}\right] \delta p_A \,\mathrm{d}V\mathrm{d}t = 0 = 0 \qquad \int_T \int_{\mathcal{B}_0} \left[p_A - \left[\frac{\partial\Psi}{\partial\tilde{J}_A} + \tilde{p}_A\right]\right] \delta\dot{\tilde{J}}_A \,\mathrm{d}V\mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} \left[\dot{\boldsymbol{H}}_A - \mathrm{cof}\dot{[}\boldsymbol{C}]\right] : \delta\boldsymbol{B}_A \,\mathrm{d}V\mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} \left[\boldsymbol{B}_A - \left[\frac{\partial\Psi}{\partial\tilde{\boldsymbol{H}}_A} + \tilde{\boldsymbol{B}}_A\right]\right] : \delta\dot{\boldsymbol{H}}_A \,\mathrm{d}V\mathrm{d}t = 0$$

Approximation — $k =$ Polynomial degree in time

- Discretization in space and time
- Lagrangian shape functions in space (N) [Wrig 08] [Bart 18]
  - Independent approximation of the different mixed fields
  - Lagrangian multiplier approximated equally as the corresponding mixed fields
- Lagrangian shape functions in time (M,M',$\tilde{M}$) [Bets 01] [Groß 18]

$$M_i(\alpha) = \prod_{\substack{j=1\\ j\neq i}}^{k+1} \frac{\alpha - \alpha_j}{\alpha_i - \alpha_j},\ 1 \leq i \leq k+1 \qquad \tilde{M}_i(\alpha) = \prod_{\substack{j=1\\ j\neq i}}^{k} \frac{\alpha - \alpha_j}{\alpha_i - \alpha_j}, 1 \leq i \leq k$$

- Time rate variables and mixed fields ($\boldsymbol{q}$,$\boldsymbol{\lambda}$,$\boldsymbol{v}$,$\boldsymbol{p}$,$\tilde{\Theta}$,Θ,η,$\tilde{\boldsymbol{C}}$,$\tilde{\boldsymbol{C}}_A$,$\tilde{\boldsymbol{H}}$,$\tilde{\boldsymbol{H}}_A$,$\tilde{J}$,$\tilde{J}_A$)

$$(\bullet)^{e,h} = \sum_{I=1}^{k+1} \sum_{A=1}^{n_{no}} M_I(\alpha) \mathrm{N}^A(\boldsymbol{\xi}) (\bullet)_I^{eA} \qquad \left((\dot{\bullet})^{e,h} = \frac{1}{h_n} \sum_{I=1}^{k+1} \sum_{A=1}^{n_{no}} M_I'(\alpha) \mathrm{N}^A(\boldsymbol{\xi}) (\bullet)_I^{eA} \right)$$

- Lagrangian multiplier and variation fields ($\boldsymbol{S}$,$\boldsymbol{S}_A$,$\boldsymbol{B}$,$\boldsymbol{B}_A$,p,p_A,$\delta_* \bullet$)

$$(\bullet)^{e,h} = \sum_{I=1}^{k} \sum_{A=1}^{n_{no_{\mathrm{u}}}} \tilde{M}_I \mathrm{N}^A (\bullet)_I^{eA}$$

Implementation

- Solve mixed fields and corresponding Lagrangian multiplier at element level
  - Discontinuous at the boundaries of spatial elements
  - Sequential solution

$$\int_T \int_{\mathcal{B}_0} \left[\dot{\tilde{J}} - \dot{J}\right] \delta p \mathrm{d}V \mathrm{d}t = 0 \quad \int_T \int_{\mathcal{B}_0} \left[p - \left[\frac{\partial \Psi}{\partial \tilde{J}} + \tilde{p}\right]\right] \delta \dot{\tilde{J}} \mathrm{d}V \mathrm{d}t = 0$$

$$\tilde{J}^e_{\text{new}} = \left[\mathbf{A}'\right]^{-1} \left(\int_0^1 \mathbf{b} \boxtimes \dot{J}^e \mathrm{d}\alpha - \mathbf{b}' \tilde{J}^e_1\right) \quad p^e_{\text{new}} = \left[\tilde{\mathbf{A}}\right]^{-1} \left(\int_0^1 \mathbf{b} \boxtimes \left[\frac{\partial \Psi}{\partial \tilde{J}} + \tilde{p}\right] \mathrm{d}\alpha\right)$$

- Eliminate $\mathbf{p}$ and η

$$\int_T \int_{\mathcal{B}_0} [\mathrm{Div}[\boldsymbol{F}\boldsymbol{S}] - \dot{\boldsymbol{p}}] \cdot \delta \dot{\boldsymbol{q}} \mathrm{d}V \mathrm{d}t = 0 \quad \int_T \int_{\mathcal{B}_0} \left[\frac{1}{\rho_0} \boldsymbol{p} - \dot{\boldsymbol{q}}\right] \cdot \delta \dot{\boldsymbol{v}} \mathrm{d}V \mathrm{d}t = 0$$

- Condensate at element level to pure displacement temperature form
  - Schur complements $\Rightarrow$ Inverted matrices are constant!

$$\begin{bmatrix} \mathbf{K}^e_{qq} & \mathbf{K}^e_{q\theta} & \mathbf{K}^e_{q\lambda} \\ \mathbf{K}^e_{\theta q} & \mathbf{K}^e_{\theta\theta} & \mathbf{0} \\ \mathbf{K}^e_{\lambda q} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{q} \\ \Delta \boldsymbol{\Theta} \\ \Delta \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{r}^e_q \\ \mathbf{r}^e_\theta \\ \mathbf{r}^e_\lambda \end{bmatrix}$$

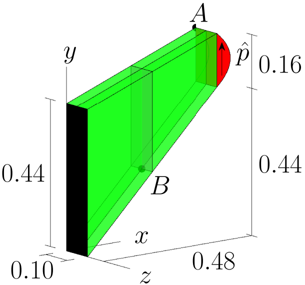
- Strain energy functions ([Schr 11] [Groß 18]):

$$\Psi_M^{iso} = \frac{\epsilon_1}{2}(\mathrm{tr}[\boldsymbol{C}])^2 + \frac{\epsilon_2}{2}(\mathrm{tr}[\mathrm{cof}[\boldsymbol{C}]])^2 - \epsilon_3 \ln(J) \quad \Psi_M^{vol} = \frac{\epsilon_4}{2}(J^{\epsilon_5} + J^{-\epsilon_5} - 2)$$

$$\Psi_M^{cap} = c_M^0(1 - \Theta_\infty c_M^1)(\Theta - \Theta_\infty - \Theta \ln \frac{\Theta}{\Theta_\infty}) - \frac{1}{2} c_M^0 c_M^1 (\Theta - \Theta_\infty)^2$$

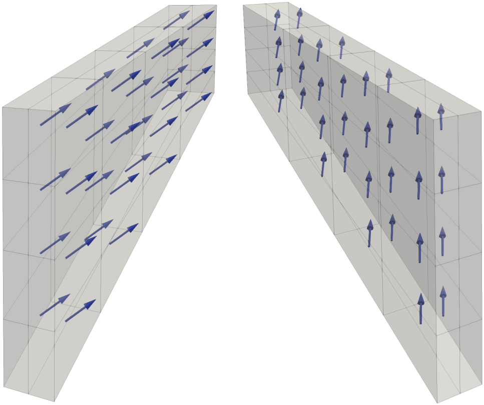
$$\Psi_{F_1}^{ela} = \epsilon_6 \left(\frac{1}{\epsilon_7 + 1}(\mathrm{tr}[\boldsymbol{C}\boldsymbol{M}_1])^{\epsilon_7+1} + \frac{1}{\epsilon_8+1}(\mathrm{tr}[\mathrm{cof}[\boldsymbol{C}]\boldsymbol{M}_1])^{\epsilon_8+1} + \frac{1}{\epsilon_9}\det[\boldsymbol{C}]^{-\epsilon_9} \right)$$

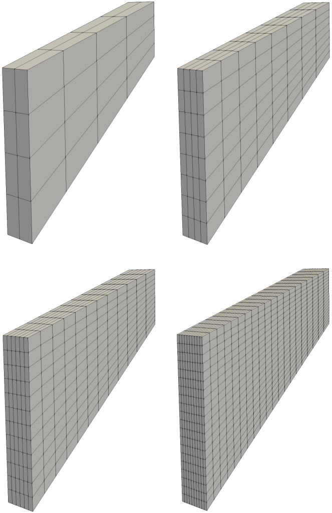
$$\Psi_{F_1}^{cap} = c_{F_1}^0(1 - \Theta_\infty c_{F_1}^1)(\Theta - \Theta_\infty - \Theta \ln \frac{\Theta}{\Theta_\infty}) - \frac{1}{2} c_{F_1}^0 c_{F_1}^1 (\Theta - \Theta_\infty)^2$$



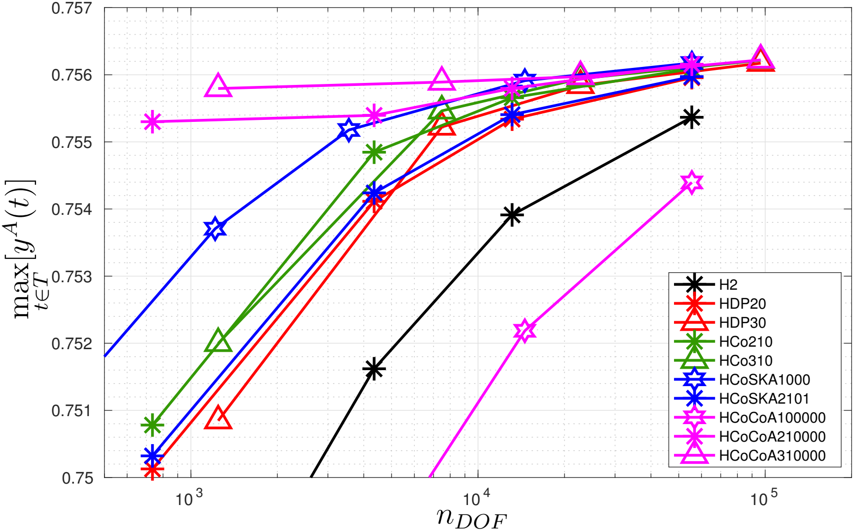
$\epsilon_1 = 0.1e6$	$k_M = 0.1$	$\rho_0 = 1000$
$\epsilon_2 = 0.1e6$	$\beta_M = 1e-6$	$\hat{p} = 1.5e6$
$\epsilon_3 = 1.8e6$	$c_M^0 = 1000$	$T = 1.0$
$\epsilon_4 = 100e6$	$c_M^1 = 0.001$	$h_n = 0.001$
$\epsilon_5 = 4$	$k_{F_1} = 0.1$	$TOL = 1e-2$
$\epsilon_6 = 10e6$	$\beta_{F_1} = 1e-6$	
$\epsilon_7 = 4$	$c_{F_1}^0 = 1000$	
$\epsilon_8 = 4$	$c_{F_1}^1 = 0.001$	
$\epsilon_9 = 1$	$\Theta_\infty = 300$	

Anisotropic direction $(\boldsymbol{a}_1^0)^T = [1\ 1\ 1]$



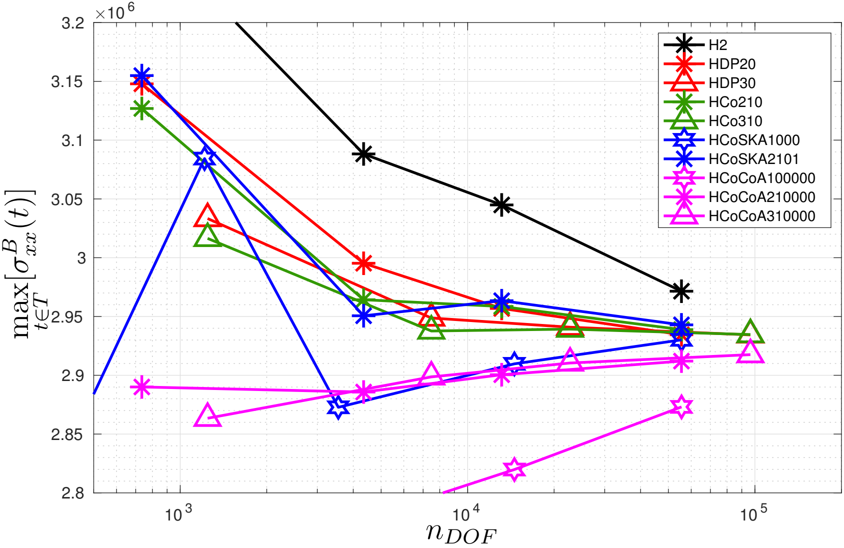


Numerical example - Cook cantilever beam
Convergence of the y-coordinate at Point A [eg(1)]



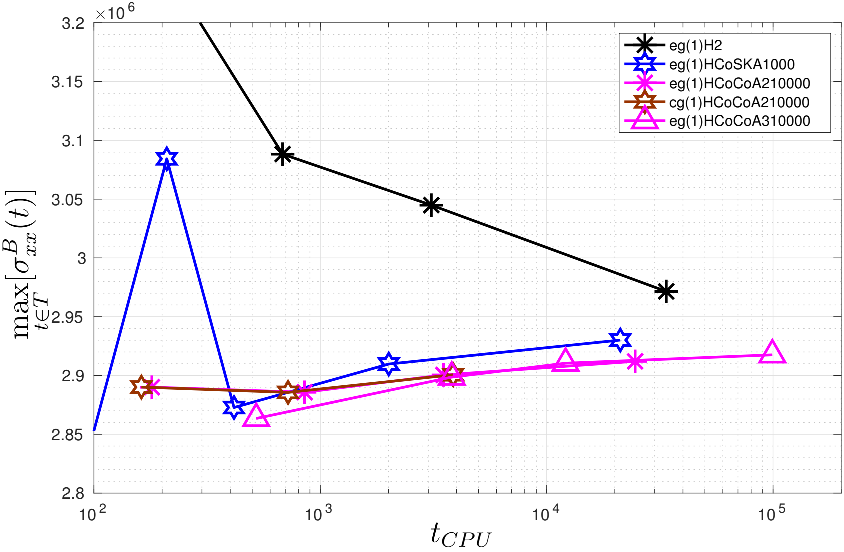
Digits (pol. degree of): H[$\boldsymbol{q}$] HDP[$\boldsymbol{q}$,$\tilde{J}$] HCo[$\boldsymbol{q}$,$\tilde{\boldsymbol{H}}$,$\tilde{J}$] HCoCoA[$\boldsymbol{q}$,$\tilde{\boldsymbol{H}}$,$\tilde{J}$,$\tilde{\boldsymbol{C}}_A$,$\tilde{\boldsymbol{H}}_A$,$\tilde{J}_A$]

Numerical example - Cook cantilever beam
Convergence of σ_{xx} at Point B [eg(1)]



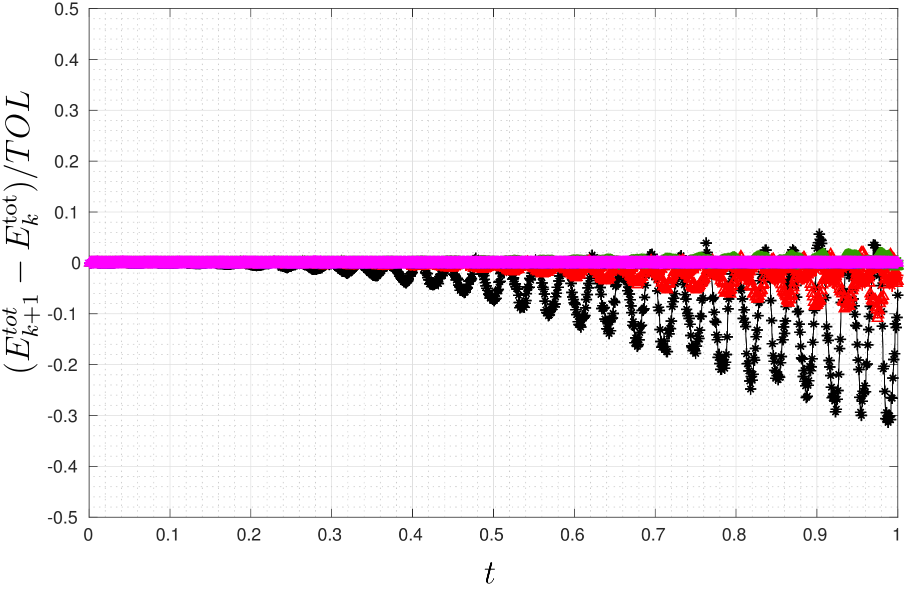
Digits (pol. degree of): H[$\boldsymbol{q}$] HDP[$\boldsymbol{q}$,$\tilde{J}$] HCo[$\boldsymbol{q}$,$\tilde{\boldsymbol{H}}$,$\tilde{J}$] HCoCoA[$\boldsymbol{q}$,$\tilde{\boldsymbol{H}}$,$\tilde{J}$,$\tilde{\boldsymbol{C}}_A$,$\tilde{\boldsymbol{H}}_A$,$\tilde{J}_A$]

Numerical example - Cook cantilever beam
CPU time

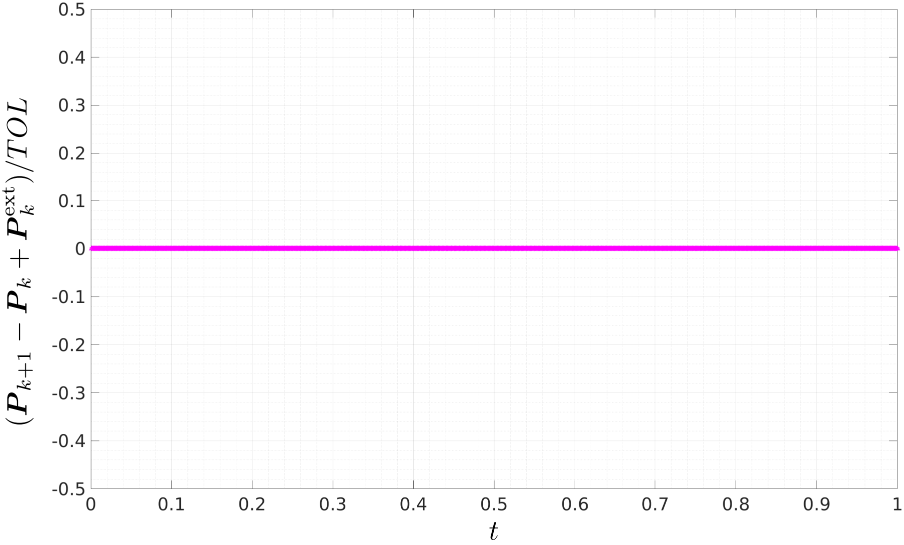


Digits (pol. degree of): H[$\boldsymbol{q}$] HDP[$\boldsymbol{q}$,$\tilde{J}$] HCo[$\boldsymbol{q}$,$\tilde{\boldsymbol{H}}$,$\tilde{J}$] HCoCoA[$\boldsymbol{q}$,$\tilde{\boldsymbol{H}}$,$\tilde{J}$,$\tilde{\boldsymbol{C}}_A$,$\tilde{\boldsymbol{H}}_A$,$\tilde{J}_A$]

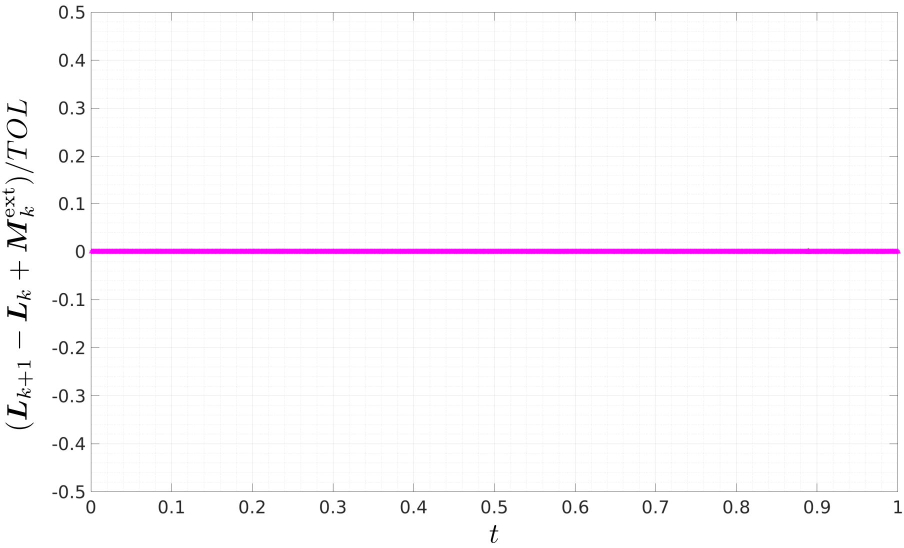
Numerical example - Cook cantilever beam
Conversation properties - Energy



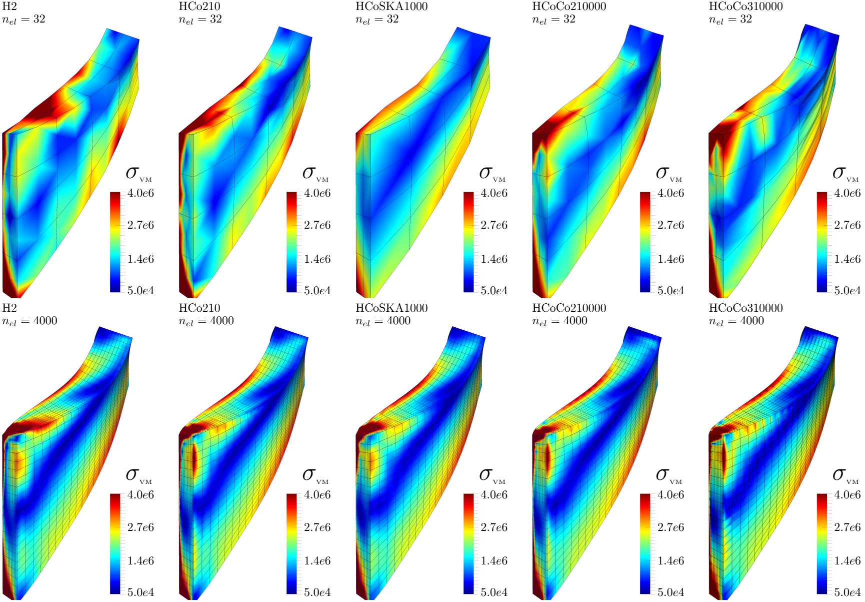
Numerical example - Cook cantilever beam
Conversation properties - Momentum [eg(1)]



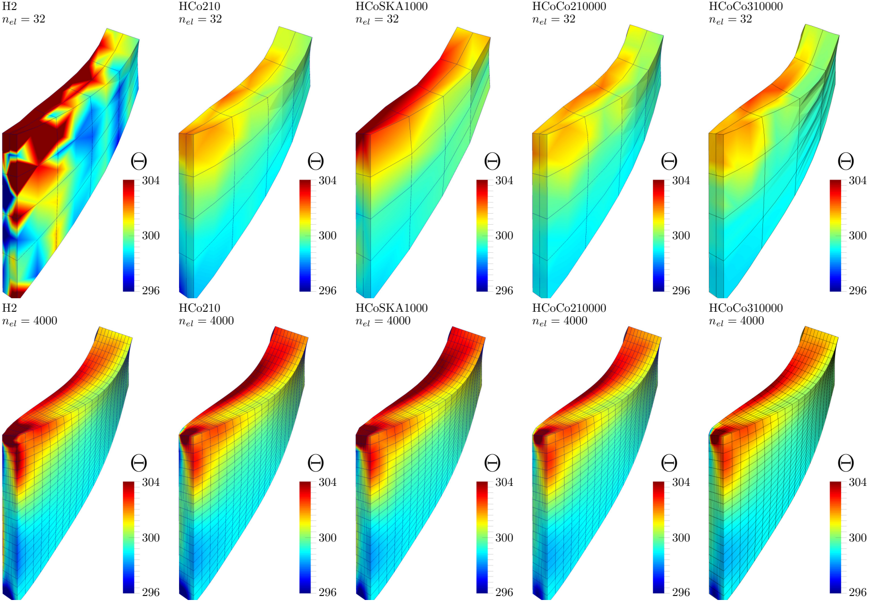
Numerical example - Cook cantilever beam
Conversation properties - Angular momentum [eg(1)]



Numerical example - Cook cantilever beam
Deformed configuration and v. Mises equivalent stress [eg(1)]

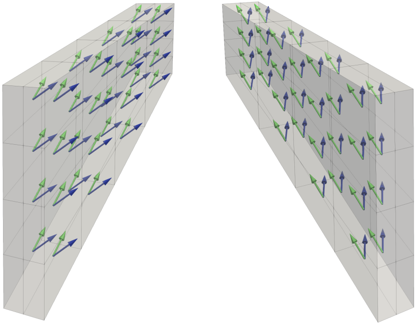


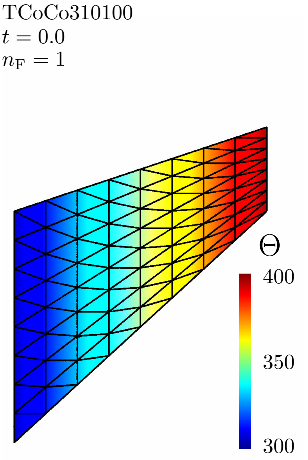
Numerical example - Cook cantilever beam
Deformed configuration and temperature distribution [eg(1)]



Fiber direction for $n_{\mathrm{F}} = 2$ and start temperature distribution

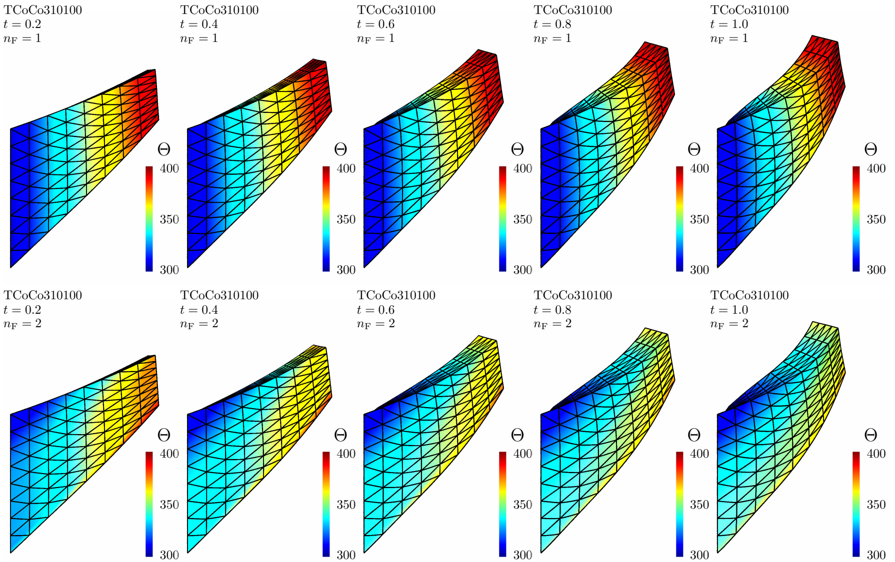
Anisotropic direction $(\boldsymbol{a}_1^0)^T = [1\ 1\ 1]$
Anisotropic direction $(\boldsymbol{a}_2^0)^T = [1\ 1\ 0]$



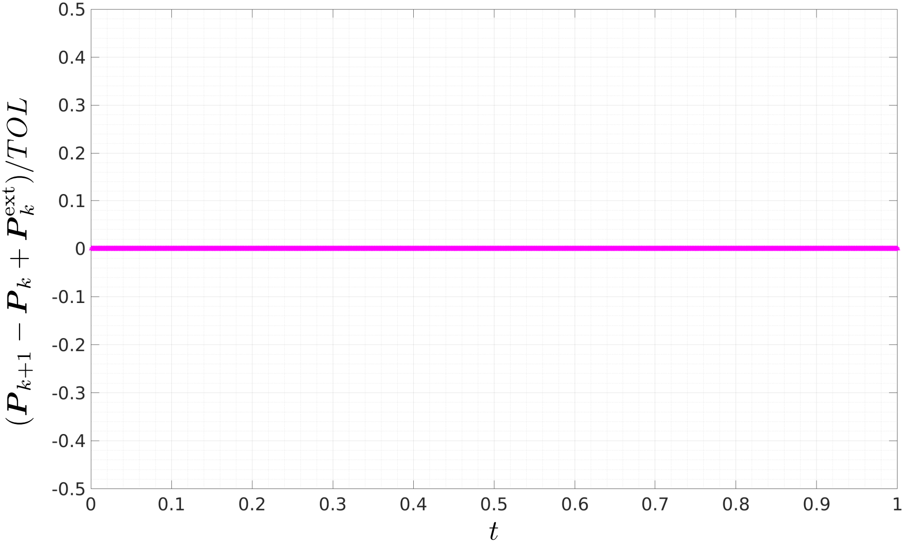


Numerical example - Cook cantilever beam

Deformed configuration and temperature distribution TCoCo310100 [eg(2)]



Numerical example - Cook cantilever beam
Conversation properties - Energy



Conclusion

- Motivation:
  - Dynamic simulations of fiber-reinforced materials in light-weight structures
- Strategy:
  - Mixed finite elements to reduce locking effect
  - Extension to a thermo-mechanical coupling
  - Higher-order energy-momentum conserving time integrators for stable and accurate simulations
- Important results:
  - Excellent performance of the mixed elements is still preserved in a thermo-mechanical context.
  - Higher-order energy-momentum time integrators conserves energy
  - Possibility to determine the mechanical and thermal properties of our model separately by using diffrent fibers
- Outlook:
  - Extend these formulations to a thermo-viscoelastic material behavior

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