A mixed finite element formulation for energy-momentum time integrations of composites with fiber bending stiffness

J. Dietzsch, M. Groß, I. Kalaimani

Professorship of Applied Mechanics and Dynamics
Department of Mechanical Engineering
Technische Universität Chemnitz

GAMM 2021 (Virtual congress) 17 March, 2021

Acknowledgments: This research is provided by DFG grant GR 3297/4 and 3297/6.

Motivation

- Dynamic simulations of fiber-reinforced materials in lightweight structures

- Keywords:
  - Nearly-incompressible material behavior
  - Anisotropic material behavior
    - Fiber stretch stiffness
    - Fiber bending stiffness
  - Long term simulations

- Solution strategy:
  1. Extend the Cauchy-Boltzmann continuum by higher order gradients to capture fiber bending stiffness
  2. Mixed finite elements to reduce locking effects
  3. Higher-order energy-momentum conserving time integrator for stable and accurate dynamic simulations

Continuum mechanics [Asmanoglo 17]

- Deformation gradient $\boldsymbol{F}$ and right Cauchy-Green tensor $\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F}$
- Fiber direction $\boldsymbol{a}_0$ and structural tensor $\boldsymbol{M} = \boldsymbol{a}_0^T \otimes \boldsymbol{a}_0$
- Gradient of fiber direction $\boldsymbol{G} = \nabla[\boldsymbol{a}_0 \cdot \boldsymbol{F}^T] = \boldsymbol{a}_0 \cdot \nabla[\boldsymbol{F}^T]$ for $\boldsymbol{a}_0 = const.$
- Referential representation $\boldsymbol{\Lambda}(\boldsymbol{F}, \nabla[\boldsymbol{F}]) = \boldsymbol{F}^T \cdot \boldsymbol{a}_0 \cdot \nabla[\boldsymbol{F}]^{T12}$

Strain energy function [Asmanoglo 17]

$$\Psi = \Psi_{\mathrm{M}}^{\mathrm{ISO}}(\boldsymbol{C}, J) + \Psi_{\mathrm{M}}^{\mathrm{VOL}}(J) + \Psi_{\mathrm{F}}^{\mathrm{ANI}}(\boldsymbol{C}, \boldsymbol{a}_0) + \Psi_{\mathrm{F}}^{\mathrm{HOG}}(\boldsymbol{C}, \boldsymbol{\Lambda})$$

- Isotropic part Ψ^{ISO} and Volumetric part Ψ^{VOL} of the matrix
  $$I_1(\boldsymbol{C}) = \boldsymbol{C} : \boldsymbol{I} \quad I_3(\boldsymbol{C}) = \det[\boldsymbol{C}] = J^2 \quad (I_2(\boldsymbol{C}) = \mathrm{cof}[\boldsymbol{C}] : \boldsymbol{I})$$
- Anisotropic part Ψ^{ANI} regarding to the stretching of the fiber
  $$I_4(\boldsymbol{C}, \boldsymbol{a}_0) = \boldsymbol{a}_0 \cdot \boldsymbol{C} \cdot \boldsymbol{a}_0 \quad \left(I_5(\boldsymbol{C}, \boldsymbol{a}_0) = \boldsymbol{a}_0 \cdot \boldsymbol{C}^2 \cdot \boldsymbol{a}_0\right)$$
- Anisotropic part Ψ^{HOG} regarding to the bending of the fiber
  $$I_6 = \boldsymbol{\kappa}_0 \cdot \boldsymbol{\kappa}_0 \quad I_7 = \boldsymbol{\kappa}_0 \cdot \boldsymbol{C} \cdot \boldsymbol{\kappa}_0 \quad \boldsymbol{\kappa}_0 = \boldsymbol{\Lambda} \cdot \boldsymbol{a}_0$$

Total internal energy Π^{int} for the mixed principle of virtual power [Groß 18]

$$\Pi^{\text{int}} = \int_{\mathcal{B}_0} \frac{1}{2}\boldsymbol{S} : (\boldsymbol{C} - \tilde{\boldsymbol{C}})\mathrm{d}V + \int_{\mathcal{B}_0} \tilde{\boldsymbol{S}} : \dot{\tilde{\boldsymbol{C}}}\mathrm{d}V + \int_{\mathcal{B}_0} p\,(J(\tilde{\boldsymbol{C}}) - \tilde{J})\mathrm{d}V + \int_{\mathcal{B}_0} \tilde{p}\dot{\tilde{J}}\mathrm{d}V$$
$$+ \int_{\mathcal{B}_0} \frac{1}{2}\boldsymbol{S}_A : (\tilde{\boldsymbol{C}} - \tilde{\boldsymbol{C}}_A)\mathrm{d}V + \int_{\mathcal{B}_0} \tilde{\boldsymbol{S}}_A : \dot{\tilde{\boldsymbol{C}}}_A\mathrm{d}V + \int_{\mathcal{B}_0} \Psi\mathrm{d}V$$

$$\boxed{+ \int_{\mathcal{B}_0} \boldsymbol{H} : (\boldsymbol{\Lambda}(\tilde{\boldsymbol{F}}, \tilde{\boldsymbol{\Gamma}}) - \tilde{\boldsymbol{\Lambda}})\mathrm{d}V + \int_{\mathcal{B}_0} \tilde{\boldsymbol{H}} : \dot{\tilde{\boldsymbol{\Lambda}}}\mathrm{d}V + \int_{\mathcal{B}_0} \boldsymbol{P} : (\boldsymbol{F} - \tilde{\boldsymbol{F}})\mathrm{d}V + \int_{\mathcal{B}_0} \boldsymbol{B} \odot_3 (\nabla(\tilde{\boldsymbol{F}}) - \tilde{\boldsymbol{\Gamma}})\mathrm{d}V}$$

$$\Psi = \left[\Psi^{\text{ISO}}_{\text{M}}(\tilde{\boldsymbol{C}}, \tilde{J}) + \Psi^{\text{VOL}}_{\text{M}}(\tilde{J}) + \Psi^{\text{ANI}}_{\text{F}}(\tilde{\boldsymbol{C}}_A, \boldsymbol{M}) + \Psi^{\text{HOG}}_{\text{F}}(\tilde{\boldsymbol{C}}_A, \boldsymbol{\Lambda})\right]$$

- Independent mixed field $\tilde{\boldsymbol{C}}$ and corresponding Lagrangian multiplier $\boldsymbol{S}$
- Superimposed stress tensor $\tilde{\boldsymbol{S}}$ to derive energy–momentum scheme [Groß 18]
- Independent mixed fields $\tilde{\boldsymbol{F}}$ and $\tilde{\boldsymbol{\Gamma}}$ for $\boldsymbol{F}$ and $\nabla(\boldsymbol{F})$ $\Big(\boldsymbol{\Lambda}(\boldsymbol{F}, \nabla(\boldsymbol{F})) \rightarrow \boldsymbol{\Lambda}(\tilde{\boldsymbol{F}}, \tilde{\boldsymbol{\Gamma}})\Big)$
- Independent mixed field $\tilde{\boldsymbol{\Lambda}}$ for $\boldsymbol{\Lambda}(\tilde{\boldsymbol{F}}, \tilde{\boldsymbol{\Gamma}})$ and corresponding superimposed field $\tilde{\boldsymbol{H}}$ for energy–momentum scheme
- Independent mixed fields for the volumetric dilatation $\tilde{J}$ and $\tilde{\boldsymbol{C}}_A$ for the anisotropic part of the strain energy function [Simo 85] [Schr 16]

Superimposed fields

[Groß 18] [Groß 20]

$$\tilde{\boldsymbol{S}} = \frac{\tilde{\Psi}(1) - \tilde{\Psi}(0) - \int \frac{\partial \tilde{\Psi}}{\partial \tilde{\boldsymbol{C}}} : \dot{\tilde{\boldsymbol{C}}}}{\dot{\tilde{\boldsymbol{C}}} : \dot{\tilde{\boldsymbol{C}}}} \dot{\tilde{\boldsymbol{C}}} \qquad \tilde{p} = \frac{\tilde{\Psi}(1) - \tilde{\Psi}(0) - \int \frac{\partial \tilde{\Psi}}{\partial \tilde{J}} \dot{\tilde{J}}}{\dot{\tilde{J}} \dot{\tilde{J}}} \dot{\tilde{J}}$$

$$\tilde{\boldsymbol{S}}_A = \frac{\tilde{\Psi}(1) - \tilde{\Psi}(0) - \int \frac{\partial \tilde{\Psi}}{\partial \tilde{\boldsymbol{C}}_A} : \dot{\tilde{\boldsymbol{C}}}_A}{\dot{\tilde{\boldsymbol{C}}}_A : \dot{\tilde{\boldsymbol{C}}}_A} \dot{\tilde{\boldsymbol{C}}}_A \qquad \tilde{\boldsymbol{H}} = \frac{\tilde{\Psi}(1) - \tilde{\Psi}(0) - \int \frac{\partial \tilde{\Psi}}{\partial \tilde{\boldsymbol{\Lambda}}} : \dot{\tilde{\boldsymbol{\Lambda}}}}{\dot{\tilde{\boldsymbol{\Lambda}}} : \dot{\tilde{\boldsymbol{\Lambda}}}} \dot{\tilde{\boldsymbol{\Lambda}}}$$

Mixed principle of virtual power

- Total power functional

$$\dot{T} + \dot{\Pi}^{\text{ext}} + \dot{\Pi}^{\text{int}} = \dot{\mathcal{H}}(\dot{\boldsymbol{q}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}, \boldsymbol{\lambda}_q, \dot{\tilde{\boldsymbol{C}}}, \dot{\tilde{J}}, \dot{\tilde{\boldsymbol{C}}}_A, \dot{\tilde{\boldsymbol{F}}}, \dot{\tilde{\boldsymbol{\Gamma}}}, \dot{\tilde{\boldsymbol{\Lambda}}}, \boldsymbol{S}, p, \boldsymbol{S}_A, \boldsymbol{P}, \boldsymbol{B}, \boldsymbol{H})$$

- Kinetic power functional

$$\dot{T}(\dot{\boldsymbol{q}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}) = \int_{\mathcal{B}_0} (\rho_0 \boldsymbol{v} - \boldsymbol{p}) \cdot \dot{\boldsymbol{v}} \mathrm{d}V + \int_{\mathcal{B}_0} \dot{\boldsymbol{p}} \cdot (\dot{\boldsymbol{q}} - \boldsymbol{v}) \mathrm{d}V + \int_{\mathcal{B}_0} \boldsymbol{p} \cdot \ddot{\boldsymbol{q}} \mathrm{d}V$$

- External power functional

$$\dot{\Pi}^{\text{ext}}(\dot{\boldsymbol{q}}, \boldsymbol{\lambda}_q) = -\int_{\partial \mathcal{B}_0} \boldsymbol{t} \cdot \dot{\boldsymbol{q}} \mathrm{d}A - \int_{\partial \mathcal{B}_0} \boldsymbol{\lambda}_q \cdot (\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}^{\text{ref}}) \mathrm{d}A$$

Variation with respect to the variables in the arguments of the total energy balance $\int_T \delta_* \dot{\mathcal{H}} \mathrm{d}t = \int_T [\delta_* \dot{T} + \delta_* \dot{\Pi}^{\mathrm{ext}} + \delta_* \dot{\Pi}^{\mathrm{int}}] \mathrm{d}t = 0$

Weak forms

$$\boxed{\int_T \int_{\mathcal{B}_0} \left[\boldsymbol{S} : \frac{1}{2} \frac{\partial \dot{\boldsymbol{C}}}{\partial \dot{\boldsymbol{q}}} + \boldsymbol{P} : \frac{\partial \dot{\boldsymbol{F}}}{\partial \dot{\boldsymbol{q}}} - \dot{\boldsymbol{p}} \right] \cdot \delta_* \dot{\boldsymbol{q}} \mathrm{d}V \mathrm{d}t = 0} \quad \int_T \int_{\partial \mathcal{B}_0} \left[-\boldsymbol{t} - \boldsymbol{\lambda}_q \right] \cdot \delta_* \dot{\boldsymbol{q}} \mathrm{d}A \mathrm{d}t = 0$$

$$\boxed{\int_T \int_{\mathcal{B}_0} \left[\frac{1}{\rho_0} \boldsymbol{p} - \dot{\boldsymbol{q}} \right] \cdot \delta_* \dot{\boldsymbol{v}} \mathrm{d}V \mathrm{d}t = 0} \quad \int_T \int_{\partial \mathcal{B}_0} \left[\dot{\tilde{\boldsymbol{q}}} - \dot{\boldsymbol{q}}^{\mathsf{ref}}(t) \right] \cdot \delta_* \boldsymbol{\lambda}_q \mathrm{d}A \mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} \left[\frac{1}{2} \boldsymbol{S} - \left(\frac{\partial \Psi}{\partial \tilde{\boldsymbol{C}}} + \tilde{\boldsymbol{S}} + \frac{p}{2J} \mathrm{cof}[\boldsymbol{C}] + \frac{1}{2} \boldsymbol{S}_A \right) \right] : \delta_* \dot{\tilde{\boldsymbol{C}}} \mathrm{d}V \mathrm{d}t = 0 \quad \int_T \int_{\mathcal{B}_0} \left[p - \left[\frac{\partial \Psi}{\partial \tilde{J}} + \tilde{p} \right] \right] \delta_* \dot{\tilde{J}} \mathrm{d}V \mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} \left[\frac{1}{2} \boldsymbol{S}_A - \boxed{\left(\frac{\partial \Psi}{\partial \tilde{\boldsymbol{C}}_A} + \tilde{\boldsymbol{S}}_A \right)} \right] : \delta_* \dot{\tilde{\boldsymbol{C}}}_A \mathrm{d}V \mathrm{d}t = 0 \quad \int_T \int_{\mathcal{B}_0} \left[\boldsymbol{P} - \boxed{\left(\boldsymbol{H} : \frac{\partial \dot{\boldsymbol{\Lambda}}}{\partial \dot{\tilde{\boldsymbol{F}}}} + \boldsymbol{B} : \frac{\partial \nabla \dot{\tilde{\boldsymbol{F}}}}{\partial \dot{\tilde{\boldsymbol{F}}}} \right)} \right] : \delta_* \dot{\tilde{\boldsymbol{F}}} \mathrm{d}V \mathrm{d}t$$

$$\int_T \int_{\mathcal{B}_0} \left[\boldsymbol{B} - \boldsymbol{H} : \frac{\partial \dot{\boldsymbol{\Lambda}}}{\partial \dot{\tilde{\boldsymbol{\Gamma}}}} \right] : \delta_* \dot{\tilde{\boldsymbol{\Gamma}}} \mathrm{d}V \mathrm{d}t = 0 \quad \int_T \int_{\mathcal{B}_0} \left[\boldsymbol{H} - \left[\frac{\partial \Psi}{\partial \tilde{\Lambda}} + \tilde{\boldsymbol{H}} \right] \right] \delta_* \dot{\tilde{\Lambda}} \mathrm{d}V \mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} \frac{1}{2} \left[\dot{\tilde{\boldsymbol{C}}} - \dot{\boldsymbol{C}} \right] : \delta_* \boldsymbol{S} \mathrm{d}V \mathrm{d}t = 0 \quad \int_T \int_{\mathcal{B}_0} \left[\dot{\tilde{J}} - \dot{J} \right] \delta_* p \mathrm{d}V \mathrm{d}t = 0 \quad \int_T \int_{\mathcal{B}_0} \frac{1}{2} \left[\dot{\tilde{\boldsymbol{C}}}_A - \dot{\tilde{\boldsymbol{C}}} \right] : \delta_* \boldsymbol{S}_A \mathrm{d}V \mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} \left[\dot{\tilde{\boldsymbol{F}}} - \dot{\boldsymbol{F}} \right] : \delta_* \boldsymbol{P} \mathrm{d}V \mathrm{d}t = 0 \quad \int_T \int_{\mathcal{B}_0} \left[\nabla(\dot{\tilde{\boldsymbol{F}}}) - \dot{\tilde{\boldsymbol{\Gamma}}} \right] \odot_3 \delta_* \boldsymbol{B} \mathrm{d}V \mathrm{d}t = 0 \quad \int_T \int_{\mathcal{B}_0} \left[\boldsymbol{\Lambda} - \tilde{\boldsymbol{\Lambda}} \right] : \delta_* \boldsymbol{H} \mathrm{d}V \mathrm{d}t = 0$$

Approximation — k = Polynomial degree in time

- Discretization in space and time
- Lagrangian shape functions in space (N) [Wrig 08] [Bart 18]
  - Independent approximation of the different mixed fields
  - Lagrangian multiplier approximated equally as the corresponding mixed fields e.g. $\tilde{\boldsymbol{C}}_A$ & $\boldsymbol{S}_A$ or $\tilde{J}$ & p
- Lagrangian shape functions in time (M,M',$\tilde{M}$) [Bets 01] [Groß 18]

$$M_i(\alpha) = \prod_{\substack{j=1 \\ j\neq i}}^{k+1} \frac{\alpha - \alpha_j}{\alpha_i - \alpha_j},\; 1 \leq i \leq k+1 \qquad \tilde{M}_i(\alpha) = \prod_{\substack{j=1 \\ j\neq i}}^{k} \frac{\alpha - \alpha_j}{\alpha_i - \alpha_j}, 1 \leq i \leq k$$

- Time rate variables and mixed fields ($\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{p}, \tilde{\boldsymbol{C}}, \tilde{J}, \tilde{\boldsymbol{C}}_A, \tilde{\boldsymbol{F}}, \tilde{\boldsymbol{\Gamma}}, \tilde{\boldsymbol{\Lambda}}$)

$$(\bullet)^{e,h} = \sum_{I=1}^{k+1} \sum_{A=1}^{n_{no}} M_I(\alpha) \mathrm{N}^A(\boldsymbol{\xi})(\bullet)_I^{eA} \qquad \left((\dot{\bullet})^{e,h} = \frac{1}{h_n} \sum_{I=1}^{k+1} \sum_{A=1}^{n_{no}} M_I'(\alpha) \mathrm{N}^A(\boldsymbol{\xi})(\bullet)_I^{eA} \right)$$

- Lagrangian multiplier and variation fields ($\boldsymbol{\lambda}_q, \boldsymbol{S}, p, \boldsymbol{S}_A, \boldsymbol{P}, \boldsymbol{B}, \boldsymbol{H}, \delta_* \bullet$)

$$(\bullet)^{e,h} = \sum_{I=1}^{k} \sum_{A=1}^{n_{no_\mathrm{u}}} \tilde{M}_I \mathrm{N}^A (\bullet)_I^{eA}$$

Implementation

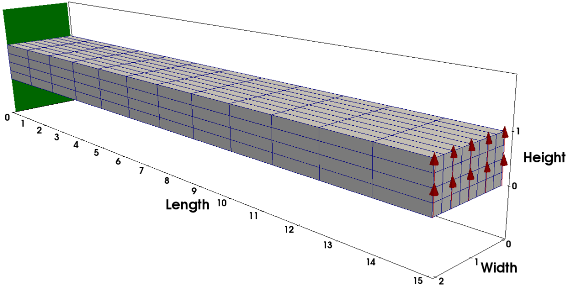
1. Staggered solution of mixed fields at element level, e.g.
   - 1. Solve values of $\tilde{J}$ with equation $\int_T \int_{\mathcal{B}_0} \left[\dot{\tilde{J}} - \dot{J}\right] \delta p \mathrm{d}V \mathrm{d}t = 0$
   - 2. Solve values of p with $\tilde{J}$ and equation $\int_T \int_{\mathcal{B}_0} \left[p - \left[\frac{\partial \Psi}{\partial \tilde{J}} + \tilde{p}\right]\right] \delta \dot{\tilde{J}} \mathrm{d}V \mathrm{d}t = 0$
2. Eliminate $\mathbf{p}$

$$\int_T \int_{\mathcal{B}_0} \left[\boldsymbol{S} : \frac{1}{2} \frac{\partial \dot{\boldsymbol{C}}}{\partial \dot{\boldsymbol{q}}} + \boldsymbol{P} : \frac{\partial \dot{\boldsymbol{F}}}{\partial \dot{\boldsymbol{q}}} - \dot{\boldsymbol{p}}\right] \cdot \delta \dot{\boldsymbol{q}} \mathrm{d}V \mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} \left[\frac{1}{\rho_0} \boldsymbol{p} - \dot{\boldsymbol{q}}\right] \cdot \delta \dot{\boldsymbol{v}} \mathrm{d}V \mathrm{d}t = 0$$

   - Rearrange first equation to $\boldsymbol{p}$ and insert in second equation
3. Condensate at element level to pure displacement temperature form
   - Schur complements $\Rightarrow$ All inverted matrices are constant

Angular momentum balance law

$$\boldsymbol{\mathcal{J}}_{n+1} - \boldsymbol{\mathcal{J}}_n = \int_{t_n}^{t_n+1} \int_{\mathcal{B}_0} \left[\left(\boldsymbol{H} : \frac{\partial \dot{\boldsymbol{\Lambda}}}{\partial \dot{\tilde{\boldsymbol{F}}}} + \boldsymbol{B} : \frac{\partial \nabla \dot{\tilde{\boldsymbol{F}}}}{\partial \dot{\tilde{\boldsymbol{F}}}}\right) \times \tilde{\boldsymbol{F}}\right] \mathrm{d}V \mathrm{d}t + \int_{t_n}^{t_n+1} \int_{\partial \mathcal{B}_0} [\boldsymbol{q} \times \boldsymbol{\lambda}_q] \mathrm{d}V \mathrm{d}t + \int_{t_n}^{t_n+1} \int_{\partial \mathcal{B}_0} [\boldsymbol{q} \times \boldsymbol{t}] \mathrm{d}V \mathrm{d}t$$

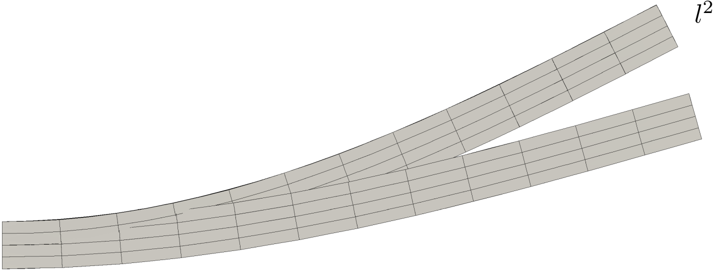


- Matrix part and fiber stretch part taken from [Dal 17]

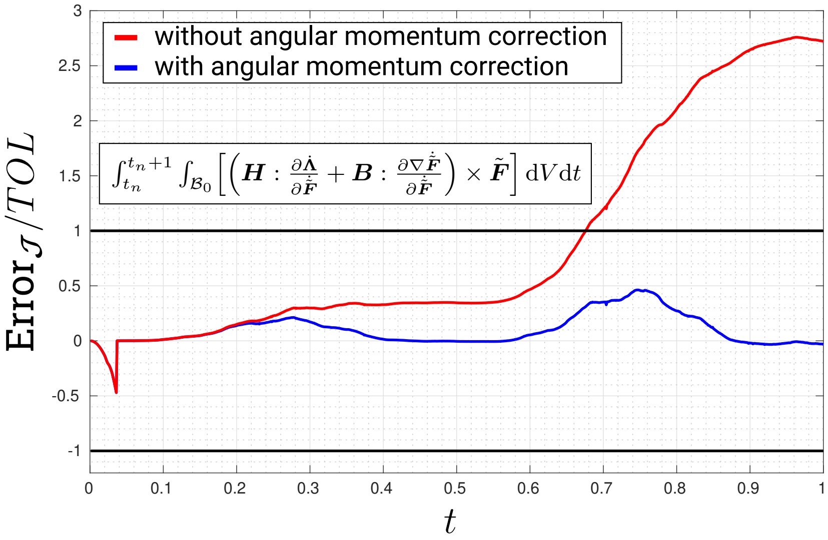
  $\Psi^{\mathrm{ISO}} = \frac{\epsilon_1}{2}(\mathrm{tr}[\boldsymbol{C}] - 3 - 2\ln(J)) \quad \Psi^{\mathrm{VOL}} = \frac{\epsilon_2}{2}\ln(J)^2 \quad \Psi^{\mathrm{ANI}} = \frac{\epsilon_3}{2}(\mathrm{tr}[\boldsymbol{C}\boldsymbol{M}] - 1)^2$
- Anisotropic direction $(\boldsymbol{a}_0)^T = [1\ 0\ 0]$
- $T = 1.0$, $h_n = 0.001$, $k = 1$ and $\hat{p} = 3.5e6 f_{\hat{p}}$ with $f_{\hat{p}} = \frac{1}{T}t$
- Quadratic serendipity mesh (20 nodes) with $n_{el} = 384$
- Linear approximation for $\tilde{J}$ and constant approximation for $\tilde{\boldsymbol{C}}_A$
- Introducing length scale l^2 with $c = \epsilon_1 l^2$ for material parameter of Ψ^{HOG}

Deflection $\epsilon_1 = 4.04e9$ $\epsilon_2 = 5.2e9$ $\epsilon_3 = 0$

$\Psi^{\mathrm{HOG}} = \epsilon_1 l^2 I_6$ [Asmanoglo 17]



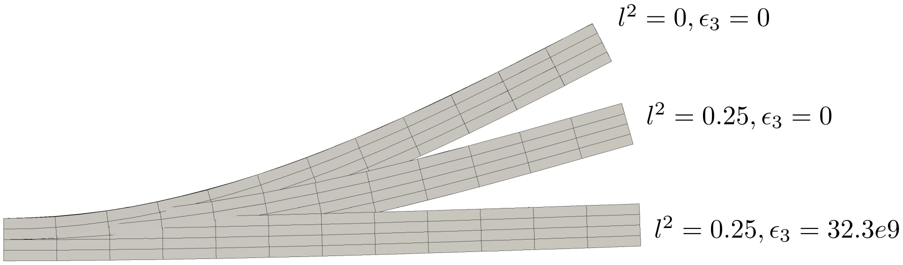
Angular momentum conservation $\epsilon_1 = 4.04e9 \quad \epsilon_2 = 5.2e9 \quad \epsilon_3 = 0 \quad l^2 = 0.25$



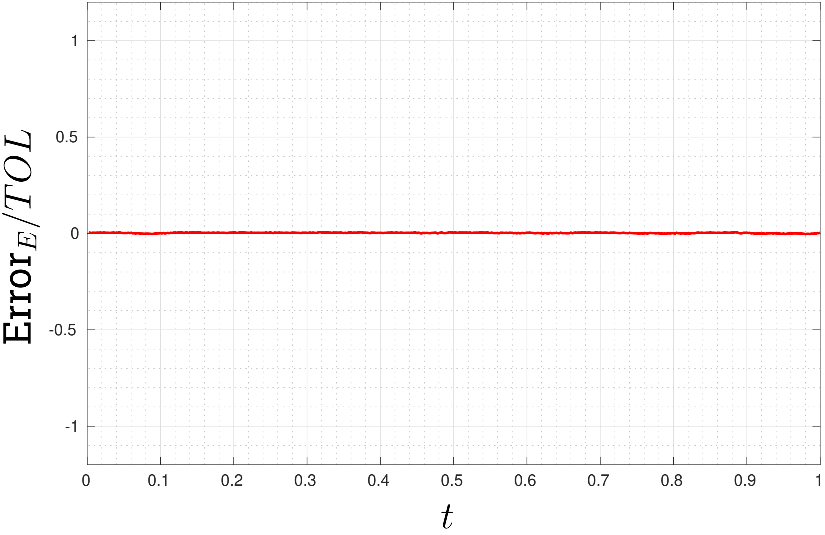
Numerical example

Deflection $\epsilon_1 = 4.04e9 \quad \epsilon_2 = 5.2e9$

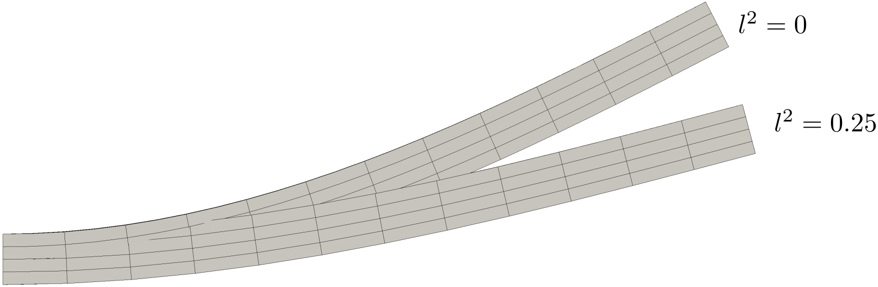
$\Psi^{\mathrm{HOG}} = \epsilon_1 l^2 I_6$



Energy conservation $\epsilon_1 = 4.04e9 \quad \epsilon_2 = 5.2e9 \quad \epsilon_3 = 32.3e9 \quad l^2 = 0.25$

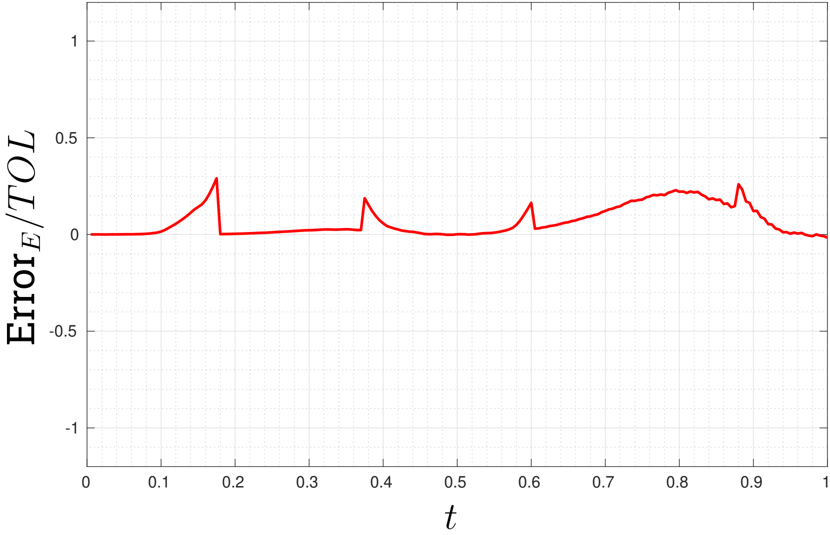


$\Psi^{\mathrm{HOG}} = \epsilon_1 l^2 (I_7 - I_6)^2$ [Asmanoglo 17]



Numerical example

Energy conservation $\epsilon_1 = 4.04e9 \quad \epsilon_2 = 5.2e9 \quad \epsilon_3 = 32.3e9 \quad l^2 = 0.25$



Conclusion

- Motivation:
  - Dynamic simulations of fiber-reinforced materials in light-weight structures
- Important results:
  - Higher-order gradients capture the fiber-bending stiffness
  - Higher-order energy-momentum time integrators conserve total energy
- Outlook:
  - Extension of the material formulation to a thermo-viscoelastic formulation
  - Investigation of locking effects

Literature I

T. Asmanoglo and A. Menzel.
"A multi-field finite element approach for the modelling of fibre-reinforced composites with fibre-bending stiffness".
Computer Methods in Applied Mechanics and Engineering, Vol. 317, pp. 1037–1067, 2017.

M. Bartelt, J. Dietzsch, and M. Groß.
"Efficient implementation of energy conservation for higher order finite elements with variational integrators".
Mathematics and Computers in Simulation, Vol. 150, pp. 83 – 121, 2018.

P. Betsch and P. Steinmann.
"Conservation properties of a time FE method—part II: Time-stepping schemes for non-linear elastodynamics".
International Journal for Numerical Methods in Engineering, Vol. 50, No. 8, pp. 1931–1955, 2001.

Literature II

H. Dal, O. Gültekin, F. Aksu Denli, and G. A. Holzapfel.
"Phase-Field Models for the Failure of Anisotropic Continua".
PAMM, Vol. 17, No. 1, pp. 91–94, 2017.

A. De Coninck, B. De Baets, D. Kourounis, F. Verbosio, O. Schenk, S. Maenhout, and J. Fostier.
"Needles: Toward Large-Scale Genomic Prediction with Marker-by-Environment Interaction".
Vol. 203, No. 1, pp. 543–555, 2016.

S. Engblom and D. Lukarski.
"Fast Matlab compatible sparse assembly on multicore computers".
Parallel Computing, Vol. 56, pp. 1–17, 2016.

Literature III

M. Groß, J. Dietzsch, and M. Bartelt.
"Thermo-viscoelastic fiber-reinforced continua simulated by variational-based higher-order energy-momentum schemes".
PAMM, Vol. 18, No. 1, p. e201800003, 2018.

D. Kourounis, A. Fuchs, and O. Schenk.
"Towards the Next Generation of Multiperiod Optimal Power Flow Solvers".
IEEE Transactions on Power Systems, Vol. PP, No. 99, pp. 1–10, 2018.

J. Schröder, N. Viebahn, D. Balzani, and P. Wriggers.
"A novel mixed finite element for finite anisotropic elasticity; the SKA-element Simplified Kinematics for Anisotropy".
Computer Methods in Applied Mechanics and Engineering, Vol. 310, No. , pp. 475 – 494, 2016.

Literature IV

J. Simo, R. Taylor, and K. Pister.
"Variational and projection methods for the volume constraint in finite deformation elasto-plasticity".
Computer Methods in Applied Mechanics and Engineering, Vol. 51, No. 1, pp. 177–208, 1985.

F. Verbosio, A. D. Coninck, D. Kourounis, and O. Schenk.
"Enhancing the scalability of selected inversion factorization algorithms in genomic prediction".
Journal of Computational Science, Vol. 22, No. Supplement C, pp. 99 – 108, 2017.

P. Wriggers.
Nonlinear finite element methods .
Vol. , Springer, Berlin, 2008.