Mixed finite element formulations and energy-momentum time integrators for thermo-mechanically coupled fiber-reinforced continua

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Motivation

- Dynamic simulations of fiber-reinforced materials in lightweight structures

- Keywords:
  - Anisotropic material behavior
  - Nearly-incompressible material behavior
  - Thermo-viscoelastic material behavior

- Solution strategy:
  1. Mixed finite elements to reduce locking effects
  2. Extension to a thermo-mechanical coupling
  3. Mixed finite elements as Viscous internal variable
  4. Higher-order energy-momentum conserving time integrator for stable and accurate dynamic simulations [eg(k)]

Continuum mechanics and material formulation

- Strain energy function (thermo-viscoelastic matrix part Ψ_{M} and n_{F} thermoelastic fiber parts Ψ_{F_i})

$$\Psi(\boldsymbol{C}, \Theta) = \Psi_{\mathrm{M}}(\boldsymbol{C}, \Theta, \boldsymbol{C}_v) + \sum_{i=1}^{n_{\mathrm{F}}} \Psi_{\mathrm{F}_i}(\boldsymbol{C}, \Theta, \boldsymbol{M}_i) \qquad \boldsymbol{M}_i = (\boldsymbol{a}_i^0)^T \otimes \boldsymbol{a}_i^0$$

$\boldsymbol{C}$ - right Cauchy-Green tensor, $\boldsymbol{C}_v$ - viscous right Cauchy-Green tensor, Θ - absolute temperature, $\boldsymbol{a}_i^0$ - fiber direction

- Components and specific dependencies ($J(\boldsymbol{C}) = \sqrt{\det[\boldsymbol{C}]}$, $\boldsymbol{\Lambda} = \boldsymbol{C}\boldsymbol{C}_v^{-1}$)

$$\Psi_{\mathrm{M}}(\boldsymbol{C}, \Theta) = \Psi_{\mathrm{M}}^{\mathrm{iso}}(\boldsymbol{C}, \mathrm{cof}[\boldsymbol{C}], J) + \Psi_{\mathrm{M}}^{\mathrm{vol}}(J) + \Psi_{\mathrm{M}}^{\mathrm{cap}}(\Theta) + \Psi_{\mathrm{M}}^{\mathrm{coup}}(\Theta, J) + \Psi_{\mathrm{M}}^{\mathrm{vis}}(\boldsymbol{\Lambda})$$

$$\Psi_{\mathrm{F}_i}(\boldsymbol{C}, \Theta, \boldsymbol{M}_i) = \Psi_{\mathrm{F}_i}^{\mathrm{ela}}(\boldsymbol{C}, \mathrm{cof}[\boldsymbol{C}], J, \boldsymbol{M}_i) + \Psi_{\mathrm{F}_i}^{\mathrm{cap}}(\Theta) + \Psi_{\mathrm{F}_i}^{\mathrm{coup}}(\Theta, \boldsymbol{C}, \boldsymbol{M}_i)$$

- Thermo-mechanical coupling [Groß 18]

$$\Psi_{\mathrm{M}}^{\mathrm{coup}} = -2 n_{\mathrm{dim}} \beta_{\mathrm{M}} (\Theta - \Theta_\infty) J \frac{\partial \Psi_{\mathrm{M}}^{\mathrm{vol}}(J)}{\partial J} \qquad \Psi_{\mathrm{F}_i}^{\mathrm{coup}} = -2 \beta_{\mathrm{F}_i} (\Theta - \Theta_\infty) \sqrt{I_4^i} \frac{\partial \Psi_{\mathrm{F}_i}^{\mathrm{ela}}(I_4^i \ldots)}{\partial I_4^i}$$

Θ_∞ - ambient temperature, $I_4^i = \mathrm{tr}[\boldsymbol{C}\boldsymbol{M}_i]$

- Polyconvex material formulation for the hyperelastic parts

Total internal energy Π^{int} for the mixed principle of virtual power [Groß 18]

$$\Pi^{\text{int}} = \Pi_{\text{HW}} + \int_{\mathcal{B}_0} \frac{1}{2}\boldsymbol{S} : (\boldsymbol{C} - \tilde{\boldsymbol{C}})\mathrm{d}V + \int_{\mathcal{B}_0} \tilde{\boldsymbol{S}} : \tilde{\boldsymbol{C}}\mathrm{d}V + \int_{\mathcal{B}_0} \eta\,(\Theta - \tilde{\Theta})\mathrm{d}V$$

- Independent mixed field $\tilde{\boldsymbol{C}}$ and corresponding Lagrangian multiplier $\boldsymbol{S}$
- Superimposed stress tensor $\tilde{\boldsymbol{S}}$ to derive energy–momentum scheme [Groß 18]
- Assumed temperature field $\tilde{\Theta}$ and the entropy density field η

Mixed elements based on Hu-Washizu functionals Π_{HW} [Simo 85] [Schr 11] [Schr 16]

D DP Co CoSKA CoCoA

$$\begin{aligned}\Pi_{\text{HW}} = {} & \int_{\mathcal{B}_0} \Psi(\dots)\mathrm{d}V + \int_{\mathcal{B}_0} p\,(J(\tilde{\boldsymbol{C}}) - \tilde{J})\mathrm{d}V + \int_{\mathcal{B}_0} \tilde{p}\tilde{J}\mathrm{d}V \\ & + \int_{\mathcal{B}_0} \boldsymbol{B} : (\operatorname{cof}[\tilde{\boldsymbol{C}}] - \tilde{\boldsymbol{H}})\mathrm{d}V + \int_{\mathcal{B}_0} \tilde{\boldsymbol{B}} : \tilde{\boldsymbol{H}}\mathrm{d}V + \int_{\mathcal{B}_0} \frac{1}{2}\boldsymbol{S}_A : (\tilde{\boldsymbol{C}} - \tilde{\boldsymbol{C}}_A)\mathrm{d}V + \int_{\mathcal{B}_0} \tilde{\boldsymbol{S}}_A : \tilde{\boldsymbol{C}}_A\mathrm{d}V \\ & + \int_{\mathcal{B}_0} \boldsymbol{B}_A : (\operatorname{cof}[\tilde{\boldsymbol{C}}] - \tilde{\boldsymbol{H}}_A)\mathrm{d}V + \int_{\mathcal{B}_0} \tilde{\boldsymbol{B}}_A : \tilde{\boldsymbol{H}}_A\mathrm{d}V + \int_{\mathcal{B}_0} p_A(J(\tilde{\boldsymbol{C}}) - \tilde{J}_A)\mathrm{d}V + \int_{\mathcal{B}_0} \tilde{p}_A\tilde{J}_A\end{aligned}$$

$$\Psi_{\text{M}}(\dots) = \Psi_{\text{M}}^{\text{iso}}(\tilde{\boldsymbol{C}}, \tilde{\boldsymbol{H}}, \tilde{J}) + \Psi_{\text{M}}^{\text{vol}}(\tilde{J}) + \Psi_{\text{M}}^{\text{cap}}(\Theta) + \Psi_{\text{M}}^{\text{coup}}(\Theta, \tilde{J}) + \Psi_{\text{M}}^{\text{vis}}(\boldsymbol{\Lambda})$$

$$\Psi_{\text{F}_i}(\dots) = \Psi_{\text{F}_i}^{\text{ela}}(\tilde{\boldsymbol{C}}_A, \tilde{\boldsymbol{H}}_A, \tilde{J}_A, \boldsymbol{M}_i) + \Psi_{\text{F}_i}^{\text{cap}}(\Theta) + \Psi_{\text{F}_i}^{\text{coup}}(\Theta, \tilde{\boldsymbol{C}}_A, \boldsymbol{M}_i)$$

Superimposed fields

- Dependencies of the strain energy functions

$$\Psi_{\mathrm{M}}(\dots) = \Psi_{\mathrm{M}}^{\mathrm{iso}}(\tilde{\boldsymbol{C}}, \tilde{\boldsymbol{H}}, \tilde{J}) + \Psi_{\mathrm{M}}^{\mathrm{vol}}(\tilde{J}) + \Psi_{\mathrm{M}}^{\mathrm{cap}}(\Theta) + \Psi_{\mathrm{M}}^{\mathrm{coup}}(\Theta, \tilde{J}) + \Psi_{\mathrm{M}}^{\mathrm{vis}}(\boldsymbol{\Lambda})$$

$$\Psi_{\mathrm{F}_i}(\dots) = \Psi_{\mathrm{F}_i}^{\mathrm{ela}}(\tilde{\boldsymbol{C}}_A, \tilde{\boldsymbol{H}}_A, \tilde{J}_A, \boldsymbol{M}_i) + \Psi_{\mathrm{F}_i}^{\mathrm{cap}}(\Theta) + \Psi_{\mathrm{F}_i}^{\mathrm{coup}}(\Theta, \tilde{\boldsymbol{C}}_A, \boldsymbol{M}_i)$$

- Superimposed stress tensor $\tilde{\boldsymbol{S}}$ with $\tilde{\Psi} = \Psi_{\mathrm{M}}^{\mathrm{iso}}(\tilde{\boldsymbol{C}}) + \Psi_{\mathrm{M}}^{\mathrm{cap}}(\Theta) + \Psi_{\mathrm{M}}^{\mathrm{vis}}(\boldsymbol{\Lambda}) + \sum_{i=1}^{n_{\mathrm{F}}}[\Psi_{\mathrm{F}_i}^{\mathrm{cap}}(\Theta)]$

$$\tilde{\boldsymbol{S}} = \frac{\tilde{\Psi}(1) - \tilde{\Psi}(0) - \int \frac{\partial \tilde{\Psi}}{\partial \tilde{\boldsymbol{C}}} : \dot{\tilde{\boldsymbol{C}}} - \int \frac{\partial \tilde{\Psi}}{\partial \Theta} \dot{\Theta} - \int \frac{\partial \tilde{\Psi}}{\partial \boldsymbol{C}_v} : \dot{\boldsymbol{C}}_v}{\dot{\tilde{\boldsymbol{C}}} : \dot{\tilde{\boldsymbol{C}}}} \dot{\tilde{\boldsymbol{C}}}$$

- Superimposed pressure $\tilde{p}$ with $\tilde{\Psi} = \Psi_{\mathrm{M}}^{\mathrm{iso}}(\tilde{J}) + \Psi_{\mathrm{M}}^{\mathrm{vol}}(\tilde{J}) + \Psi_{\mathrm{M}}^{\mathrm{coup}}(\Theta, \tilde{J})$

$$\tilde{p} = \frac{\tilde{\Psi}(1) - \tilde{\Psi}(0) - \int \frac{\partial \tilde{\Psi}}{\partial \tilde{J}} \dot{\tilde{J}} - \int \frac{\partial \tilde{\Psi}}{\partial \Theta} \dot{\Theta}}{\dot{\tilde{J}} \dot{\tilde{J}}} \dot{\tilde{J}}$$

- Superimposed stress tensor $\tilde{\boldsymbol{S}}_A$ with $\tilde{\Psi} = \sum_{i=1}^{n_{\mathrm{F}}}[\Psi_{\mathrm{F}_i}^{\mathrm{ela}}(\tilde{\boldsymbol{C}}_A, \boldsymbol{M}_i) + \Psi_{\mathrm{F}_i}^{\mathrm{coup}}(\Theta, \tilde{\boldsymbol{C}}_A, \boldsymbol{M}_i)$

$$\tilde{\boldsymbol{S}}_A = \frac{\tilde{\Psi}(1) - \tilde{\Psi}(0) - \int \frac{\partial \tilde{\Psi}}{\partial \tilde{\boldsymbol{C}}_A} : \dot{\tilde{\boldsymbol{C}}}_A - \int \frac{\partial \tilde{\Psi}}{\partial \Theta} \dot{\Theta}}{\dot{\tilde{\boldsymbol{C}}}_A : \dot{\tilde{\boldsymbol{C}}}_A} \dot{\tilde{\boldsymbol{C}}}_A$$

- The remaining superimposed fields be designed in the same manner

Mixed principle of virtual power of CoCoA Element

- Total energy balance law in functional form $\dot{\mathcal{H}} = \dot{T} + \dot{\Pi}^{\mathrm{ext}} + \dot{\Pi}^{\mathrm{int}}$

$$\dot{\mathcal{H}}(\dot{\boldsymbol{q}}, \boldsymbol{\lambda}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}, \tilde{\Theta}, \dot{\Theta}, \dot{\eta}, \dot{\boldsymbol{C}}_v, \dot{\tilde{\boldsymbol{C}}}, \dot{\tilde{\boldsymbol{C}}}_A, \dot{\tilde{\boldsymbol{H}}}, \dot{\tilde{\boldsymbol{H}}}_A, \dot{\tilde{J}}, \dot{\tilde{J}}_A, \boldsymbol{S}, \boldsymbol{S}_A, \boldsymbol{B}, \boldsymbol{B}_A, p, p_A) = 0$$

- Kinetic power functional

$$\dot{T}(\dot{\boldsymbol{q}}, \dot{\boldsymbol{v}}, \dot{\boldsymbol{p}}) = \int_{\mathcal{B}_0} (\rho_0 \boldsymbol{v} - \boldsymbol{p}) \cdot \dot{\boldsymbol{v}} \mathrm{d}V + \int_{\mathcal{B}_0} \dot{\boldsymbol{p}} \cdot (\dot{\boldsymbol{q}} - \boldsymbol{v}) \mathrm{d}V + \int_{\mathcal{B}_0} \boldsymbol{p} \cdot \ddot{\boldsymbol{q}} \mathrm{d}V$$

- External power functional

$$\dot{\Pi}^{\mathrm{ext}}(\dot{\boldsymbol{q}}, \boldsymbol{\lambda}, \tilde{\Theta}, \dot{\boldsymbol{C}}_v) = -\int_{\partial\mathcal{B}_0} \boldsymbol{t} \cdot \dot{\boldsymbol{q}} \mathrm{d}A - \int_{\partial\mathcal{B}_0} \boldsymbol{\lambda} \cdot (\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}^{\mathrm{ref}}) \mathrm{d}A + \int_{\mathcal{B}_0} \nabla \left(\frac{\tilde{\Theta}}{\Theta} \right) \cdot \boldsymbol{Q} \mathrm{d}V$$

$$+ \int_{\mathcal{B}_0} \frac{\tilde{\Theta}}{\Theta} D^{\mathrm{int}} \mathrm{d}V + \int_{\mathcal{B}_0} \frac{1}{2} \dot{\boldsymbol{C}}_v : \mathbb{V}(\boldsymbol{C}_v) : \dot{\boldsymbol{C}}_v \mathrm{d}V \quad \boldsymbol{Q} = -\left[\sum_{i=1}^{n_{\mathrm{F}}} J \frac{k_{\mathrm{F}_i} - k_{\mathrm{M}}}{\tilde{\boldsymbol{C}}_A : \boldsymbol{M}_i} \boldsymbol{M}_i + k J \tilde{\boldsymbol{C}}^{-1} \right] \nabla\Theta$$

- Viscous dissipation D^{int} and positive-definite viscosity tensor $\mathbb{V}$

$$D^{\mathrm{int}} = \dot{\boldsymbol{C}}_v : \mathbb{V}(\boldsymbol{C}_v) : \dot{\boldsymbol{C}}_v \quad \mathbb{V}(\boldsymbol{C}_v) = \frac{1}{4} \left(V_{\mathrm{vol}} - \frac{V_{\mathrm{dev}}}{n_{\mathrm{dim}}} \right) \boldsymbol{C}_v^{-1} \otimes \boldsymbol{C}_v^{-1} + \frac{V_{\mathrm{dev}}}{4} \mathbb{I}_{\mathrm{s}} : \boldsymbol{C}_v^{-1} \otimes \boldsymbol{C}_v^{-1}$$

- Variation with respect to the variables in the arguments of the total energy balance $\int_T [\delta_* \dot{T} + \delta_* \dot{\Pi}^{\mathrm{ext}} + \delta_* \dot{\Pi}^{\mathrm{int}}] \mathrm{d}t = 0$

Weak formulation of CoCoA Element

$$\int_T\int_{\mathcal{B}_0}[\mathrm{Div}[\boldsymbol{F}\boldsymbol{S}]-\dot{\boldsymbol{p}}]\cdot\delta\dot{\boldsymbol{q}}\,\mathrm{d}V\mathrm{d}t=0 \qquad \int_T\int_{\mathcal{B}_0}\left[\frac{1}{\rho_0}\boldsymbol{p}-\dot{\boldsymbol{q}}\right]\cdot\delta\dot{\boldsymbol{v}}\,\mathrm{d}V\mathrm{d}t=0 \qquad \int_T\int_{\partial\mathcal{B}_0}[-\boldsymbol{t}-\boldsymbol{\lambda}]\cdot\delta\dot{\boldsymbol{q}}\,\mathrm{d}A\mathrm{d}t=0$$

$$\int_T\int_{\partial\mathcal{B}_0}\left[\dot{\boldsymbol{q}}-\dot{\boldsymbol{q}}^{\mathrm{ref}}(t)\right]\cdot\delta\boldsymbol{\lambda}\,\mathrm{d}A\mathrm{d}t=0 \qquad \int_T\int_{\mathcal{B}_0}\left[\Theta-\tilde{\Theta}\right]\delta\dot{\eta}\,\mathrm{d}V\mathrm{d}t=0 \qquad \int_T\int_{\mathcal{B}_0}\left[\eta+\frac{\partial\Psi}{\partial\Theta}\right]\delta\dot{\Theta}\,\mathrm{d}V\mathrm{d}t=0$$

$$\int_T\int_{\mathcal{B}_0}\left[\frac{\mathrm{Div}[\boldsymbol{Q}]}{\Theta}+\frac{D^{\mathrm{int}}}{\Theta}+\dot{\eta}\right]\delta\dot{\tilde{\Theta}}\,\mathrm{d}V\mathrm{d}t=0 \qquad \int_T\int_{\mathcal{B}_0}\left[\frac{\partial\Psi}{\partial\boldsymbol{C}_v}+\dot{\boldsymbol{C}}_v:\mathbb{V}(\boldsymbol{C}_v)\right]:\delta\dot{\boldsymbol{C}}_v\,\mathrm{d}V\mathrm{d}t=0$$

$$\int_T\int_{\mathcal{B}_0}\left[\frac{1}{2}\boldsymbol{S}-\left(\frac{\partial\Psi}{\partial\tilde{\boldsymbol{C}}}+\tilde{\boldsymbol{S}}+\boldsymbol{B}:\mathbb{P}+\frac{p}{2J}\mathrm{cof}[\boldsymbol{C}]+\frac{1}{2}\boldsymbol{S}_A+\boldsymbol{B}_A:\mathbb{P}+\frac{p_A}{2J}\mathrm{cof}[\boldsymbol{C}]\right)\right]:\delta\dot{\tilde{\boldsymbol{C}}}\,\mathrm{d}V\mathrm{d}t=0 \qquad \mathbb{P}=\frac{\partial\mathrm{cof}[\boldsymbol{C}]}{\partial\boldsymbol{C}}$$

$$\int_T\int_{\mathcal{B}_0}\frac{1}{2}\left[\dot{\tilde{\boldsymbol{C}}}_A-\dot{\tilde{\boldsymbol{C}}}\right]:\delta\boldsymbol{S}_A\,\mathrm{d}V\mathrm{d}t=0 \qquad \int_T\int_{\mathcal{B}_0}\left[\frac{1}{2}\boldsymbol{S}_A-\left[\frac{\partial\Psi}{\partial\tilde{\boldsymbol{C}}_A}+\tilde{\boldsymbol{S}}_A\right]\right]:\delta\dot{\tilde{\boldsymbol{C}}}_A\,\mathrm{d}V\mathrm{d}t=0$$

$$\int_T\int_{\mathcal{B}_0}\left[\dot{\tilde{J}}-\dot{J}\right]\delta p\,\mathrm{d}V\mathrm{d}t=0 \qquad \int_T\int_{\mathcal{B}_0}\left[p-\left[\frac{\partial\Psi}{\partial\tilde{J}}+\tilde{p}\right]\right]\delta\dot{\tilde{J}}\,\mathrm{d}V\mathrm{d}t=0 \qquad \int_T\int_{\mathcal{B}_0}\frac{1}{2}\left[\dot{\tilde{\boldsymbol{C}}}-\dot{\boldsymbol{C}}\right]:\delta\boldsymbol{S}\,\mathrm{d}V\mathrm{d}t=0$$

$$\int_T\int_{\mathcal{B}_0}\left[\dot{\tilde{\boldsymbol{H}}}-\mathrm{co}\dot{\mathrm{f}}[\boldsymbol{C}]\right]:\delta\boldsymbol{B}\,\mathrm{d}V\mathrm{d}t=0 \qquad \int_T\int_{\mathcal{B}_0}\left[\boldsymbol{B}-\left[\frac{\partial\Psi}{\partial\tilde{\boldsymbol{H}}}+\tilde{\boldsymbol{B}}\right]\right]:\delta\dot{\tilde{\boldsymbol{H}}}\,\mathrm{d}V\mathrm{d}t=0$$

$$\int_T\int_{\mathcal{B}_0}\left[\dot{\tilde{J}}_A-\dot{J}\right]\delta p_A\,\mathrm{d}V\mathrm{d}t=0=0 \qquad \int_T\int_{\mathcal{B}_0}\left[p_A-\left[\frac{\partial\Psi}{\partial\tilde{J}_A}+\tilde{p}_A\right]\right]\delta\dot{\tilde{J}}_A\,\mathrm{d}V\mathrm{d}t=0$$

$$\int_T\int_{\mathcal{B}_0}\left[\dot{\boldsymbol{H}}_A-\mathrm{co}\dot{\mathrm{f}}[\boldsymbol{C}]\right]:\delta\boldsymbol{B}_A\,\mathrm{d}V\mathrm{d}t=0 \qquad \int_T\int_{\mathcal{B}_0}\left[\boldsymbol{B}_A-\left[\frac{\partial\Psi}{\partial\tilde{\boldsymbol{H}}_A}+\tilde{\boldsymbol{B}}_A\right]\right]:\delta\dot{\boldsymbol{H}}_A\,\mathrm{d}V\mathrm{d}t=0$$

Approximation $k =$ Polynomial degree in time

- Discretization in space and time
- Lagrangian shape functions in space (N) [Wrig 08] [Bart 18]
  - Independent approximation of the different mixed fields
  - Lagrangian multiplier approximated equally as the corresponding mixed fields e.g. $\tilde{\boldsymbol{C}}_A$ & $\boldsymbol{S}_A$, $\tilde{J}$ & p or $\tilde{\boldsymbol{H}}$ & $\boldsymbol{B}$
- Lagrangian shape functions in time (M,M',$\tilde{M}$) [Bets 01] [Groß 18]

$$M_i(\alpha) = \prod_{\substack{j=1 \\ j\neq i}}^{k+1} \frac{\alpha - \alpha_j}{\alpha_i - \alpha_j},\ 1 \leq i \leq k+1 \qquad \tilde{M}_i(\alpha) = \prod_{\substack{j=1 \\ j\neq i}}^{k} \frac{\alpha - \alpha_j}{\alpha_i - \alpha_j}, 1 \leq i \leq k$$

- Time rate variables and mixed fields ($\boldsymbol{q}$,$\boldsymbol{\lambda}$,$\boldsymbol{v}$,$\boldsymbol{p}$,$\tilde{\Theta}$,Θ,η,$\boldsymbol{C}_v$,$\tilde{\boldsymbol{C}}$,$\tilde{\boldsymbol{C}}_A$,$\tilde{\boldsymbol{H}}$,$\tilde{\boldsymbol{H}}_A$,$\tilde{J}$,$\tilde{J}_A$)

$$(\bullet)^{e,h} = \sum_{I=1}^{k+1} \sum_{A=1}^{n_{no}} M_I(\alpha)\mathrm{N}^A(\boldsymbol{\xi})(\bullet)_I^{eA} \qquad \left((\dot{\bullet})^{e,h} = \frac{1}{h_n} \sum_{I=1}^{k+1} \sum_{A=1}^{n_{no}} M_I'(\alpha)\mathrm{N}^A(\boldsymbol{\xi})(\bullet)_I^{eA} \right)$$

- Lagrangian multiplier and variation fields ($\boldsymbol{S}$,$\boldsymbol{S}_A$,$\boldsymbol{B}$,$\boldsymbol{B}_A$,p,p_A,$\delta_*\bullet$)

$$(\bullet)^{e,h} = \sum_{I=1}^{k} \sum_{A=1}^{n_{no_\mathrm{u}}} \tilde{M}_I \mathrm{N}^A (\bullet)_I^{eA}$$

Implementation

- Solve mixed fields at element level, e.g.

$$\int_T \int_{\mathcal{B}_0} \left[\dot{\tilde{J}} - \dot{J}\right] \delta p \mathrm{d}V \mathrm{d}t = 0 \quad \int_T \int_{\mathcal{B}_0} \left[p - \left[\frac{\partial \Psi}{\partial \tilde{J}} + \tilde{p}\right]\right] \delta \dot{\tilde{J}} \mathrm{d}V \mathrm{d}t = 0$$

$$\tilde{J}^e_{\text{new}} = \left[\mathbf{A}'\right]^{-1} \left(\int_0^1 \mathbf{b} \boxtimes \dot{J}^e \mathrm{d}\alpha - \mathbf{b}' \tilde{J}^e_1\right) \quad p^e_{\text{new}} = \left[\tilde{\mathbf{A}}\right]^{-1} \left(\int_0^1 \mathbf{b} \boxtimes \left[\frac{\partial \Psi}{\partial \tilde{J}} + \tilde{p}\right] \mathrm{d}\alpha\right)$$

  - Discontinuous at the boundaries of spatial elements
- Iterative solution of $\boldsymbol{C}_v$ is calculated on element level not on Gauss point level
- Eliminate $\mathbf{p}$ and η, e.g.

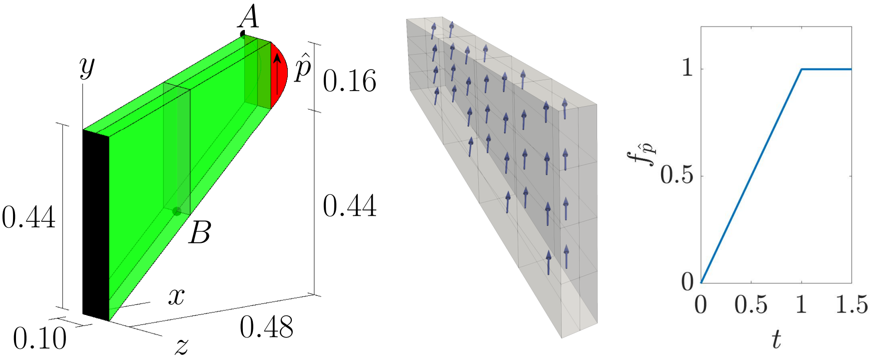
$$\int_T \int_{\mathcal{B}_0} [\mathrm{Div}[\boldsymbol{F}\boldsymbol{S}] - \dot{\boldsymbol{p}}] \cdot \delta \dot{\boldsymbol{q}} \mathrm{d}V \mathrm{d}t = 0 \quad \int_T \int_{\mathcal{B}_0} \left[\frac{1}{\rho_0}\boldsymbol{p} - \dot{\boldsymbol{q}}\right] \cdot \delta \dot{\boldsymbol{v}} \mathrm{d}V \mathrm{d}t = 0$$

- Condensate at element level to pure displacement temperature form
  - Schur complements $\Rightarrow$ Except $\mathbf{K}_{\boldsymbol{C}_v \boldsymbol{C}_v}$, all inverted matrices are constant!

$$\begin{bmatrix} \mathbf{K}^e_{qq} & \mathbf{K}^e_{q\theta} & \mathbf{K}^e_{q\lambda} \\ \mathbf{K}^e_{\theta q} & \mathbf{K}^e_{\theta\theta} & \mathbf{0} \\ \mathbf{K}^e_{\lambda q} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{q} \\ \Delta \boldsymbol{\Theta} \\ \Delta \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{r}^e_q \\ \mathbf{r}^e_\theta \\ \mathbf{r}^e_\lambda \end{bmatrix}$$

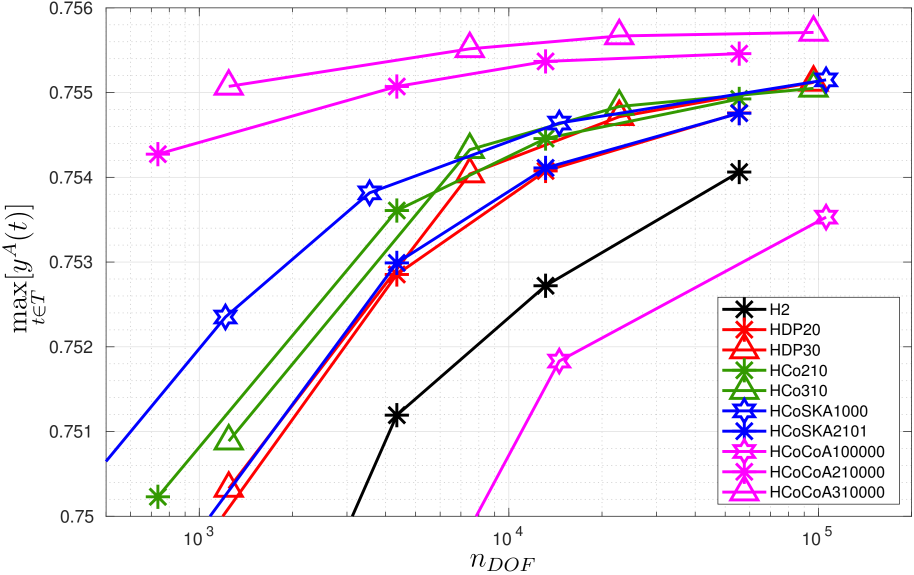
Numerical example - Cook cantilever beam for $n_{\mathrm{F}} = 1$
Load Case

- High Poisson ratio ($\nu \approx 0.4995$), stiff fiber ($\approx 10 \cdot 10^6$), $\rho_0 = 1000$
- $T = 1.5$, $h_n = 0.001$, $\hat{p} = 1.5e6 f_{\hat{p}}$, $TOL = 1e^{-2}$
- Anisotropic direction $(\boldsymbol{a}_1^0)^T = [1\ 1\ 1]$



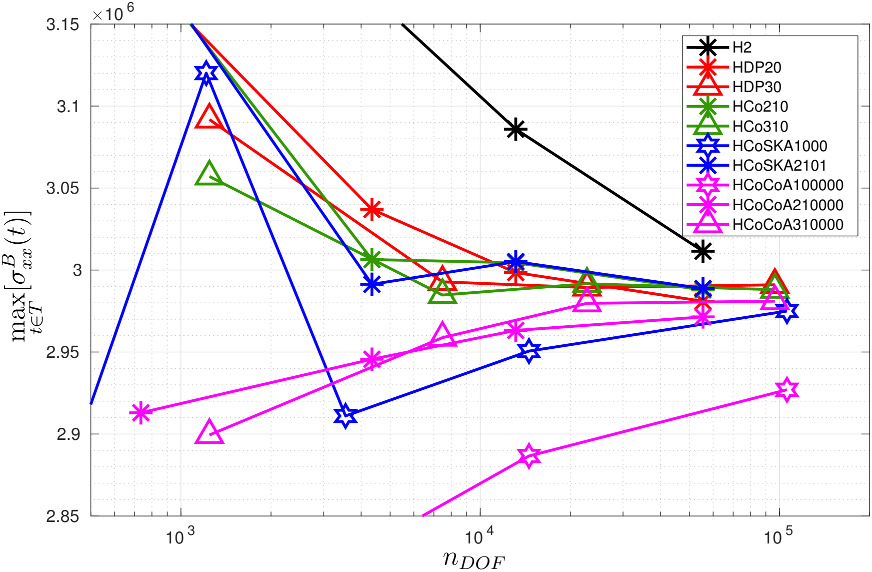
Numerical example - Cook cantilever beam for $n_{\mathrm{F}} = 1$

Convergence of the y-coordinate at Point A [eg(1)]



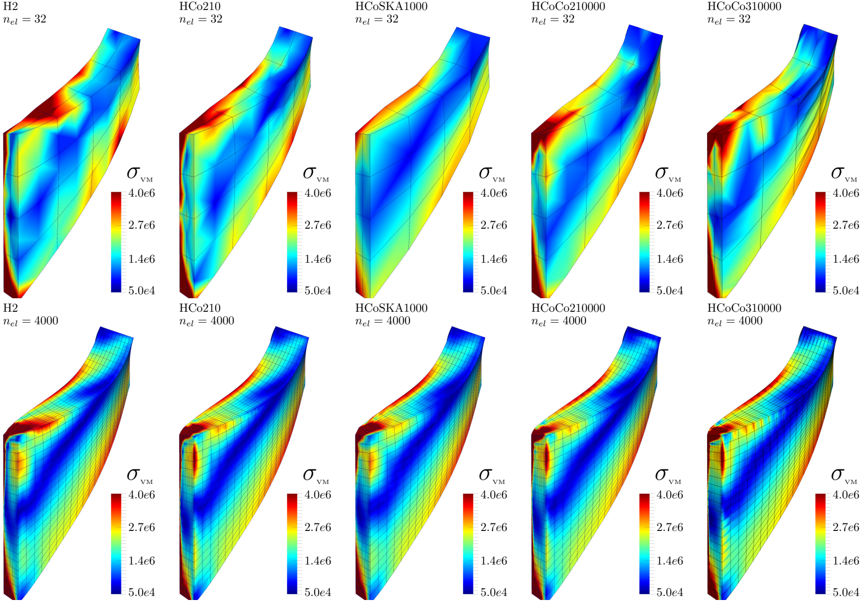
Digits (pol. degree of): H[$\boldsymbol{q}$] HDP[$\boldsymbol{q}$,$\tilde{J}$] HCo[$\boldsymbol{q}$,$\tilde{\boldsymbol{H}}$,$\tilde{J}$] HCoSKA/HCoCoA[$\boldsymbol{q}$,$\tilde{\boldsymbol{H}}$,$\tilde{J}$,$\tilde{\boldsymbol{C}}_A$,$\tilde{\boldsymbol{H}}_A$,$\tilde{J}_A$]

Numerical example - Cook cantilever beam for $n_{\mathrm{F}} = 1$
Convergence of σ_{xx} at Point B [eg(1)]

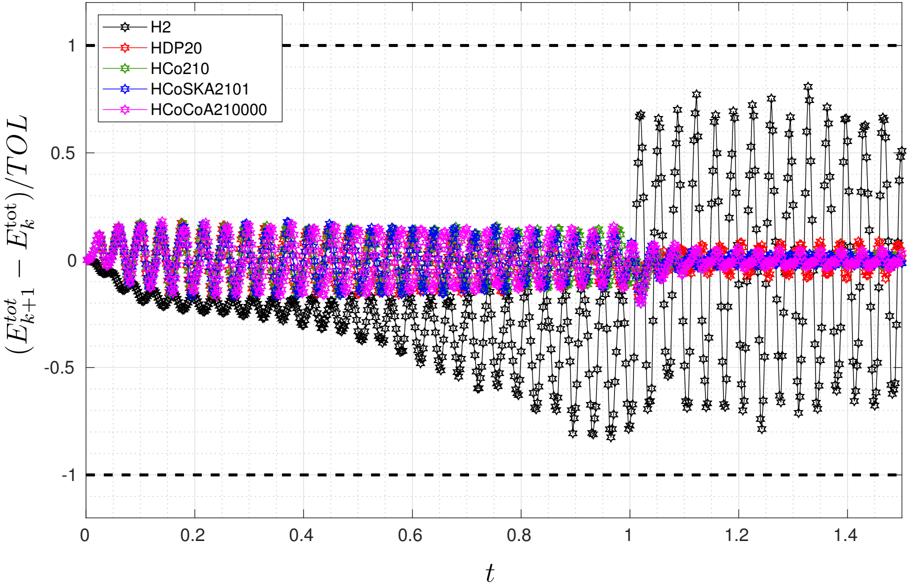


Digits (pol. degree of): H[$\boldsymbol{q}$] HDP[$\boldsymbol{q}$,$\tilde{J}$] HCo[$\boldsymbol{q}$,$\tilde{\boldsymbol{H}}$,$\tilde{J}$] HCoSKA/HCoCoA[$\boldsymbol{q}$,$\tilde{\boldsymbol{H}}$,$\tilde{J}$,$\tilde{\boldsymbol{C}}_A$,$\tilde{\boldsymbol{H}}_A$,$\tilde{J}_A$]

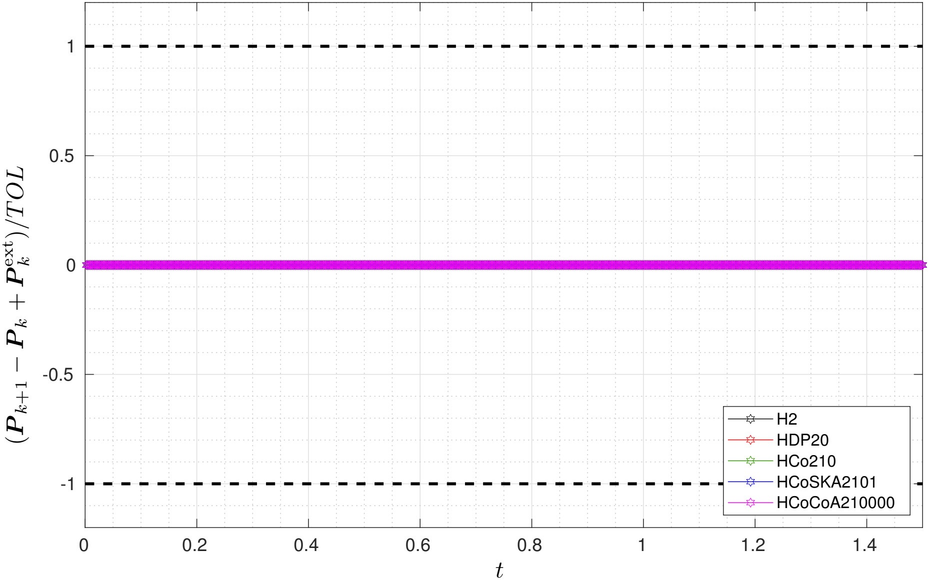
Numerical example - Cook cantilever beam for $n_{\mathrm{F}} = 1$
Deformed configuration and v. Mises equivalent stress [eg(1)]



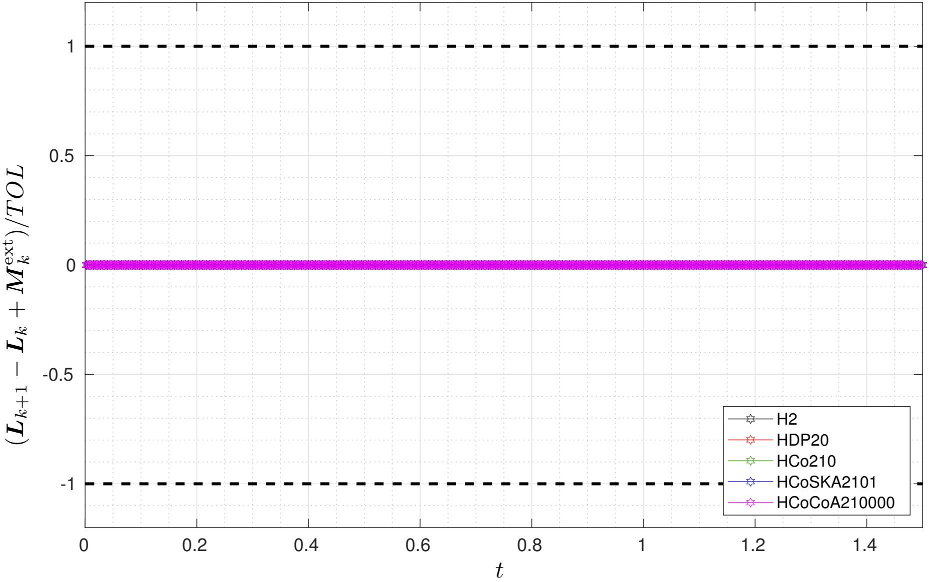
Numerical example - Cook cantilever beam for $n_{\mathrm{F}} = 1$
Conservation properties - Energy [eg(1)]



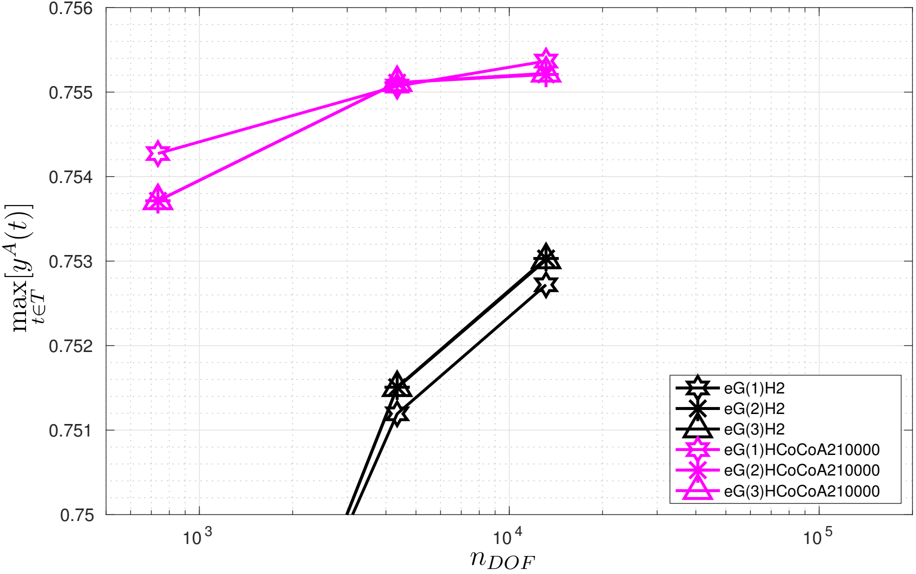
Numerical example - Cook cantilever beam for $n_{\mathrm{F}} = 1$
Conservation properties - Momentum [eg(1)]



Numerical example - Cook cantilever beam for $n_{\mathrm{F}} = 1$
Conservation properties - Angular momentum [eg(1)]

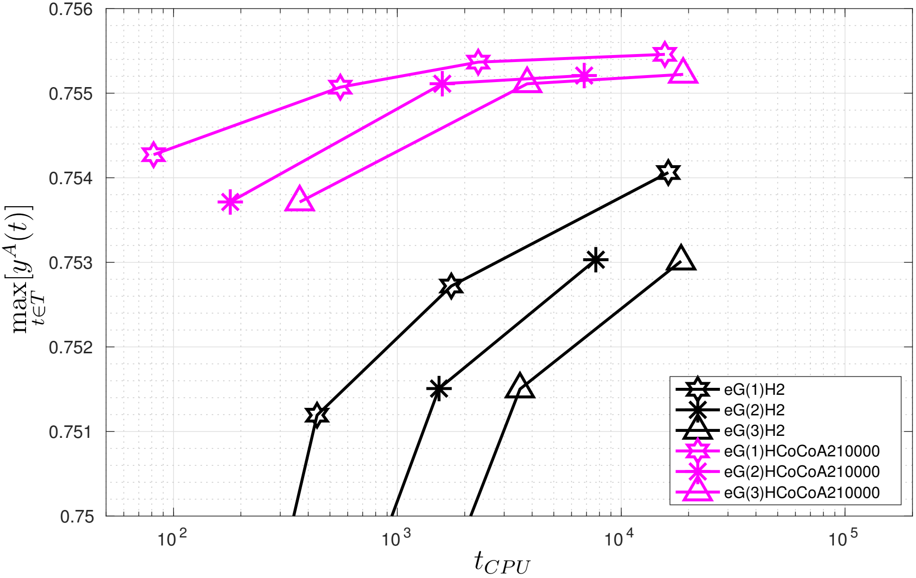


Numerical example - Cook cantilever beam for $n_{\mathrm{F}} = 1$
Convergence of the y-coordinate at Point A [eg(1),eg(2),eg(3)]



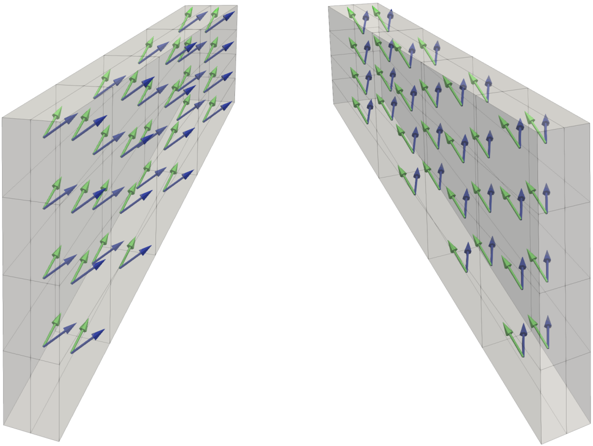
Digits (pol. degree of): H[$\boldsymbol{q}$] HCoCoA[$\boldsymbol{q}$,$\tilde{\boldsymbol{H}}$,$\tilde{J}$,$\tilde{\boldsymbol{C}}_A$,$\tilde{\boldsymbol{H}}_A$,$\tilde{J}_A$]

Numerical example - Cook cantilever beam for $n_{\mathrm{F}} = 1$
CPU time [eg(1),eg(2),eg(3)]

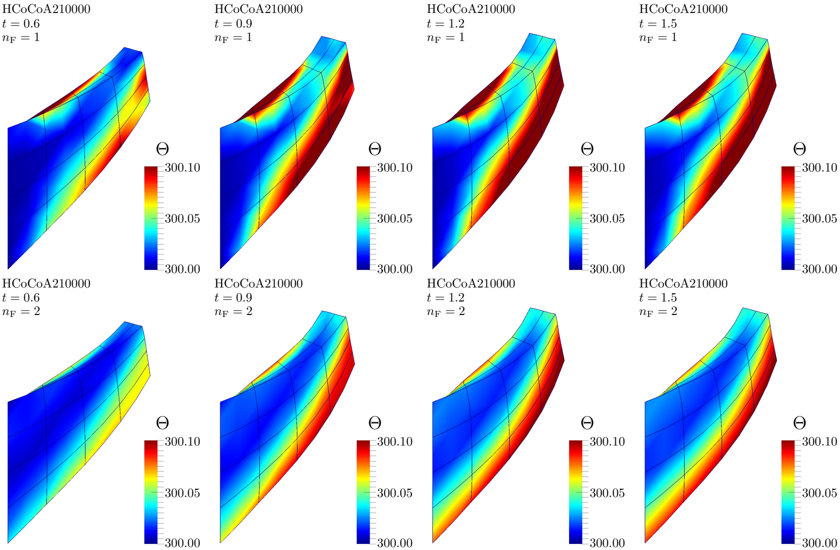


Digits (pol. degree of): H[$\boldsymbol{q}$] HCoCoA[$\boldsymbol{q}$,$\tilde{\boldsymbol{H}}$,$\tilde{J}$,$\tilde{\boldsymbol{C}}_A$,$\tilde{\boldsymbol{H}}_A$,$\tilde{J}_A$]

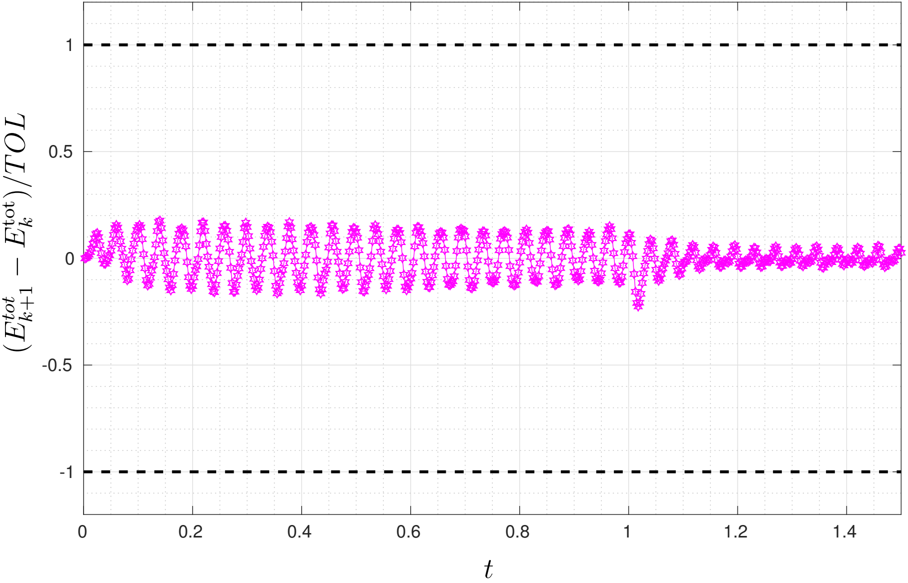
- Anisotropic direction $(\boldsymbol{a}_1^0)^T = [1\ 1\ 1]$ and $(\boldsymbol{a}_2^0)^T = [1\ 1\ 0]$
- Second fiber with high thermal conductivity, low stiffness



Numerical example - Cook cantilever beam for $n_{\mathrm{F}} = 2$
Deformed configuration and temperature distribution HCoCoA210000 [eg(2)]



Numerical example - Cook cantilever beam for $n_{\mathrm{F}} = 2$
Conservation properties - Energy HCoCoA210000 [eg(2)]



Conclusion

- Motivation:
  - Dynamic simulations of fiber-reinforced materials in light-weight structures
- Strategy:
  - Mixed finite elements to reduce locking effect
  - Extension to a thermo-mechanical coupling
  - Mixed finite elements as Viscous internal variable
  - Higher-order energy-momentum conserving time integrators for stable and accurate simulations
- Important results:
  - Excellent performance of the mixed elements is still preserved in a thermo-viscoelastic context.
  - Huge computing time reduction, especially in the context of the iterative calculation of the internal variable
  - Higher-order energy-momentum time integrators conserves energy
- Outlook:
  - Extend these formulations to a thermo-viscoelastic material behavior of the fibers

$$\Psi_{\mathrm{M}}^{\mathrm{iso}} = \frac{\epsilon_1}{2}(\mathrm{tr}[\boldsymbol{C}])^2 + \frac{\epsilon_2}{2}(\mathrm{tr}[\mathrm{cof}[\boldsymbol{C}]])^2 - \epsilon_3 \ln(J) \quad \Psi_{\mathrm{M}}^{\mathrm{vol}} = \frac{\epsilon_4}{2}(J^{\epsilon_5} + J^{-\epsilon_5} - 2)$$

$$\Psi_{\mathrm{M}}^{\mathrm{vis}} = \frac{\epsilon_1^{\mathrm{vis}}}{2}(\mathrm{tr}[\boldsymbol{C}])^2 + \frac{\epsilon_2^{\mathrm{vis}}}{2}(\mathrm{tr}[\mathrm{cof}[\boldsymbol{C}]])^2 - \epsilon_3^{\mathrm{vis}} \ln(J) + \frac{\epsilon_4^{\mathrm{vis}}}{2}(J^{\epsilon_5^{\mathrm{vis}}} + J^{-\epsilon_5^{\mathrm{vis}}} - 2)$$

$$\Psi_{\mathrm{F}_1}^{\mathrm{ela}} = \epsilon_6 \left(\frac{1}{\epsilon_7 + 1}(\mathrm{tr}[\boldsymbol{C}\boldsymbol{M}_1])^{\epsilon_7+1} + \frac{1}{\epsilon_8 + 1}(\mathrm{tr}[\mathrm{cof}[\boldsymbol{C}]\boldsymbol{M}_1])^{\epsilon_8+1} + \frac{1}{\epsilon_9}\det[\boldsymbol{C}]^{-\epsilon_9} \right)$$

$$\Psi_{\mathrm{F}_2}^{\mathrm{ela}} = \frac{\epsilon_{10}}{2}(\mathrm{tr}[\boldsymbol{C}\boldsymbol{M}_2] - 1)^2$$

$$\Psi_X^{cap} = c_X^0 (1 - \Theta_\infty c_X^1)(\Theta - \Theta_\infty - \Theta \ln \frac{\Theta}{\Theta_\infty}) - \frac{1}{2} c_X^0 c_X^1 (\Theta - \Theta_\infty)^2$$

$\epsilon_1 = 0.1e6$	$V_{\text{dev}} = 0.1e4$	$k_{\text{F}_1} = 0.1$
$\epsilon_2 = 0.1e6$	$V_{\text{vol}} = 0.2e4$	$\beta_{F_1} = 1e-10$
$\epsilon_3 = 1.8e6$	$\epsilon_6 = 10e6$	$c^0_{\text{F}_1} = 1500$
$\epsilon_4 = 100e6$	$\epsilon_7 = 4$	$c^1_{\text{F}_1} = 0.003$
$\epsilon_5 = 4$	$\epsilon_8 = 4$	$k_{\text{F}_2} = 50$
$\epsilon_1^{\text{vis}} = 0.1e5$	$\epsilon_9 = 1$	$\beta_{\text{F}_2} = 1e-10$
$\epsilon_2^{\text{vis}} = 0.1e5$	$\epsilon_{10} = 0.1e6$	$c^0_{\text{F}_2} = 3000$
$\epsilon_3^{\text{vis}} = 1.8e5$	$k_{\text{M}} = 0.1$	$c^1_{\text{F}_2} = 0.006$
$\epsilon_4^{\text{vis}} = 100e5$	$\beta_{\text{M}} = 1e-10$	$\Theta_\infty = 300$
$\epsilon_5^{\text{vis}} = 4$	$c^0_{\text{M}} = 1500$	$\hat{p} = 1.5e6$
$\rho_0 = 1000$	$c^1_{\text{M}} = 0.003$	
$h_n = 0.001$	$T = 1.5$	$TOL = 1e-2$

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