Mixed finite element formulations for the Galerkin-based time integration of finite anisotropic elastodynamics

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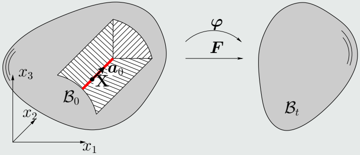
Motivation

- Dynamic simulations of fiber-reinforced materials in lightweight structures

- Keywords:
  - Anisotropic material behavior
  - Near-incompressible material behavior
  - Hyperelastic material behavior
  - Long-term simulations

- Solution strategy:
  1. Mixed finite elements for finite anisotropic elastodynamics to reduce locking effect [polyconvex material formulations]
  2. Higher-order time integrators to increase accuracy [continuous Galerkin cG(k)]
  3. Energy-momentum conserving time integrators for stable long-term simulations [Discrete gradient eG(k)]

Anisotropic material



Continuum mechanics

$$\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F} \qquad \boldsymbol{M} = \boldsymbol{a}_0 \otimes \boldsymbol{a}_0$$

$$J = I_3(\boldsymbol{F}) = \det[\boldsymbol{F}] = \sqrt{\det[\boldsymbol{C}]}$$

$\boldsymbol{F}$ ... Deformation gradient
$\boldsymbol{C}$ ... Right Cauchy-Green tensor
$\boldsymbol{a}_0$ ... Fiber direction

Hyperelastic, anisotropic, polyconvex material formulation

- Additive split of strain energy function

$$\begin{aligned}
\Psi(\boldsymbol{C},\boldsymbol{M}) &= \Psi^{iso}(\boldsymbol{C}) + \Psi^{ani}(\boldsymbol{C},\boldsymbol{M}) \\
\Psi(\boldsymbol{C},\mathrm{cof}[\boldsymbol{C}],J,\boldsymbol{M}) &= \Psi^{iso}(\boldsymbol{C},\mathrm{cof}[\boldsymbol{C}],J) + \Psi^{ani}(\boldsymbol{C},\mathrm{cof}[\boldsymbol{C}],J,\boldsymbol{M}) \\
&= \Psi^{iso}_{\boldsymbol{C}}(\boldsymbol{C}) + \Psi^{iso}_{\mathrm{cof}[\boldsymbol{C}]}(\mathrm{cof}[\boldsymbol{C}]) + \Psi^{iso}_{J}(J) \\
&\quad + \Psi^{ani}_{\boldsymbol{C}}(\boldsymbol{C},\boldsymbol{M}) + \Psi^{ani}_{\mathrm{cof}[\boldsymbol{C}]}(\mathrm{cof}[\boldsymbol{C}],\boldsymbol{M}) + \Psi^{ani}_{J}(J)
\end{aligned}$$

- Mixed elements based on Hu-Washizu functionals

Displacement Element

$$\Pi_{HW}^{D}(\boldsymbol{q}) = \int_{\mathcal{B}_0} \Psi(\boldsymbol{C}(\boldsymbol{q}))\mathrm{d}V \quad \text{with} \quad \Psi = \Psi^{iso}(\boldsymbol{C}, \mathrm{cof}[\boldsymbol{C}], J) + \Psi^{ani}(\boldsymbol{C}, \mathrm{cof}[\boldsymbol{C}], J, \boldsymbol{M})$$

Displacement-Pressure Element [Simo 85]

$$\Pi_{HW}^{DP}(\boldsymbol{q}, \Theta, p) = \Pi_{HW}^{D} + \int_{\mathcal{B}_0} p(J(\boldsymbol{q}) - \Theta)\mathrm{d}V$$

with $\Psi = \Psi^{iso}(\boldsymbol{C}, \mathrm{cof}[\boldsymbol{C}], \Theta) + \Psi^{ani}(\boldsymbol{C}, \mathrm{cof}[\boldsymbol{C}], \Theta, \boldsymbol{M})$

CoFEM Element [Schr 11]

$$\Pi_{HW}^{CoFEM}(\boldsymbol{q}, \dots, \boldsymbol{H}, \boldsymbol{B}) = \Pi_{HW}^{DP} + \int_{\mathcal{B}_0} \boldsymbol{B} : (\mathrm{cof}[\boldsymbol{C}(\boldsymbol{q})] - \boldsymbol{H})\mathrm{d}V$$

with $\Psi = \Psi^{iso}(\boldsymbol{C}, \boldsymbol{H}, \Theta) + \Psi^{ani}(\boldsymbol{C}, \boldsymbol{H}, \Theta, \boldsymbol{M})$

SKA Element [Schr 16]

$$\Pi_{HW}^{CoSKA}(\boldsymbol{q},\ldots,\boldsymbol{C}_A,\boldsymbol{S}_A) = \Pi_{HW}^{CoFEM} + \int_{\mathcal{B}_0} \frac{1}{2}\boldsymbol{S}_A(\boldsymbol{C}(\boldsymbol{q}) - \boldsymbol{C}_A)\mathrm{d}V$$

with $\Psi = \Psi^{iso}(\boldsymbol{C},\boldsymbol{H},\Theta) + \Psi^{ani}(\boldsymbol{C}_A, \mathrm{cof}[\boldsymbol{C}_A], \sqrt{\det[\boldsymbol{C}_A]}, \boldsymbol{M})$

CoCoA Element

$$\Pi_{HW}^{CoCoA}(\ldots) = \Pi_{HW}^{CoSKA} + \int_{\mathcal{B}_0} \boldsymbol{B}_A : (\mathrm{cof}[\boldsymbol{C}(\boldsymbol{q})] - \boldsymbol{H}_A)\mathrm{d}V + \int_{\mathcal{B}_0} p_A(J(\boldsymbol{q}) - \Theta_A)\mathrm{d}V$$

with $\Psi = \Psi^{iso}(\boldsymbol{C},\boldsymbol{H},\Theta) + \Psi^{ani}(\boldsymbol{C}_A, \boldsymbol{H}_A, \Theta_A, \boldsymbol{M})$

Veubeke-Hu-Washizu functional

$$\Pi_{VHW}(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{v},\boldsymbol{p},\ldots) = \boxed{\int_T\int_{\mathcal{B}_0} \frac{1}{2}\rho_0 \boldsymbol{v}^T\boldsymbol{v}\mathrm{d}V\mathrm{d}t + \int_T\int_{\mathcal{B}_0} \boldsymbol{p}(\dot{\boldsymbol{q}} - \boldsymbol{v})\mathrm{d}V\mathrm{d}t} - \int_T \Pi_{HW}(\boldsymbol{q},\ldots)\mathrm{d}t$$
$$+ \int_{\mathcal{B}_0} \rho_0 \boldsymbol{g}\cdot\boldsymbol{q}\mathrm{d}V + \int_{\partial\mathcal{B}_0} \boldsymbol{t}\cdot\boldsymbol{q}\mathrm{d}A + \int_{\partial\mathcal{B}_0} \boldsymbol{\lambda}[\boldsymbol{q} - \boldsymbol{q}^{\mathrm{ref}}]\mathrm{d}A$$

- Dirichlet boundary conditions are modelled by Lagrange multipliers $\boldsymbol{\lambda}$ [Bets 02]

- Variation with respect to all unknowns
- Apply Discrete Gradient

Weak formulation of CoCoA Element

$$\int_T \int_{\mathcal{B}_0} (\mathrm{div}[\boldsymbol{F}\boldsymbol{S}] - \dot{\boldsymbol{p}})\delta \boldsymbol{q} \mathrm{d}V \mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} (\frac{1}{\rho_0}\boldsymbol{p} - \dot{\boldsymbol{q}})\delta \boldsymbol{p} \mathrm{d}V \mathrm{d}t = 0 \qquad \mathbb{P} = \frac{\partial \mathrm{cof}[\boldsymbol{C}]}{\partial \boldsymbol{C}}$$

$$\int_T \int_{\mathcal{B}_0} \frac{1}{2}(\boldsymbol{C}_A - \boldsymbol{C})\delta \boldsymbol{S}_A \mathrm{d}V \mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} (\frac{1}{2}\boldsymbol{S}_A - \boxed{\frac{\partial \Psi}{\partial \boldsymbol{C}_A}})\delta \boldsymbol{C}_A \mathrm{d}V \mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} (\Theta - J)\delta p \mathrm{d}V \mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} (\boldsymbol{H} - \mathrm{cof}[\boldsymbol{C}])\delta \boldsymbol{B} \mathrm{d}V \mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} (p - \boxed{\frac{\partial \Psi}{\partial \Theta}})\delta \Theta \mathrm{d}V \mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} (\boldsymbol{B} - \boxed{\frac{\partial \Psi}{\partial \boldsymbol{H}}})\delta \boldsymbol{H} \mathrm{d}V \mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} (\Theta_A - J)\delta p_A \mathrm{d}V \mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} (\boldsymbol{H}_A - \mathrm{cof}[\boldsymbol{C}])\delta \boldsymbol{B}_A \mathrm{d}V \mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} (p_A - \boxed{\frac{\partial \Psi}{\partial \Theta_A}})\delta \Theta_A \mathrm{d}V \mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} (\boldsymbol{B}_A - \boxed{\frac{\partial \Psi}{\partial \boldsymbol{H}_A}})\delta \boldsymbol{H}_A \mathrm{d}V \mathrm{d}t = 0$$

with $\boldsymbol{S} = (2\boxed{\frac{\partial \Psi}{\partial \boldsymbol{C}}} + 2\boldsymbol{B} : \mathbb{P} + pJ^{-1}\mathrm{cof}[\boldsymbol{C}] + \boldsymbol{S}_A + 2\boldsymbol{B}_A : \mathbb{P} + p_A J^{-1}\mathrm{cof}[\boldsymbol{C}])$

- Additive split of strain energy function

$$
\begin{aligned}
\Psi(\boldsymbol{C}, \operatorname{cof}[\boldsymbol{C}], J, \boldsymbol{M}) &= \Psi^{iso}_{\boldsymbol{C}}(\boldsymbol{C}) + \Psi^{iso}_{\operatorname{cof}[\boldsymbol{C}]}(\operatorname{cof}[\boldsymbol{C}]) + \Psi^{iso}_{J}(J) \\
&\quad + \Psi^{ani}_{\boldsymbol{C}}(\boldsymbol{C}, \boldsymbol{M}) + \Psi^{ani}_{\operatorname{cof}[\boldsymbol{C}]}(\operatorname{cof}[\boldsymbol{C}], \boldsymbol{M}) + \Psi^{ani}_{J}(J) \\
\Psi(\boldsymbol{C}, \boldsymbol{H}, \Theta, \boldsymbol{C}_A, \boldsymbol{H}_A, \Theta_A, \boldsymbol{M}) &= \Psi^{iso}_{\boldsymbol{C}}(\boldsymbol{C}) + \Psi^{iso}_{\operatorname{cof}[\boldsymbol{C}]}(\boldsymbol{H}) + \Psi^{iso}_{J}(\Theta) \\
&\quad + \Psi^{ani}_{\boldsymbol{C}}(\boldsymbol{C}_A, \boldsymbol{M}) + \Psi^{ani}_{\operatorname{cof}[\boldsymbol{C}]}(\boldsymbol{H}_A, \boldsymbol{M}) + \Psi^{ani}_{J}(\Theta_A)
\end{aligned}
$$

	$\partial\Psi(\boldsymbol{C}, \boldsymbol{H}, \Theta, \boldsymbol{C}_A, \boldsymbol{H}_A, \Theta_A)$					
$\partial\boldsymbol{C}$	$\frac{\partial\Psi^{iso}_{\boldsymbol{C}}(\boldsymbol{C})}{\partial\boldsymbol{C}}$	0	0	0	0	0
$\partial\boldsymbol{H}$	0	$\frac{\partial\Psi^{iso}_{\operatorname{cof}[\boldsymbol{C}]}(\boldsymbol{H})}{\partial\boldsymbol{H}}$	0	0	0	0
$\partial\Theta$	0	0	$\frac{\partial\Psi^{iso}_{J}(\Theta)}{\partial\Theta}$	0	0	0
$\partial\boldsymbol{C}_A$	0	0	0	$\frac{\partial\Psi^{ani}_{\boldsymbol{C}}(\boldsymbol{C}_A)}{\partial\boldsymbol{C}_A}$	0	0
$\partial\boldsymbol{H}_A$	0	0	0	0	$\frac{\partial\Psi^{ani}_{\operatorname{cof}[\boldsymbol{C}]}(\boldsymbol{H}_A)}{\partial\boldsymbol{H}_A}$	0
$\partial\Theta_A$	0	0	0	0	0	$\frac{\partial\Psi^{ani}_{J}(\Theta_A)}{\partial\Theta_A}$

- Discrete Gradients are independent!

Discrete Gradient [Bets 18]

- Scalar: $\frac{\partial \Psi(a)}{\partial a} = \frac{\partial \Psi(a)}{\partial a} + \frac{\Psi(1) - \Psi(0) - \int \frac{\partial \Psi(a)}{\partial a} \dot{a}}{\int \dot{a}\dot{a}} \dot{a}$

  e.g. $\frac{\partial \Psi_J^{iso}(\Theta)}{\partial \Theta} = \frac{\partial \Psi_J^{iso}(\Theta)}{\partial \Theta} + \frac{\Psi_J^{iso}(1) - \Psi_J^{iso}(0) - \int \frac{\partial \Psi_J^{iso}(\Theta)}{\partial a} \dot{\Theta}}{\int \dot{\Theta}\dot{\Theta}} \dot{\Theta}$

- Second rank tensor: $\frac{\partial \Psi(\boldsymbol{A})}{\partial \boldsymbol{A}} = \frac{\partial \Psi(\boldsymbol{A})}{\partial \boldsymbol{A}} + \frac{\Psi(1) - \Psi(0) - \int \frac{\partial \Psi(\boldsymbol{A})}{\partial \boldsymbol{A}} : \dot{\boldsymbol{A}}}{\int \dot{\boldsymbol{A}} : \dot{\boldsymbol{A}}} \dot{\boldsymbol{A}}$

  e.g. $\frac{\partial \Psi^{ani}_{\mathrm{cof}[\boldsymbol{C}]}(\boldsymbol{H}_A)}{\partial \boldsymbol{H}_A} = \frac{\partial \Psi^{ani}_{\mathrm{cof}[\boldsymbol{C}]}(\boldsymbol{H}_A)}{\partial \boldsymbol{H}_A} + \frac{\Psi^{ani}_{\mathrm{cof}[\boldsymbol{C}]}(1) - \Psi^{ani}_{\mathrm{cof}[\boldsymbol{C}]}(0) - \int \frac{\partial \Psi^{ani}_{\mathrm{cof}[\boldsymbol{C}]}(\boldsymbol{A})}{\partial \boldsymbol{A}} : \dot{\boldsymbol{H}}_A}{\int \dot{\boldsymbol{H}}_A : \dot{\boldsymbol{H}}_A} \dot{\boldsymbol{H}}_A$

Hidden Constraints [Bets 02]

- Discrete Gradient needs time derivative $\Rightarrow$ Fullfills constraints at velocity level, e.g.

$$\int_T \int_{\mathcal{B}_0} (\Theta - J)\delta p \mathrm{d}V \mathrm{d}t \Rightarrow \int_T \int_{\mathcal{B}_0} (\dot{\Theta} - \dot{J})\delta p \mathrm{d}V \mathrm{d}t = \int_T \int_{\mathcal{B}_0} (\dot{\Theta} - \frac{1}{2J}\mathrm{cof}[\boldsymbol{C}] : \dot{\boldsymbol{C}})\delta p \mathrm{d}V \mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} (\boldsymbol{H} - \mathrm{cof}[\boldsymbol{C}])\delta \boldsymbol{B} \mathrm{d}V \mathrm{d}t \Rightarrow \int_T \int_{\mathcal{B}_0} (\dot{\boldsymbol{H}} - \mathrm{cof}\dot{[\boldsymbol{C}]})\delta \boldsymbol{B} \mathrm{d}V \mathrm{d}t = \int_T \int_{\mathcal{B}_0} (\dot{\boldsymbol{H}} - \mathbb{P} : \dot{\boldsymbol{C}})\delta \boldsymbol{B} \mathrm{d}V \mathrm{d}t = 0$$

- Discrete Gradient and Constrains at velocity level

Modified Weak formulation of CoCoA Element

$$\int_T \int_{\mathcal{B}_0} (\mathrm{div}[\boldsymbol{F}\boldsymbol{S}] - \dot{\boldsymbol{p}})\delta\boldsymbol{q}\,\mathrm{d}V\mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} (\frac{1}{\rho_0}\boldsymbol{p} - \dot{\boldsymbol{q}})\delta\boldsymbol{p}\,\mathrm{d}V\mathrm{d}t = 0 \qquad \mathbb{P} = \frac{\partial \mathrm{cof}[\boldsymbol{C}]}{\partial \boldsymbol{C}}$$

$$\int_T \int_{\mathcal{B}_0} \frac{1}{2}\boxed{(\dot{\boldsymbol{C}}_A - \dot{\boldsymbol{C}})}\delta\boldsymbol{S}_A\mathrm{d}V\mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} (\frac{1}{2}\boldsymbol{S}_A - \boxed{\frac{\partial \Psi^{ani}_{\boldsymbol{C}}(\boldsymbol{C}_A)}{\partial \boldsymbol{C}_A}})\delta\boldsymbol{C}_A\mathrm{d}V\mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} \boxed{(\dot{\Theta} - \frac{1}{2J}\mathrm{cof}[\boldsymbol{C}] : \dot{\boldsymbol{C}})}\delta p\,\mathrm{d}V\mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} \boxed{(\dot{\boldsymbol{H}} - \mathbb{P} : \dot{\boldsymbol{C}})}\delta\boldsymbol{B}\mathrm{d}V\mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} (p - \boxed{\frac{\partial \Psi^{iso}_J(\Theta)}{\partial \Theta}})\delta\Theta\mathrm{d}V\mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} (\boldsymbol{B} - \boxed{\frac{\partial \Psi^{iso}_{\mathrm{cof}[\boldsymbol{C}]}(\boldsymbol{H})}{\partial \boldsymbol{H}}})\delta\boldsymbol{H}\mathrm{d}V\mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} \boxed{(\dot{\Theta}_A - \frac{1}{2J}\mathrm{cof}[\boldsymbol{C}] : \dot{\boldsymbol{C}})}\delta p_A\mathrm{d}V\mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} \boxed{(\dot{\boldsymbol{H}}_A - \mathbb{P} : \dot{\boldsymbol{C}})}\delta\boldsymbol{B}_A\mathrm{d}V\mathrm{d}t = 0$$

$$\int_T \int_{\mathcal{B}_0} (p_A - \boxed{\frac{\partial \Psi^{ani}_J(\Theta_A)}{\partial \Theta_A}})\delta\Theta_A\mathrm{d}V\mathrm{d}t = 0 \qquad \int_T \int_{\mathcal{B}_0} (\boldsymbol{B}_A - \boxed{\frac{\partial \Psi^{ani}_{\mathrm{cof}[\boldsymbol{C}]}(\boldsymbol{H}_A)}{\partial \boldsymbol{H}_A}})\delta\boldsymbol{H}_A\mathrm{d}V\mathrm{d}t = 0$$

with $\boldsymbol{S} = (2\boxed{\frac{\partial \Psi^{iso}_{\boldsymbol{C}}(\boldsymbol{C})}{\partial \boldsymbol{C}}} + 2\boldsymbol{B} : \mathbb{P} + pJ^{-1}\mathrm{cof}[\boldsymbol{C}] + \boldsymbol{S}_A + 2\boldsymbol{B}_A : \mathbb{P} + p_A J^{-1}\mathrm{cof}[\boldsymbol{C}])$

Approximation $k =$ Polynomial degree in time

- Lagrangian shape functions in space (N) [Wrig 08]
- Lagrangian shape functions in time (M,M',$\tilde{M}$) [Bets 01]

$$M_i(\alpha) = \prod_{\substack{j=1\\ j\neq i}}^{k+1} \frac{\alpha - \alpha_j}{\alpha_i - \alpha_j},\ 1 \leq i \leq k+1 \qquad \tilde{M}_i(\alpha) = \prod_{\substack{j=1\\ j\neq i}}^{k} \frac{\alpha - \alpha_j}{\alpha_i - \alpha_j}, 1 \leq i \leq k$$

- $\boldsymbol{q}$, $\boldsymbol{C}$, $\boldsymbol{H}$, Θ, $\boldsymbol{C}_A$, $\boldsymbol{H}_A$, Θ_A with $(\dots)^{e,h} = \sum_{I=1}^{k+1} \sum_{A=1}^{n_{no}} M_I(\alpha) \mathrm{N}^A(\boldsymbol{\xi}) (\dots)_I^{eA}$
- $\dot{\boldsymbol{q}}$, $\dot{\boldsymbol{C}}$, $\dot{\boldsymbol{H}}$, $\dot{\Theta}$, $\dot{\boldsymbol{C}}_A$, $\dot{\boldsymbol{H}}_A$, $\dot{\Theta}_A$ with $(\dots)^{e,h} = \sum_{I=1}^{k+1} \sum_{A=1}^{n_{no}} M'_I(\alpha) \mathrm{N}^A(\boldsymbol{\xi}) (\dots)_I^{eA}$
- $\boldsymbol{B}$, p, $\boldsymbol{S}_A$, $\boldsymbol{B}_A$, p_A, $\delta \boldsymbol{q}^{e,h}, \dots$ with $(\dots)^{e,h} = \sum_{J=1}^{k} \sum_{A=1}^{n_{no_\mathrm{u}}} \tilde{M}_J \mathrm{N}^A (\dots)_I^{eA}$

- Solve Θ^e_{new}, p^e_{new}, ... at element level, e.g.

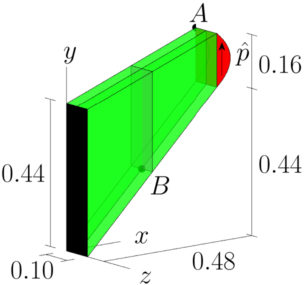
$$\Theta^e_{\text{new}} = \left[\mathbf{A}'\right]^{-1} \left(\int_0^1 \mathbf{b} \boxtimes J^e \mathrm{d}\alpha - \mathbf{b}' \Theta_1^e \right) \quad p^e_{\text{new}} = \left[\tilde{\mathbf{A}}\right]^{-1} \left(\int_0^1 \mathbf{b} \boxtimes \frac{\partial \Psi_J^{iso}(\Theta)}{\partial \Theta} \mathrm{d}\alpha \right)$$

- Eliminate $\mathbf{p}$ and condensate at element level to pure displacement form

- Compare the non-standard mixed elements with the standard elements for hexahedral elements (serendipity formulation)
- Anisotropic direction $\boldsymbol{a}_0^T = [1\ 1\ 1]$
- Strain energy function ([Schr 11]):

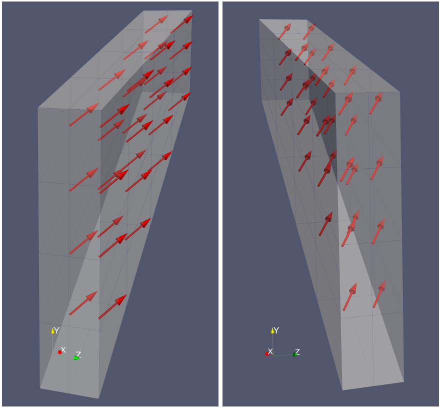
$$\Psi^{iso} = \frac{\epsilon_1}{2}(\mathrm{tr}[\boldsymbol{C}])^2 + \frac{\epsilon_2}{2}(\mathrm{tr}[\mathrm{cof}[\boldsymbol{C}]])^2 - \epsilon_3 \ln(J) + \frac{\epsilon_4}{2}(J^{\epsilon_5} + J^{-\epsilon_5} - 2)$$

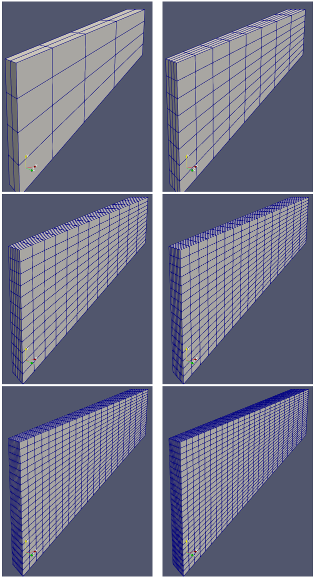
$$\Psi^{ani} = \epsilon_6 \left(\frac{1}{\epsilon_7 + 1}(\mathrm{tr}[\boldsymbol{C}\boldsymbol{M}])^{\epsilon_7+1} + \frac{1}{\epsilon_8 + 1}(\mathrm{tr}[\mathrm{cof}[\boldsymbol{C}]\boldsymbol{M}])^{\epsilon_8+1} + \frac{1}{\epsilon_9} J^{-\epsilon_9} \right)$$



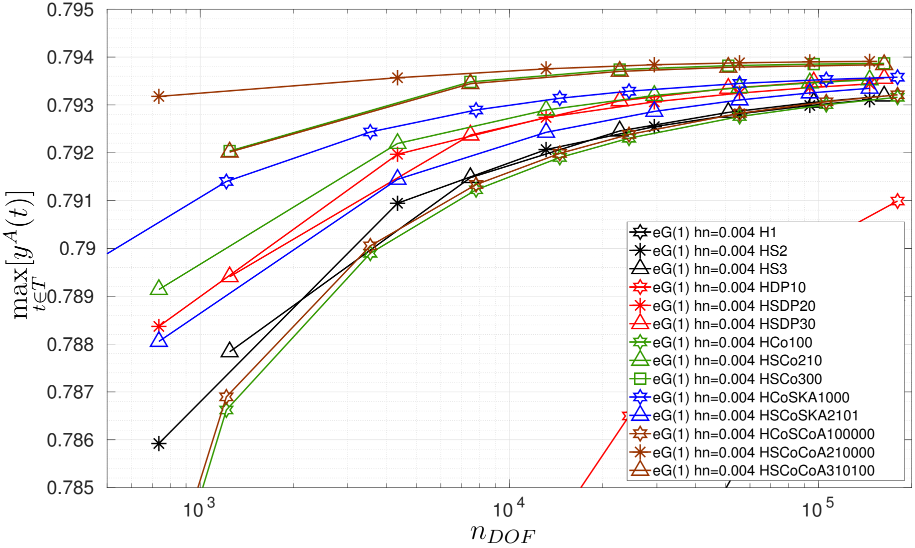
$\epsilon_1 = 16e6$	$\epsilon_8 = 8$
$\epsilon_2 = 32e6$	$\epsilon_9 = 2$
$\epsilon_3 = 480e6$	$\rho_0 = 1000$
$\epsilon_4 = 80e6$	$T = 1.1$
$\epsilon_5 = 20$	$h_n = 0.005$
$\epsilon_6 = 1200e6$	$TOL = 1e-4$
$\epsilon_7 = 4$	$\hat{p} = 4.7e8$

Numerical example - Cook cantilever beam
Fiber direction and refinement levels

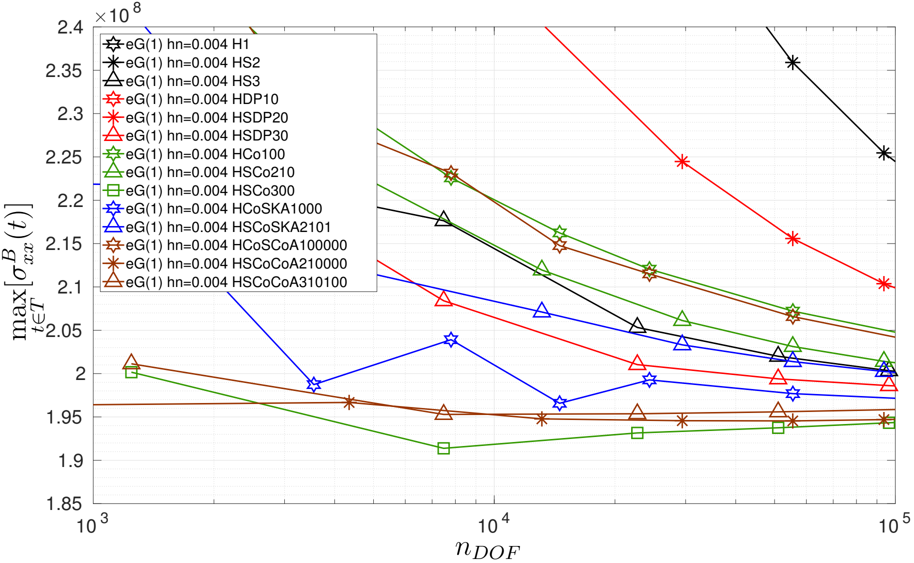




Numerical example - Cook cantilever beam
Convergence of the y-coordinate at Point A

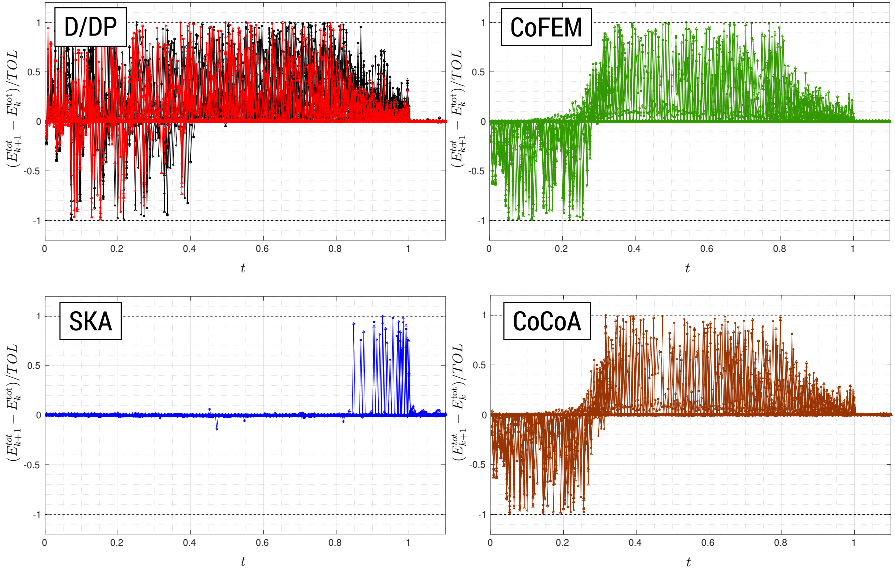


Digits (pol. degree of): H[q] HDP[q,Θ] HCo[q,$\boldsymbol{H}$,Θ] HCoCoA[q,$\boldsymbol{H}$,Θ,$\boldsymbol{C}_A$,$\boldsymbol{H}_A$,Θ_A]

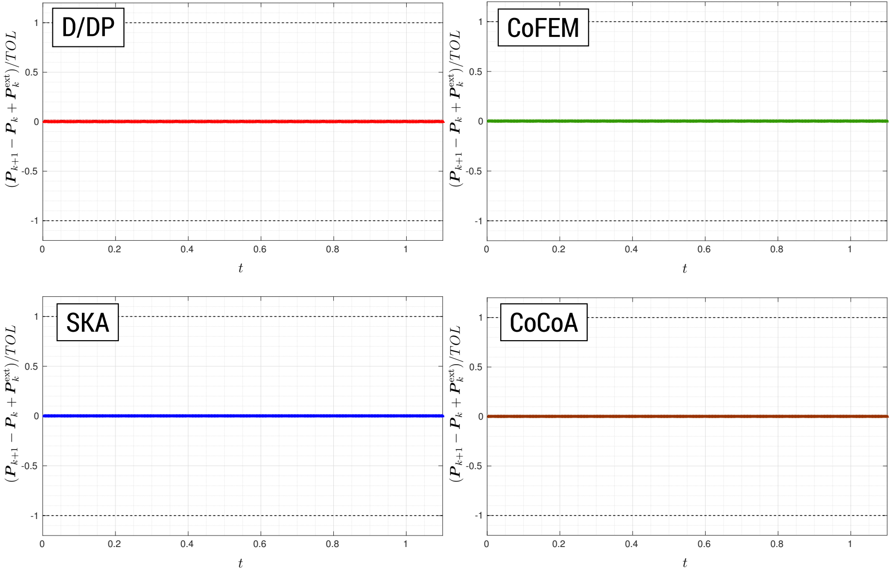


Digits (pol. degree of): H[q] HDP[q,Θ] HCo[q,H,Θ] HCoCoA[q,H,Θ,C_A,H_A,Θ_A]

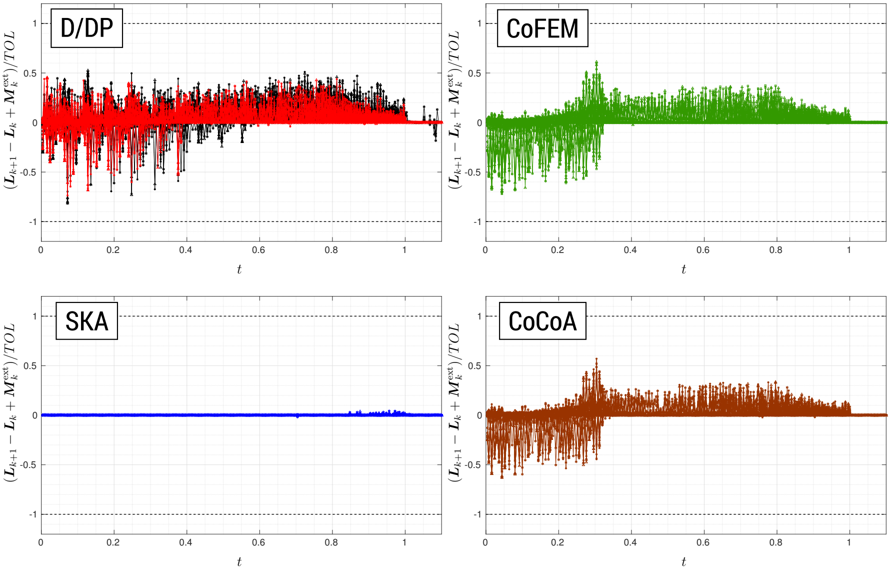
Numerical example - Cook cantilever beam
Conversation properties - Energy



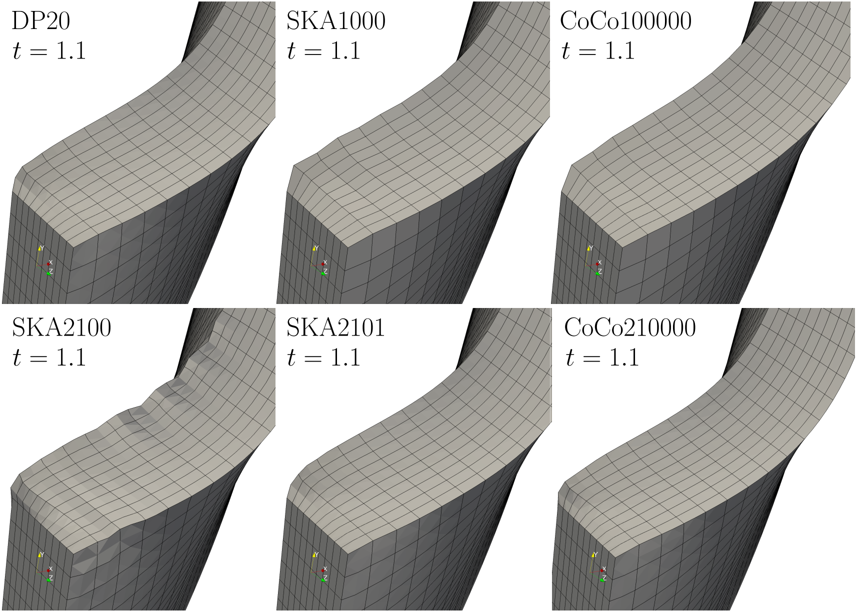
Numerical example - Cook cantilever beam
Conversation properties - Momentum



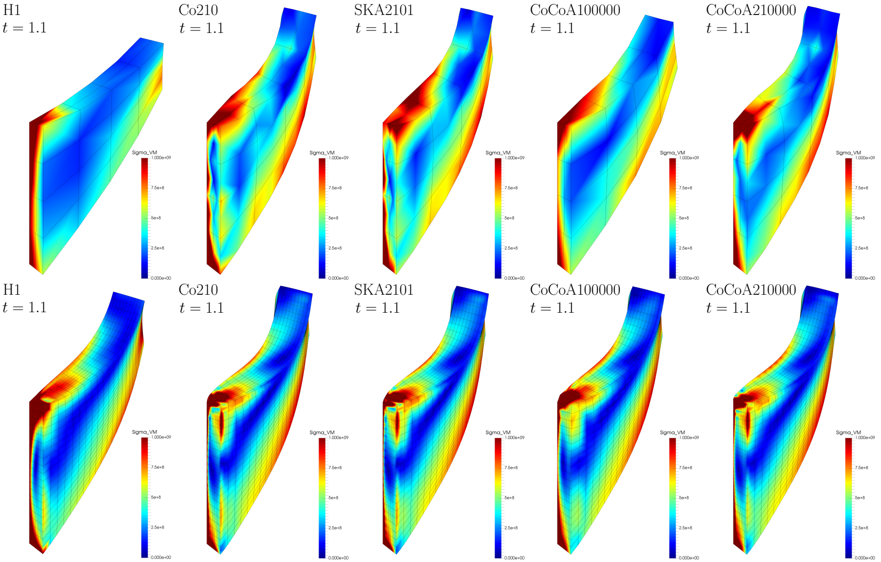
Numerical example - Cook cantilever beam
Conversation properties - Angular momentum



Numerical example - Cook cantilever beam
SKA element



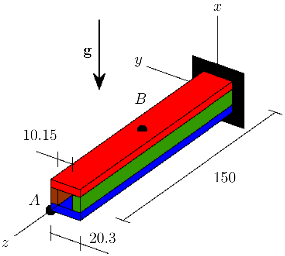
Numerical example - Cook cantilever beam
Deformed configuration and v. Mises equivalent stress for $n_{el} = 32$ and $n_{el} = 4000$



- Compare the non-standard mixed elements with the standard elements for tetrahedral elements
- Oscillating in a gravity field $\boldsymbol{g} = \begin{bmatrix} -9.81 & 0 & 0 \end{bmatrix}^T$
- Different fiber directions
- Strain energy function ([Schr 11],[Holz 00]):

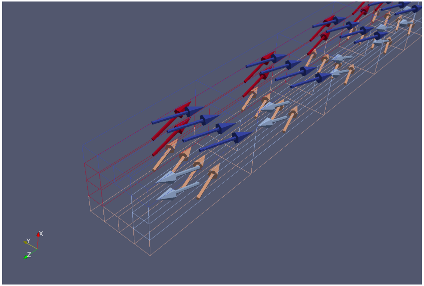
$$\Psi^{iso} = \frac{\epsilon_1}{2}(\mathrm{tr}[\boldsymbol{C}])^2 + \frac{\epsilon_2}{2}(\mathrm{tr}[\mathrm{cof}[\boldsymbol{C}]])^2 - \epsilon_3 \ln(J) + \frac{\epsilon_4}{2}(J^{\epsilon_5} + J^{-\epsilon_5} - 2)$$

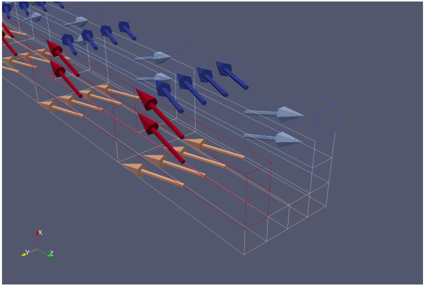
$$\Psi^{ani} = \frac{\epsilon_6}{2\epsilon_7}\left(e^{\epsilon_7 (\mathrm{tr}[\boldsymbol{C}\boldsymbol{M}])^2} - 1\right)$$

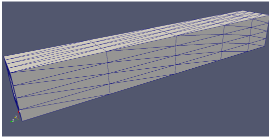


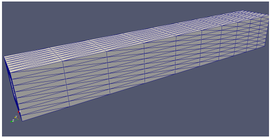
$\epsilon_1 = 16e6$	$\rho_0 = 4e5$
$\epsilon_2 = 32e6$	$T = 3$
$\epsilon_3 = 480e6$	$h_n = 0.004$
$\epsilon_4 = 80e6$	$TOL = 1e-4$
$\epsilon_5 = 20$	$\boldsymbol{a}_0 = [0 \;\; -0.3095 \;\; -0.9509]$
$\epsilon_6 = 10e10$	$\boldsymbol{a}_0 = [0.3095 \; 0 \; 0.9509]$
$\epsilon_7 = 10$	$\boldsymbol{a}_0 = [0 \; 0.3095 \;\; -0.9509]$
	$\boldsymbol{a}_0 = [0.3095 \; 0 \;\; -0.9509]$

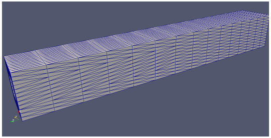
Numerical example - Oscillating cantilever beam
Fiber direction and refinement levels

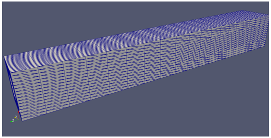




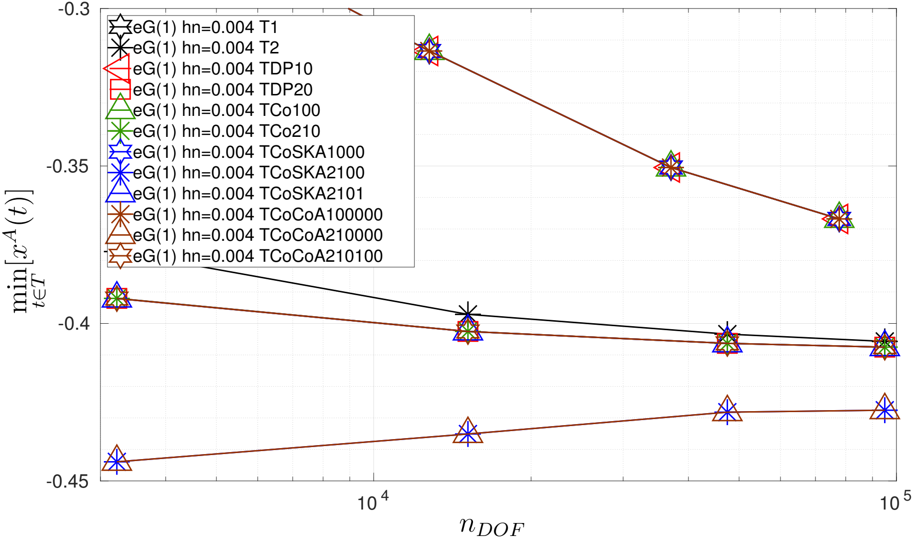






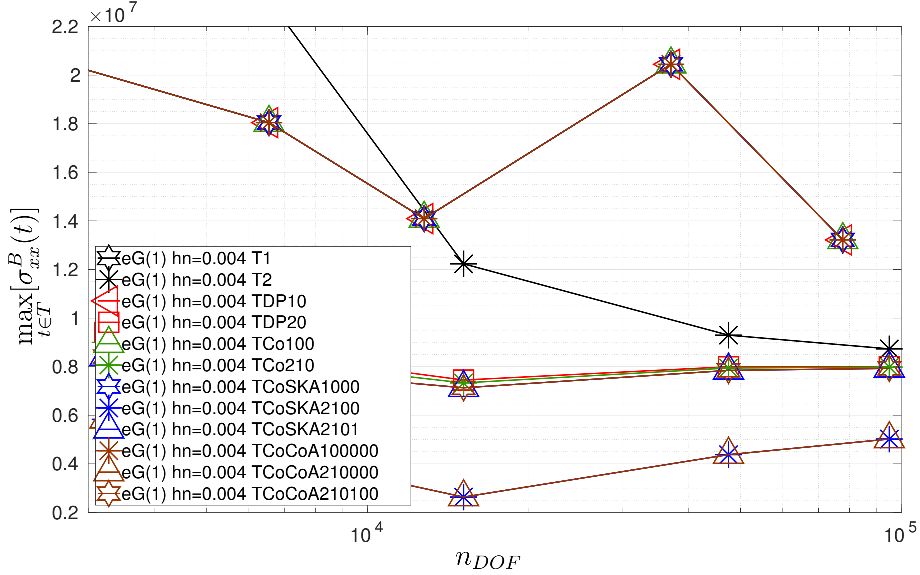


Numerical example - Oscillating cantilever beam
Convergence of the x-coordinate at Point A



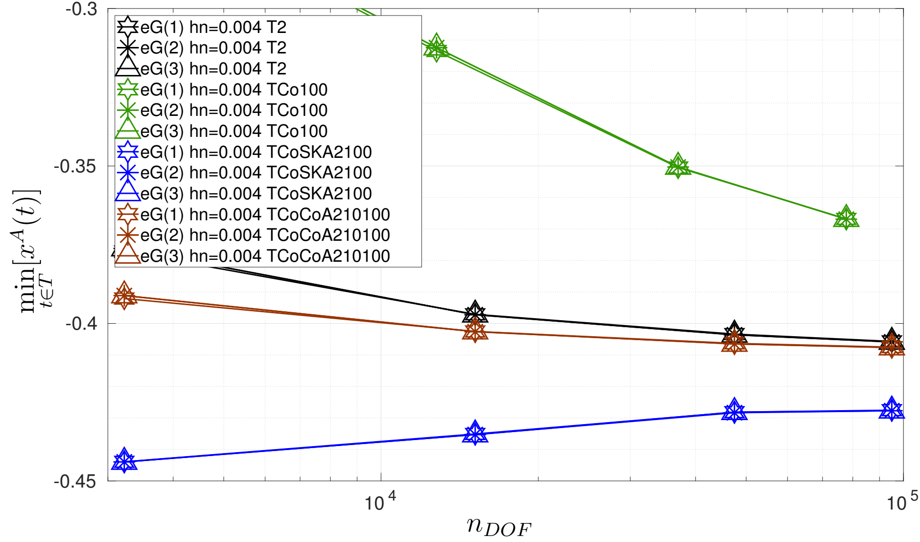
Digits (pol. degree of): T[q] TDP[q,Θ] TCo[q,H,Θ] TCoCoA[q,H,Θ,C_A,H_A,Θ_A]

Numerical example - Oscillating cantilever beam
Convergence of σ_{xx} at Point B



Digits (pol. degree of): T[q] TDP[q,Θ] TCo[q,H,Θ] TCoCoA[q,H,Θ,C_A,H_A,Θ_A]

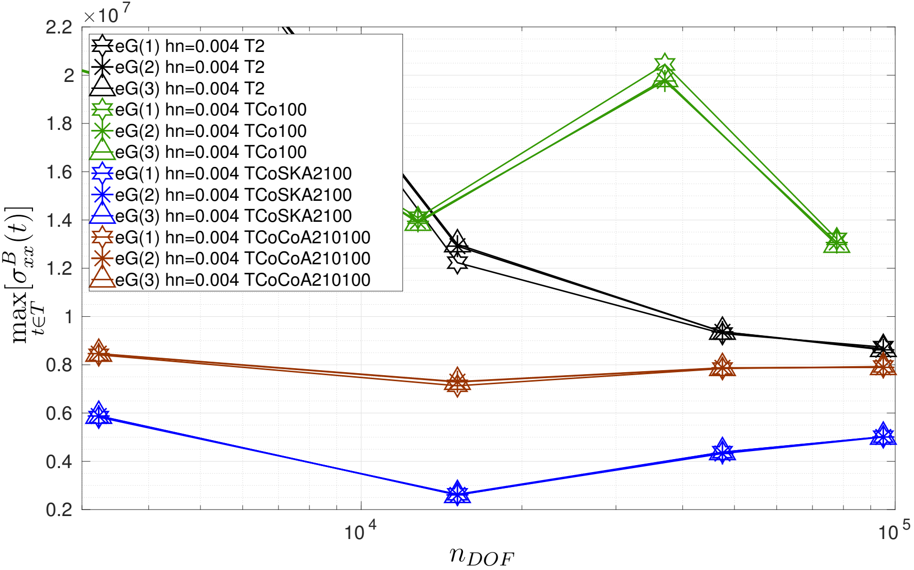
Numerical example - Oscillating cantilever beam
Convergence of the x-coordinate at Point A



Digits (pol. degree of): T[q] TDP[q,Θ] TCo[q,H,Θ] TCoCoA[q,H,Θ,C_A,H_A,Θ_A]

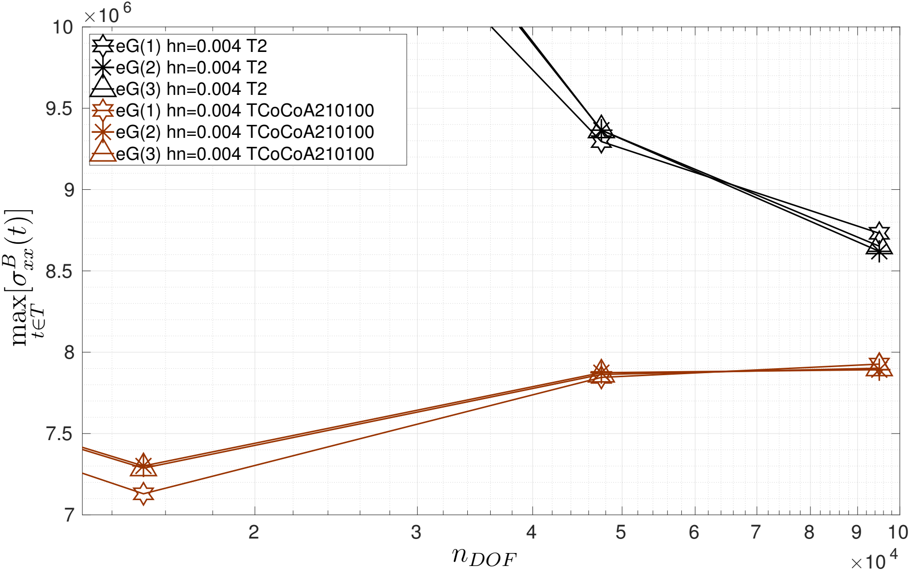
Numerical example - Oscillating cantilever beam

Convergence of σ_{xx} at Point B



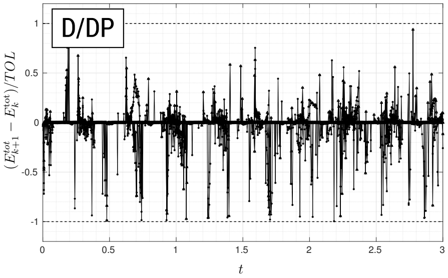
Digits (pol. degree of): T[q]   TDP[q,Θ]   TCo[q,H,Θ]   TCoCoA[q,H,Θ,C_A,H_A,Θ_A]

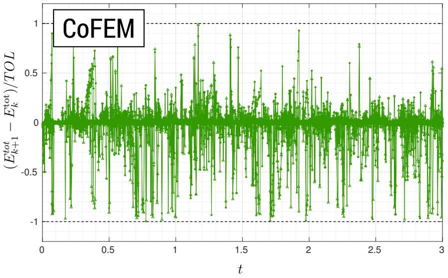
Numerical example - Oscillating cantilever beam
Convergence of σ_{xx} at Point B - Detail

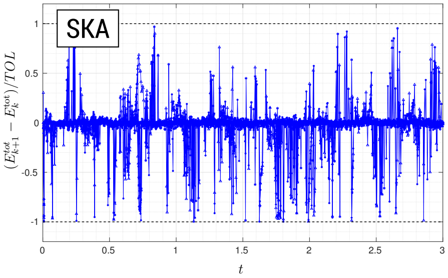


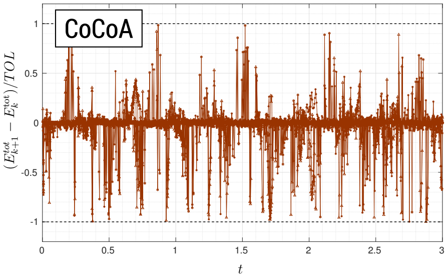
Digits (pol. degree of): T[q] TDP[q,Θ] TCo[q,H,Θ] TCoCoA[q,H,Θ,C_A,H_A,Θ_A]

Numerical example - Oscillating cantilever beam
Conversation properties - Energy

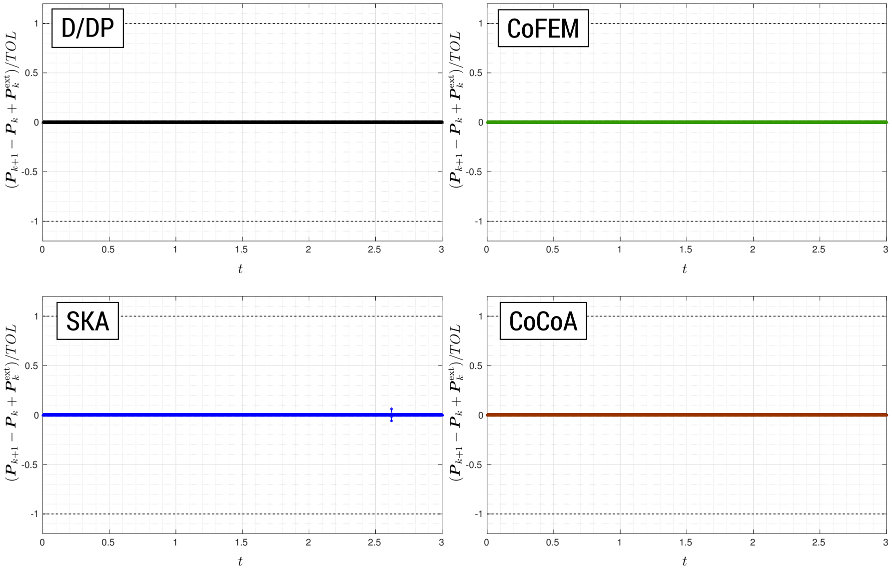




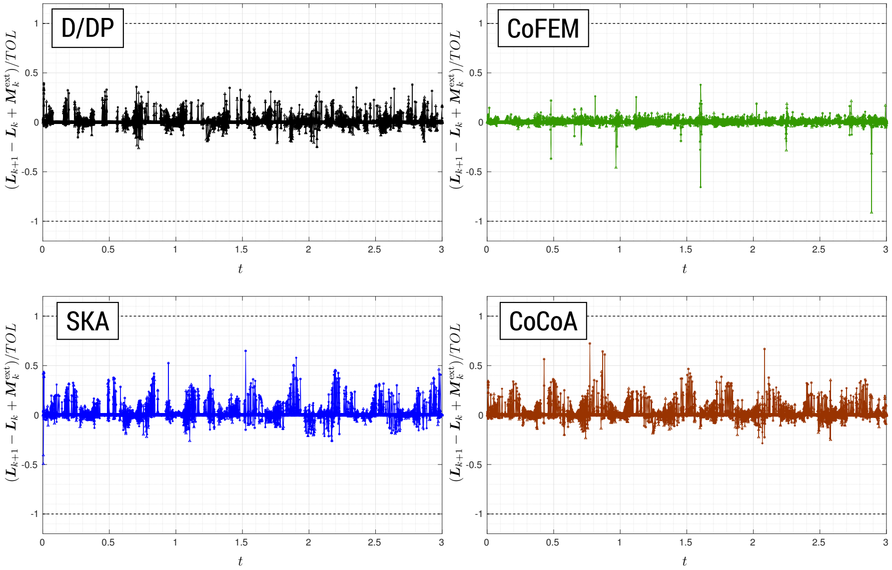




Numerical example - Oscillating cantilever beam
Conversation properties - Momentum

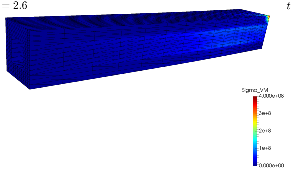


Numerical example - Oscillating cantilever beam
Conversation properties - Angular momentum

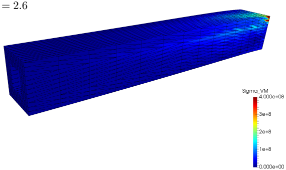


Numerical example - Oscillating cantilever beam
Deformed configuration and v. Mises equivalent stress for $n_{el} = 10169$

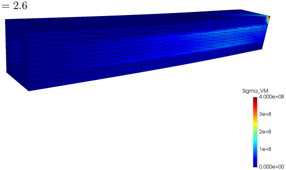
T2
$t = 2.6$



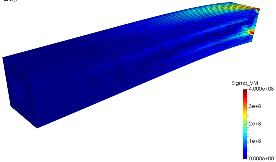
SKA2100
$t = 2.6$



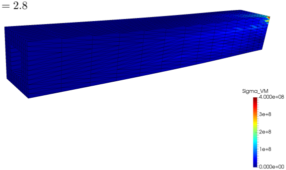
CoCoA210100
$t = 2.6$



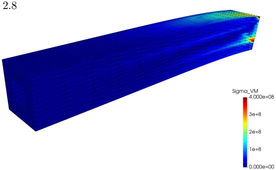
T2
$t = 2.8$



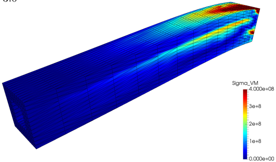
SKA2100
$t = 2.8$



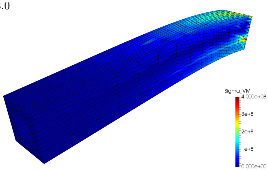
CoCoA210100
$t = 2.8$



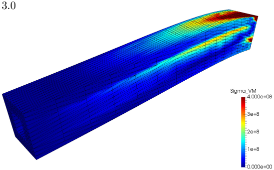
T2
$t = 3.0$



SKA2100
$t = 3.0$



CoCoA210100
$t = 3.0$



Conclusion

- Motivation:
  - Dynamic simulations of fiber-reinforced materials in light-weight structures
- Strategy:
  - Mixed finite elements for finite anisotropic elastodynamics
  - Higher-order time integrators [continuous Galerkin cG(k)]
  - Energy-momentum conserving time integrators [Discrete gradient eG(k)]
- Important results:
  - Mixed element reduce locking effect
  - Higher-order time integrators increase accuracy
  - Discrete gradient conserves energy
- Outlook:
  - Extend these formulations to a thermo-viscoelastic material behavior

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