Prof. Dr. Vladimir Shikhman
Professur für Wirtschaftsmathematik
Technische Universität Chemnitz

Übungsleiter: David Müller
david.mueller@mathematik.tu-chemnitz.de

Mathematical Foundation of Big Data Analytics (SS 2019)
Online Learning I

Ex. 1 A function $u : \mathcal{X} \to \mathbb{R}$ defined on an convex set $\mathcal{X} \subset \mathbb{R}^n$ is called concave, if for all $x, y \in \mathcal{X}$ and $\lambda \in [0, 1]$

$$u(\lambda x + (1 - \lambda)y) \geq \lambda u(x) + (1 - \lambda)u(y).$$

a) Show that for $u : \mathcal{X} \to \mathbb{R}$ differentiable, the function is concave if and only if for all $x, y \in \mathcal{X}$

$$u(y) \leq u(x) + \nabla^T u(x)(y - x).$$

b Show that a twice differentiable function $u : \mathcal{X} \to \mathbb{R}$ defined on an open convex set $\mathcal{X} \subset \mathbb{R}^n$ is concave if and only if the Hessian matrix $\nabla^2 u(x)$ is negative semidefinite for all $x \in \mathcal{X}$.

Ex. 2 The euclidean projection $\pi(x)$ of a point $x \in \mathbb{R}^n$ on a closed convex subset C of $\mathbb{R}^n$ is defined by

$$\pi(x) = \arg\inf_{y \in C} \frac{1}{2}||x - y||_2^2$$

a) Show that such an element $\pi(x) \in C$ exists for every $x \in \mathbb{R}^n$.

b) Prove the following variational characterization

$$(x - \pi(x))^T(y - \pi(x)) \leq 0 \quad \forall y \in C.$$

c) Show that the projection $\pi(x)$ is unique.

Ex. 3 Recall the definition of the projected gradient algorithm for minimizing a function on a closed convex and bounded set C

$$y(l + 1) = u(x(l), p(l)) - \alpha \nabla_x^T u(x(l), p(l)) \quad l = 1, \ldots, \tag{1}$$

$$x(l + 1) = \arg\inf_{x \in C} \frac{1}{2}||y(l + 1) - x||_2^2 \tag{2}$$

a) Suppose you have to maximize the logarithmic utility function $u(x, p)$ where the decision variables x are elements of the simplex $\triangle$. Verify the above conditions for the simplex $\triangle$ and show concavity of the objective function.

b) The optimal learning rate for the online investment problem is $\alpha = \frac{2 \cdot R}{G \cdot \sqrt{t}}$. Assume that prices of the assets are bounded below and above, i.e. $0 < \underline{p} \leq p^{(i)} \leq \overline{p}$ for $i = 1, \ldots, n$. Derive the update rule (1) for the problem.

c) Derive the rule for (2), i.e calculate the euclidean projection on the simplex.