Equilibrium Selection

Multicriteria Optimization and Semi-infinite Programming

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Karlsruhe Institute of Technology (KIT)

UNIVERS Winter School on Optimization, Games and Markets
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Big thanks go to

- Didier Aussel and Vladimir Shikhman

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Agenda

1. Introduction and goal of this mini-course
2. Optimality notions in multicriteria optimization
3. Methods for computing nondominated points
4. Functional descriptions of equilibrium sets

Nash equilibria

We consider players $\nu \in \{1, \ldots, N\}$ who aim to choose their decision vectors x^ν as minimal points of

$$Q_\nu(x^{-\nu}): \qquad \min_{x^\nu} \overset{f_\nu}{\theta_\nu}(x^\nu, x^{-\nu}) \quad \text{s.t.} \quad x^\nu \in \overset{M_\nu}{X_\nu}(x^{-\nu}),$$

given the vector $x^{-\nu}$ of all other players' decisions.

Let $S_\nu(x^{-\nu})$ denote the minimal point set of $Q_\nu(x^{-\nu})$. Then $x^\star = (x^{1,\star}, \ldots, x^{N,\star})$ is called a (generalized) Nash equilibrium iff

$$x^{\nu,\star} \in S_\nu(x^{-\nu,\star}), \quad \nu = 1, \ldots, N,$$

holds.

No cooperation between the players is assumed.

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holds.

No cooperation between the players is assumed.

Example 1

Let $N = 2,\ n_1 = n_2 = 1$

(so that $x^1 = x^{-2} = x_1$ and $x^2 = x^{-1} = x_2$),

$$\theta_1(x) \underline{= x_1,} \quad g_1^1(x) = -2x_1 + x_2,$$

$$\theta_2(x) \underline{= x_2,} \quad g_1^2(x) = x_1^2 + x_2^2 - 1, \quad g_2^2(x) = -x_1 - x_2, \ \text{that is,}$$

$$\rightarrow Q_1(x_2): \quad \min_{x_1} x_1 \quad \text{s.t.} \quad -2x_1 + x_2 \le 0,$$

$$Q_2(x_1): \quad \min_{x_2} x_2 \quad \text{s.t.} \quad x_1^2 + x_2^2 - 1 \le 0,\ -x_1 - x_2 \le 0.$$

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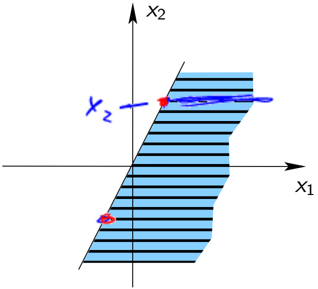

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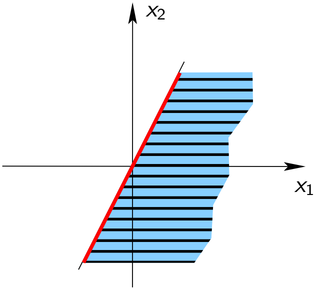

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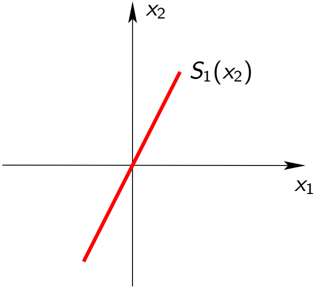

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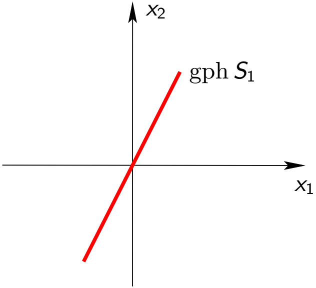

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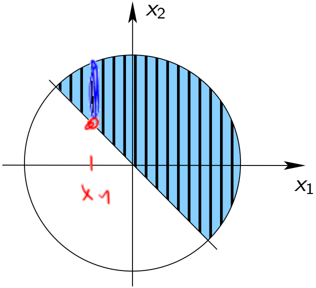

$$Q_2(x_1): \quad \min_{x_2} \; x_2 \quad \text{s.t.} \quad x_1^2 + x_2^2 - 1 \leq 0, \; -x_1 - x_2 \leq 0$$

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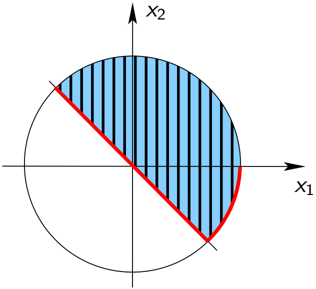

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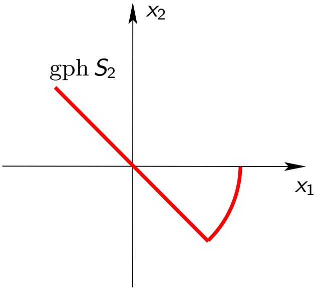

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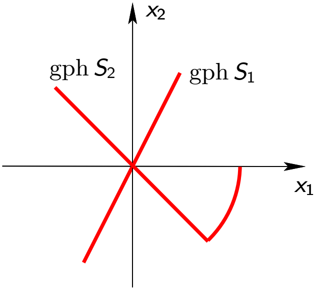

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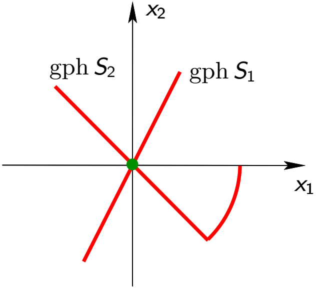

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The set of equilibria

With the graph $\operatorname{gph} S_\nu$ of the set-valued mapping S_ν

$$E \ := \ \bigcap_{\nu=1}^{N} \operatorname{gph} S_\nu$$

forms the set of all generalized Nash equilibria (GNEs).

The task to identify an element of E is called generalized Nash equilibrium problem (GNEP).

E may be empty, a singleton, or a non-singleton set.

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How it all started ...

- J. von Neumann, *Zur Theorie der Gesellschaftsspiele,* Mathematische Annalen, Vol. 100 (1928), 295-320. English translation: *On the Theory of Games of Strategy,* in: A.W. Tucker and R.D. Luce (eds.), Contributions to the Theory of Games, Vol. IV, Annals of the Mathematics Studies 40. Princeton University Press.
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- G. Debreu, *A social equilibrium existence theorem,* Proceedings of the National Academy of Sciences, Vol. 38 (1952), 886–893.
- K.J. Arrow, G. Debreu, *Existence of an equilibrium for a competitive economy,* Econometrica, Vol. 22 (1954), 265–290.

Prisoners' dilemma

$X_1 = X_2 = \{\text{c (confession)}, \text{s (silence)}\}$

Player 2

Player 1

θ_1, θ_2	c	s
c	$8, 8$	$0, 10$
s	$10, 0$	$\underline{3, 3}$

Exercise 1

Is there a Nash equilibrium? And if yes, how many?

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Modified prisoners' dilemma

$X_1 = X_2 = \{\text{c (confession)}, \text{s (silence)}\}$

θ_1, θ_2	c	s
c	$8, 8$	$5, 10$
s	$10, 5$	$3, 3$

Exercise 2

Is there a Nash equilibrium? And if yes, how many?

Modified prisoners' dilemma

$X_1 = X_2 = \{\text{c (confession)}, \text{s (silence)}\}$

θ_1, θ_2	c	s
c	$8, 8$	$5, 10$
s	$10, 5$	$3, 3$

Exercise 2

Is there a Nash equilibrium? And if yes, how many?

Equilibrium selection

In the case $|E| > 1$ the players may prefer some equilibria over others.

Their preferences may be explained by refined equilibrium concepts

$\rightsquigarrow$ Equilibrium selection / Nash refinement

J.C. Harsanyi, R. Selten, *A General Theory of Equilibrium Selection in Games*, MIT Press Books, Cambridge, 1988.

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Risk dominance

Two main concepts for equilibrium selection:

1. A Nash equilibrium is called risk dominant if it has the largest basin of attraction (i.e. is less risky).

Risk dominance takes a dynamic/evolutionary point of view, while we keep the static point of view.

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Payoff dominance

Two main concepts for equilibrium selection:

2. A Nash equilibrium is called payoff dominant if it is Pareto superior to all other Nash equilibria in the game. When faced with a choice among equilibria, all players would agree on a payoff dominant equilibrium since it offers to each player at least as much payoff as the other Nash equilibria.

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Cost dominance

3. A Nash equilibrium is called cost dominant if it is Pareto superior to all other Nash equilibria in the game. When faced with a choice among equilibria, all players would agree on the cost dominant equilibrium since it offers to each player at most the costs as the other Nash equilibria.

Modified prisoners' dilemma

$X_1 = X_2 = \{\text{c (confession)}, \text{s (silence)}\}$

θ_1, θ_2	c	s
c	$8, 8$	$5,$ ~~1~~0 8
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Exercise 3

Is one of the Nash equilibria cost dominant? And if yes, which one?

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Example 2

Let $N = 2$, $n_1 = n_2 = 1$, $q_1,\ q_2 : \mathbb{R}^1 \to \mathbb{R}^1$ convex quadratic,

$$\theta_1(x) = \underline{x_1,} \quad g_1^1(x) = q_1(x_2) - x_1,$$

$$\theta_2(x) = \underline{x_2,} \quad g_1^2(x) = q_2(x_1) - x_2, \text{ that is,}$$

$$Q_1(x_2): \quad \min_{x_1} x_1 \quad \text{s.t.} \quad \underline{q_1(x_2) \leq x_1,}$$

$$Q_2(x_1): \quad \min_{x_2} x_2 \quad \text{s.t.} \quad q_2(x_1) \leq x_2.$$

The functions θ_1, θ_2, g_1^1 and g_1^2 are convex in (x_1, x_2), i.e., not only with respect to the player variables. This is called complete convexity.

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$g^1(x_1,x_2) = q_1(x_2) - x_1 \le 0$

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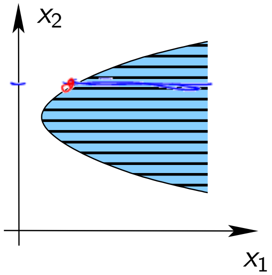

$$Q_1(x_2): \quad \min_{x_1} \; x_1 \quad \text{s.t.} \quad q_1(x_2) \le x_1$$

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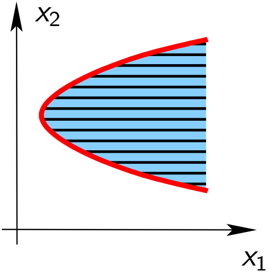

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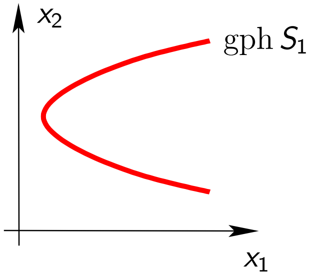

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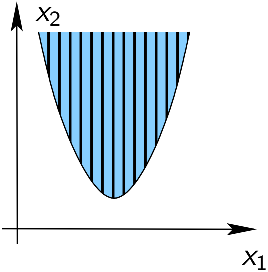

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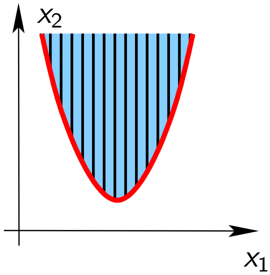

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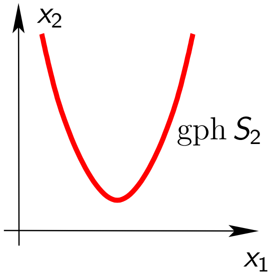

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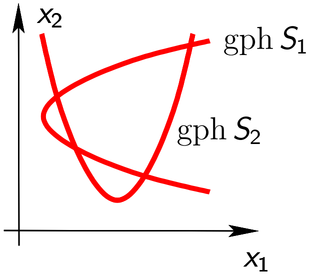

$$x^{1,\star} \in S_1(x^{2,\star}), \quad x^{2,\star} \in S_2(x^{1,\star})$$

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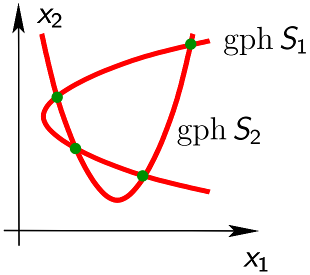

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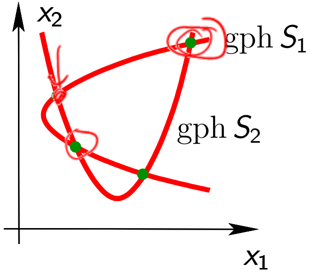

Exercise 4

Are any of the GNEs cost dominant? And if yes, which ones?

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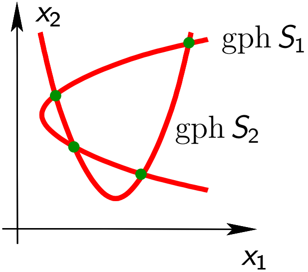

Exercise 4

Are any of the GNEs cost dominant? And if yes, which ones?
None!

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Let $N = 2$, $n_1 = n_2 = 1$,

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$$Q_1(x_2): \quad \min_{x_1} x_1 \quad \text{s.t.} \quad x_1^2 + x_2^2 \leq 1,$$

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The function g_1 appears simultaneously as a constraint of player 1 and of player 2. This is called a shared constraint.

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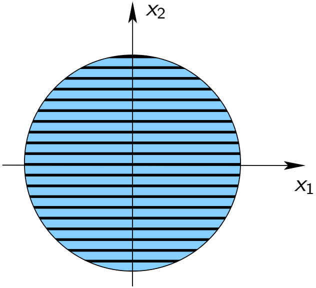

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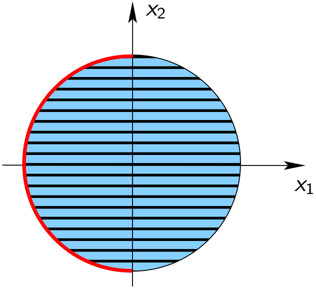

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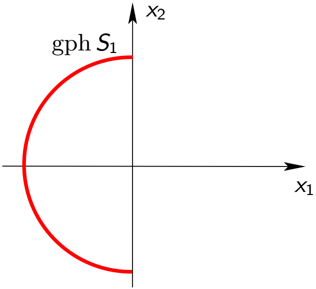

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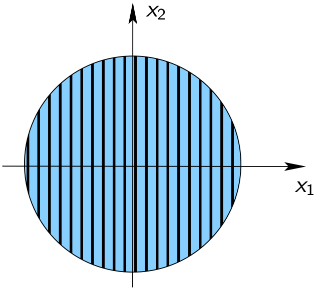
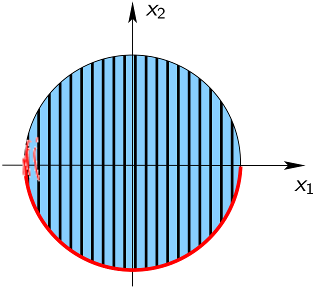


$$Q_2(x_2): \quad \min_{x_2} x_1 \quad \text{s.t.} \quad x_1^2 + x_2^2 \leq 1$$

Example 3



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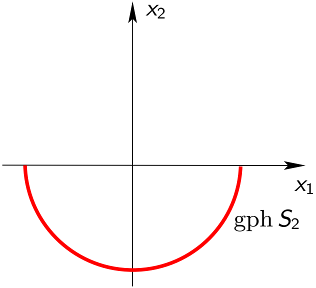

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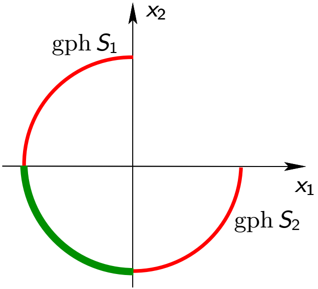
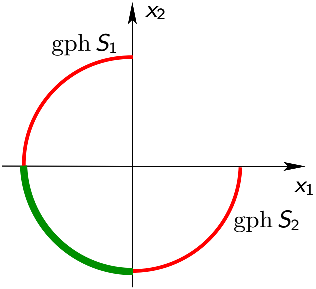


$$x^{1,\star} \in S_1(x^{2,\star}), \quad x^{2,\star} \in S_2(x^{1,\star})$$

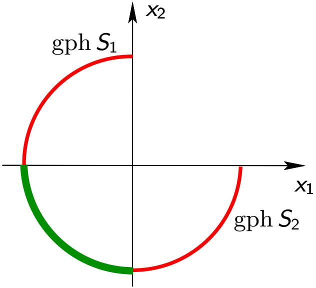
Example 3



Exercise 5

Are any of the GNEs cost dominant? And if yes, which ones?

Example 3



Exercise 5

Are any of the GNEs cost dominant? And if yes, which ones?
None!

Pareto superiority vs. Pareto noninferiority

Recall:

A Nash equilibrium is called cost dominant if it is Pareto superior to all other Nash equilibria in the game. When faced with a choice among equilibria, all players would agree on a cost dominant equilibrium since it offers to each player at most the costs as the other Nash equilibria.

Alternative and more appropriate concept:

A Nash equilibrium is called cost nondominated if it is Pareto noninferior to all other Nash equilibria in the game. When faced with a choice among equilibria, the players would not agree on a cost dominated equilibrium, since this would offer at least one player lower costs when moving to the dominating equilibrium, while none of the other players face higher costs.

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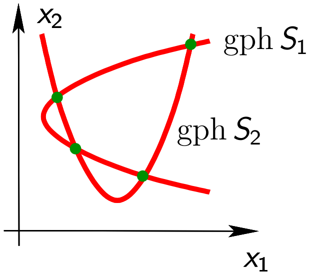
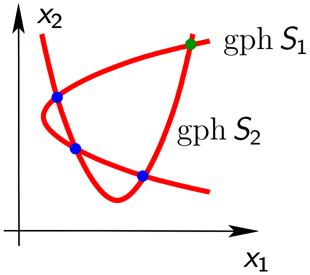
Example 2



Exercise 6

Are any of the GNEs cost nondominated? And if yes, which ones?

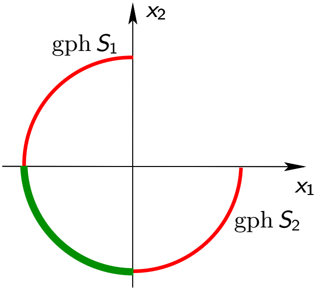
Example 2



Exercise 6

Are any of the GNEs cost nondominated? And if yes, which ones?

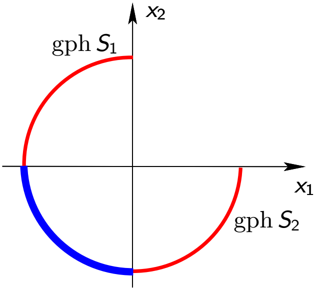
Example 3



Exercise 7

Are any of the GNEs cost nondominated? And if yes, which ones?

Example 3



Exercise 7

Are any of the GNEs cost nondominated? And if yes, which ones?

Goal and agenda of this mini-course

Goal:

Design a method to compute (all) cost nondominated GNE(s) under possibly mild assumptions.

Agenda:

- Proper definition of Pareto superiority/noninferiority
- Methods for finding (all) Pareto noninferior points of a multicriteria problem
- Functional descriptions of the equilibrium set E
- Methods for finding (all) cost nondominated GNE(s)

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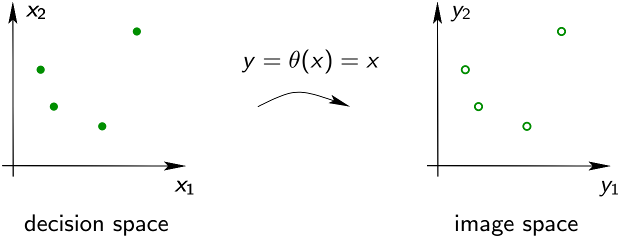
The image space

So far, all our graphical examples used the data $n = n_1 + n_2 = 2$, $N = 2$, $\theta_1(x_1, x_2) = x_1$, $\theta_2(x_1, x_2) = x_2$.

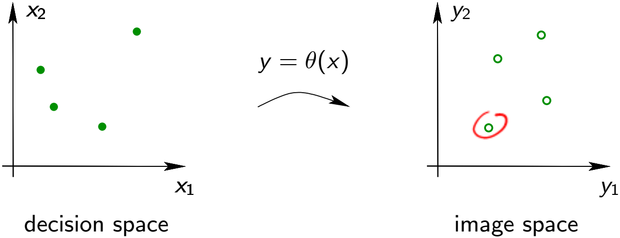
This results in $\theta(x) = x$ so that, in particular, the position of equilibria x and their image points $y = \theta(x)$ are identical.



The image space

However, the Pareto properties of equilibria x depend on the position of $y = \theta(x)$ in the image space,

and more general functions θ usually lead to image space positions which are different from the decision space positions.



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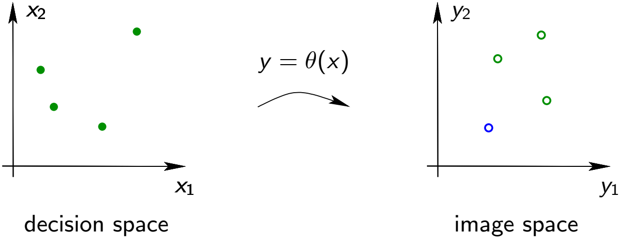

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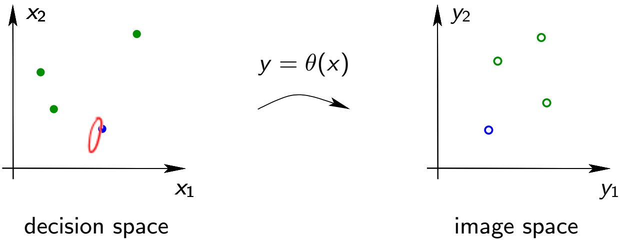
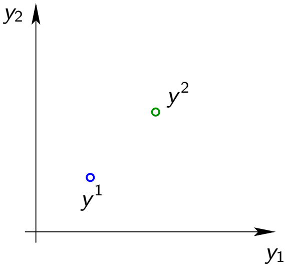
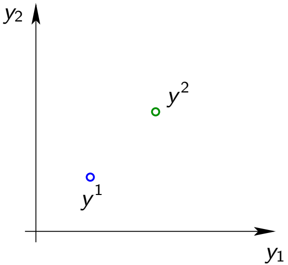


Domination



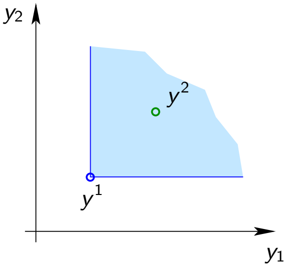
For $y^1 \leq y^2$ the point y^1 dominates y^2, and y^2 is dominated by y^1.

Domination



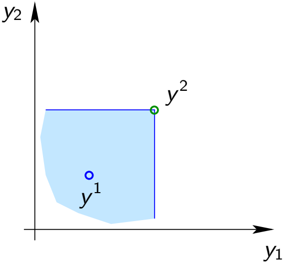
For $y^1 \leq y^2$ the point y^1 dominates y^2, and y^2 is dominated by y^1.
$y^1 \neq y^2$

Domination



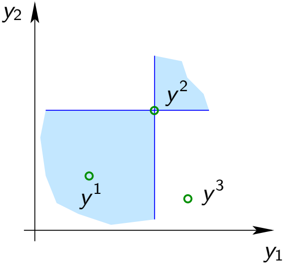
y^1 dominates all points in the set $y^1 + \underline{\mathbb{R}^N_+} = \{y \in \mathbb{R}^N |\ y^1 \leq y\}$, except for y^1 itself.

Domination



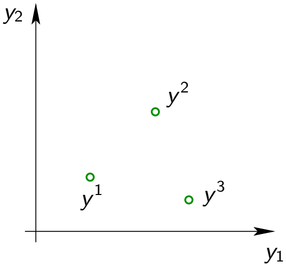
y^2 is dominated by all points in $y^2 - \mathbb{R}^N_+ = \{y \in \mathbb{R}^N |\ y \leq y^2\}$ except by y^2 itself.

Domination



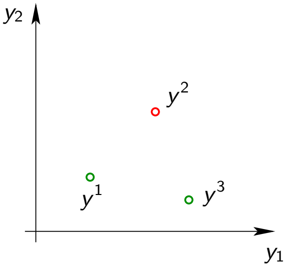
For $N > 1$ not all points in $\mathbb{R}^N$ can be mutually compared by domination.

Domination



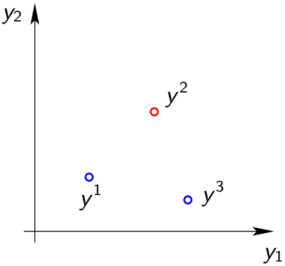
Still, one may dispose of all points which are “not interesting” ...

Domination



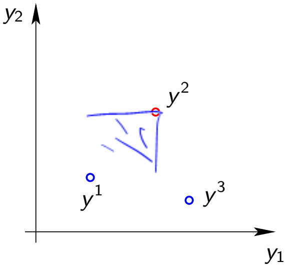
… that is, the dominated ones, …

Domination



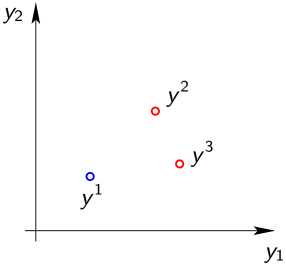
… and rather concentrate one the nondominated points.

Domination



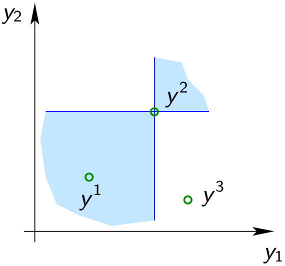
For $Y \subseteq \mathbb{R}^N$ a point $\bar{y} \in Y$ is called a nondominated point of Y if there is no $\underbrace{y \in Y \text{ with } y \leq \bar{y},\ y \neq \bar{y}}$.

Domination



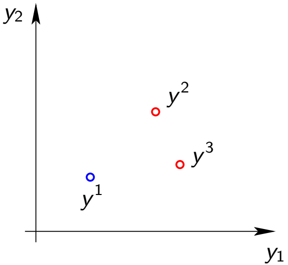
For $Y \subseteq \mathbb{R}^N$ a point $\bar{y} \in Y$ is called a dominant point of Y if all $y \in \underline{Y \setminus \{\bar{y}\}}$ satisfy $\bar{y} \leq y$.

Domination



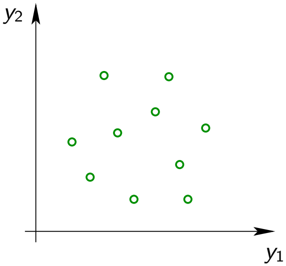
Since domination is not the opposite of nondomination, the concepts of nondominated and dominant points are different.

Domination



While a dominant point is always nondominated ...

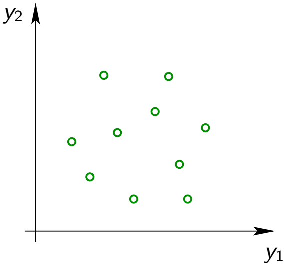
Domination



... even simple sets Y do not possess a dominant point.

But they possess nondominated points under mild assumptions.

Domination



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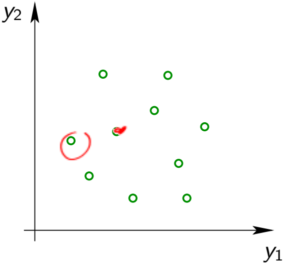
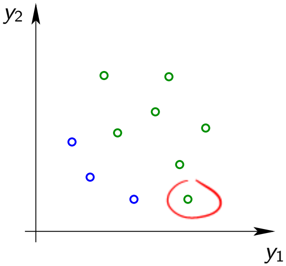


Exercise 8

Determine the nondominated points of the above set.

Domination



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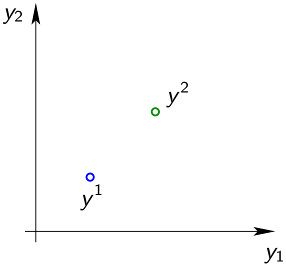
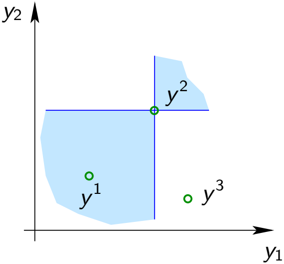
Determine the nondominated points of the above set.

Domination and Pareto terminology



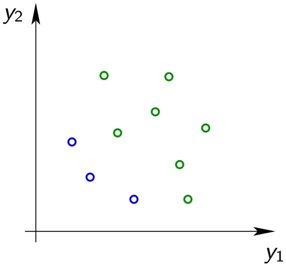
For $y^1 \leq y^2$, $y^1 \neq y^2$, y^1 is also called Pareto superior to y^2, and y^2 Pareto inferior to y^1.

Domination and Pareto terminology



Therefore, the concepts of Pareto superior points and Pareto noninferior points are different ...

Domination and Pareto terminology

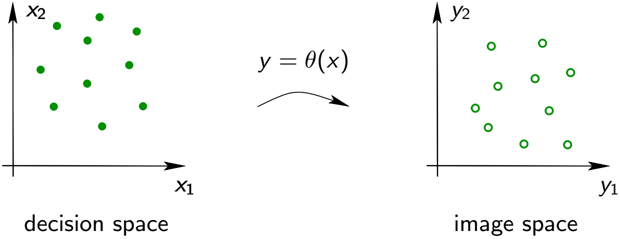


... and even simple sets do not possess Pareto superior points, but they often possess Pareto noninferior points.

Decision space and image space

Exercise 9

In equilibrium selection, of which set Y are we interested in nondominated / Pareto noninferior points?



Decision space and image space

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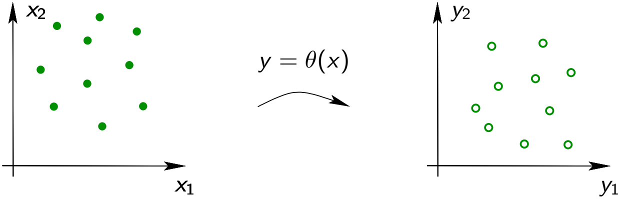
In equilibrium selection, of which set Y are we interested in nondominated / Pareto noninferior points?

$Y := \theta(E)$, the image set of E under the vector function θ.



decision space

image space

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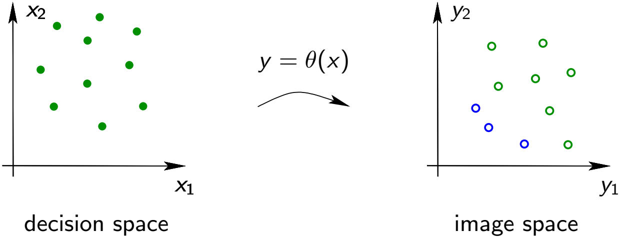
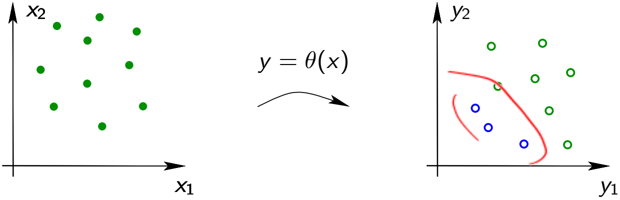


Decision space and image space

Exercise 10

What can we say about the positions of the interesting equilibria?



decision space

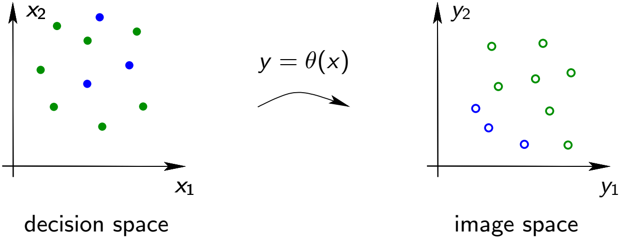
image space

Decision space and image space

Exercise 10

What can we say about the positions of the interesting equilibria?

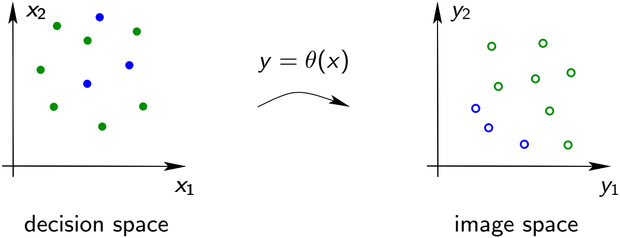
As preimages of the nondominated points of Y under θ, they do not possess any special positions.



Efficient points

Nondominance / Pareto noninferiority is an image space concept.

The preimages x of nondominated / Pareto noninferior points are called efficient points of θ on E.

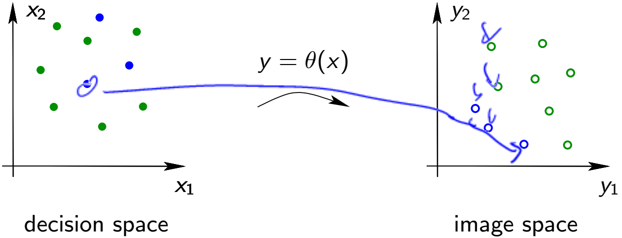


Efficient points

An efficient point $\bar{x}$ of θ on E cannot be strictly improved in one objective function θ_ν by moving to some $x \in E \setminus \{\bar{x}\}$, without strictly worsening another objective function θ_μ, i.e.

negation $\forall \mu : \theta_\mu(x) \leq \theta_\mu(\bar{x})$

$$\underline{\theta_\nu(x) < \theta_\nu(\bar{x})} \quad \Rightarrow \quad \underline{\exists\, \mu} \in \{1, \ldots, N\} : \ \theta_\mu(x) > \theta_\mu(\bar{x}).$$



Recall: Pareto superiority vs. Pareto noninferiority

Superiority concept:

A point $\bar{x} \in E$ is called cost dominant if it is Pareto superior to all other $x \in E$. When faced with a choice among equilibria, all players would agree on a cost dominant equilibrium $\bar{x}$ since it offers to each player at most the costs as the other Nash equilibria.

Noninferiority concept:

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Recall our goal

Goal:

Design a method to compute (all) cost nondominated GNE(s) under possibly mild assumptions.

With the introduced terminology this means:

compute (all) efficient points of θ on E,

i.e. (all) preimages of (all) nondominated points of $\theta(E)$.

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Multicriteria optimality notions

In the case $N = 1$, $\bar{y} \in \theta(E)$ is a nondominated point of $\theta(E)$ if

there is no $y \in \theta(E)$ with $y \leq \bar{y},\ y \neq \bar{y}$

$\Updownarrow$

there is no $y \in \theta(E)$ with $y < \bar{y}$

$\Updownarrow$

there is no $x \in E$ with $\theta(x) < \theta(\bar{x})$
(where $\bar{x} \in E$ is any preimage of $\bar{y}$ under θ)

$\Updownarrow$

$\bar{x}$ is a minimal point of θ on E with minimal value $\bar{y}$.

Multicriteria optimality notions

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(where $\bar{x} \in E$ is any preimage of $\bar{y}$ under θ)

$\Updownarrow$

$\bar{x}$ is a minimal point of θ on E with minimal value $\bar{y}$.

Multicriteria optimality notions

Exercise 11

In the case $N = 1$, $\bar{y} \in \theta(E)$ is a dominant point of $\theta(E)$ if

all $\underline{y \in \theta(E) \setminus \{\bar{y}\}}$ satisfy $\bar{y} \leq y$

$\Updownarrow$

$\theta(\bar{x}) \leq \theta(x) \quad \forall x \in E$

???

Multicriteria optimality notions

In the case $N = 1$,

$\bar{x} \in E$ is a minimal point of θ on E with minimal value $\bar{y} = \theta(\bar{x})$

$\Updownarrow$

there is no $x \in E$ with $\theta(x) < \bar{y}$

$\Updownarrow$

there is no $y \in \theta(E)$ with $y < \bar{y}$

$\Updownarrow$

$\bar{y} \in \theta(E)$ is a weakly nondominated point of $\theta(E) =: Y$.

Multicriteria optimality notions

In the case $N = 1$,

$\bar{x} \in E$ is a minimal point of θ on E with minimal value $\bar{y} = \theta(\bar{x})$

$\Updownarrow$

there is no $x \in E$ with $\theta(x) < \bar{y}$

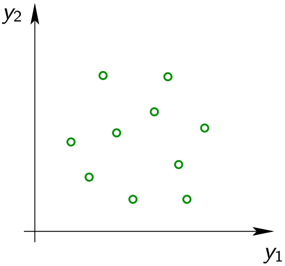
$\Updownarrow$

there is no $y \in \theta(E)$ with $y < \bar{y}$

$\Updownarrow$

$\bar{y} \in \theta(E)$ is a weakly nondominated point of $\theta(E) =: Y$.

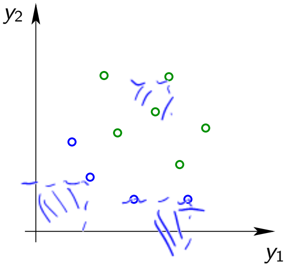
Weakly nondominated points



Exercise 12

Determine the weakly nondominated points of the above set.

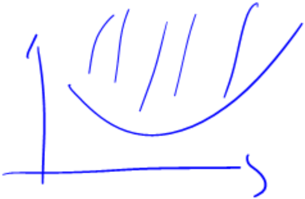
Weakly nondominated points

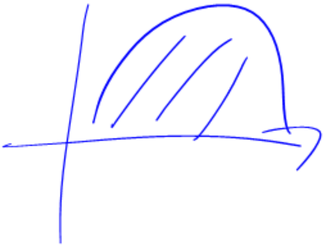


Exercise 12

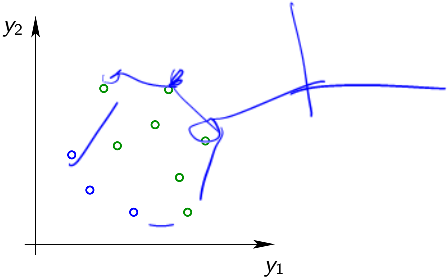
Determine the weakly nondominated points of the above set.

$$\min \underset{\substack{\uparrow\\ \text{convex}}}{f(x)} \quad \text{s.t.} \quad x \in \underset{\substack{\uparrow\\ \text{convex}}}{M}$$



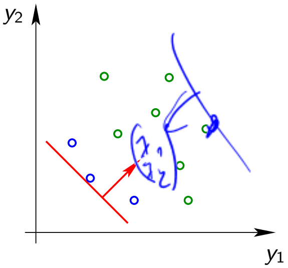


Computation of nondominated points



How can we compute (all) nondominated points of some set $Y \subseteq \mathbb{R}^N$?

Computation of nondominated points



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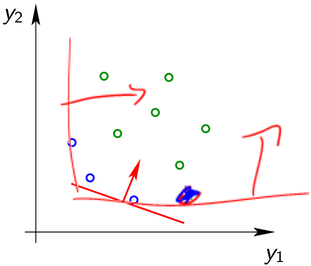

How can we compute (all) nondominated points of some set $Y \subseteq \mathbb{R}^N$?

The weighted sum method

Lemma

Let $Y \subseteq \mathbb{R}^N$ and $\lambda \in \mathbb{R}^N$ with $\lambda > 0$. Then any minimal point of

$$P(\lambda): \qquad \min \underbrace{\langle \lambda, y \rangle}_{\sum \lambda_i y_i} \quad s.t. \quad y \in Y$$

is a nondominated point of Y.

Proof: Assume $\bar{y}$ is a minimal point of $P(\lambda)$,
but a dominated point of Y.

$\Rightarrow \quad \exists\, y \in Y: \quad y \leq \bar{y},\ y \neq \bar{y}$

$\Rightarrow \quad \langle \lambda, y \rangle - \langle \lambda, \bar{y} \rangle = \langle \lambda, y - \bar{y} \rangle = \sum_{j=1}^{N} \lambda_j (y_j - \bar{y}_j) < 0$

$\Rightarrow \quad \bar{y}$ is not minimal ↯

The weighted sum method

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Proof: Assume $\underline{\bar{y}}$ is a minimal point of $P(\lambda)$, but a dominated point of Y.

$$\Rightarrow \quad \underline{\exists\, y \in Y: \quad y \leq \bar{y},\ y \neq \bar{y}}$$

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$$\Rightarrow \quad \bar{y} \text{ is not minimal } ↯$$

The weighted sum method

Exercise 13

Let $Y \subseteq \mathbb{R}^N$ and $\lambda \in \mathbb{R}^N$ with $\underline{\lambda \geq 0,\ \lambda \neq 0.}$ Then any minimal point of

$$\min \langle \lambda, y \rangle \quad \text{s.t.} \quad y \in Y$$

is ???

The weighted sum method

Theorem

Let $\underline{\lambda \in \mathbb{R}^N}$ with $\underline{\lambda > 0}$. Then any minimal point $\bar{x}$ of

$$ES_{WSM}(\lambda): \qquad \min_x \langle \lambda, \underline{\theta(x)} \rangle \quad s.t. \quad x \in E$$

is an efficient point of θ on E (i.e., a cost nondominated GNE), and $\underline{\theta(\bar{x})}$ is a nondominated point of $\theta(E)$ (i.e. a Pareto noninferior point of $\theta(E)$).

Method for finding some cost nondominated GNE $\bar{x}$

Choose some $\lambda \in \mathbb{R}^N$ with $\lambda > 0$ and compute some optimal point $\bar{x}$ of $ES_{WSM}(\lambda)$.

The weighted sum method

Theorem

Let $\lambda \in \mathbb{R}^N$ *with* $\lambda > 0$. *Then any minimal point* $\bar{x}$ *of*

$$ES_{WSM}(\lambda): \qquad \min_x \langle \lambda, \theta(x) \rangle \quad s.t. \quad x \in E$$

is an efficient point of θ *on* E *(i.e., a cost nondominated GNE), and* $\theta(\bar{x})$ *is a nondominated point of* $\theta(E)$ *(i.e. a Pareto noninferior point of* $\theta(E)$*).*

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Choose some $\lambda \in \mathbb{R}^N$ with $\lambda > 0$ and compute some optimal point $\bar{x}$ of $ES_{WSM}(\lambda)$.

The weighted sum method

Conjecture

For any nonempty and compact set $Y \subseteq \mathbb{R}^N$ the set

$$\bigcup_{\lambda>0} \text{Argmin}\{\langle\lambda, y\rangle |\ y \in Y\}$$

coincides with the set of all nondominated points of Y.

Exercise 14

Is this true or false?

The weighted sum method

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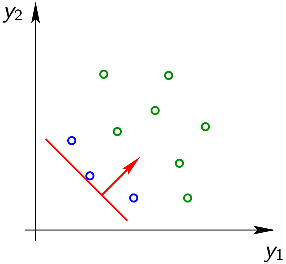
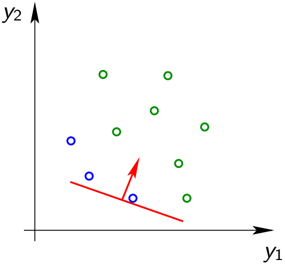
Exercise 14

Is this true or false?

Computation of nondominated points



Computation of nondominated points



The weighted sum method

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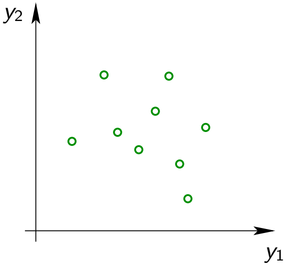
Exercise 14

Is this true or false? **false!**

The weighted sum method

The WSM does not necessarily find all nondominated points of Y



The weighted sum method

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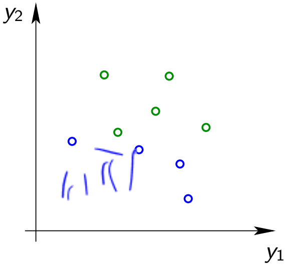

The weighted sum method

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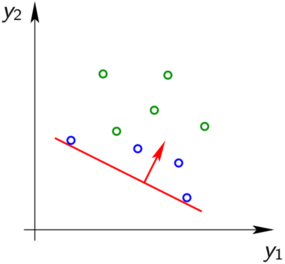
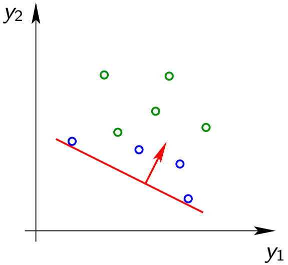


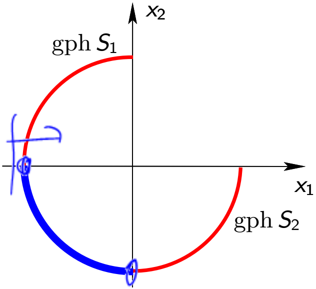
The weighted sum method

The WSM does not necessarily find all nondominated points of Y but only those in the convex hull of Y ...



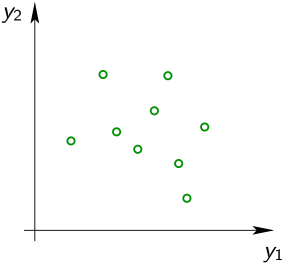
The weighted sum method for Example 3

The WSM does not necessarily find all nondominated points of Y but only those in the convex hull of Y, which are proper.



The weighted Chebyshev norm method

Let $\widehat{y} \in \mathbb{R}^N$ be a point with $\widehat{y} < y$ for all $y \in Y$.



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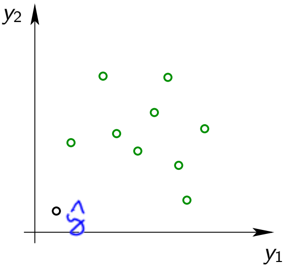

The weighted Chebyshev norm method

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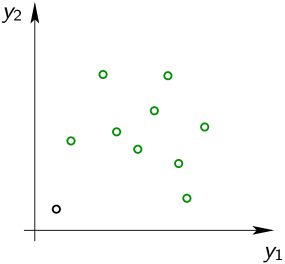

Then for any weight vector $\lambda \in \mathbb{R}^N$, $\lambda > 0$, and any $y \in Y$

$$\max_{j=1,\ldots,N} \lambda_j(y_j - \widehat{y}_j) \;=\; \|y - \widehat{y}\|_{\infty,\lambda}$$

is a weighted Chebyshev norm of $y - \widehat{y}$.

The weighted Chebyshev norm method

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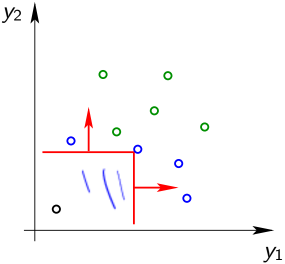

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The weighted Chebyshev norm method

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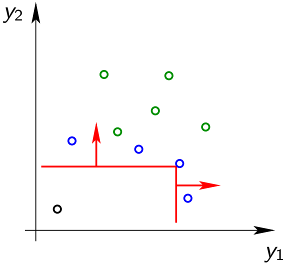

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The weighted Chebyshev norm method

Lemma

Let $Y \subseteq \mathbb{R}^N$. Then $\bar{y}$ is a nondominated point of Y if and only if there exists some $\bar{\lambda} > 0$ such that $\bar{y}$ is a strictly minimal point of

$$P_\infty(\bar{\lambda}): \qquad \min \|y - \widehat{y}\|_{\infty,\bar{\lambda}} \quad s.t. \quad y \in Y.$$

Proof: Assume $\bar{y}$ is a strictly minimal point of $P_\infty(\bar{\lambda})$ for some $\bar{\lambda} > 0$, but a dominated point of Y.

$\Rightarrow \; \exists\, y \in Y: \quad y \leq \bar{y},\; y \neq \bar{y}$

$\Rightarrow \; \forall j: \; \bar{\lambda}_j(y_j - \widehat{y}_j) \leq \bar{\lambda}_j(\bar{y}_j - \widehat{y}_j) \; \Rightarrow \; \|y - \widehat{y}\|_{\infty,\bar{\lambda}} \leq \|\bar{y} - \widehat{y}\|_{\infty,\bar{\lambda}}$

$\Rightarrow \; \bar{y}$ is not strictly minimal ϟ

The weighted Chebyshev norm method

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On the other hand, let $\bar{y}$ be a nondominated point of Y.

With $\bar{\lambda}_j := (\bar{y}_j - \widehat{y}_j)^{-1}$, $j = 1, \ldots, N$, we have $\|\bar{y} - \widehat{y}\|_{\infty,\bar{\lambda}} = 1$

and $\|y - \widehat{y}\|_{\infty,\bar{\lambda}} > 1$ for all $y \in Y \setminus \{\bar{y}\}$,

since any $y \in Y \setminus \{\bar{y}\}$ with $\|y - \widehat{y}\|_{\infty,\bar{\lambda}} \leq 1$ would dominate $\bar{y}$.

The weighted Chebyshev norm method

Lemma

Let $E \subseteq \mathbb{R}^n$. Then $\bar{x} \in E$ is an efficient point of E if and only if there exists some $\underline{\bar{\lambda} > 0}$ such that $\bar{x}$ is a strictly minimal point of

$$ES_{WCM}(\bar{\lambda}): \qquad \min \|\theta(x) - \widehat{y}\|_{\infty,\bar{\lambda}} \quad s.t. \quad x \in E,\ \theta(x) \neq \theta(\bar{x}).$$

This is due to the strict minimality requirement in the image space. While for efficient $\bar{x}$, any $x \in E$ with $x \neq \bar{x}$ and $\theta(x) = \theta(\bar{x})$ is also efficient, $\bar{x}$ would not be strictly minimal without the $\neq$-constraint.

Exercise 15

What are drawbacks of the constraint $\theta(x) \neq \theta(\bar{x})$?

The weighted Chebyshev norm method

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Let $E \subseteq \mathbb{R}^n$. Then $\bar{x} \in E$ is an efficient point of E if and only if there exists some $\bar{\lambda} > 0$ such that $\bar{x}$ is a strictly minimal point of

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Exercise 15

What are drawbacks of the constraint $\theta(x) \neq \theta(\bar{x})$?

It defines an open set, and it depends on the unknown $\bar{x}$.

The weighted Chebyshev norm method

Theorem

Let θ be injective on E. Then the efficient points of θ on E (i.e., the cost nondominated GNEs) form the union over all $\lambda > 0$ of the sets of strictly minimal points $\bar{x}$ of

$$ES_{WCM}(\lambda): \qquad \min \; \|\theta(x) - \widehat{y}\|_{\infty,\lambda} \quad s.t. \quad x \in E.$$

Method for finding all cost nondominated GNEs $\bar{x}$ for injective θ

For all $\lambda > 0$ compute all strictly minimal points $\bar{x}$ of $ES_{WCM}(\lambda)$.

The weighted Chebyshev norm method

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Let θ be injective on E. Then the efficient points of θ on E (i.e., the cost nondominated GNEs) form the union over all $\lambda > 0$ of the sets of strictly minimal points $\bar{x}$ of

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Method for finding all cost nondominated GNEs $\bar{x}$ for injective θ

For all $\lambda > 0$ compute all strictly minimal points $\bar{x}$ of $ES_{WCM}(\lambda)$.

The weighted Chebyshev norm method

Note that, by the epigraphical reformulation,

$$ES_{WCM}(\lambda): \qquad \min \|\theta(x) - \widehat{y}\|_{\infty,\lambda} \quad \text{s.t.} \quad x \in E$$

is equivalent to

$$\min_{x,\alpha} \alpha \quad \text{s.t.} \quad x \in E, \;\; \lambda_j \left(\theta_j(x) - \widehat{y}_j\right) \; \leq \; \alpha, \; j = 1, \ldots, N.$$

Recall: The weighted sum method

Method for finding some cost nondominated GNE $\bar{x}$

Choose $\lambda \in \mathbb{R}^N$ with $\lambda > 0$ and compute an optimal point $\bar{x}$ of

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Recall: The weighted sum method

Method for finding some cost nondominated GNE $\bar{x}$

Choose $\lambda \in \mathbb{R}^N$ with $\lambda > 0$ and compute an optimal point $\bar{x}$ of

$$MPEC_{ES}(\lambda): \qquad \min_x \langle \lambda, \theta(x) \rangle \quad \text{s.t.} \quad x \in E$$

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Reformulation of the equilibrium condition

$$x \in \bigcap_{\nu=1}^{N} \mathrm{gph}\, S_\nu$$

$$\Updownarrow$$

For all ν the vector x^ν is an optimal point of $Q_\nu(x^{-\nu})$.

The weighted sum method and a bilevel formulation

Method for finding some cost nondominated GNE $\bar{x}$

Choose $\lambda \in \mathbb{R}^N$ with $\lambda > 0$ and compute an optimal point $\bar{x}$ of

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The weighted sum method and a bilevel formulation

Method for finding some cost nondominated GNE $\bar{x}$

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$$BL_{ES}(\lambda): \quad \min_x \langle \lambda, \theta(x) \rangle \quad \text{s.t.} \quad x^\nu \text{ opt.pt. of } Q_\nu(x^{-\nu}), \\ \nu = 1, \ldots, N$$

The weighted sum method and a bilevel formulation

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This is a bilevel problem with N followers.

The weighted sum method and a bilevel formulation

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This is a bilevel problem with N followers.

For its algorithmic treatment it is helpful if all player problems $Q_\nu(x^{-\nu})$ are convex.

Player convexity

Player convexity

For each $\nu \in \{1, \ldots, N\}$ and each $\underline{x^{-\nu}}$ the set $X_\nu(\underline{x^{-\nu}})$ and the function $\underline{\theta_\nu(\cdot, x^{-\nu})} : X_\nu(x^{-\nu}) \to \mathbb{R}$ are convex.

Exercise 16

For functional descriptions

$$X_\nu(x^{-\nu}) \;=\; \{y^\nu |\, g^\nu(y^\nu, x^{-\nu}) \leq 0\}, \quad \nu = 1, \ldots, N,$$

of the strategy sets with $g^\nu : \mathbb{R}^n \to \mathbb{R}^{m_\nu}$, what is a sufficient condition for their convexity?

Player convexity

Player convexity

For each $\nu \in \{1, \dots, N\}$ and each $x^{-\nu}$ the set $X_\nu(x^{-\nu})$ and the function $\theta_\nu(\cdot, x^{-\nu}) : X_\nu(x^{-\nu}) \to \mathbb{R}$ are convex.

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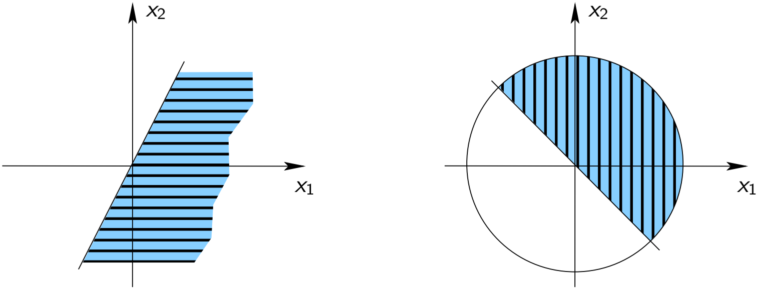
of the strategy sets with $g^\nu : \mathbb{R}^n \to \mathbb{R}^{m_\nu}$, what is a sufficient condition for their convexity?

Like θ_ν, also g_i^ν, $i = 1, \ldots, m_\nu$, only need to be (quasi-)convex in the player variable ν. Problems with convex θ_ν and g_i^ν in all variables are called completely convex.

Player convexity in Example 1

All our graphical examples so far are even completely convex.



Player convexity in Example 2

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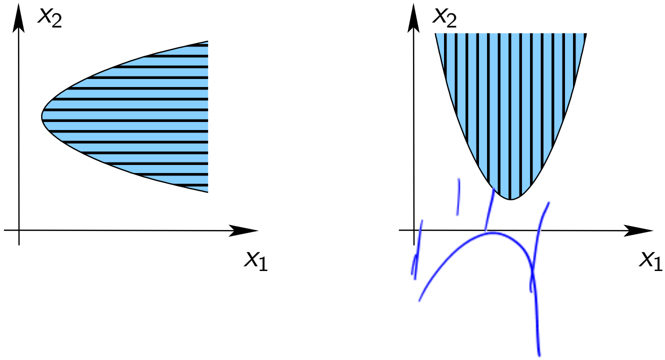
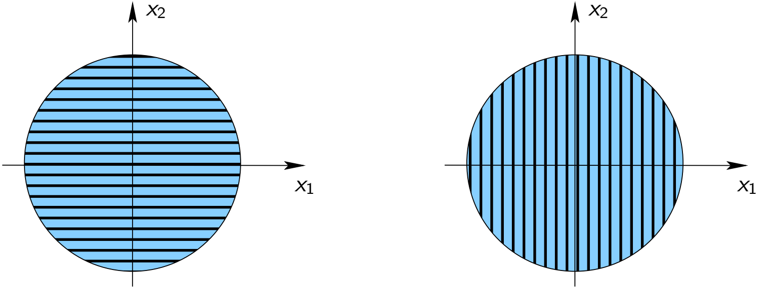


Player convexity in Example 3

All our graphical examples so far are even completely convex.



KKT reformulation

If all θ_ν and g_i^ν are also differentiable in the player variable, one may define the Lagrangian

$$L_\nu(x^\nu, x^{-\nu}, \gamma^\nu) \;=\; \theta_\nu(x^\nu, x^{-\nu}) \underset{\sim}{+} (\gamma^\nu)^\mathsf{T} g^\nu(x^\nu, x^{-\nu})$$

of $Q_\nu(x^{-\nu})$ and consider the KKT system

$$\begin{array}{rcl} \nabla_{x^\nu} L_\nu(x^\nu, x^{-\nu}, \gamma^\nu) & = & 0, \\ 0 \;\le\; \gamma^\nu \;\perp\; -g^\nu(x^\nu, x^{-\nu}) & \ge & 0. \end{array}$$

KKT reformulation

Under some CQ, like Slater's condition for each appearing set $X_\nu(x^{-\nu})$, we obtain

$$\begin{aligned} E \;=\; \{x|\ \exists\, \underline{\gamma}:\ & \nabla_{x^\nu} L_\nu(x^\nu, x^{-\nu}, \gamma^\nu) = 0, \\ & 0 \;\le\; \gamma^\nu \;\perp\; -g^\nu(x^\nu, x^{-\nu}) \;\ge\; 0,\ \underline{\nu} = 1, \ldots, N\} \end{aligned}$$

as well as the MPCC reformulation of $MPEC_{ES}(\lambda)$

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Drawback: Slater's condition is necessarily violated at the boundaries of the domains of X_ν, $\nu = 1, \ldots, N$.

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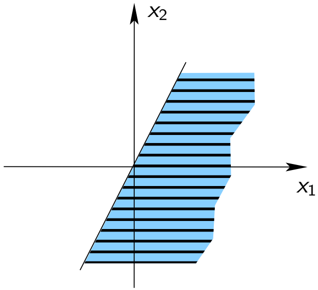
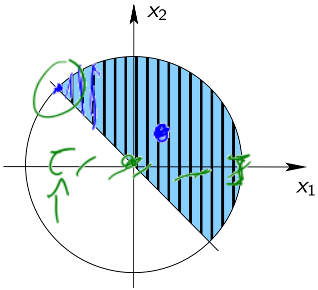
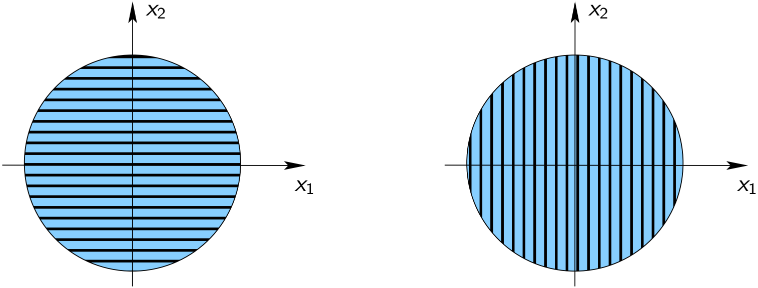
Violated lower level CQ in Example 1





Violated lower level CQ in Example 3

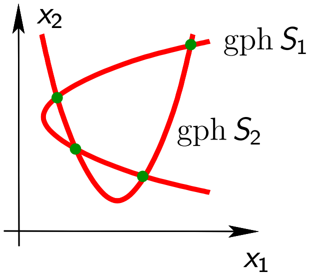


Violated upper level CQ

Additional problem:

MPCCs intrinsically violate the MFCQ, so that tailored solution methods should be employed.

KKT approach for Example 2



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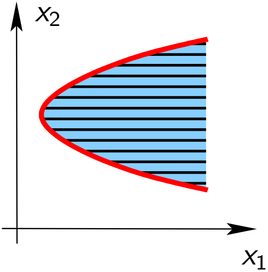

$$Q_1(x_2): \quad \min_{x_1} x_1 \quad \text{s.t.} \quad q_1(x_2) \le x_1$$

$$L_1(x_1,x_2,\gamma_1) = x_1 + \gamma_1(q_1(x_2) - x_1)$$

$$KKT_1(x_2): \quad 1 - \gamma_1 = 0$$

$$\gamma_1 \ge 0,\ q_1(x_2) - x_1 \le 0,\ \gamma_1(q_1(x_2) - x_1) = 0$$

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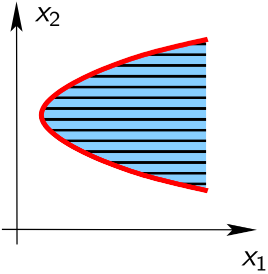

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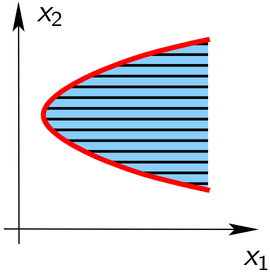

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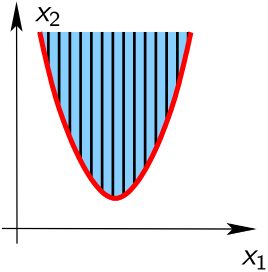

$$Q_2(x_1): \quad \min_{x_2} x_2 \quad \text{s.t.} \quad q_2(x_1) \le x_2$$

$$L_2(x_2, x_1, \gamma_2) = x_2 + \gamma_2(q_2(x_1) - x_2)$$

$$KKT_2(x_1): \quad 1 - \gamma_2 = 0$$

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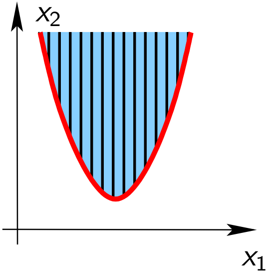

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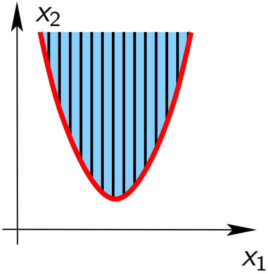

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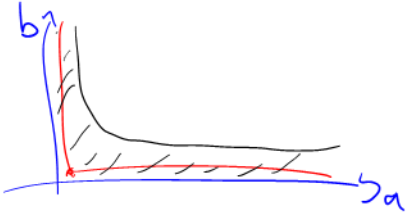
\end{aligned}
$$

a, b complementary $\Leftrightarrow$

$a \cdot b \leq t,\ a \geq 0,\ b \geq 0$ $\Leftrightarrow$ $\min(a,b)=0$



KKT approach for Example 2, regularized

$$\begin{aligned} MPCC_{ES}(\lambda): \quad \min_{x,\gamma} \langle \lambda, x \rangle \quad \text{s.t.} \quad & 1-\gamma_1=0 \\ & \gamma_1 \geq 0,\ q_1(x_2)-x_1 \leq 0,\ \gamma_1(q_1(x_2)-x_1) = -t \\ & 1-\gamma_2=0 \\ & \gamma_2 \geq 0,\ q_2(x_1)-x_2 \leq 0,\ \gamma_2(q_2(x_1)-x_2) = -t \end{aligned}$$

KKT approach for Example 2, Scholtes regularized

$$
\begin{aligned}
MPCC_{ES}(\lambda): \quad \min_{x,\gamma}\, \langle \lambda, x\rangle \quad \text{s.t.} \quad & 1-\gamma_1 = 0\\
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& 1-\gamma_2 = 0\\
\gamma_2 \geq 0,\ q_2(x_1)-x_2 \leq 0,\ & \gamma_2(q_2(x_1)-x_2) \leq -t
\end{aligned}
$$

.

Reformulation of the equilibrium condition

$$x \in \bigcap_{\nu=1}^{N} \operatorname{gph} S_\nu$$

$$\Updownarrow$$

For all ν the vector x^ν is an optimal point of $Q_\nu(x^{-\nu})$.

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$$\forall\, \nu : x^\nu \in X_\nu(x^{-\nu})$$

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Reformulation of the equilibrium condition

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$$\Updownarrow$$

$x^{\frown}$

$x^{\nu} =$

$$x \; \in \; X_1(x^{-1}) \times \ldots \times X_N(x^{-N}) \; =: \; Y(x)$$

$$\Updownarrow$$

$$x \; \in \; \operatorname{fix} Y$$

$$\Downarrow$$

$$x \; \in \; \operatorname{dom} Y \; = \; \{x \in \mathbb{R}^n \,|\; Y(x) \neq \emptyset\}.$$

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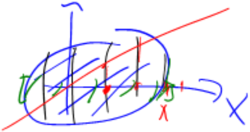

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With functional descriptions

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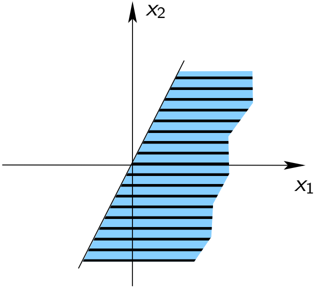
of the strategy sets one obtains

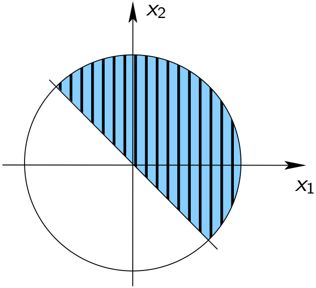
$$\mathrm{fix}\, Y \;=\; \{x \in \mathbb{R}^n |\; g^\nu(x) \le 0,\; \nu = 1, \ldots, N\}.$$

$$x \in \mathrm{fix}\, Y \iff \underset{\forall \nu}{x^\nu \in X_\nu(x^{-\nu})} \iff \underset{\forall \nu}{g^\nu}(\underbrace{x^\nu, x^{-\nu}}_{x}) \le 0$$

Example 1

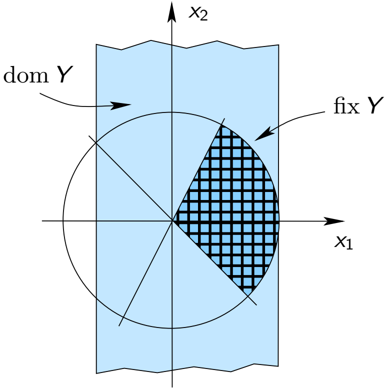


$$x_2 \;\leq\; 2x_1$$



$$x_1^2 + x_2^2 \;\leq\; 1, \quad x_2 \;\geq -x_1$$

Example 1 – domain and fixed point set



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The weighted sum method and a semi-infinite formulation

Method for finding some cost nondominated GNE $\bar{x}$

Choose $\lambda \in \mathbb{R}^N$ with $\lambda > 0$ and compute an optimal point $\bar{x}$ of

$$MPEC_{ES}(\lambda) : \quad \min_{x} \langle \lambda, \theta(x) \rangle \quad \text{s.t.} \quad x \in E$$

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These are generalized semi-infinite constraints.

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These are generalized semi-infinite constraints.

For their algorithmic treatment it is helpful if all player problems $Q_\nu(x^{-\nu})$ are convex or if the index sets X_ν do not depend on $x^{-\nu}$.

$$g(x,y) \le 0 \quad \forall y \in Y(x)$$

$$\Leftrightarrow \sup_{y \in Y(x)} g(x,y) \le 0$$

$\uparrow$

optimal value of

$$\max_{y} \; g(x,y) \quad \text{s.t.} \quad y \in Y(x)$$

Standard NEPs

A GNEP with constant strategy sets

$$X_\nu(x^{-\nu}) \ \equiv X_\nu\,, \quad \nu = 1, \ldots, N,$$

is called standard Nash equilibrium problem (NEP).

Standard NEPs satisfy $Y(x) = Y := X_1 \times \ldots \times X_N$

and $\operatorname{fix} Y = \{x \in \mathbb{R}^n |\ x \in Y(x) = Y\} = Y$.

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Standard NEPs

Thus, for a standard NEP the problem

$$SIP_{ES}(\lambda): \quad \min_x \langle \lambda, \theta(x) \rangle \quad \text{s.t.} \quad \begin{array}{l} x \in Y \\ \theta_\nu(x) - \theta_\nu(y^\nu, x^{-\nu}) \leq 0 \\ \forall\, y^\nu \in X_\nu,\ \nu = 1, \ldots, N \end{array}$$

is a standard semi-infinite optimization problem.

$\rightsquigarrow$ Solve $SIP_{ES}(\lambda)$ by, e.g., an adaptive discretization method.

Main disadvantage of the semi-infinite approach:
the feasible set of $SIP_{ES}(\lambda)$ violates Slater's condition.

Exercise 18: Why?

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Nikaido-Isoda approach

$$\forall \nu : \ \underline{\forall} y^{\nu} \in X_{\nu}(x^{-\nu}) : \ \theta_{\nu}(x^{\nu}, x^{-\nu}) \ \underline{\leq} \ \theta_{\nu}(\underline{y}^{\nu}, x^{-\nu})$$

$$\Updownarrow \quad (x \in Y(x))$$

$$\forall \nu : \ \theta_{\nu}(x^{\nu}, x^{-\nu}) \ = \ \min_{\underline{y^{\nu} \in X_{\nu}(x^{-\nu})}} \underline{\theta_{\nu}(y^{\nu}, x^{-\nu})} \ =: \ \varphi_{\nu}(x^{-\nu})$$

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Reformulation of the equilibrium condition

$$x \in \bigcap_{\nu=1}^{N} \operatorname{gph} S_\nu$$

$$\Updownarrow$$

For all ν the vector x^ν is an optimal point of $Q_\nu(x^{-\nu})$.

$$\Updownarrow$$

$$x \in \operatorname{fix} Y$$

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$$V(x) = 0$$

The weighted sum method

Method for finding some cost nondominated GNE $\bar{x}$

Choose $\lambda \in \mathbb{R}^N$ with $\lambda > 0$ and compute an optimal point $\bar{x}$ of

$$ES_{WSM}(\lambda): \qquad \min_x \langle \lambda, \theta(x) \rangle \quad \text{s.t.} \quad x \in E$$

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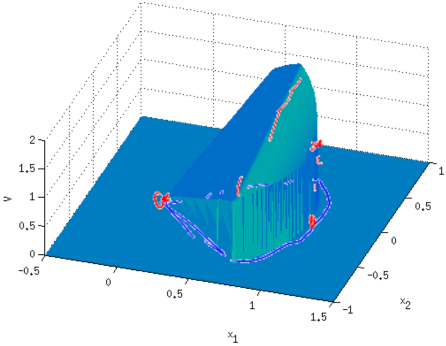
$\Rightarrow$ for all $x \in \operatorname{fix} Y$: $V(x) \geq 0$ ($\rightsquigarrow$ V is called gap function)

Example 1 – gap function

$$V(x) \;=\; x_1 + x_2 + \min\left\{x_1, \sqrt{1-x_1^2}\right\} - \frac{x_2}{2}$$



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This is a purely hierarchical bilevel problem, where the evaluation of V involves more embedded optimization problems, i.e., it is a trilevel problem!

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The Nikaido-Isoda function

For all $x \in \text{fix}\, Y \subseteq \text{dom}\, Y$:

$$\begin{aligned} V(x) &= \sum_{\nu=1}^{N} \Big(\theta_\nu(x^\nu, x^{-\nu}) - \inf_{y^\nu \in X_\nu(x^{-\nu})} \theta_\nu(y^\nu, x^{-\nu})\Big) \\ &= \sup_{y \in X_1(x^{-1}) \times \ldots \times X_N(x^{-N})} \sum_{\nu=1}^{N} \big(\theta_\nu(x^\nu, x^{-\nu}) - \theta_\nu(y^\nu, x^{-\nu})\big) \\ &= \sup_{y \in Y(x)} \sum_{\nu=1}^{N} \big(\theta_\nu(x^\nu, x^{-\nu}) - \theta_\nu(y^\nu, x^{-\nu})\big) \\ &= \sup_{y \in Y(x)} \psi(x, y) \end{aligned}$$

with the Nikaido-Isoda function ψ (aka Ky-Fan function).

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A semi-infinite optimization problem

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This is again a generalized semi-infinite inequality (namely an aggregation of the former N single generalized semi-infinite inequalities), which violates Slater's condition.

Reformulation of the equilibrium condition

$$x \;\in\; \bigcap_{\nu=1}^{N} \operatorname{gph} S_\nu$$

$$\Updownarrow$$

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Quasi-variational inequalities

$$\forall\,\nu:\ \ \forall\,y^\nu\in X_\nu(x^{-\nu}):\ \ \theta_\nu(x^\nu,x^{-\nu})\ \le\ \theta_\nu(y^\nu,x^{-\nu})$$

$\Updownarrow$ (player convexity, differentiability of θ_ν, $\nu=1,\ldots,N$)

$$\forall\,\nu:\ \ \langle\underline{\nabla_{x^\nu}\theta_\nu(x^\nu,x^{-\nu})},y^\nu-x^\nu\rangle\ge 0\quad\forall\ y^\nu\in X_\nu(x^{-\nu})$$

$\Updownarrow$ ($x\in Y(x)$)

$$\langle F(x),y-x\rangle\ge 0\ \ \forall\,y\in Y(x),\quad F(x):=\begin{pmatrix}\nabla_{x^1}\theta_1(x^1,x^{-1})\\ \vdots\\ \nabla_{x^N}\theta_N(x^N,x^{-N})\end{pmatrix}$$

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Quasi-variational inequality reformulation

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$$\langle F(x), y - x \rangle \geq 0 \quad \forall\, y \in Y(x)$$

Advantage: The lower level problem has a linear objective function.

If the underlying equilibrium problem is a standard NEP, then the QVI becomes a VI, fix $Y = Y$, and

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Tikhonov-like method

The solution method for nested VIs, covering OPVICs, from

Lampariello/Neumann/Ricci/Sagratella/St., *An explicit Tikhonov algorithm for nested variational inequalities,* COAP, Vol. 77 (2020), 335-350.

has been successfully applied to some $SIP_{ES,VI}$ in

Lampariello/Neumann/Ricci/Sagratella/St., *Equilibrium selection for multi-portfolio optimization,* EJOR (2021), DOI: 10.1016/j.ejor.2021.02.033.

Standard NEPs with polyhedral strategy sets

Consider the special case of an underlying standard NEP with polytopes X_ν, $\nu = 1, \ldots, N$. Then Y is a polytope, and

$$SIP_{ES,VI}: \quad \min_x \langle \lambda, \theta(x) \rangle \quad \text{s.t.} \quad x \in Y,\ \langle F(x), y - x \rangle \geq 0 \quad \forall\, y \in Y$$

is a standard SIP with linear lower level problem.

While the vertex theorem of linear programming yields the equivalent finite problem

$$P_{ES,VI}: \quad \min_x \langle \lambda, \theta(x) \rangle \quad \text{s.t.} \quad x \in Y,\ \langle F(x), y - x \rangle \geq 0 \quad \forall\, y \in \text{vert}\, Y$$

this may be difficult to solve due to a vast set vert Y.

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this may be difficult to solve due to a vast set vert Y.

Example for a vast vertex set

In Lampariello/Neumann/Ricci/Sagratella/St., *Equilibrium selection for multi-portfolio optimization*, we have

$$X_\nu \; = \; \{x^\nu \in \mathbb{R}^K \,|\; x^\nu \geq 0,\; \langle e, x^\nu \rangle = 1\}, \quad \nu = 1, \ldots, 25$$

and

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$$Y \; = \; \{x \in \mathbb{R}^{25K} \,|\, x \geq 0,\; \langle e, x^\nu \rangle = 1,\; \nu = 1, \ldots, 25\}$$

with $K = 10$ ($\Rightarrow$ $|\text{vert } Y| = 10^{25}$)
and $K = 29$ ($\Rightarrow$ $|\text{vert } Y| = 29^{25}$).

Benders-like method

```
$Y_d \leftarrow \emptyset$
repeat
    Compute an optimal point $\bar{x}$ of    $\min_{y \in Y} \langle F(x), y-x \rangle \geq 0$
        $P_{ES,VI,d}: \min_x f(x)$ s.t. $x \in Y$, $\underline{\langle F(x), y-x \rangle \geq 0 \ \forall y \in Y_d}$.
    Compute an optimal vertex $\bar{y}$ of
        $LP: \min_y \langle F(\bar{x}), y \rangle$ s.t. $y \in Y$.
    $Y_d \leftarrow Y_d \cup \{\bar{y}\}$
until $\langle F(\bar{x}), \bar{y} - \bar{x} \rangle \geq -\varepsilon$;
```

Numerical results

λ	$\|\|x^\star - \overline{x}^\star\|\|_1$	iterations	run time [s]
λ^1	0.666517	9	22.796807
λ^2	0.658469	10	28.127697
λ^3	0.578055	11	43.600155
λ^4	0.774364	10	24.194173

Table: data set SX5E, $N = 25$, $K = 10$, $\varepsilon = 10^{-4}$

λ	$\|\|x^\star - \overline{x}^\star\|\|_1$	iterations	run time [s]
λ^1	2.658054	7	214.447290
λ^2	3.123464	8	306.947685
λ^3	2.850787	7	209.581576
λ^4	1.941140	8	290.675761

Table: data set DJ, $N = 25$, $K = 29$, $\varepsilon = 10^{-3}$

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- involves notions and methods from multicriteria optimization
- leads to MPCCs or semi-infinite optimization problems
- the latter do not enjoy standard CQs and need to be handled with care

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