Multi-Leader-Follower Games: non cooperative and hierarchical/bilevel interactions

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UNIVERS Winter school - November 14th-17th, 2021

- Professor in Applied Mathematics at Univ. of Perpignan

Perpignan, France

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    - Industrial Eco-Parks (IEP)
    - Demand-side management
    - and others....(management of renewable energy plants)

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  - Variational and quasi-variational inequalities
  - Quasiconvex optimization
- Research lab.: PROMES (CNRS)

What do I work on?

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Optimization /Math. programming

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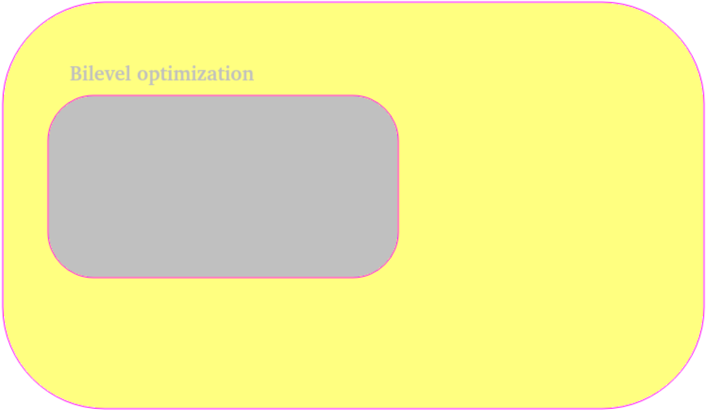

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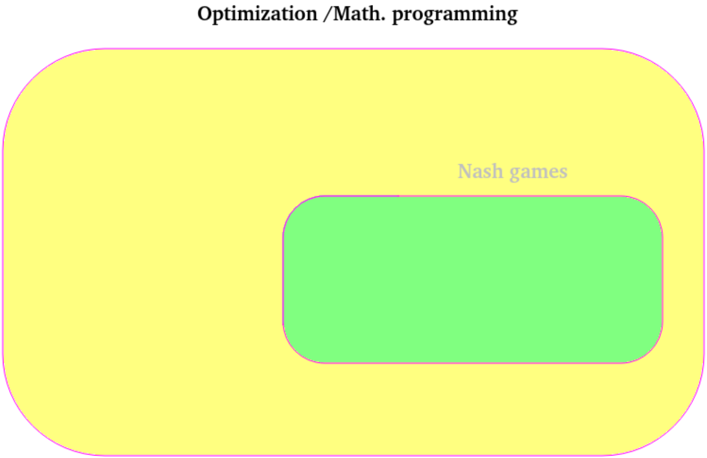

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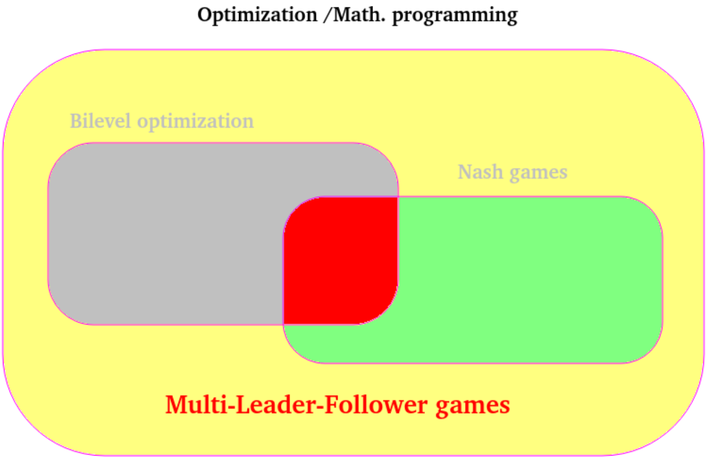
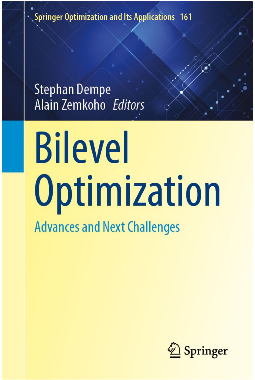


Hell zone!!!!!

An advertisement

A short state of art on Multi-Leader-Follower games, D.A. and A. Svensson, in a book dedicated to Stackelberg, editors A. Zemkoho and S. Dempe, Springer Ed. (2021)



Menu (for the three days...)

- Lecture 1: Definitions, motivations and well posedness
- Lecture 2: Motivations and existence
- Lecture 3: Reformulations (differentiable and non differentiable cases)

Lecture 1

- *Definitions of different MLMFG*

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- *Definitions of different MLMFG*
- *Is this well-posed? Is it meaningfull?*

Generalized Nash game (GNEP):

$$\boxed{\boxed{\begin{array}{ll} \min\limits_{x_1} & \theta_1(x_1, x_{-1}) \\ \text{s.t.} & \{\ x_1 \in K_1(x_{-1}) \end{array}} \quad \dots \quad \boxed{\begin{array}{ll} \min\limits_{x_n} & \theta_n(x_n, x_{-n}) \\ \text{s.t.} & \{\ x_n \in K_n(x_{-n}) \end{array}}}$$

Generalized Nash game (GNEP):

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So we have n players and they are interacting in a **non cooperative way** (join venture is forbidden or impossible...)

$\bar{x}$ is a Generalized Nash Equilibrium
if and only if
in case a player i would decide to unilaterally deviate from $\bar{x}_i$
(say to $\tilde{x}_i$) then

"he will loose" := "$\theta_i(\tilde{x}_i, \bar{x}_{-i}) \geq \theta_i(\bar{x}_i, \bar{x}_{-i})$" !!!!

Generalized Nash game (GNEP):

$$\boxed{\begin{array}{ll} \min\limits_{x_1} & \theta_1(x_1, x_{-1}) \\ \text{s.t.} & \{ \; x_1 \in K_1(x_{-1}) \end{array}} \quad \ldots \quad \boxed{\begin{array}{ll} \min\limits_{x_n} & \theta_n(x_n, x_{-n}) \\ \text{s.t.} & \{ \; x_n \in K_n(x_{-n}) \end{array}}$$

For this problem you are new experts in:

- Existence
- First order conditions
- Qualification conditions (at a point or generically)

Bilevel problem:

$$\begin{array}{|ll|} \hline \text{"}\min_x\text{"} & \theta(x,y) \\ \text{s.t.} & \left\{ \begin{array}{l} x \in X(y) \\ \min_y \quad \phi(x,y) \\ \quad \text{s.t.} \quad y \in Y(x) \end{array} \right. \\ \hline \end{array}$$

For this problem you are new experts in:

- Existence
- First order conditions
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Single-Leader-Multi-Follower-Game (SLMFG):

$$\begin{array}{ll} \text{" min}_x\text{ "} & \theta(x,y) \\ \text{s.t.} & \begin{cases} x \in X(y) \\ y \in GNEP(x) \end{cases} \end{array}$$

$$\downarrow\uparrow$$

$$\begin{array}{ll} \min_{y_1} & \phi_1(x,y) \\ \text{s.t.} & \{\ y_1 \in K_1(x,y_{-1}) \end{array} \quad \cdots \quad \begin{array}{ll} \min_{y_n} & \phi_n(x,y) \\ \text{s.t.} & \{\ y_n \in K_n(x,y_{-n}) \end{array}$$

Multi-Leader-Single-Follower-Game (MLSFG):

$$\begin{array}{ll} \text{" }\min_{x_1}\text{ "} & \theta_1(x,y) \\ \text{s.t.} & \left\{ \begin{array}{l} x_1 \in X_1(x_{-1},y) \\ y \in S(x) \end{array}\right. \end{array} \quad \dots \quad \begin{array}{ll} \text{" }\min_{x_p}\text{ "} & \theta_p(x,y) \\ \text{s.t.} & \left\{ \begin{array}{l} x_p \in X_p(x_{-p},y) \\ y \in S(x) \end{array}\right. \end{array}$$

$$\downarrow\uparrow$$

$$\begin{array}{ll} \min_y & \phi(x,y) \\ \text{s.t.} & \left\{ \; y \in K(x) \right. \end{array}$$

Multi-Leader-Multi-Follower-Game (MLFG):

$$
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$$

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\downarrow\uparrow \qquad\qquad \downarrow\uparrow
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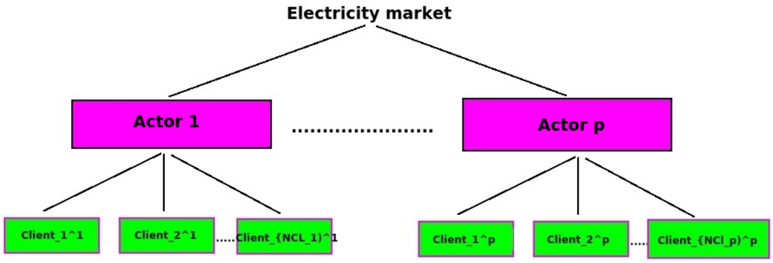
$$

Problems we decided to tackle...

For the Demand-side management, we recently introduced the Multi-Leader-Disjoint-Follower game (MLDFG)



see D.A., G. Bouza and S. Dempe (SIOPT 21) where we prove some genericity of the constraints qualification

Actually there is a lot of different Single-Leader-Multi-Follower-Game (SLMFG) with different characteristics:

$$\text{" }\min_x\text{ "} \quad \theta_1(x,y) \quad \text{s.t.} \quad \begin{cases} G(x,y) \leq 0 \\ y \in GNEP(x) \end{cases}$$

$$\downarrow\uparrow$$

$$\min_{y_1} \quad \phi_1(x,y) \quad \text{s.t.} \quad g_1(y_1,\dots,y_n,x) \leq 0 \qquad \dots \qquad \min_{y_n} \quad \phi_n(x,y) \quad \text{s.t.} \quad g_n(y_1,\dots,y_n,x) \leq 0$$

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Then the lower level is a (parametrized) Nash Equilibrium Problem (NEP)

Actually there is a lot of different Single-Leader-Multi-Follower-Game (SLMFG) with different characteristics:

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$$\begin{array}{ll} \min_{y_1} & \phi_1(x,y) \\ \text{s.t.} & g_{sc}(y_1,\dots,y_n,x) \leq 0 \end{array} \quad \dots \quad \begin{array}{ll} \min_{y_n} & \phi_n(x,y) \\ \text{s.t.} & g_{sc}(y_1,\dots,y_n,x) \leq 0 \end{array}$$

Thus now the lower level game is a GNEP with a shared constraint!!

Let us stop and think about the well-posedness of a SLMFG....

BL = Single-Leader-Single-Follower game

A Bilevel Problem consists in an **upper-level/leader's problem**

$$\begin{array}{rl} \text{“}\min_{x\in\mathbb{R}^n}\text{”} & F(x,y) \\ s.t. & \left\{ \begin{array}{l} x \in X(y) \\ y \in S(x) \end{array} \right. \end{array}$$

where $\emptyset \neq X \subset \mathbb{R}^n$ and $S(x)$ stands for the solution set of its **lower-level/follower's problem**

$$\begin{array}{rl} \min_{y\in\mathbb{R}^m} & f(x,y) \\ s.t & g(x,y) \leq 0 \end{array}$$

A trivial example

Consider the following simple bilevel problem

$$\begin{array}{cl} \text{“}\min_{x\in\mathbb{R}}\text{”} & x \\ s.t. & \left\{ \begin{array}{l} x \in [-1,1] \\ y \in S(x) \end{array} \right. \end{array}$$

with $S(x) =$ “y solving

$$\begin{array}{cl} \min_{y\in\mathbb{R}} & -xy \\ s.t & x^2(y^2-1) \leq 0 \text{ ”} \end{array}$$

A trivial example

Lower level problem:

$$\min_{y\in\mathbb{R}} \quad -x.y$$
$$s.t \quad x^2(y^2-1) \leq 0 \text{ ”}$$

Note that the solution map of this convex problem is

$$S(x) := \begin{cases} \{1\} & x < 0 \\ \{-1\} & x > 0 \\ \mathbb{R} & x = 0 \end{cases}$$

Thus for each $x \neq 0$ there is a unique associated solution of the lower level problem

A trivial example

Lower level problem:

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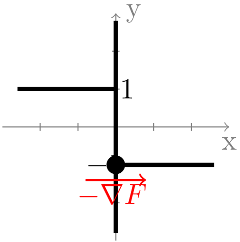

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Ambiguity: Optimistic approach

An *Optimistic Bilevel Problem* consists in an **upper-level/leader's problem**

$$\begin{array}{rl} \min_{x\in\mathbb{R}^n} \min_{y\in\mathbb{R}^m} & F(x,y) \\ s.t. & \left\{ \begin{array}{l} x \in X(y) \\ y \in S(x) \end{array} \right. \end{array}$$

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$$\begin{array}{rl} \min_{y\in\mathbb{R}^m} & f(x,y) \\ s.t & g(x,y) \leq 0 \end{array}$$

Ambiguity: Pessimistic approach

An *Pessimistic Bilevel Problem* consists in an
upper-level/leader's problem

$$\min_{x\in\mathbb{R}^n} \max_{y\in\mathbb{R}^m} \quad F(x,y) \qquad s.t. \quad \begin{cases} x \in X(y) \\ y \in S(x) \end{cases}$$

where $\emptyset \neq X \subset \mathbb{R}^n$ and $S(x)$ stands for the solution set of its
lower-level/follower's problem

$$\min_{y\in\mathbb{R}^m} \quad f(x,y) \qquad s.t \quad g(x,y) \leq 0$$

And of course the "confortable situation" corresponds to the case of a unique response

$$\forall\, x \in X, \quad S(x) = \{y(x)\}.$$

Then

$$\begin{array}{rl} \min_{x \in \mathbb{R}^n} & F(x, y(x)) \\ s.t. & \{\; x \in X(y(x)) \end{array}$$

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$$\begin{array}{rl} \min_{x\in\mathbb{R}^n} & F(x, y(x)) \\ s.t. & \{\ x \in X(y(x)) \end{array}$$

For example when

for any x, $g(x,\cdot)$ *is quasiconvex and* $f(x,\cdot)$ *is strictly convex.*

Ambiguity: Selection approach

An *"Selection-type" Bilevel Problem* consists in an **upper-level/leader's problem**

$$\begin{array}{ll} \min_{x\in\mathbb{R}^n} & F(x, y(x)) \\ s.t. & \left\{ \begin{array}{l} x \in X(y(x)) \\ y(x) \textit{ is a uniquely determined selection of } S(x) \end{array} \right. \end{array}$$

J. Escobar & A. Jofré, *Equilibrium Analysis of Electricity Auctions* (2011)

Recently, D.Salas and A. Svensson proposed a **probabilistic approach**:

- *Consider a probability on the different possible follower's reactions*
- *Minimize the expectation of the leader(s)*

An alternative point of view

Instead of considering the previous (optimistic) formulation of BL:

$$\begin{array}{rl} \min_{x\in\mathbb{R}^n} \min_{y\in\mathbb{R}^m} & F(x,y) \\ s.t. & \begin{cases} x \in X \\ y \in S(x) \end{cases} \end{array}$$

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one can define the (optimistic) value function

$$\varphi_{min}(x) = \min_y \{F(x,y) : g(x,y) \leq 0\} \tag{1}$$

and the BL problem becomes

$$\begin{array}{rl} \min_{x\in\mathbb{R}^n} & \varphi_{min}(x) \\ s.t. & x \in X \end{array}$$

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Instead of considering the previous (pessimistic) formulation of BL:

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one can define the (pessimistic) value function

$$\varphi_{max}(x) = \max_y \{F(x,y) : g(x,y) \le 0\} \tag{2}$$

and the Bl problem becomes

$$\begin{array}{rl} \min_{x\in\mathbb{R}^n} & \varphi_{max}(x) \\ s.t. & x \in X \end{array}$$

An alternative point of view

This is the point of view presented in Stephan Dempe's book:

$$\min_{x\in\mathbb{R}^n} \min / \max_{y\in\mathbb{R}^m} \quad F(x,y) \qquad s.t. \quad \begin{cases} x \in X \\ y \in S(x) \end{cases}$$

vs

$$\min_{x\in\mathbb{R}^n} \quad \varphi_{min/max}(x) \qquad s.t. \quad x \in X$$

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$$\begin{array}{ll} \min_{x\in\mathbb{R}^n} \min/\max_{y\in\mathbb{R}^m} & F(x,y) \\ s.t. & \begin{cases} x \in X \\ y \in S(x) \end{cases} \end{array} \quad \textbf{vs} \quad \begin{array}{ll} \min_{x\in\mathbb{R}^n} & \varphi_{min/max}(x) \\ s.t. & x \in X \end{array}$$

It immediately raises the question

What is a solution??

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It immediately raises the question

What is a solution??

- *an optimal* x *= leader's optimal strategy?*
- *an optimal couple* (x, y) *= couple of strategies of leader and follower?*

Optimistic Single-Leader-Multi-Follower-Game (SLMFG):

$$\min_x \min_{y_1,\dots,y_n} \quad \theta(x,y)$$
$$\text{s.t.} \quad \begin{cases} G(x,y) \le 0 \\ y \in GNEP(x) \end{cases}$$

$$\downarrow\uparrow$$

$$\min_{y_1} \quad \phi_1(x,y) \qquad \text{s.t.} \quad g_1(y_1,\dots,y_n,x) \le 0$$

$$\dots$$

$$\min_{y_n} \quad \phi_n(x,y) \qquad \text{s.t.} \quad g_n(y_1,\dots,y_n,x) \le 0$$

All followers are "friends" of the leader!!!

Pessimistic Single-Leader-Multi-Follower-Game (SLMFG):

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All followers are "ennemies" of the leader!!!

Mix Single-Leader-Multi-Follower-Game (SLMFG):

$$\min_x \min_{y_1,\ldots,y_p} \max_{y_{p+1},\ldots,y_n} \quad \theta(x,y)$$
$$\text{s.t.} \quad \begin{cases} G(x,y) \le 0 \\ y \in GNEP(x) \end{cases}$$

$$\downarrow\uparrow$$

$$\begin{array}{lll} \min_{y_1} & \phi_1(x,y) \\ \text{s.t.} & g_1(y_1,\ldots,y_n,x) \le 0 \end{array} \quad \ldots \quad \begin{array}{lll} \min_{y_n} & \phi_n(x,y) \\ \text{s.t.} & g_n(y_1,\ldots,y_n,x) \le 0 \end{array}$$

Some are "friends" and some are "ennemies"!!!

Usually when considering SLMFG

$$\min_{x\in\mathbb{R}^n} \min_{y\in\mathbb{R}^m} \quad F(x,y)$$
$$s.t. \quad \begin{cases} x \in X \\ y \in GNEP(x) \end{cases}$$

people says

- *Step A: the leader plays first*
- *Step B: the followers react*

But in real life it's a little bit more complex....

Real life...

Actually in real life, when considering SLMFG

$$\min_{x\in\mathbb{R}^n}\min_{y\in\mathbb{R}^m} \quad F(x,y)$$
$$s.t. \quad \begin{cases} x \in X \\ y \in GNEP(x) \end{cases}$$

We only work for the leader!!

Real life...

Actually in real life, when considering SLMFG

$$\min_{x\in\mathbb{R}^n} \min_{y\in\mathbb{R}^m} \quad F(x,y) \quad s.t. \quad \begin{cases} x \in X \\ y \in GNEP(x) \end{cases}$$

We only work for the leader!! Indeed

- *Step 0: the leader has a model of the follower's reaction: optimistic or pessimistic or ...*
- *Step 1: we compute a solution $\bar{x}$ or $(\bar{x}, \bar{y})$ of the SLMFG model*
- *Step 2: the leader plays $\bar{x}$*
- *Step 3: the follower decides to play...*

Real life...

Actually in real life, when considering SLMFG

$$\min_{x \in \mathbb{R}^n} \min_{y \in \mathbb{R}^m} \quad F(x,y)$$
$$s.t. \quad \begin{cases} x \in X \\ y \in GNEP(x) \end{cases}$$

We only work for the leader!! Indeed

- *Step 0: the leader has a model of the follower's reaction: optimistic or pessimistic or ...*
- *Step 1: we compute a solution $\bar{x}$ or $(\bar{x}, \bar{y})$ of the SLMFG model*
- *Step 2: the leader plays $\bar{x}$*
- *Step 3: the follower decides to play...whatever he wants!!!*

Consider an Optimistic Single-Leader-Multi-Follower-Game (SLMFG):

$$\begin{array}{ll} \min_x \min_{y_1,\ldots,y_n} & \theta_1(x,y) \\ \text{s.t.} & \left\{ \begin{array}{l} G(x,y) \le 0 \\ y \in GNEP(x) \end{array} \right. \end{array}$$

$$\downarrow\uparrow$$

$$\begin{array}{ll} \min_{y_1} & \phi_1(x,y) \\ \text{s.t.} & g_1(y_1,\ldots,y_n,x) \le 0 \end{array} \quad \ldots \quad \begin{array}{ll} \min_{y_n} & \phi_n(x,y) \\ \text{s.t.} & g_n(y_1,\ldots,y_n,x) \le 0 \end{array}$$

Actually while modeling a real life application, one can also consider this alternative Optimistic Single-Leader-Multi-Follower-Game (SLMFG):

$$\begin{array}{cc} \min_x \min_{y_1,\dots,y_n} & \theta_1(x,y) \\ \text{s.t.} & y \in GNEP(x) \end{array}$$

$$\downarrow\uparrow$$

$$\begin{array}{ccc} \begin{array}{cl} \min_{y_1} & \phi_1(x,y) \\ \text{s.t.} & \left\{ \begin{array}{l} G(x,y) \le 0 \\ g_1(y_1,\dots,y_n,x) \le 0 \end{array} \right. \end{array} & \cdots & \begin{array}{cl} \min_{y_n} & \phi_n(x,y) \\ \text{s.t.} & \left\{ \begin{array}{l} G(x,y) \le 0 \\ g_n(y_1,\dots,y_n,x) \le 0 \end{array} \right. \end{array} \end{array}$$

Actually while modeling a real life application, one can also consider this alternative Optimistic Single-Leader-Multi-Follower-Game (SLMFG):

$$\begin{array}{cc} \min_x \min_{y_1,\ldots,y_n} & \theta_1(x,y) \\ \text{s.t.} & y \in GNEP(x) \end{array}$$

$$\downarrow\uparrow$$

$$\begin{array}{|c|c|c|} \hline \begin{array}{cc} \min_{y_1} & \phi_1(x,y) \\ \text{s.t.} & \left\{ \begin{array}{l} G(x,y) \leq 0 \\ g_1(y_1,\ldots,y_n,x) \leq 0 \end{array} \right. \end{array} & \cdots & \begin{array}{cc} \min_{y_n} & \phi_n(x,y) \\ \text{s.t.} & \left\{ \begin{array}{l} G(x,y) \leq 0 \\ g_n(y_1,\ldots,y_n,x) \leq 0 \end{array} \right. \end{array} \\ \hline \end{array}$$

Do these two models generate the same solutions???

Consider for example the following situation:

- three retailors of electricity are sharing a set of clients of an eco-district and are buying electricity to the same provider. The retailor 1, 2 and 3 plan to buy respectively x, y_1 and y_2 (MW);
- retailor 1 is a leader of the market and he want to minimize the total cost of energy given by $(x + y_1 + y_2)^2$;
- both retailors 2 and 3 want to maximize the sum of his purchase with the one of the leader;
- but retailor 2 cannot buy (for budget reasons) more that $1/2MW$ while retailor 3 is forces to buy less that retailor 1;

What's the leader problem???

What's the follower problem???

knowing that the producer cannot produce more than $1MW$;

$$\begin{array}{ll} \min_{x_1,y} & \theta_1(x_1,x_2,y) = x_1.y \\ s.t. & \left\{ \begin{array}{l} x_1 \in [0,1] \\ y \in S(x_1,x_2) \end{array} \right. \end{array} \qquad \begin{array}{ll} \min_{x_2,y} & \theta_1(x_1,x_2,y) = -x_2.y \\ s.t. & \left\{ \begin{array}{l} x_2 \in [0,1] \\ y \in S(x_1,x_2) \end{array} \right. \end{array}$$

with

$$\begin{array}{ll} \min_y & f(x_1,x_2,y) = \frac{1}{3}y^3 - (x_1+x_2)^2 y \\ s.t. & y \in \mathbb{R} \end{array}$$

Exercise: Please analyse this small example...

Solution of the exercise

MLSF game: ill-posedness

$$\min_{x_1,y} \quad \theta_1(x_1,x_2,y) = x_1.y \qquad\qquad \min_{x_2,y} \quad \theta_1(x_1,x_2,y) = -x_2.y$$

$$s.t. \quad \left\{ \begin{array}{l} x_1 \in [0,1] \\ y \in S(x_1,x_2) \end{array} \right. \qquad\qquad s.t. \quad \left\{ \begin{array}{l} x_2 \in [0,1] \\ y \in S(x_1,x_2) \end{array} \right.$$

with

$$\begin{array}{ll} \min_y & f(x_1,x_2,y) = \frac{1}{4}y^4 - \frac{(x_1+x_2)^2}{2}y^2 \\ s.t. & y \in \mathbb{R} \end{array}$$

The follower problem first

$$
\begin{array}{ll}
\min_y & f(x_1, x_2, y) = \frac{1}{4}y^4 - \frac{(x_1+x_2)^2}{2}y^2 \\
s.t. & y \in \mathbb{R}
\end{array}
$$

MLSF game: ill-posedness

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The solution map of this follower problem is

$$S(x_1,x_2) = \{y_1 = x_1 + x_2, y_2 = -x_1 - x_2\}.$$

The solution map of this follower problem is

$$S(x_1, x_2) = \{y_1 = x_1 + x_2, y_2 = -x_1 - x_2\}.$$

The leader 1 problem

$$\theta_1(x, y) = x_1.y = \begin{cases} x_1^2 + x_1.x_2 & \textit{if } y = y_1 \\ -x_1^2 - x_1.x_2 & \textit{if } y = y_2 \end{cases}$$

MLSF game: ill-posedness

The solution map of this follower problem is

$$S(x_1, x_2) = \{y_1 = x_1 + x_2, y_2 = -x_1 - x_2\}.$$

The leader 1 problem

$$\theta_1(x, y) = x_1.y = \begin{cases} x_1^2 + x_1.x_2 & \textit{if } y = y_1 \\ -x_1^2 - x_1.x_2 & \textit{if } y = y_2 \end{cases}$$

Thus the response function of player 1 is

$$\mathbb{R}_1(x_2) = \begin{cases} \{0\} & \textit{if } y = y_1 \textit{ with a payoff} = 0 \\ \{1\} & \textit{if } y = y_2 \textit{ with a payoff} = -1 - x_2 \end{cases}$$

MLSF game: ill-posedness

The solution map of this follower problem is

$$S(x_1, x_2) = \{y_1 = x_1 + x_2, y_2 = -x_1 - x_2\}.$$

The leader 2 problem

$$\theta_1(x, y) = -x_2.y = \begin{cases} -x_1^2 - x_1.x_2 & \text{if } y = y_1 \\ x_1^2 + x_1.x_2 & \text{if } y = y_2 \end{cases}$$

The solution map of this follower problem is

$$S(x_1, x_2) = \{y_1 = x_1 + x_2, y_2 = -x_1 - x_2\}.$$

The leader 2 problem

$$\theta_1(x, y) = -x_2.y = \begin{cases} -x_1^2 - x_1.x_2 & \textit{if } y = y_1 \\ x_1^2 + x_1.x_2 & \textit{if } y = y_2 \end{cases}$$

Thus the response function of player 1 is

$$\mathbb{R}_2(x_1) = \begin{cases} \{1\} & \textit{if } y = y_1 \textit{ with a payoff} = -1 - x_1 \\ \{0\} & \textit{if } y = y_2 \textit{ with a payoff} = 0 \end{cases}$$

MLSF game: ill-posedness

$$\mathbb{R}_1(x_2) = \begin{cases} \{(0, y = y_1)\} & \textit{with a payoff} = 0 \\ \{(1, y = y_2)\} & \textit{with a payoff} = -1 - x_2 \end{cases}$$

$$\mathbb{R}_2(x_1) = \begin{cases} \{(1, y = y_1)\} & \textit{with a payoff} = -1 - x_1 \\ \{(0, y = y_2)\} & \textit{with a payoff} = 0 \end{cases}$$

MLSF game: ill-posedness

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$$\mathbb{R}_2(x_1) = \begin{cases} \{(1, y = y_1)\} & \textit{with a payoff} = -1 - x_1 \\ \{(0, y = y_2)\} & \textit{with a payoff} = 0 \end{cases}$$

So the Nash equilibrium will be $(x_1, x_2) = (1, 1)$ but....

Just one last example [Pang-Fukushima 05]

Let us consider a 2-leader-single-follower game:

$$min_{x_1,y} \quad \tfrac{1}{2}x_1 + y \qquad\qquad min_{x_2,y} \quad -\tfrac{1}{2}x_2 - y$$

$$\left\{\begin{array}{l} x_1 \in [0,1] \\ y \in S(x_1,x_2) \end{array}\right. \qquad\qquad \left\{\begin{array}{l} x_2 \in [0,1] \\ y \in S(x_1,x_2) \end{array}\right.$$

where $S(x_1, x_2)$ is the solution map of

$$min_{y\geq 0} \quad y(-1 + x_1 + x_2) + \frac{1}{2}y^2$$

Just one example [Pang-Fukushima 05]

Let us consider a 2-leader-single-follower game:

$$min_{x_1,y_1} \quad \frac{1}{2}x_1 + y_1 \qquad\qquad min_{x_2,y_2} \quad -\frac{1}{2}x_2 - y_2$$

$$\left\{ \begin{array}{l} x_1 \in [0,1] \\ y_1 \in S(x_1,x_2) \end{array} \right. \qquad\qquad \left\{ \begin{array}{l} x_2 \in [0,1] \\ y_2 \in S(x_1,x_2) \end{array} \right.$$

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$$min_{y\geq 0} \quad y(-1 + x_1 + x_2) + \frac{1}{2}y^2$$

Actually $S(x_1, x_2) = \max\{0, 1 - x_1 - x_2\}$ thus the problem becomes

$$min_{x_1,y_1} \quad \frac{1}{2}x_1 + y_1 \qquad \left\{ \begin{array}{l} x_1 \in [0,1] \\ y_1 = \max\{0, 1 - x_1 - x_2\} \end{array} \right.$$

$$min_{x_2,y_2} \quad -\frac{1}{2}x_2 - y_2 \qquad \left\{ \begin{array}{l} x_2 \in [0,1] \\ y_2 = \max\{0, 1 - x_1 - x_2\} \end{array} \right.$$

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$$min_{x_2,y_2} \quad -\frac{1}{2}x_2 - y_2 \qquad \left\{ \begin{array}{l} x_2 \in [0,1] \\ y_2 = \max\{0, 1 - x_1 - x_2\} \end{array} \right.$$

Then the Response maps are

$$\mathcal{R}_1(x_2) = \{1 - x_2\} \quad \text{and} \quad \mathcal{R}_2(x_1) = \left\{ \begin{array}{ll} \{0\} & x_1 \in [0, \frac{1}{2}[\\ \{0,1\} & x_1 = \frac{1}{2} \\ \{1\} & x_1 \in]\frac{1}{2}, 1] \end{array} \right.$$

and thus there is no Nash equilibrium.......

But let us consider the slightly modified problem.......

$$min_{x_1,y_1} \quad \frac{1}{2}x_1 + y_1 \qquad\qquad min_{x_2,y_2} \quad -\frac{1}{2}x_2 - y_2$$

$$\left\{\begin{array}{l} x_1 \in [0,1] \\ y_1 = \max\{0, 1 - x_1 - x_2\} \\ y_2 = \max\{0, 1 - x_1 - x_2\} \end{array}\right. \qquad\qquad \left\{\begin{array}{l} x_2 \in [0,1] \\ y_1 = \max\{0, 1 - x_1 - x_2\} \\ y_2 = \max\{0, 1 - x_1 - x_2\} \end{array}\right.$$

But let us consider the slightly modified problem.......

$$min_{x_1,y_1} \quad \frac{1}{2}x_1 + y_1 \qquad \left\{\begin{array}{l} x_1 \in [0,1] \\ y_1 = \max\{0, 1 - x_1 - x_2\} \\ y_2 = \max\{0, 1 - x_1 - x_2\} \end{array}\right.$$

$$min_{x_2,y_2} \quad -\frac{1}{2}x_2 - y_2 \qquad \left\{\begin{array}{l} x_2 \in [0,1] \\ y_1 = \max\{0, 1 - x_1 - x_2\} \\ y_2 = \max\{0, 1 - x_1 - x_2\} \end{array}\right.$$

that can be proved to have a (unique) Nash equilibrium namely $(x_1, x_2) = (0, 1)$ with $y_1 = y_2 = 0$!!!!

The kind of "trick" is called "All Equilibrium approach" and has been introduced in A.A. Kulkarni & U.V. Shanbhag, *A Shared-Constraint Approach to Multi-Leader Multi-Follower Games*, Set-Valued Var. Anal (2014).

They proved that every Nash equilibirum (initial problem) is a Nash equilibrium for the "all equilibrium" formulation.

It corresponds to the case where each leader takes into account the conjectures regarding the follower decision made by all other leaders....

Some of my works...

- D. Aussel, A. Svensson, *Towards Tractable Constraint Qualifications for Parametric Optimisation Problems and Applications to Generalised Nash Games*, **J. Optim. Theory Appl.** 182 (2019), 404-416.
- D. Aussel, L. Brotcorne, S. Lepaul, L. von Niederhäusern, *A Trilevel Model for Best Response in Energy Demand Side Management*, **Eur. J. Operations Research** 281 (2020), 299-315.
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- D. Aussel & A. Svensson, *Some remarks on existence of equilibria, and the validity of the EPCC reformulation for multi-leader-follower games*, **J. Nonlinear Convex Anal.** 19 (2018), 1141-1162.
- E. Allevi, D. Aussel & R. Riccardi, *On a equilibrium problem with complementarity constraints formulation of pay-as-clear electricity market with demand elasticity*, **J. Global Optim.** 70 (2018), 329-346.
- D. Aussel, P. Bendotti and M. Pištěk, *Nash Equilibrium in Pay-as-bid Electricity Market : Part 1 - Existence and Characterisation*, **Optimization** 66:6 (2017), 1013-1025.
- D. Aussel, P. Bendotti and M. Pištěk, *Nash Equilibrium in Pay-as-bid Electricity Market : Part 2 - Best Response of Producer*, **Optimization** 66:6 (2017), 1027-1053.

Some of my works...

- M. Ramos, M. Boix, D. Aussel, L. Montastruc, S. Domenech, *Water integration in Eco-Industrial Parks Using a Multi-Leader-Follower Approach*, **Computers & Chemical Engineering** 87 (2016) 190-207.
- M. Ramos, M. Boix, D. Aussel, L. Montastruc, S. Domenech, *Optimal Design of Water Exchanges in Eco-Inductrial Parks Through a Game Theory Approach*, **Computers Aided Chemical Engineering** 38 (2016) 1177-1183.
- M. Ramos, M. Rocafull, M. Boix, D. Aussel, L. Montastruc & S. Domenech, *Utility Network Optimization in Eco-Industrial Parks by a Multi-Leader-Follower game Methodology*, **Computers & Chemical Engineering** 112 (2018), 132-153.
- D. Salas, Cao Van Kien, D. Aussel, L. Montastruc, *Optimal design of exchange networks with blind inputs and its application to Eco-Industrial parks*, Computers & Chemical Engineering 143 (2020), 18 pp, published online.
- Aussel, K. Cao Van, *Control-input approach of of water exchange in Eco-Industrial Parks*, submitted (2020).

Multi-Leader-Follower Games: non cooperative and hierarchical/bilevel interactions
Talk 2

Didier Aussel

Lab. Promes UPR CNRS 8521, University of Perpignan, France

UNIVERS Winter school - November 14th-17th, 2021

Menu (for the three days...)

- Lecture 1: Definitions, motivations and well posedness
- Lecture 2: Motivations and existence
- Lecture 3: Reformulations (differentiable and non differentiable cases)

Lecture 2

- *Motivation: why to plan a travel to hell?*

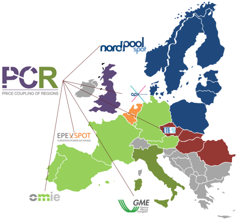
Lecture 2

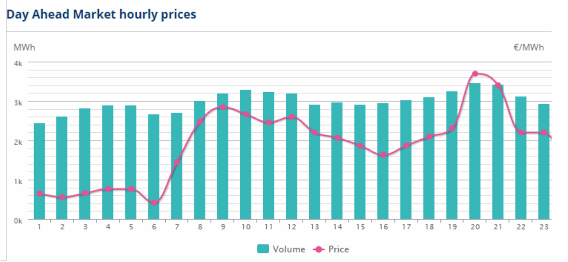
- *Motivation: why to plan a travel to hell?*
  ....exploration of energy management
- *Existence for GNEP and SLMFG*

I- Some motivation examples

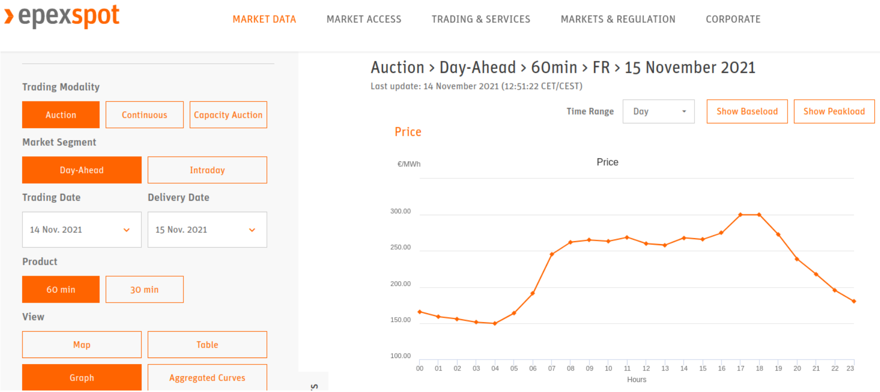
Electricity markets

A short introduction to electricity markets





A short introduction to electricity markets



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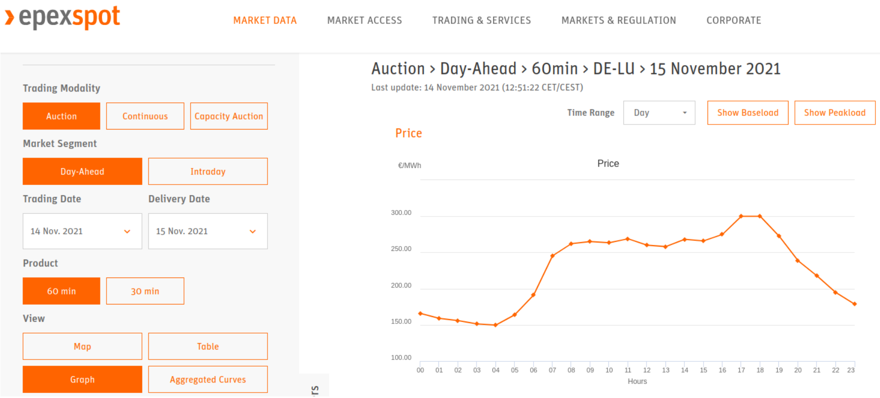
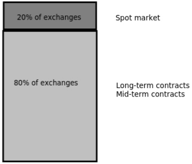


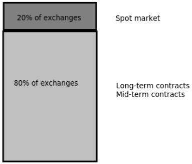
A short introduction to electricity markets (cont.)

Volume of exchanges

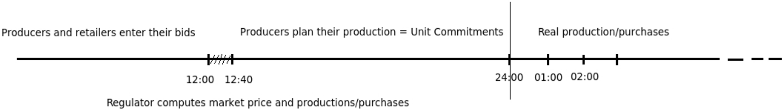


A short introduction to electricity markets (cont.)

Volume of exchanges



Bid schedule of the spot market (EPEX-FR/DE)



Modeling an Electricity Markets

- electricity market consists of
  - i) generators/consumers $i \in \mathcal{N}$, each of them taking care of his benefit by trying to find the "better/best" bids
  - ii) market operator (ISO) who maintain energy generation and load balance, and maximize social welfare

- the ISO has to consider:
  - ii) quantities q_i of generated/consumed electricity
  - iii) electricity dispatch t_e with respect to transmission capacities between bidding zones

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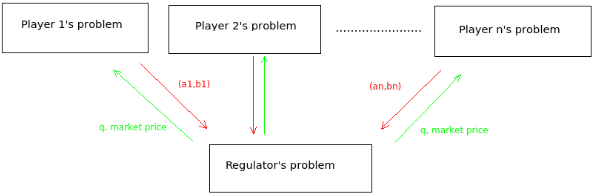
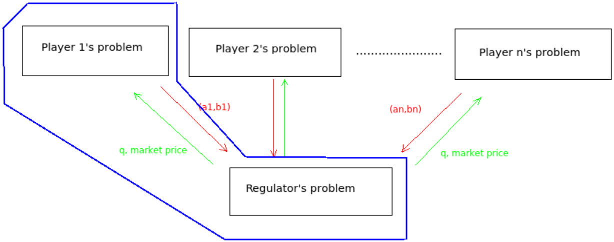
- since 1990s, Generalized Nash equilibrium problem is the most popular way of modeling spot electricity markets or, more precisely, Multi-Leader-Single-Follower game

Multi-Leader-Common-Follower game



Multi-Leader-Common-Follower game

A classical problem (of a producer) is the best response search



A simplified model of electricity market

Let consider a fixed hour "for tomorrow" and denote

- $D > 0$ be the overall energy demand of all consumers
- $\mathcal{N}$ be the set of producers
- $q_i \geq 0$ be the production of i-th producer, $i \in \mathcal{N}$

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We assume that producer $i \in \mathcal{N}$ provides to the ISO a quadratic bid function $a_i q_i + b_i q_i^2$ given by $a_i, b_i \geq 0$.

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- $q_i \geq 0$ be the production of i-th producer, $i \in \mathcal{N}$

We assume that producer $i \in \mathcal{N}$ provides to the ISO a quadratic bid function $a_i q_i + b_i q_i^2$ given by $a_i, b_i \geq 0$.

Similarly, let $A_i q_i + B_i q_i^2$ be the true production cost of i-th producer with $A_i \geq 0$ and $B_i > 0$ reflecting the increasing marginal cost of production.

Associated Multi-Leader-Single-Follower game

Peculiarity of electricity markets is their bi-level structure:

$$P_i(a_{-i}, b_{-i}, D) \qquad \max_{a_i, b_i} \max_{q_i} \quad a_i q_i + b_i q_i^2 - (A_i q_i + B_i q_i^2)$$
$$\text{such that} \quad \begin{cases} a_i, b_i \geq 0 \\ (q_j)_{j \in \mathcal{N}} \in Q(a,b) \end{cases}$$

where set-valued mapping $Q(a,b)$ denotes solution set of

$$ISO(a,b,D) \qquad Q(a,b) = \underset{q}{argmin} \quad \sum_{i \in \mathcal{N}} (a_i q_i + b_i q_i^2)$$
$$\text{such that} \quad \begin{cases} q_i \geq 0 \ , \ \forall i \in \mathcal{N} \\ \displaystyle\sum_{i \in \mathcal{N}} q_i = D \end{cases}$$

Some references:

- **Electricity markets without transmission losses:**
  X. Hu & D. Ralph, Using EPECs to Model Bilevel Games in Restructured Electricity Markets with Locational Prices, *Operations Research (2007). bid-on-a-only*

Some references:

- **Electricity markets without transmission losses:**
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- **Electricity markets with transmission losses:**
  - *Henrion, R., Outrata, J. & Surowiec, T.,* Analysis of M-stationary points to an EPEC modeling oligopolistic competition in an electricity spot market, *ESAIM: COCV (2012). M-stationary points*
  - *D. A., R. Correa & M. Marechal* Spot electricity market with transmission losses, *J. Industrial Manag. Optim (2013). existence of Nash equil., case of a two island model*
  - *D.A., M. Cervinka & M. Marechal,* Deregulated electricity markets with thermal losses and production bounds, *RAIRO (2016) production bounds, well-posedness of model*

Some references on the topic (cont.)

- **Best response in electricity markets:**
  - *E. Anderson and A. Philpott,* Optimal Offer Construction in Electricity Markets, *Mathematics of Operations Research (2002). Linear bid function - necessary optimality cond. for local best response in time dependent case*
  - *D. Aussel, P. Bendotti and M. Pištěk, Nash Equilibrium in Pay-as-bid Electricity Market : Part 2 - Best Response of Producer, Optimization (2017) linear unit bid function, explicit formula for best response*

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- **Explicit formula for equilibria**
  D. Aussel, P. Bendotti and M. Pištěk, Nash Equilibrium in Pay-as-bid Electricity Market : Part 1 - Existence and Characterisation, Optimization (2017) explicit formula for equilibria

But also...

- **Non a priori structured bid functions**
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  - *Escobar, J.F. and Jofré, A., Equilibrium analysis of electricity auctions, preprint (2014)*
  - *E. Anderson, P. Holmberg and A. Philpott, Mixed strategies in discriminatory divisible-good auctions, The RAND Journal of Economics (2013). necessary optimality cond. for local best response*
- **Robustness analysis**
  Kramer, A., Krebs, V. and Schmidt, M., Strictly and ?-robust counterparts of electricity market models: Perfect competition and Nash-Cournot equilibria. Oper. Res. Perspect. (2021)
- **Optimal design for network expansion**
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Some motivation examples

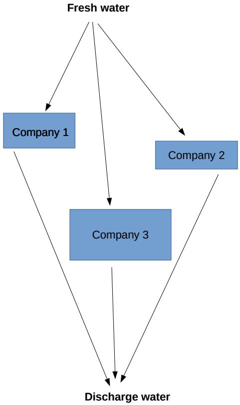
Industrial Eco-Parks

What is an « Eco-park » ?

Example of water management

- In a geographical area, there are different companies $1, \ldots, n$
- Each of them is buying fresh water (high price) for their production processes
- Each company generates some "dirty water" and have to pay for discharge

Stand alone situation



How does it work ?

The aims in designing Industrial Eco-park (IEP) are

a) Reduce cost of production of each company

b) Reduce the environmental impact of the whole production

Thus "Eco" of IEP is at the same time Economical and ecological

What is an « Eco-park » ?

Example of water management

How to reach these aims?

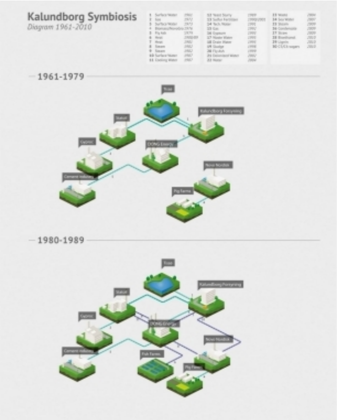
a) create a network (water tubes) between the companies

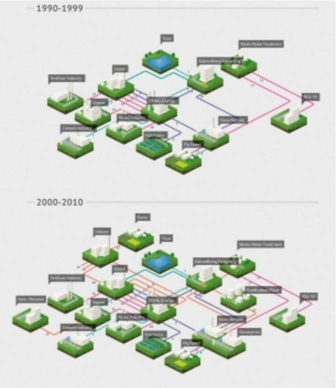
b) Eventually install some regeneration unit (cleaning of the water)

It is important to understand that this approach is not limited to water. It can be applied to vapor, gas, coaling fluids, human resources...

Kalundborg (Danemark)

An symbolic example of Industrial eco-park is Kalundborg (Danemark)





Definition

What is an « Eco-park » ?

In order to convince companies to participate to the Ecopark, our model should guarantee that:

a) **each company** will have a lower cost of production in Eco-park organization than in stand-alone organization

b) the eco-park organization must generate a **lower freshwater consumption** than with a stand-alone organization

MOO classical treatment

The Eco-park design was done through Multi-objective Optimization by the evaluation of Pareto fronts (Gold programming algorithms, scalarization...).

$$\min \quad \begin{cases} \textit{Fresh water consumption} \\ \textit{Individual costs of producer } 1 \\ \vdots \\ \textit{Individual costs of producer } n \end{cases}$$

$$s.t. \quad \begin{cases} \textit{Water balances} \\ \textit{Topological constraints} \\ \textit{Water quality criteria} \end{cases}$$

MOO classical treatment

Stand-alone structure

Enterprise		1	2	3	Total
Water flowrate (tonne/hr)	Fresh	98.33	54.64	186.67	339.64
Cost (MMUSD/year)	Freshwater+discharge	0.28	0.15	0.52	0.95
	Reused water	0.01	0.01	0.02	0.03
	Total	0.28	0.16	0.54	0.98

Eco-park structure : MOO approach

Enterprise		1	2	3	Total
Water flowrate (tonne/hr)	Fresh	88.33	20.00	206.02	314.36
	Shared	76.67	61.04	82.00	219.71
Cost (MMUSD/year)	Freshwater+ Discharge	0.18	0.11	0.59	0.88
	Reused water	0.01	0.02	0.02	0.06
	Total	0.20	0.13	0.61	0.94

Alternative approach

The needed change :

. . . to have an independant designer/regulator

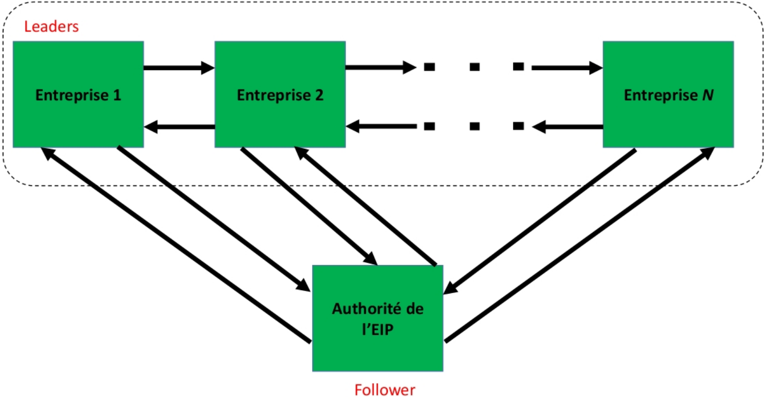
. . . to have fair solutions for the companies

Thus we propose to use two different possible models:

- Hierarchical optimisation (bi-level optim.)
- Nash game concept between the companies

M. Ramos, M. Boix, D. Aussel, L. Montastruc, S. Domenech, Water integration in Eco-Industrial Parks Using a Multi-Leader-Follower Approach, Computers & Chemical Engineering (2016)

Multi-Leader-Single-Follower game



Here leaders are the companies and the follower is the designer of the EIP.

Multi-Leader-Single-Follower game

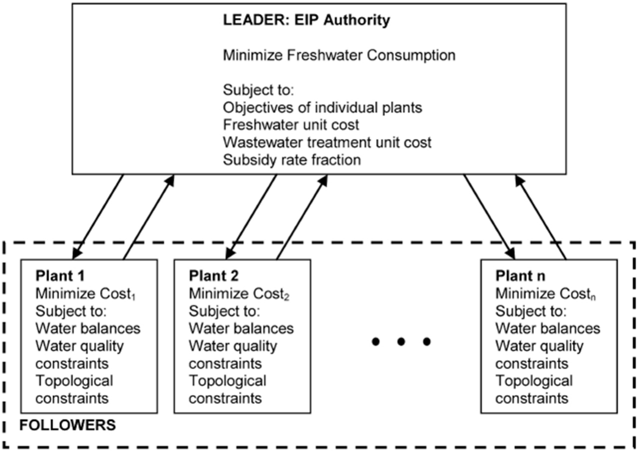
Stand-alone implementation with regeration units

Enterprise		1	2	3	Total
Water flowrate (tonne/hr)	Fresh	98.33	22.00	97.50	217.83
	Regenerated	0.00	38.17	111.46	149.63
Cost (MMUSD/year)	Freshwater+discharge	0.28	0.06	0.27	0.61
	Reused water	0.01	0.02	0.05	0.08
	Regenerated water	0.00	0.08	0.19	0.27
	Total	0.28	0.17	0.51	0.96

Nash equilibrium (MLCFG) with regeneration units

Enterprise		1	2	3	Total
Water flowrate (tonne/hr)	Freshwater (tonne/hr)	77.10	48.14	94.38	219.62
	Shared	86.38	63.56	124.93	274.87
	Regenerated	23.95	0.00	96.30	120.24
Cost (MMUSD/year)	Freshwater+Discharge	0.17	0.13	0.31	0.61
	Reused water	0.03	0.01	0.04	0.09
	Regenerated water	0.05	0.00	0.11	0.15
	Total	0.24	0.14	0.44	0.83

Single-Leader-Multi-Follower game



Numerical treatment

This very difficult problem is treated as follows:

- first we replace the lower-level (convex) optimization problem by their KKT systems; the resulting problem is an Mathematical Programming with Complementarity Constraints (MPCC);
- second the MPCC problem is solved by penalization methods

Numerical results have been obtained with Julia meta-solver coupled with Gurobi, IPOPT and Baron.

Single-Leader-Multi-Follower game

Stand-alone implementation with regeration units

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	Regenerated water	0.00	0.08	0.19	0.27
	Total	0.28	0.17	0.51	0.96

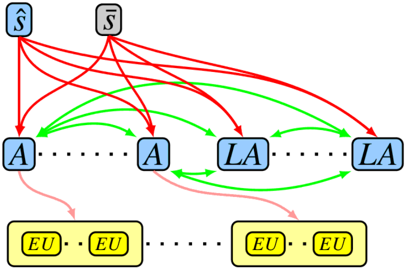
Nash equilibrium (SLMFG) with regeneration units

Enterprise		1	2	3	Total
Water flowrate (tonne/hr)	Freshwater (tonne/hr)	20.00	20.00	20.00	60.00
	Shared	126.49	149.54	226.66	502.69
	Regenerated	100.62	64.67	166.64	331.93
Cost (MMUSD/year)	Freshwater+Discharge	0.04	0.02	0.11	0.17
	Reused water	0.04	0.03	0.08	0.15
	Regenerated water	0.12	0.08	0.19	0.39

Some motivation examples

Demand-side Management

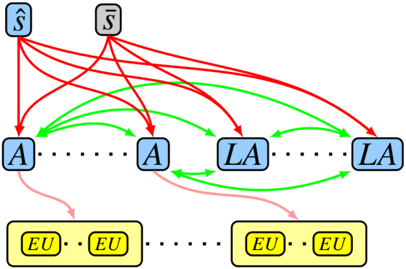
Our study case: **Contract problem with Demand-Side Management**



where

- S is the electricity producer (big company)
- A are the aggregators (bying/saling electricity but not producing)
- LA are the local agents (managers of smart grids)
- EU are the end-users (smart/active consumers)

Our study case: **Contract problem with Demand-Side Management**



a **THREE level model**

where

- S is the electricity producer (big company)
- A are the aggregators (bying/saling electricity but not producing)
- LA are the local agents (managers of smart grids)
- EU are the end-users (smart/active consumers)

End User's problem

$$(P_i) \qquad \max_{\mathbf{d}_i} \sum_{h\in H} r_{ai}^h d_i^h - V_i^h\left(d_i^h\right)$$
$$s.t. \begin{cases} \sum_{h\in H} d_i^h = W_i \\ d_i^h \geq 0 \qquad\qquad \forall h \in H, \end{cases}$$

where d_i^h denotes i's demand at time h, W_i denotes i's overall need in electricity and $V_i^h\left(d_i^h\right) = v_i^h\left(d_i^h - d_i^{h,0}\right)^2$ is the inconvenience caused by the load shifting ($v_i^h > 0$ is fixed). As for the local agents, $\left(d_i^{h,0}\right)_h$ stands for the *a priori* demand vector of end user i.

Aggregator's problem

$$\min_{\mathbf{e}_{a\cdot},\mathbf{r}_a,\mathbf{p}_{a\cdot},\boldsymbol{\alpha}_{a\cdot}} \min_{\mathbf{d^a}} \sum_{h\in H}\Bigg(\sum_{s\in\mathcal{S}} p^h_{sa}e^h_{as} + \sum_{\ell\in\mathcal{L}}\left(p^h_{\ell a}e^h_{a\ell} - p^h_{a\ell}e^h_{\ell a}\right) + \sum_{a'\neq a}\left(p^h_{a'a}e^h_{aa'} - p^h_{aa'}e^h_{a'a}\right)$$
$$+ \textstyle\sum_{i\in\mathcal{I}_a} r^h_{ai}d^h_i\Bigg)$$

$$s.t. \begin{cases} \sum_{s\in\mathcal{S}} e^h_{as} + \sum_{\ell\in\mathcal{L}}\left(e^h_{a\ell} - e^h_{\ell a}\right) + \sum_{a'\neq a}\left(e^h_{aa'} - e^h_{a'a}\right) = \sum_{i\in\mathcal{I}_a} d^h_i & \forall h\in H \\ \mathbf{d}^a_i \in \operatorname{argmax}(P_i) & \forall i\in\mathcal{I}_a \\ p^h_{ax} \leq \alpha^h_{axs}p^h_{sx} + \left(1-\alpha^h_{axs}\right)p^h_{sa} & \forall h\in H, x\in\mathcal{L}\cup\mathcal{A}, s\in\mathcal{S} \\ r^h_{ai}, e^h_{ax} \geq 0 & \forall h\in H, x\in\mathcal{L}\cup\mathcal{A}, i\in\mathcal{I}_a \\ \alpha^h_{axs}\in[0,1] & \forall h\in H, x\in\mathcal{L}\cup\mathcal{A}, s\in\mathcal{S} \end{cases}$$

where $\mathbf{d}^a$ stands for the vector $\mathbf{d}^a = (\mathbf{d}^a_i)_{i\in\mathcal{I}_a}$, with $\mathbf{d}^a_i = \left(d^h_i\right)_{h\in H}$, (P_i) is the end user i's problem and $\mathcal{I}_a$ denotes the set of end users who are in contract with aggregator a. It is here assumed that each end user is in contract with only one aggregator.

Local agent's problem

$$\min_{\mathbf{e}_{\ell\cdot},\mathbf{p}_{\ell\cdot},\boldsymbol{\alpha}_{\ell\cdot}} \sum_{h\in H}\left(\sum_{s\in\mathcal{S}} p^h_{s\ell}e^h_{\ell s} + \sum_{a\in\mathcal{A}}\left(p^h_{a\ell}e^h_{\ell a} - p^h_{\ell a}e^h_{a\ell}\right) + \sum_{\ell'\neq\ell}\left(p^h_{\ell'\ell}e^h_{\ell\ell'} - p^h_{\ell\ell'}e^h_{\ell'\ell}\right) + V^h_\ell\left(d^h_\ell\right)\right)$$

$$s.t.\begin{cases}\sum_{h\in H} d^h_\ell = W_\ell & \\ e^h_{\ell x}\geq 0 & \forall h\in H, x\in\mathcal{L}\cup\mathcal{A}\cup\mathcal{S},\\ p^h_{\ell x}\leq \alpha^h_{\ell x s}p^h_{sx} + \left(1-\alpha^h_{\ell x s}\right)p^h_{s\ell} & \forall h\in H, x\in\mathcal{L}\cup\mathcal{A}, s\in\mathcal{S}\\ \alpha^h_{\ell x s}\in[0,1] & \forall h\in H, x\in\mathcal{L}\cup\mathcal{A}, s\in\mathcal{S},\end{cases}$$

where $V^h_\ell\left(d^h_\ell\right) = v^h_\ell\left(d^h_\ell - d^{h,0}_\ell\right)^2$ is the inconvenience caused by the load shifting and $v^h_\ell > 0$ is a fixed inconvenience coefficient. Note that, for any h, the notation d^h_ℓ stands for the demand value and thus $d^h_\ell \overset{not.}{=} \sum_{s\in\mathcal{S}} e^h_{\ell s} + \sum_{a\in\mathcal{A}}\left(e^h_{\ell a} - e^h_{a\ell}\right) + \sum_{\ell'\neq\ell}\left(e^h_{\ell\ell'} - e^h_{\ell'\ell}\right)$ since no storage is considered in this model. This function has nice mathematical properties, like convexity and differentiability, and it adequately models the real inconvenience that the consumers are undergoing. A small shift of the consumption will not represent a significant inconvenience, whereas an important shift will have strong repercussions on the consumer's comfort thanks to the square power.

Supplier's problem

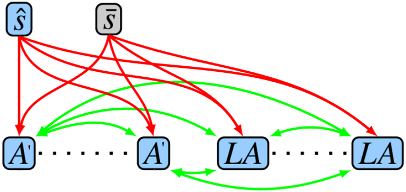
$$(P_{\hat{s}}) \qquad \max_{\mathbf{p}_{\hat{s}\cdot}} \max_{\mathbf{e},\mathbf{r},\mathbf{p}_{x\cdot},\boldsymbol{\alpha}} \sum_{h\in H} \left(\sum_{a\in\mathcal{A}} p^h_{\hat{s}a} e^h_{a\hat{s}} + \sum_{\ell\in\mathcal{L}} p^h_{\hat{s}\ell} e^h_{\ell\hat{s}} - c^h_{\hat{s}} \left(\sum_{a\in\mathcal{A}} e^h_{a\hat{s}} + \sum_{\ell\in\mathcal{L}} e^h_{\ell\hat{s}} \right) \right)$$

We assume here that the function $c^h_{\hat{s}} : t \mapsto c^h_{\hat{s}}(t)$ is increasing and convex for all $h \in H$.

According to the previous notations, (P_a) and (P_ℓ) respectively denote the optimization problems of the aggregator $a \in \mathcal{A}$ and the local agent $\ell \in \mathcal{L}$.

First simplification

Well since the EU's problem is simple...



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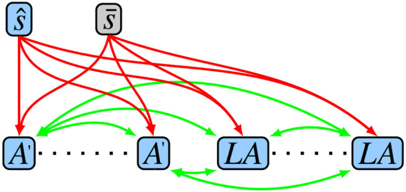


But even so, it is still a Single-Leader-Multi-Follower game.....

A smart strategy

- *Concentrate on a specific concept of equilibriums/solutions (called here revisited solutions)*
- *Generate from this specific solution several "classical solutions*

A smart strategy

The following proposition ensures that we can restrain the search of an optimal GNE in the set $GNE^o(\mathbf{p})$, which is the set of GNEs at the intermediary level where all energy exchanges among ILAs are equal to zero.

Assume few "technical assumptions". Then one can construct:

- *a leader price profile* $\mathbf{p}^*$
- *a GNE* $S^*(\mathbf{p}^*)$

such that

$$\mathbf{e}^*_{xy} = 0, \text{ for all } x, y \in \mathcal{L} \cup \mathcal{A}$$

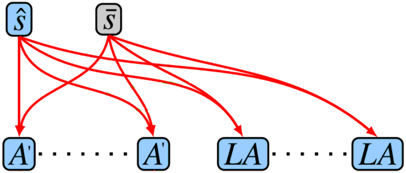
and

$$z(S(\mathbf{p})) = z(S^*(\mathbf{p}^*))$$

where z denotes the objective function of $\hat{s}$. That is, $\mathbf{p}^*$ is an optimal price profile for the leader too.

Better simplification

So finally leading to the more simple problem...



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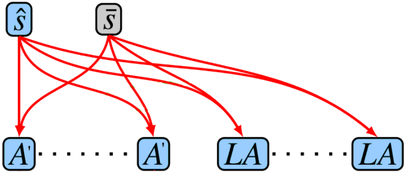
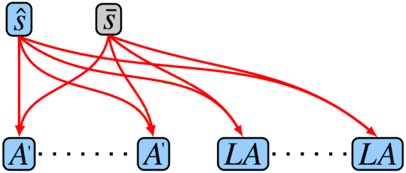


$$\max_{\mathbf{p}_{\hat{s}x}} \max_{\mathbf{e}_x, \mathbf{d}^a, \hat{\boldsymbol{\lambda}}^a} \sum_{h \in H} \left(\sum_{x \in \mathcal{L} \cup \mathcal{A}} p^h_{\hat{s}x} e^h_{x\hat{s}} - c^h_{\hat{s}} \left(\sum_{x \in \mathcal{L} \cup \mathcal{A}} e^h_{x\hat{s}} \right) \right)$$

$$s.t. \begin{cases} p^h_{\hat{s}x} \geq 0 & \forall h \in H, x \in \mathcal{L} \cup \mathcal{A} \\ (\mathbf{e}_{\ell\hat{\mathbf{s}}}, \mathbf{e}_{\ell\bar{s}}) \in \operatorname{argmax}\left(P^{el}_{\ell}\right) & \forall \ell \in \mathcal{L} \\ \left(\mathbf{e}_{\mathbf{a}\hat{\mathbf{s}}}, \mathbf{e}_{a\bar{s}}, \mathbf{d}^a, \hat{\boldsymbol{\lambda}}^a\right) \in \operatorname{argmax}\left(P^{el}_{a}\right) & \forall a \in \mathcal{A}, \end{cases}$$

Better simplification

So finally leading to the more simple problem...



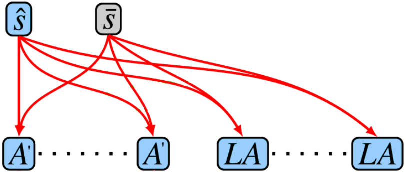
$$\max_{\mathbf{p}_{\hat{s}x}} \max_{\mathbf{e}_x,\mathbf{d}^a,\hat{\boldsymbol{\lambda}}^a} \sum_{h\in H} \left(\sum_{x\in\mathcal{L}\cup\mathcal{A}} p^h_{\hat{s}x} e^h_{x\hat{s}} - c^h_{\hat{s}} \left(\sum_{x\in\mathcal{L}\cup\mathcal{A}} e^h_{x\hat{s}} \right)\right)$$

$$s.t. \begin{cases} p^h_{\hat{s}x} \geq 0 & \forall h \in H, x \in \mathcal{L}\cup\mathcal{A} \\ (\mathbf{e}_{\ell\hat{\mathbf{s}}}, \mathbf{e}_{\ell\bar{s}}) \in \operatorname{argmax}\left(P^{el}_\ell\right) & \forall \ell \in \mathcal{L} \\ \left(\mathbf{e}_{\mathbf{a}\hat{\mathbf{s}}}, \mathbf{e}_{a\bar{s}}, \mathbf{d}^a, \hat{\boldsymbol{\lambda}}^a\right) \in \operatorname{argmax}\left(P^{el}_a\right) & \forall a \in \mathcal{A}, \end{cases}$$

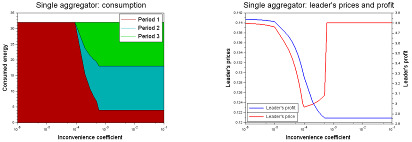
Then it can be tackled with classical tools: Julia with Path solver,...

Better simplification

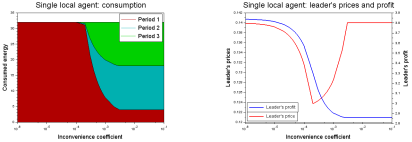
So finally leading to the more simple problem...



- For small instances (3 time slots, 2 local agents, 1 aggregator in contract with 2 end users), the difference between both methods might not be significant (282 variables, 342 constraints for the classical method versus 45 variables, 51 constraints for the revisited optimistic formulation)
- But the classical method becomes intractable as soon as instances grow larger (24 time slots, 10 aggregators with 1 end user each, 10 local agents give 79710 variables and 99150 constraints, whereas the revisited optimistic formulation only needs 1710 variables and 1950 constraints).



Situation with one aggregator: at left, the shift induced by the leader's prices. At right, the price offered by the leader during the first time period and the leader's profit.



Situation with one local agent: at left, the shift induced by the leader's prices. At right, the price offered by the leader during the first time period and the leader's profit.

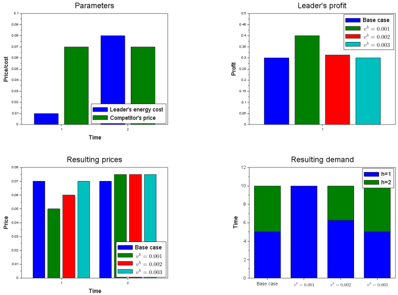


Figure: On top left, the data for the example where the leader is not always competitive: the production cost and the competitor's prices. On top right, the resulting profit of the leader for the four various cases: no optimization (i.e. copying the competitor's prices), and optimization for the three possible values of v^h.
At bottom left, the optimal prices of the leader for the example where the leader is not always competitive. At bottom right, the follower's demand resulting of these prices.

II- About existence of solutions

Problems we want to tackle...

Single-Leader-Multi-Follower-Game (SLMFG):

$$\text{" }\min_x\text{ "} \quad \theta(x,y) \qquad \text{s.t.} \quad \begin{cases} x \in X(y) \\ y \in GNEP(x) \end{cases}$$

$$\downarrow\uparrow$$

$$\min_{y_1} \quad \phi_1(x,y) \qquad \text{s.t.} \quad \{\ y_1 \in K_1(x, y_{-1})$$

$$\dots$$

$$\min_{y_n} \quad \phi_n(x,y) \qquad \text{s.t.} \quad \{\ y_n \in K_n(x, y_{-n})$$

Problems we want to tackle...

Single-Leader-Multi-Follower-Game (SLMFG):

$$\text{"}\min_x\text{"} \quad \theta(x,y) \quad \text{s.t.} \quad \begin{cases} x \in X(y) \\ y \in GNEP(x) \end{cases}$$

$$\downarrow\uparrow$$

$$\min_{y_1} \quad \phi_1(x,y) \quad \text{s.t.} \quad \{\; y_1 \in K_1(x,y_{-1}) \quad \ldots \quad \min_{y_n} \quad \phi_n(x,y) \quad \text{s.t.} \quad \{\; y_n \in K_n(x,y_{-n})$$

Would be interesting to consider:

- *Existence of solution upper level problem* $\Leftrightarrow$ *SLMFG*
- *Existence results for lower level problem* $\Leftrightarrow$ *GNEP*

An existence results for SLMFG

Theorem

Assume that

- F *is lower semi-continuous;*
- *for each follower* $\nu = 1, ..., n$*, the objective* $f_\nu(\cdot, \cdot\cdot)$ *is continuous;*
- *for each follower* $\nu = 1, ..., n$*,* $(x, y_{-\nu}) \mapsto K_\nu(x, y_{-\nu}) := \{y_\nu \mid g_\nu(x, y) \leq 0\}$ *is a lower semi-continuous set-valued map which has nonempty compact graph;*
- *the graph of* $GNEP$ *is nonempty.*

Then the SLMF game admits an optimistic solution of a $SLMFG$

D. Aussel & A. Svensson, Some remarks on existence of equilibria, and the validity of the EPCC reformulation for multi-leader-follower games, J. Nonlinear Convex Anal. (2018)

An existence results for SLMFG (cont.)

Proof.

We will prove first that the set-valued map $GNEP$ is graph closed, thus defining a closed constraint set for the leader. Let us observe that we can write $GNEP(x) = \bigcap_{\nu=1}^{n} S_\nu(x)$ with

$$S_\nu(x) := \left\{ (y_\nu, y_{-\nu}) \mid y_\nu \in \arg\min_{\tilde{y}_\nu} \{ f_\nu(x, \tilde{y}_\nu, y_{-\nu}) \mid \tilde{y}_\nu \in K_\nu(x, y_{-\nu}) \} \right\}$$

thus it is sufficient to prove that each of these maps has closed graph. Let us fix ν and take sequences x_k and y_k in $\mathbb{R}^p$ and $\mathbb{R}^q$ converging respectively to x and y, and such that $y_k \in S_\nu(x_k)$ for all $k \in \mathbb{N}$. We want to prove that then $y \in S_\nu(x)$. □

An existence results for SLMFG (cont.)

Proof.

Since

$$S_\nu(x) := \left\{ (y_\nu, y_{-\nu}) \mid y_\nu \in \arg\min_{\tilde{y}_\nu} \{ f_\nu(x, \tilde{y}_\nu, y_{-\nu}) \mid \tilde{y}_\nu \in K_\nu(x, y_{-\nu}) \} \right\}$$

is a subset of the closed graph map K_ν, one has $y \in K_\nu(x)$. Take $\tilde{y}_\nu \in K_\nu(x, y_{-\nu})$. By lower semi-continuity of the set-valued map K_ν we know that (up to subsequences) there exist $\tilde{y}_{\nu,k} \in K_\nu(x_k, y_{-\nu,k})$ such that $\tilde{y}_{\nu,k} \to \tilde{y}_\nu$. Since, for any k, $y_k \in S_\nu(x_k)$ then

$$f_\nu(x_k, y_{\nu,k}, y_{-\nu,k}) \le f_\nu(x_k, \tilde{y}_{\nu,k}, y_{-\nu,k}), \quad \forall k \in \mathbb{N},$$

and taking the limit, since f_ν is continuous it gives $f_\nu(x, y_\nu, y_{-\nu}) \le f_\nu(x, \tilde{y}_\nu, y_{-\nu})$. Since $\tilde{y}_\nu$ was arbitrarily chosen from $K_\nu(x, y_{-\nu})$ we conclude that $y \in S_\nu(x)$. Thus S_ν is closed.
From the assumption we deduce that the graph of $GNEP$ is nonempty and compact. We conclude that there exists a minimiser since F is lower semi-continuous.

An existence results for GNEP: continuous cost functions

Theorem (Ichiishi 83)

Assume that

- *for every player* ν*, the loss function* θ_ν *is continuous on* $\mathbb{R}^n$ *and quasiconvex with respect to the* ν*-th variable.*
- *the set-valued map* $X = \prod_\nu X_\nu$ *is Upper Semicontinuous and Lower Semicontinuous with nonempty convex compact values*

then the Generalized Nash equilibrium problem admits a solution.

T. Ichiishi (1983), Game theory for economic analysis - Academic Press

Initially Arrow-Debreu (1954), Econometrica

An existence results for GNEP: discontinuous cost functions

Proposition (Reny 2016)

For any $\nu \in I$, let C_ν be a nonempty, compact and convex subset of $\mathbb{R}^{n_\nu}$, $\theta_\nu : \mathbb{R}^N \to \mathbb{R}$. Then, the $NEP(\theta, C)$ admits a Nash Equilibrium if

1. *For every $x_{-\nu} \in C_{-\nu}$, the function $\theta_\nu(\cdot, x_{-\nu})$ is quasiconvex and lower semi-continuous;*
2. *the game is* better-reply secure.

P.J. Reny (2016), Economic. Theory
Bich and R. Laraki (2017), Econ. Theory Bull.

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2. *the game is* better-reply secure.

Let us recall, adapting the notations of Laraki-Bich, that the game satisfies the *Better-reply secure* if, for any

$$(x, \lambda) := ((x_\nu)_\nu, (\lambda_\nu)_\nu)) \in cl \left\{ ((u_\nu)_\nu, (\theta_\nu(u_\nu))_\nu) \;:\; (u_\nu)_\nu \in C = \prod_\nu C_\nu \right\}$$

with (x, λ) not a Nash equilibrium of the game, then for at least one of the players ν, there exists $d_\nu \in C_\nu$ such that

$$\bar{\theta}_\nu(d_\nu, x_{-\nu}) := \limsup_{x'_{-\nu} \to x_{-\nu}} \theta_\nu(d_\nu, x'_{-\nu}) < \lambda_\nu. \tag{1}$$

Another existence result for GNEP with discontinuous cost functions

Theorem (Tian (95), J. Math. Econ.)

Assume that any X_ν is a non-empty convex compact subset in a locally convex topological vector space.

- *for every player ν, the loss function θ_ν is quasiconvex with respect to the ν-th variable and quasi-transfer lower continuous in x with respect to K_ν,*
- *the set-valued maps K_ν are non-empty convex compact valued and upper semicontinuous,*
- *for every $x_\nu \in X_{-\nu}$, the function u_ν is transfer lower continuous in x_ν on $K_\nu(x_{-\nu})$,*

then the Generalized Nash equilibrium problem admits a solution.

A function f is said to be transfer lower continuous on X if for points $x, y \in X$, $f(y) > f(x)$ implies that there exists a point $x' \in X$ and a neighbourhood $N(y)$ of y such that $f(z) > f(x')$ for all $z \in N(y)$.

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using QVI to prove existence of GNEP

GNEP: QVI reformulation

$$\min_{x_1} \ \theta_1(x_1, x_{-1}) \quad \text{s.t.} \ \{ \ x_1 \in K_1(x_{-1}) \qquad \ldots \qquad \min_{x_n} \ \theta_n(x_n, x_{-n}) \quad \text{s.t.} \ \{ \ x_n \in K_n(x_{-n})$$

Suppose that for any ν and any $x^{-\nu} \in \mathbb{R}^{n^{-\nu}}$, function $\theta_\nu(\cdot, x^{-\nu})$ is continuously differentiable and convex and $X_\nu(x^{-\nu})$ is convex.

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Denoting by
and

$$X(x) = \prod X_\nu(x^{-\nu}), \quad \forall\, x \in \mathbb{R}^n$$

$$F(x) = (\nabla_{x^1}\theta_1(x), \ldots, \nabla_{x^N}\theta_N(x)) \quad \in \mathbb{R}^n$$

we have the reformulation

$$\bar{x} \textit{ gene. Nash equil.} \Leftrightarrow \begin{cases} \bar{x} \in X(\bar{x}) \textit{ and} \\ \langle F(\bar{x}), y - \bar{x}\rangle \geq 0, \quad \forall\, y \in X(\bar{x}) \end{cases}$$

that is a quasi-variational inequality.

GNEP: reformulation (cont.)

$$\boxed{\min_{x_1} \ \theta_1(x_1, x_{-1}) \ \text{s.t.} \ \{ x_1 \in K_1(x_{-1}) } \quad \dots \quad \boxed{\min_{x_n} \ \theta_n(x_n, x_{-n}) \ \text{s.t.} \ \{ x_n \in K_n(x_{-n}) }$$

Suppose that for any ν and any $x^{-\nu} \in \mathbb{R}^{n^{-\nu}}$, function $\theta_\nu(\cdot, x^{-\nu})$ is lower semicontinuous and quasiconvex and $X_\nu(x^{-\nu})$ is convex.

Denoting by

$$X(x) = \prod_\nu X_\nu(x^{-\nu}), \quad \forall x \in \mathbb{R}^n$$

and

$$F(x) = (N^a_{\theta_1}(x), \dots, N^a_{\theta_N}(x)) \quad \in \mathbb{R}^n$$

we have the reformulation *(with some add. hypotheses)*

$$\bar{x} \ \textit{gene. Nash equil.} \ " \Leftrightarrow " \begin{cases} \bar{x} \in X(\bar{x}) \ \textit{and} \\ \langle F(\bar{x}), y - \bar{x} \rangle \geq 0, \quad \forall y \in X(\bar{x}) \end{cases}$$

Quasiconvexity

- A function $f : X \to \mathbb{R}_\infty$ is said to be *quasiconvex* on K if,

for all x, $y \in K$ *and all* $t \in [0, 1]$,

$$f(tx + (1 - t)y) \leq \max\{f(x), f(y)\}.$$

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f *differentiable*

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or

$$f \text{ is quasiconvex} \iff \partial f \text{ is quasimonotone}$$

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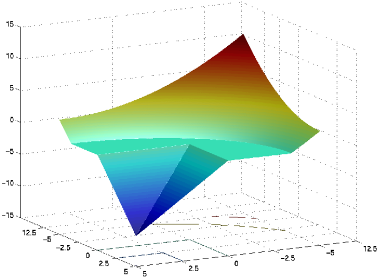
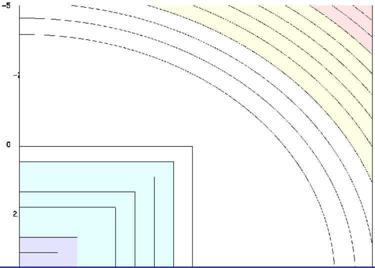
- A function $f : X \to \mathbb{R}_\infty$ is said to be *semistrictly quasiconvex* on K if, f is quasiconvex and for any $x, y \in K$,

$$f(x) < f(y) \Rightarrow f(z) < f(y), \quad \forall\, z \in [x, y[.$$





Adjusted normal operator

Adjusted sublevel set:

For any $x \in \mathrm{dom} f$, we define

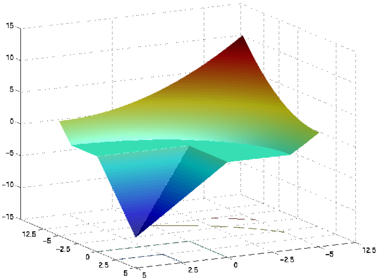
$$S_f^a(x) = S_{f(x)} \cap \overline{B}(S_{f(x)}^<, \rho_x)$$

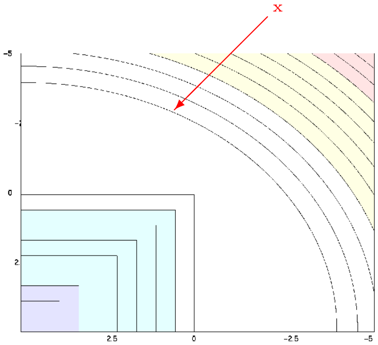
where $\rho_x = dist(x, S_{f(x)}^<)$, if $S_{f(x)}^< \neq \emptyset$.

Ajusted normal operator:

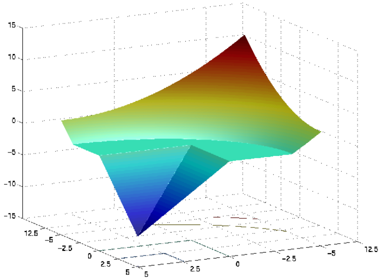
$$N_f^a(x) = \{x^* \in X^* \ : \ \langle x^*, y - x\rangle \leq 0, \quad \forall\, y \in S_f^a(x)\}$$

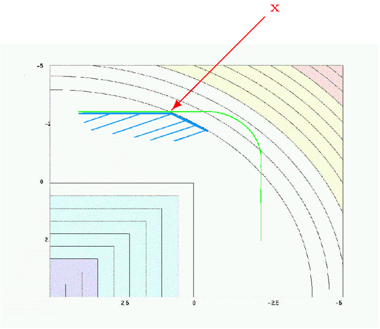
Example





Example





$\overline{B}(S^{<}_{f(x)}, \rho_x)$

$S^a_f(x) = S_f(x) \cap \overline{B}(S^{<}_{f(x)}, \rho_x)$

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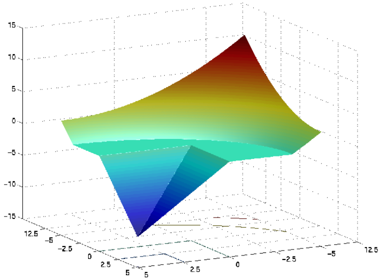

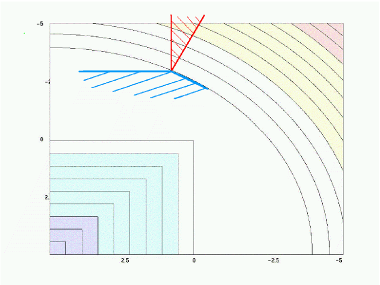

$$S_f^a(x) = S_f(x) \cap \overline{B}(S^<_{f(x)}, \rho_x)$$

$$N_f^a(x) = \{x^* \in X^* \ : \ \langle x^*, y - x \rangle \leq 0, \quad \forall\, y \in S_f^a(x)\}$$

GNEP: reformulation (cont.)

$\min_{x_1} \ \theta_1(x_1, x_{-1})$ s.t. $\{ \ x_1 \in K_1(x_{-1})$	...	$\min_{x_n} \ \theta_n(x_n, x_{-n})$ s.t. $\{ \ x_n \in K_n(x_{-n})$

Suppose that for any ν and any $x^{-\nu} \in \mathbb{R}^{n^{-\nu}}$, function $\theta_\nu(\cdot, x^{-\nu})$ is lower semicontinuous and quasiconvex and $X_\nu(x^{-\nu})$ is convex.

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we have the reformulation *(with some add. hypotheses)*

$$\bar{x} \textit{ gene. Nash equil. } " \Leftrightarrow " \begin{cases} \bar{x} \in X(\bar{x}) \textit{ and} \\ \langle T(\bar{x}), y - \bar{x} \rangle \geq 0, \quad \forall\, y \in X(\bar{x}) \end{cases}$$

Normal operator: properties

Definition (sub-boundarily constant functions)

A function $f : \mathbb{R}^p \to \mathbb{R}$ is said to be *sub-boundarily constant* on a subset C if, for every $x \in C$, one has that

$$f(y) < f(x) \Longrightarrow [y, x[\cap \text{int} S_f^a(x) \neq \emptyset. \tag{2}$$

Remark

One can easily verify the following observations concerning the above definition:

i) *Note that due to the special structure of the adjusted sublevel sets* S_f^a*, the subset* $S_f^a(x) \setminus \text{int} S_f^a(x)$ *has nothing to do in general with the level set* $L_f(x) = \{y \in \mathbb{R}^n \ : \ f(y) = f(x)\}$*, even when* f *is quasiconvex.*

ii) *If* f *is radially continuous then* f *is sub-boundarily constant on* $\text{dom} f$*.*

iii) *Also, if* $p = 1$*, that is* f *is defined over* $\mathbb{R}$*, and if* f *is quasiconvex, then* f *must be sub-boundarily constant on its domain.*

iv) *Note that, if* f *is sub-boundarily constant, then for every* $\lambda > \inf_{\mathbb{R}^p} f$*, the sublevel sets* $S_\lambda(f)$ *must have nonempty interior.*

Normal operator: properties

A sufficient optimality condition

Proposition

Let $f : \mathbb{R}^p \to \mathbb{R}$ be a quasiconvex and sub-boundarily constant function, and $C \subset \mathbb{R}^p$ be a nonempty set. Then, any solution of the Stampacchia variational inequality defined by the operator $N_f^a \setminus \{0\}$ on C is a global minimizer of f over C, that is

$$S\left(N_f^a \setminus \{0\}, C\right) \subset \arg\min_C f.$$

D. Aussel, D. Salas, K. Cao Van, Existence results for generalized Nash equilibrium problems under continuity-like properties of sublevel sets., SIAM J. Optim. (2019), Vol. 29, No. 2, pp. 1558-1577.

Normal operator: properties

Proof.

Let $\bar{x} \in S\left(N_f^a \setminus \{0\}, C\right)$ and $x^* \in N_f^a(\bar{x}) \setminus \{0\}$ be such that

$$\langle x^*, y - \bar{x} \rangle \geq 0, \quad \forall y \in C. \tag{3}$$

Now, assume, for a contradiction, that there exists $y \in C$ such that $f(y) < f(\bar{x})$. Then, $y \in S_f^<(\bar{x}) \subset S_f^a(\bar{x})$. Combining the definition of $N_f^a(x) \setminus \{0\}$ together with (3), one immediately have $\langle x^*, y - \bar{x} \rangle = 0$. Now since f is sub-boundarily constant on dom$f$, there exists $z \in [y, x[$ such that z is also an element of int$S_f^a(\bar{x})$. □

Normal operator: properties

Proof.

Thus there exists $\varepsilon > 0$ such that $B(z, \varepsilon)$ is included into $S_f^a(\bar{x})$. Since $x^* \neq 0$ there exists $d \in \mathbb{R}^n$ such that $\langle x^*, d \rangle > 0$. Then for $t > 0$ small enough, $w = z + td$ is an element of $B(z, \varepsilon)$ and thus of $S_f^a(\bar{x})$. Therefore since $x^* \in N_f^a(\bar{x})$, one gets $\langle x^*, z - \bar{x} \rangle \leq 0$. But this is impossible since

$$\begin{aligned} \langle x^*, w - \bar{x} \rangle &= \langle x^*, z - \bar{x} \rangle + t \langle x^*, d \rangle \\ &= \frac{\|z - \bar{x}\|}{\|y - \bar{x}\|} \langle x^*, y - \bar{x} \rangle + t \langle x^*, d \rangle \\ &= t \langle x^*, d \rangle > 0. \end{aligned}$$

As a conclusion and since y is arbitrary on C, this proves that $f(\bar{x}) = \min_C f$. □

Normal operator: properties

Proposition

N_f^a is always quasimonotone, that is, for any x, y,

$$\exists\, x^* \in N_f^a(x) \text{ such that } \langle x^*, y-x\rangle > 0 \Rightarrow \forall\, y^* \in N_f^a(x), \langle y^*, y-x\rangle \geq 0.$$

Normal operator: properties

Definition (Upper sign-continuity)

Let C be a nonempty convex subset of $\mathbb{R}^p$ and let $T : C \rightrightarrows \mathbb{R}^p$ be a set-valued map with nonempty values. We say that T is *upper sign-continuous* on C if for every $x, y \in C$, the following implication holds:

$$\left(\forall t \in]0,1[,\ \inf_{x_t^* \in T(x_t)} \langle x_t^*, y - x\rangle \geq 0\right) \implies \sup_{x^* \in T(x)} \langle x^*, y - x\rangle \geq 0,$$

where $x_t := (1-t)x + ty$.

And T is said to be *locally upper sign-continuous* at x if there exist a neighbourhood V_x of x and an upper sign-continuous operator $S_x : V_x \cap K \mapsto 2^X \setminus \{\emptyset\}$ with convex, compact values satisfying $S_x(y) \subset T(y)$, for all $y \in V_x \cap K$.

Normal operator: properties

Proposition

If f is lower semicontinuous such that for every $\lambda > \inf_{\mathbb{R}^n} f$, $int(S_\lambda) \neq \emptyset$ then N_f^a is locally upper sign-continuous on $\operatorname{dom} f \setminus \arg\min_{\mathbb{R}^n} f$.

GNEP: reformulation (cont.)

$$\begin{array}{ll} \min\limits_{x_1} & \theta_1(x_1, x_{-1}) \\ \text{s.t.} & \{ \; x_1 \in K_1(x_{-1}) \end{array} \quad \cdots \quad \begin{array}{ll} \min\limits_{x_n} & \theta_n(x_n, x_{-n}) \\ \text{s.t.} & \{ \; x_n \in K_n(x_{-n}) \end{array}$$

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$\Rightarrow$ Need of an existence result for quasimonotone $QVI(T, X)$ on product spaces.

Existence for QVI

Theorem (Tan, 1985)

Let X be a locally convex Hausdorff space, C be a nonempty convex compact subset of X, and $T : C \rightrightarrows X^*$ and $K : C \rightrightarrows C$ be two set-valued maps such that

(i) *K is lower semicontinuous with nonempty convex compact values;*

(ii) *T is upper semicontinuous with nonempty convex compact values.*

Then $QVI(T, K)$ is nonempty.

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Problem: $T(x) = (N^a_{\theta_1}(x), \ldots, N^a_{\theta_N}(x))$ is **not USC**

Existence for QVI

Proposition (D. A. and J. Cotrina (2013))

Let $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $K : C \rightrightarrows C$ be two set-valued maps with C being a convex compact subset of X. Let us suppose that the following properties hold:

i) *the map K is closed and lower semicontinuous with convex values and $Int(K(x)) \neq \emptyset$, for all $x \in C$,*

ii) *T is quasimonotone, locally upper sign-continuous and dually lower semicontinuous.*

Then $\mathrm{QVI}^*(T, K)$ admits at least a solution.

D. A. and J. Cotrina (2013), J. Optim. Theory Appli.

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Problem: $T(x) = (N^a_{\theta_1}(x), \ldots, N^a_{\theta_N}(x))$ is **not quasimonotone**, even if each $N^a_{\theta_\nu}(x)$ is clearly quasimonotone!!!!

new existence results for product-type QVI

The setting

Let I be a finite index set, that is, $I = \{1, 2, \ldots, n\}$. For each $i \in I$, let X_i be a Banach space with dual X_i^*, and C_i be a nonempty subset of X_i. We denote

$$C = \prod_{i \in I} C_i; \quad C_{-i} = \prod_{j \neq i, j \in I} C_j; \quad X = \prod_{i \in I} X_i; \quad X^* = \prod_{i \in I} X_i^*.$$

For each $i \in I$ and each $x_{-i} \in C_{-i}$, let $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ and $K_i(\cdot, x_{-i}) : C_i \rightrightarrows C_i$ be two set-valued maps. We set

$$T(x) = \prod_{i \in I} T_i(x_i, x_{-i}), \quad \textit{and} \quad K(x) = \prod_{i \in I} K_i(x_i, x_{-i}).$$

Counterexample

Problem: the product of quasimonotone maps is not quasimonotone

Example

Let $C_1 = [-2,2]$, $C_2 = [-2,2]$ and $C = [-2,2] \times [-2,2]$.
Define

$$\begin{array}{rccl} T_1(\cdot, x_2): & C_1 & \rightrightarrows & \mathbb{R} \\ & x_1 & \mapsto & \{x_1^2\} \end{array} \quad \textit{and} \quad \begin{array}{rccl} T_2(x_1, \cdot): & C_2 & \rightrightarrows & \mathbb{R} \\ & x_2 & \mapsto & \{1 + x_2^2\} \end{array}$$

Then, both component operators are quasimonotone, but the product operator

$$\begin{array}{rccl} T: & C & \rightrightarrows & \mathbb{R}^2 \\ & x & \mapsto & T(x) = \{x_1^2\} \times \{1 + x_2^2\} \end{array}$$

is not quasimonotone.

QVI on product spaces: What do we need?

$$\Downarrow$$

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$QVI(T,K)$ **with each** T_ν **quasimonotone....**

QVI on product spaces: What do we need?

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on product spaces $X = \prod_\nu X_\nu$

Some previous works:

- Q. H. Ansari and Z. Khan, Densely relative pseudomonotone variational inequalities over product of sets, J. Nonlinear Convex Anal. (2006)
- D. Inoan, Existence and behavior of solutions for variational inequalities over products of sets, Math. Inequal. Appl. (2009)
- I. V. Konnov, Relatively monotone variational inequalities over product sets, Oper. Res. Lett. (2001)

One of our existence results

Theorem

For each $i \in I$, let C_i be a nonempty weakly compact convex subset of X_i, $T_i : C_i \times C_{-i} \rightrightarrows X_i^*$ be a set-valued map with nonempty convex values and $K_i : C_i \times C_{-i} \rightrightarrows C_i$ be a set-valued map with nonempty values. Consider T and K defined as product maps.
Assume that

(i) *for each $i \in I$, the set-valued map $K_i : C_i \times C_{-i} \rightrightarrows C_i$ is weakly closed with nonempty interior and convex values.*

(ii) *for each $i \in I$ and each $x_{-i} \in C_{-i}$, $T_i(\cdot, x_{-i}) : C_i \rightrightarrows X_i^*$ is quasimonotone and locally upper sign-continuous.*

(iii) *for each $i \in I$, the pair of set-valued maps $(T_i, \text{int}K_i)$ is weakly net-lower-sign continuous with respect to the parameter pair (C_i, C_{-i}).*

Then $QVI^*(T, K)$ is nonempty.

Net-lower-sign continuity

Definition

Let (U, τ_U) and (Λ, τ_Λ) be two topological spaces, Y be a Banach space, and τ_Y be a locally convex topology consistent with the duality $\langle Y, Y^* \rangle$. Let $T : Y \times \Lambda \rightrightarrows Y^*$ and $K : U \times \Lambda \rightrightarrows Y$ be two set-valued maps. The pair (T, K) is said to be *$(\tau_U \times \tau_\Lambda)$-$\tau_Y$ net-lower-sign continuous with respect to the parameter pair (U, Λ) at $(\mu, \lambda) \in U \times \Lambda$ and $y \in K(\mu, \lambda)$* if for every net $(\mu_\alpha, \lambda_\alpha)_\alpha \subseteq U \times \Lambda$ converging to (μ, λ), every $z \in \overline{K(\mu, \lambda)}^{\tau_Y}$ and every selection $(z_\alpha)_\alpha$ of $\left(\overline{K(\mu_\alpha, \lambda_\alpha)}^{\tau_Y}\right)_\alpha$ τ_Y-converging to z, the following condition holds:

$$\begin{cases} \text{If for every subnet } (\mu_\beta, \lambda_\beta)_\beta \text{ of } (\mu_\alpha, \lambda_\alpha)_\alpha \text{ and every selection} \\ (y_\beta)_\beta \text{ of } (K(\mu_\beta, \lambda_\beta))_\beta \ \tau_Y\text{-converging to } y \text{ one has that} \\ \qquad \limsup_\beta \sup_{y^*_\beta \in T(y_\beta, \lambda_\beta)} \langle y^*_\beta, z_\beta - y_\beta \rangle \le 0, \qquad (4) \\ \text{then, } \sup_{y^* \in T(y, \lambda)} \langle y^*, z - y \rangle \le 0, \end{cases}$$

where $(z_\beta)_\beta$ is the corresponding subnet of $(z_\alpha)_\alpha$ induced by the index set

back to GNEP

Theorem

For any $\nu \in I$, let C_ν be a nonempty, compact and convex subset of $\mathbb{R}^{n_\nu}$, $\theta_\nu : \mathbb{R}^n \to \mathbb{R}$ and $K_\nu : C_{-\nu} \rightrightarrows C_\nu$. Then, the $GNEP(\theta, K)$ admits a generalized Nash equilibrium if

(i). *for every $x_{-\nu} \in C_{-\nu}$, the function $\theta_\nu(\cdot, x_{-\nu})$ is quasiconvex, lower semicontinuous and sub-boundarily constant;*

(ii). *the set-valued map $K_\nu : C_{-\nu} \rightrightarrows C_\nu$ is closed with compact convex values;*

(iii). *for every sequence $(x^n_{-\nu})_{n\in\mathbb{N}} \subseteq C_{-\nu}$ converging to $x_{-\nu}$ and every $y_\nu \in K_\nu(x_{-\nu})$ such that*

$$N^a_{\theta_\nu(\cdot, x_{-\nu})}(y_\nu) \subseteq \operatorname*{Limsup}_{K_\nu(x^n_{-\nu}) \ni y^n_\nu \to y_\nu} N^a_{\theta_\nu(\cdot, x^n_{-\nu})}(y^n_\nu).$$

A function $f : \mathbb{R}^p \to \mathbb{R}$ is said to be *sub-boundarily constant* on a subset C if, for every $x \in C$, one has that

$$f(y) < f(x) \Longrightarrow [y, x[\cap \operatorname{int} S^a_f(x) \neq \emptyset.$$

and in a more simplier case...

Assume that the GNEP has a specific structure:

(ν-separability)

each of the cost function θ_ν is ν-separable, that is, for any ν, there exists two functions $\alpha^\nu_\nu : \mathbb{R}^{n_\nu} \to \mathbb{R}$ and $\alpha^\nu_{-\nu} : \mathbb{R}^{n_{-\nu}} \to \mathbb{R}$ such that

$$\theta_\nu(x_\nu, x_{-\nu}) = \alpha^\nu_\nu(x_\nu) + \alpha^\nu_{-\nu}(x_{-\nu}), \quad \forall x \in \mathbb{R}^N.$$

and in a more simplier case...

Assume that the GNEP has a specific structure:

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each of the cost function θ_ν *is* ν-separable, *that is, for any* ν*, there exists two functions* $\alpha^\nu_\nu : \mathbb{R}^{n_\nu} \to \mathbb{R}$ *and* $\alpha^\nu_{-\nu} : \mathbb{R}^{n_{-\nu}} \to \mathbb{R}$ *such that*

$$\theta_\nu(x_\nu, x_{-\nu}) = \alpha^\nu_\nu(x_\nu) + \alpha^\nu_{-\nu}(x_{-\nu}), \quad \forall x \in \mathbb{R}^N.$$

(Separable Inequality Constrained (SIC))

for any $x_{-\nu} \in C_{-\nu}$*, the set* $K_\nu(x_{-\nu})$ *is described by a finite set of inequalities, that*

$$K_\nu(x_{-\nu}) = \left\{ y_\nu \in C_\nu \;:\; g^\nu_j(y_\nu) \leq h^\nu_j(x_{-\nu}), \quad j \in \{1, \ldots, J_\nu\} \right\}$$

where $J_\nu \in \mathbb{N}$ *and, for* $j = 1, \ldots, J_\nu$, g^ν_j *and* h^ν_j *are respectively defined from* C_ν *to* $\mathbb{R}$ *and from* $C_{-\nu}$ *to* $\mathbb{R}$.

and in a more simplier case...

Proposition

For any $\nu \in I$, let C_ν be a nonempty, compact and convex subset of $\mathbb{R}^{n_\nu}$, $\theta_\nu : \mathbb{R}^N \to \mathbb{R}$ and $K_\nu : C_{-\nu} \rightrightarrows C_\nu$. Then, the $GNEP(\theta, K)$ admits a generalized Nash Equilibrium if, for any ν,

- *for every $x_{-\nu} \in C_{-\nu}$, the function $\theta_\nu(\cdot, x_{-\nu})$ is sub-boundarily constant and ν-separable with the function α_ν^ν being quasiconvex and lower semicontinuous;*
- *the set-valued map $K_\nu : C_{-\nu} \rightrightarrows C_\nu$ are defined accordingly to (SIC) with*
  - *each of the functions g_j^ν being continuous and semi-strictly quasiconvex;*
  - *each of the functions h_j^ν being continuous;*
  - *values $K_\nu(x_{-\nu})$ are compact sets with nonempty interior;*

Reformulation: MPCC for BL

Reformulation: MPCC for BL

Replacing the lower-level problem by its KKT conditions, gives place to a Mathematical Program with Complementarity Constraints.

Bilevel

$$\text{“}\min_{x \in X}\text{”} F(x,y)$$
$$s.t.\;\; y \in S(x)$$

with $S(x) =$ “y solving

$$\min_{y \in \mathbb{R}^m} \; f(x,y)$$
$$s.t \; g(x,y) \leq 0 \text{ ”}$$

MPCC

$$\text{“}\min_{x \in X}\text{”} F(x,y)$$
$$s.t.\;\; (y,u) \in KKT(x)$$

with $KKT(x) =$ “(y,u) solving

$$\begin{cases} \nabla_y f(x,y) + u^T \nabla_y g(x,y) = 0 \\ 0 \leq u \perp -g(x,y) \geq 0 \text{ ”} \end{cases}$$

We write $\Lambda(x,y)$ for the set of u satisfying $(y,u) \in KKT(x)$.

Example 1

Consider the following Bilevel problem and its MPCC reformulation

Bilevel

$$\text{“}\min_{x\in\mathbb{R}}\text{”} \; -x \qquad s.t. \;\; y \in S(x)$$

with $S(x) =$ “y solving

$$\min_{y\in\mathbb{R}} \quad xy \qquad s.t \; x^2(y^2-1) \leq 0 \text{ ”}$$

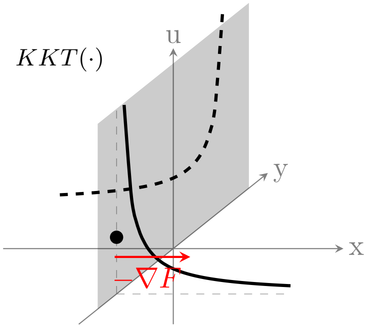
MPCC

$$\text{“}\min_{x\in\mathbb{R}}\text{”} \; -x \qquad s.t. \;\; (y,u) \in KKT(x)$$

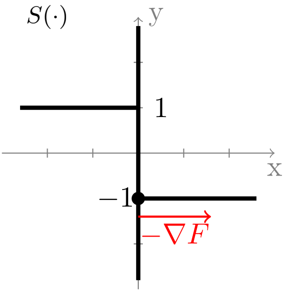
with $KKT(x) =$ “(y,u) solving

$$\begin{cases} x + u \cdot 2yx^2 = 0 \\ 0 \leq u \perp -x^2(y^2-1) \geq 0 \end{cases} \text{ ”}$$

1. $(0,-1,u)$ is a local solution of “MPCC”, for any $u \in \Lambda(0,-1) = \mathbb{R}_+$
2. $(0,-1)$ is NOT a local solution of “Bilevel”



(a) $(0, -1, u)$ is a local solution of MPCC, $\forall u \in \mathbb{R}_+$.



(b) $(0, -1)$ isn't a local solution of the Bilevel problem.

Reformulation: MPCC for BL

The optimistic Bilevel (OB) is

$$\min_x \min_y F(x,y)$$
$$s.t.\ \ y \in S(x), x \in X.$$

The pessimistic Bilevel (PB) is

$$\min_x \max_y F(x,y)$$
$$s.t.\ \ y \in S(x), x \in X.$$

Reformulation: MPCC for BL

The optimistic Bilevel (OB) is

$$\min_x \min_y F(x,y) \quad s.t. \; y \in S(x), x \in X.$$

The optimistic MPCC (OMPCC):

$$\min_x \min_y F(x,y) \quad s.t. \; (y,u) \in KKT(x), x \in X.$$

The pessimistic Bilevel (PB) is

$$\min_x \max_y F(x,y) \quad s.t. \; y \in S(x), x \in X.$$

The pessimistic MPCC (PMPCC):

$$\min_x \max_y F(x,y) \quad s.t. \; (y,u) \in KKT(x), x \in X.$$

Optimistic approach

Is bilevel programming a special case of a MPCC?
Dempe-Dutta (2012 Math. Prog.)

$$\min_{x} \min_{y} F(x, y)$$
$$\text{s.t. } y \in S(x), x \in X.$$

Local solutions for in optimistic approach

Definition

A **local (resp. global) solution** of (OB) is a point $(\bar{x}, \bar{y}) \in Gr(S)$ if there exists $U \in \mathcal{N}(\bar{x}, \bar{y})$ (resp. $U = \mathbb{R}^n \times \mathbb{R}^m$) such that

$$F(\bar{x}, \bar{y}) \leq F(x, y), \quad \forall (x, y) \in U \cap Gr(S).$$

Definition

A **local (resp. global) solution** for (OMPCC) is a triplet $(\bar{x}, \bar{y}, \bar{u}) \in Gr(KKT)$ such that there exists $U \in \mathcal{N}(\bar{x}, \bar{y}, \bar{u})$ (resp. $U = \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$) with

$$F(\bar{x}, \bar{y}) \leq F(x, y), \quad \forall (x, y, u) \in U \cap Gr(KKT).$$

Results for the optimistic case

In Dempe-Dutta it was considered the Slater type constraint qualification for a parameter $x \in X$:

$$\textbf{Slater:}\ \exists y(x) \in \mathbb{R}^m \text{ s.t. } g_i(x, y(x)) < 0,\ \forall i = 1, .., p.$$

Results for the optimistic case

Theorem 1 Dempe-Dutta (2012)

Assume the convexity condition and Slater's CQ at $\bar{x}$.

1. If $(\bar{x}, \bar{y})$ is a local solution for (OB), then for each $\bar{u} \in \Lambda(\bar{x}, \bar{y})$, $(\bar{x}, \bar{y}, \bar{u})$ is a local solution for (OMPCC).
2. Conversely, assume that Slater's CQ holds on a neighbourhood of $\bar{x}$, $\Lambda(\bar{x}, \bar{y}) \neq \emptyset$, and $(\bar{x}, \bar{y}, u)$ is a local solution of (OMPCC) for every $u \in \Lambda(\bar{x}, \bar{y})$. Then $(\bar{x}, \bar{y})$ is a local solution of (OB).

Under the convexity assumption and some CQ ensuring $KKT(x) \neq \emptyset$, $\forall x \in X$:

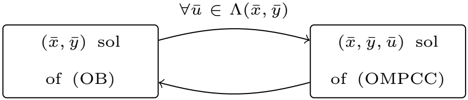


Figure: Global solution comparison in optimistic approach

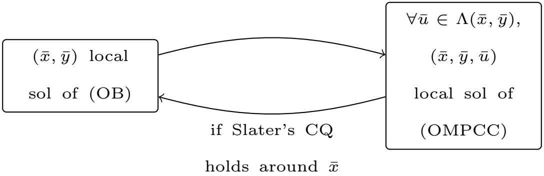


Figure: Local solution comparison in optimistic approach

Example 1 (optimistic)

Consider the following optimistic Bilevel problem

$$\min_x \min_y \; -x$$
$$\text{s.t. } y \in S(x),\; x \in \mathbb{R}$$

with lower-level

$$\min_y xy$$
$$\text{s.t } x^2(y^2-1) \le 0.$$

1. $(0,-1,u)$ is a local solution of (OMPCC), for any $u \in \Lambda(0,-1) = \mathbb{R}_+$
2. $(0,-1)$ is NOT a local solution of (OB).

Pessimistic Approach

D. Aussel & A. Svensson, Is Pessimistic Bilevel Programming a Special Case of a Mathematical Program with Complementarity Constraints?, J. Optim. Theory Appl. (2019)

$$\min_{x} \max_{y} F(x,y)$$
$$\text{s.t. } y \in S(x), x \in X.$$

Definition

A pair $(\bar{x}, \bar{y})$ is said to be a *local (resp. global) solution* for (PB), if $(\bar{x}, \bar{y}) \in Gr(S_p)$ and $\exists U \in \mathcal{N}(\bar{x}, \bar{y})$ such that

$$F(\bar{x}, \bar{y}) \leq F(x, y), \quad \forall (x, y) \in U \cap Gr(S_p). \tag{5}$$

where $S_p(x) := argmax_y \{F(x, y) \mid y \in S(x)\}$.

Definition

A triplet $(\bar{x}, \bar{y}, \bar{u})$ is said to be a *local (resp. global) solution* for (PMPCC), if $(\bar{x}, \bar{y}, \bar{u}) \in Gr(KKT_p)$ and $\exists U \in \mathcal{N}(\bar{x}, \bar{y}, \bar{u})$ such that

$$F(\bar{x}, \bar{y}) \leq F(x, y), \quad \forall (x, y, u) \in U \cap Gr(KKT_p). \tag{6}$$

where $KKT_p(x) := argmax_{y,u} \{F(x, y) \mid (y, u) \in KKT(x)\}$.

Results for the pessimistic case

Theorem 2

Assume the convexity condition and that $KKT(x) \neq \emptyset, \forall x \in X$.

1. If $(\bar{x}, \bar{y})$ is a local solution for (PB), then for each $\bar{u} \in \Lambda(\bar{x}, \bar{y})$, $(\bar{x}, \bar{y}, \bar{u})$ is a local solution for (PMPCC).
2. Conversely, assume that one of the following condition are satisfied:
    1. The multifunction KKT_p is LSC around $(\bar{x}, \bar{y}, \bar{u})$ and $(\bar{x}, \bar{y}, \bar{u})$ is a local solution of (PB).
    2. Slater's CQ holds on a neighbourhood of $\bar{x}$, $\Lambda(\bar{x}, \bar{y}) \neq \emptyset$, and for every $u \in \Lambda(\bar{x}, \bar{y})$, $(\bar{x}, \bar{y}, u)$ is a local solution of (PMPCC).

    Then $(\bar{x}, \bar{y})$ is a local solution of (PB).

Example 1 (pessimistic)

Consider the following pessimistic Bilevel problem

$$\min_x \max_y \; -x$$
$$\text{s.t. } y \in S(x),\; x \in \mathbb{R}$$

with lower-level

$$\min_y xy$$
$$\text{s.t } x^2(y^2-1) \le 0.$$

1. $(0,-1,u)$ is a local solution of (PMPCC), for any $u \in \Lambda(0,-1) = \mathbb{R}_+$
2. $(0,-1)$ is NOT a local solution of (PB).

Example 2

Consider the following Bilevel problem

$$\text{“}\min_x\text{”} x$$
$$s.t. \quad y \in S(x)$$

with $S(x)$ the solution of the lower-level problem

$$\min_y \{-y \mid x + y \leq 0, y \leq 0\}$$

Even though Slater's CQ holds, we have

1. $(0, 0, u_1, u_2)$ with $(u_1, u_2) \in \Lambda(0, 0) = \{(\lambda, 1 - \lambda) \mid \lambda \in [0, 1]\}$ is a local solution of “(MPCC)”, iff $u_1 \neq 0$,
2. $(0, 0)$ is NOT a local solution for “(B)”.

Under the convexity assumption and some (CQ) ensuring $KKT(x) \neq \emptyset$, $\forall x \in X$:

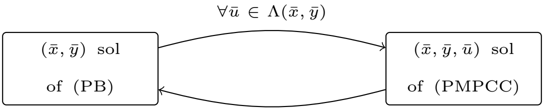


Figure: Global solution comparison in pessimistic approach

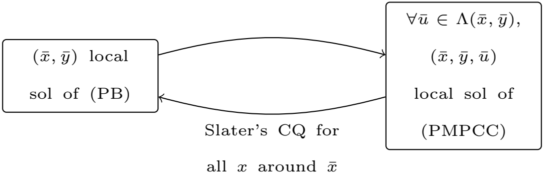


Figure: Local solutions comparison in pessimistic approach

Concept of global solution

Optimistic bilevel optimization:

$$\varphi_o(x) = \min_y \{F(x,y) : y \in \Psi(x)\} \qquad (7)$$

and

$$\min_x \{\varphi_o(x) : G(x) \leq 0\} \qquad (8)$$

Cooperation of the follower.

Concept of global solution

Optimistic bilevel optimization:

$$\varphi_o(x) = \min_y \{F(x,y) : y \in \Psi(x)\} \tag{7}$$

and

$$\min_x \{\varphi_o(x) : G(x) \leq 0\} \quad (8)$$

Cooperation of the follower. Pessimistic bilevel optimization

$$\varphi_p(x) = \max_y \{F(x,y) : y \in \Psi(x)\} \tag{9}$$

and

$$\min_x \{\varphi_p(x) : G(x) \leq 0\} \quad (10)$$

Leader needs to bound the damage resulting from "bad" selection of the follower.

Concepts of global solution (cont.)

For any given $x \in X$, let us defined the *(partial) pessimistic value function* by

$$\varphi_p(x) := \max_{y \in S(x)} F(x, y)$$

Let us now describe two different, but at the same time natural, definitions of global solution to (10). The first one has been considered in Dempe ('14) and corresponds to saying that to solve (10) the leader has to choose an x that minimizes the worst value $\varphi_p(x)$.

Definition

A point $\bar{x} \in \mathbb{R}^n$ is an *original solution* of (10), if $\bar{x} \in X$ and for all $x \in X$

$$\varphi_p(\bar{x}) \leq \varphi_p(x).$$

Concepts of global solution (cont.)

A second one, that was considered in the reference monograph Dempe ('02). It involves at the same time both the decision vectors of the leader and of the follower and we here call it "conventional solution". The terms "original" and "conventional" are taken as the names given to the corresponding optimistic problems Dempe ('12).

Definition

A pair $(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ is a *conventional solution* of (10), if $\bar{x} \in X$, $\bar{y} \in S(\bar{x})$ and

$$F(\bar{x}, \bar{y}) \geq F(\bar{x}, y), \quad \forall y \in S(\bar{x})$$

and

$$\varphi_p(\bar{x}) \leq \varphi_p(x), \quad \forall x \in X.$$

Equivalently one can say that a pair $(\bar{x}, \bar{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ is a conventional solution of (10) if $(\bar{x}, \bar{y})$ is in the graph of the set-valued map S_p and $\bar{x}$ minimizes φ_p over X.
It is clear from the definition that if $(\bar{x}, \bar{y})$ is a conventional solution, then the first coordinate $\bar{x}$ is an original solution of (10). *Conversely, if $\bar{x}$ is an original solution, then for any $\bar{y} \in S_p(\bar{x})$ the couple $(\bar{x}, \bar{y})$ is a conventional solution of* (10).

Reformulation: MPCC for SLMFG

D. A.& A. Svensson, Towards Tractable Constraint Qualifications for Parametric Optimisation Problems and Applications to Generalised Nash Games, *J. Optim. Theory Appl. (2019)*

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D. A. & A. Svensson, Chapter "A short state of the art on Multi-Leader-Follower Games" *In: Dempe S., Zemkoho A. (eds) Bilevel Optimization. Springer (2021)*

MPCC for SLMFG

Thus the MPCC reformulation of the SLMFG consists of the following optimization problem

$$\min_{x,y,\mu} \quad F(x,y)$$

$$\left\{ \begin{array}{l} x \in X, \\ \quad (y,\mu) \in KKT(x) \end{array} \right.$$

MPCC for SLMFG

Thus the MPCC reformulation of the SLMFG consists of the following optimization problem

$$\min_{x,y,\mu} \quad F(x,y)$$
$$\begin{cases} x \in X, \\ \quad (y,\mu) \in KKT(x) \end{cases}$$

We make the following basic hypotheses:

(H_1) (Follower's differentiability) For any $j \in J$ and any $(x, y_{-j}) \in X \times \mathbb{R}^{m_{-j}}$, $f_j(x,\cdot,y_{-j})$ and $g_j(x,\cdot,y_{-j})$ are differentiable;

(H_2) (Follower's player convexity) For any $j \in J$ and any $(x, y_{-j}) \in X \times \mathbb{R}^{m_{-j}}$, $f_j(x,\cdot,y_{-j})$ is convex and the components of $g_j(x,\cdot,y_{-j})$ are quasiconvex functions.

MPCC for SLMFG

Theorem

Assume (H_1) and (H_2). The relation between solutions of the SLMFG and its MPCC reformulation are as follows.

1. *If $(\bar{x}, \bar{y}) \in$ SLMFG and $\bar{\mu} \in \Lambda(\bar{x}, \bar{y})$, then $(\bar{x}, \bar{y}, \bar{\mu}) \in$ (MPCC).*
2. *Assume that for each leader's strategy $x \in X$, for each follower $j \in J$, and for each joint strategy $y = (y_j, y_{-j})$ which is feasible for all followers the Guignard's CQ holds for the constraint "$g_j(x, \cdot, y_{-j}) \leq 0$" at the point y_j. If $(\bar{x}, \bar{y}, \bar{\mu}) \in$ (MPCC), then $(\bar{x}, \bar{y}) \in$ SLMFG.*

Guignard CQ

Let us recall that the Guignard's CQ holds for problem

$$\inf_{g_i(x,p)\leq 0, i} f(x,p)$$

at a point (x,p) if $T(x,p)^\circ = L(x,p)^\circ$, i.e., if the polar of the tangent $T(x,p)$ equals the polar to the linearised cone L
where

- the tangent cone $T(x,p)$ stands for the Bouligand tangent cone at x to the parametrised feasible set $\mathcal{F}(p)$
- the linearised cone $L(x,p)$ is given by
  $L(x,p) := \{d \,:\, \nabla_x g_i(x,p)^T.d \leq 0, \text{ for } i \text{ such that } g_i(x,p) = 0\}$.

MPCC for SLMFG

Definition

Let $j \in J$. An opponent strategy $(\hat{x}, \hat{y}_{-j}) \in \mathbb{R}^n \times \mathbb{R}^{m_j}$ is said to be

- an *admissible opponent strategy* (for player j) if $(\hat{x}, \hat{y}_{-j}) \in \mathcal{A}_j := \text{dom}Y_j$, that is, such that there exists $y_j \in X$ with $g_j(\hat{x}, y_j, \hat{y}_{-j}) \leq 0$;
- an *interior opponent strategy* if it is in $int(\mathcal{A}_j)$;
- a *boundary opponent strategy* if it is in $bd(\mathcal{A}_j)$.

MPCC for SLMFG

Theorem

Assume (H_1), (H_2) *and that for each* $j \in J$, *the three following properties hold:*

(1) *(Joint Convexity) Each* g_{jk} *is jointly convex with respect to* (x, y);

(2) *(Joint Slater's CQ) There exists a joint strategy* $(\tilde{x}(j), \tilde{y}(j))$ *such that* $g_j(\tilde{x}(j), \tilde{y}(j)) < 0$;

(3) *(Guignard's CQs for boundary opponent strategies) For any boundary opponent strategy* $(\hat{x}, \hat{y}_{-j}) \in bd(\mathcal{A}_j)$ *Guignard's CQ is satisfied at any feasible point* $y_j \in Y_j(\hat{x}, \hat{y}_{-j})$.

If $(\bar{x}, \bar{y}, \bar{\mu}) \in$ *(MPCC), then* $(\bar{x}, \bar{y}) \in$ *SLMFG.*

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