70 Years of Fractal Projections

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Caveat

Throughout this talk we work in $\mathbb{R}^2$

– Many results, but not all, have higher dimensional analogues.

All sets considered are assumed to be ‘reasonable’

i.e. Borel or analytic, non-empty and bounded.

John Marstrand's 1954 paper

SOME FUNDAMENTAL GEOMETRICAL PROPERTIES OF PLANE SETS OF FRACTIONAL DIMENSIONS

By J. M. MARSTRAND

[Received 27 March 1953.—Read 16 April 1953]

THEOREM II. *Any s-set whose dimension does not exceed unity projects into a set of dimension s in almost all directions.*

Proof. It will be understood throughout the proof that s is a fixed positive number such that $s \leqslant 1$. Suppose then that E is any s-set, and suppose that t is any positive number such that $t < s$. It follows from Lemma 9 that at almost all points (x, y) of E, for all sufficiently small positive numbers d and any positive number δ such that $\delta < d$,

$$\int_0^{2\pi} \delta^{-t}\Lambda^s ER\, d\theta < K_s\,\delta^{s-t}\left(1+\log\frac{d}{\delta}\right) = o\left(\delta^{\frac{1}{2}(s-t)}\right) \quad \text{as } \delta \to 0.$$

Proceedings of the London Mathematical Society(3),**4** (1954), 257-302

John Marstrand's 1954 paper

- Projection theorems

  $+$ much more ...

- Intersection with lines – almost every line through almost every point of an s-set E $(s > 1)$ intersects E in a set of dimension $s - 1$.

- Radial projections, i.e. projection of sets from points

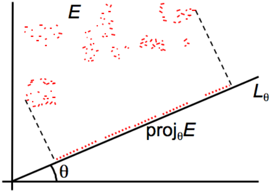
- Examples to show results are best possible

- The density $\lim_{r \to 0} \mathcal{H}^s(E \cap B(x,r))/(2r)^s$ of an s-set $E \subset \mathbb{R}^2$ can only exist and equal 1 on a set of positive $\mathcal{H}^s$-measure if $s = 0, 1$ or 2

- Bounds on angular densities (i.e. densities restricted to a sector)

- Discussion of weak tangents to sets

Marstrand's projection theorems



Theorem (Marstrand 1954) Let $E \subset \mathbb{R}^2$ be Borel.

(i) For all $\theta \in [0, \pi)$ $\dim_{\rm H} \text{proj}_\theta E \leq \min\{\dim_{\rm H} E, 1\}$ with equality for almost all $\theta \in [0, \pi)$,

(ii) If $\dim_{\rm H} E > 1$, $\mathcal{L}(\text{proj}_\theta E) > 0$ for almost all $\theta \in [0, \pi)$.

[proj_θ denotes orthogonal projection onto the line L_θ, $\dim_{\rm H}$ is Hausdorff dimension, $\mathcal{L}$ is Lebsegue measure on L_θ.]

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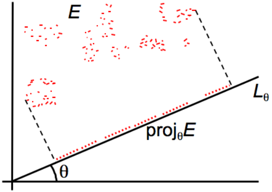

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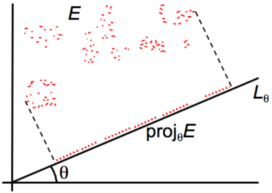

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That $\dim_{\mathrm{H}} \mathrm{proj}_\theta E \leq \min\{\dim_{\mathrm{H}} E, 1\}$ for all θ follows since projection is a Lipschitz map which cannot increase dimension. Marstrand's lower bound proof was geometric and intricate.

Capacities and Hausdorff dimension of projections

Kaufman's (1968) potential theoretic proof has become a standard approach for such problems.

Marstrand's lower bound may be derived from the capacity characterisation of Hausdorff dimension. Let $\mathcal{M}(E)$ be the set of probability measures on E. With the capacity $C^s(E)$ of $E \subset \mathbb{R}^n$ given by

$$\frac{1}{C^s(E)} = \inf_{\mu \in \mathcal{M}(E)} \int\int \frac{d\mu(x)d\mu(y)}{|x-y|^s},$$

$$\dim_{\mathrm{H}} E = \sup\left\{s : C^s(E) > 0\right\}.$$

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Let μ_θ be the projection of μ onto line in direction θ. If $0 < s < 1$

$$\int_0^\pi \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\mu_\theta(t)d\mu_\theta(u)}{|t-u|^s}\right]d\theta = \int_0^\pi \left[\int_E\int_E \frac{d\mu(x)d\mu(y)}{|x\cdot\theta - y\cdot\theta|^s}\right]d\theta$$

$$\leq c\int_E\int_E \frac{d\mu(x)d\mu(y)}{|x-y|^s} < \infty$$

Length of projections when $\dim_H E = 1$

What can we say about $\mathcal{L}(\text{proj}_\theta E)$ when $\dim_H E = 1$? In fact almost anything can happen!

Theorem (Davies 1952) Given a Borel $E \subset \mathbb{R}^2$ of finite area, there is a set of lines $\mathbb{L}$ such that $E \subset \bigcup_{L\in\mathbb{L}}$ and $\text{area}\big(\bigcup_{L\in\mathbb{L}} \setminus E\big) = 0$.

Dualising we get:

Theorem Given a Borel set E_θ for each $\theta \in [0,\pi)$ (+ measurability condition), there exists a Borel set $E \subset \mathbb{R}^2$ such that $\mathcal{L}(E_\theta \triangle \text{proj}_\theta E) = 0$ for almost all directions θ.

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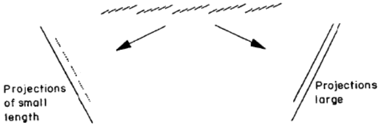
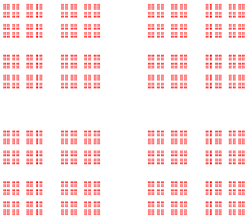
Alternatively, there is a direct 'iterated venetian blind' construction.



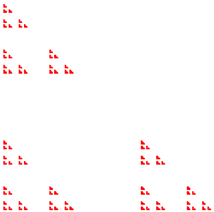
Higher dimensional analogues are also valid.

Exceptional directions

Marstrand's theorem tells nothing about which particular directions may have projections with dimension or measure smaller than typical, i.e. when $\dim_{\mathrm{H}} \mathrm{proj}_\theta E < \min\{\dim_{\mathrm{H}} E, 1\}$, or $\dim_{\mathrm{H}} E > 1$ and $\mathcal{L}(\mathrm{proj}_\theta E) = 0$.



Dimension $\log 4/\log(5/2) = 1.51$, some projections of dimension < 1.



1-dimensional Sierpinski triangle, properties of $\mathrm{proj}_\theta E$ depend on (p,q) where slope $\theta = p/q$.

Exceptional directions

The set of exceptional directions can't be 'too big':

Theorem (Kaufman, 1968) If $E \subseteq \mathbb{R}^2$ and $\dim_{\mathrm{H}} E \leq 1$,

$$\dim_{\mathrm{H}}\{\theta : \dim_{\mathrm{H}}\mathrm{proj}_\theta E < \dim_{\mathrm{H}} E\} \leq \dim_H E.$$

– This follows from a minor modification of the Kaufman's potential theoretic argument.

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Theorem (F, 1982) If $E \subseteq \mathbb{R}^2$ and $\dim_{\mathrm{H}} E > 1$,

$$\dim_{\mathrm{H}}\{\theta : \mathcal{L}(\mathrm{proj}_\theta E) = 0\} \leq 2 - \dim_{\mathrm{H}} E.$$

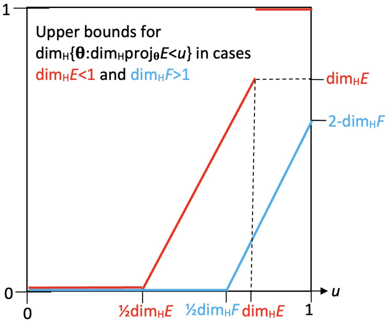
– Fourier transform proof.

Marstrand, Mattila, Falconer, Davies, Kaufman
(Photo: Tuomas Sahlsten, Bristol, 2014)

Exceptional directions

Improvements by Oberlin, Bourgain, He, Orponen, Guth, Shmerkin, Wang ...

Oberlin's Conjecture (2012)
If $E \subseteq \mathbb{R}^2$ and
$0 \leq u \leq \min\{\dim_H E, 1\}$,
then
$$\dim_H\{\theta : \dim_H \text{proj}_\theta E < u\} \leq \max\{2u - \dim_H E, 0\}.$$



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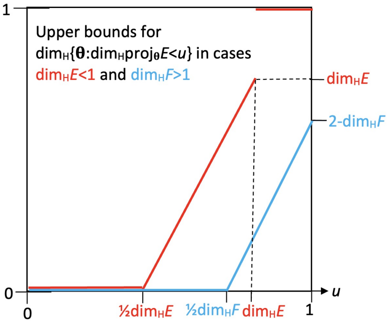
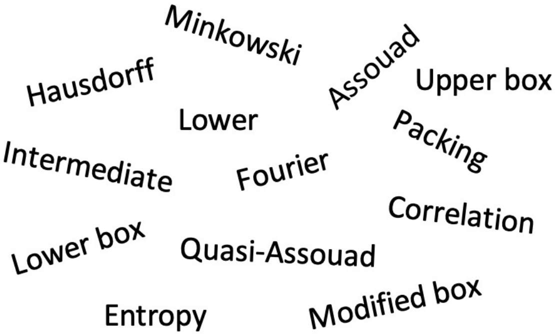


Proved by Ren & Wang (2023+) as a corollary of their proof of the Furstenberg set conjecture.

Other fractal dimensions



Are there ‘Marstrand-type’ theorems for such definitions of dimension?

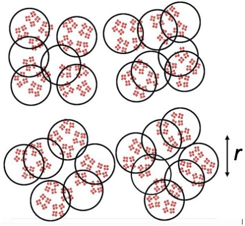
Box-counting dimension

The box-counting dimension of a non-empty and compact $E \subset \mathbb{R}^2$ is

$$\dim_B E = \lim_{r \to 0} \frac{\log N_r(E)}{-\log r}$$

where $N_r(E)$ is the least number of sets of diameter r covering E.

[Taking lower/upper limits gives the lower/upper box dimensions.]



Box-counting dimensions of projections

Is there a Marstrand-type theorem for box-dimensions?

For $E \subset \mathbb{R}^2$

$$\frac{\dim_{\mathrm{B}} E}{1 + \frac{1}{2}\dim_{\mathrm{B}} E} \leq \dim_{\mathrm{B}} \mathrm{proj}_\theta E \leq \min\{\dim_{\mathrm{B}} E, 1\} \quad \text{for almost all } \theta,$$

and examples show that these bounds are best possible.

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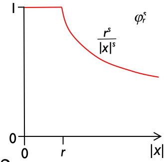
and examples show that these bounds are best possible.

Even so, we do get a ‘Marstrand-type’ theorem: $\underline{\dim}_{\mathrm{B}} \mathrm{proj}_\theta E$ and $\overline{\dim}_{\mathrm{B}} \mathrm{proj}_\theta E$ must be constant for almost all θ, (F & Howroyd, 1996) using a messy argument to get an indirectly defined value.

Using capacities things become much simpler.

Box-counting dimensions of projections

Define kernels $\phi_r^s(x)$ for $s > 0,\ 0 < r < 1,\ x \in \mathbb{R}^2$ by $\phi_r^s(x) = \min\left\{1, \left(\frac{r}{|x|}\right)^s\right\}$.



The capacity $C_r^s(E)$ of a compact $E \subset \mathbb{R}^2$ w.r.t. ϕ_r^s is

$$\frac{1}{C_r^s(E)} = \inf_{\mu \in \mathcal{M}(E)} \int\int \phi_r^s(x-y)d\mu(x)d\mu(y),$$

where $\mathcal{M}(E)$ are the probability measures on E. Then for $E \subset \mathbb{R}^2$

$$c_1 C_r^s(E) \le N_r(E) \le \begin{cases} c_2 \log(1/r)\ C_r^s(E) & \text{if } s = 2 \\ c_2\ C_r^s(E) & \text{if } s > 2 \end{cases} \qquad (1),$$

(c_1, c_2 depend on $s, \mathrm{diam} E$). In particular for $E \subset \mathbb{R}^2$

$$\lim_{r\to 0} \frac{\log C_r^2(E)}{-\log r} = \lim_{r \to 0} \frac{\log N_r(E)}{-\log r} = \dim_{\mathrm{B}} E =: \dim_{\mathrm{B}}^2 E$$

(can replace $\dim_{\mathrm{B}}$ and lim by $\underline{\dim}_{\mathrm{B}}$ and $\underline{\lim}$, or $\overline{\dim}_{\mathrm{B}}$ and $\overline{\lim}$).

Box-counting dimensions of projections

Theorem (F, 2019) Let $E \subset \mathbb{R}^2$ be non-empty compact.
For all $\theta \in [0,\pi)$

$$\dim_B \text{proj}_\theta E \ \leq \ \lim_{r\to 0} \frac{\log C_r^1(E)}{-\log r} \ =: \dim_B^1 E.$$

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We call

$$\dim_{\mathrm{B}}^s E := \lim_{r \to 0} \frac{\log C_r^s(E)}{-\log r} \quad (E \subset \mathbb{R}^2 \text{ or } \mathbb{R}^n),$$

using capacity with respect to the kernel $\phi_r^s(x) = \min\left\{1, \left(\frac{r}{|x|}\right)^s\right\}$, the s-box-dimension profile of E, which should be thought of as the 'box-dimension of E when regarded from an s-dimensional viewpoint'.

Packing dimensions of projections

For $E \subset \mathbb{R}^2$ and $s > 0$ we define

$$\dim_{\mathrm{P}}^s E = \inf\left\{ \sup_i \overline{\dim}_{\mathrm{B}}^s E_i : E \subset \bigcup_{i=1}^{\infty} E_i \text{ with each } E_i \text{ compact}\right\}.$$

Then $\dim_{\mathrm{P}} E = \dim_{\mathrm{P}}^2 E$.

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One can get bounds for the Hausdorff dimension of the set of exceptional directions of projections for box and packing dimensions, for example:

If $E \subseteq \mathbb{R}^2$ and $0 \leq s < 1$,

$$\dim_{\mathrm{H}}\{\theta : \dim_{\mathrm{P}} \mathrm{proj}_\theta E < \dim_{\mathrm{P}}^s E\} \leq s.$$

Intermediate dimensions

(Fraser, Kempton, F, 2020)
Let $E \subseteq \mathbb{R}^2$ be non-empty and bounded. For $0 \leq \alpha \leq 1$ define the upper α-intermediate dimension of E by

$$\overline{\dim}_\alpha E = \inf \Big\{ s \geq 0 : \text{ for all } \epsilon > 0 \text{ and all sufficiently small } \delta > 0$$
$$\text{there is a cover } \{U_i\} \text{ of } E \text{ s.t. } \delta^{1/\alpha} \leq |U_i| \leq \delta \text{ and } \sum |U_i|^s \leq \epsilon \Big\}.$$

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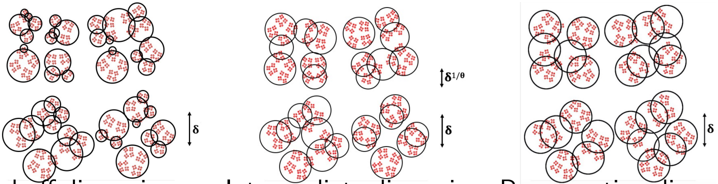


Hausdorff dimension α-Intermediate dimension Box-counting dimension

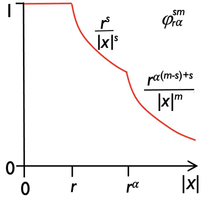
Then $\dim_\alpha E$ interpolates between Hausdorff and box dimensions. Thus $\dim_\alpha E$ is increasing for $\alpha \in [0,1]$ and

$$\dim_{\mathrm{H}} E = \dim_0 E \le \dim_\alpha E \le \dim_1 E = \dim_{\mathrm{B}} E.$$

Intermediate dimensions of projections

For projections, α-intermediate dimensions behave like box-dimensions. Here we use kernels of the form

$$\phi^{s,m}_{r,\alpha}(x) = \begin{cases} 1 & 0 \le |x| < r \\ \left(\frac{r}{|x|}\right)^s & r \le |x| < r^\alpha \\ \frac{r^{\alpha(m-s)+s}}{|x|^m} & r^\alpha \le |x| \end{cases}$$



For $E \subset \mathbb{R}^2$ and $s \ge 0$ we define $\underline{\dim}^s_\alpha E, \overline{\dim}^s_\alpha E$ in terms of capacities w.r.t. this kernel. Then

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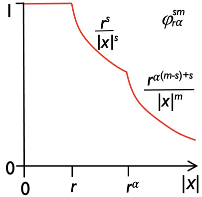

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Theorem (Burrell, Fraser, F, 2021) Let $E \subset \mathbb{R}^2$. Then for almost all θ,

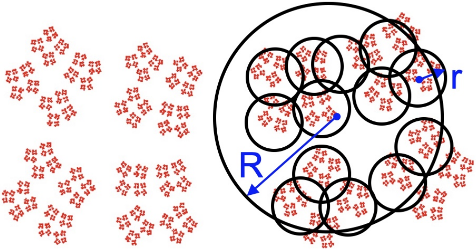
$$\underline{\dim}_\alpha \text{proj}_\theta E = \underline{\dim}^1_\alpha F \quad \text{and} \quad \overline{\dim}_\alpha \text{proj}_\theta E = \overline{\dim}^1_\alpha F$$

for all $\alpha \in [0, 1]$.

Assouad dimension

The Assouad dimension of $E \subset \mathbb{R}^2$ is given by

$$\dim_A E = \inf \Big\{ \alpha : \text{ there exists } c > 0 \text{ s.t. for all } 0 < r < R \text{ and } x \in E,\ N_r(B(x,R) \cap F) \leq c\left(\frac{R}{r}\right)^{\alpha} \Big\}.$$



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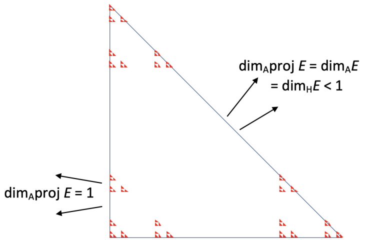
Example (Fraser & Orponen 2017)
Let s be slightly less than 1. The s-dimensional right Sierpiński triangle has $\dim_A \mathrm{proj}_\theta E = s < 1$ if $\theta \in (-\epsilon, \epsilon)$ and $\dim_A \mathrm{proj}_\theta E = 1$ if $\theta \in (\pi/4 - \epsilon, \pi/4 + \epsilon)$.



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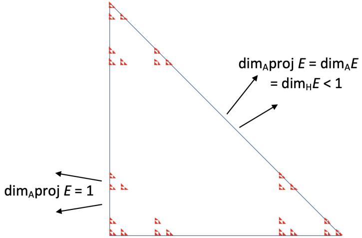

Theorem (Fraser & Orponen 2017, Orponen 2021) Let $E \subset \mathbb{R}^2$. Then for almost all θ,

$$\dim_{\mathrm{A}}\mathrm{proj}_\theta E \geq \min\{1, \dim_{\mathrm{A}} E\}.$$

Moreover

$$\dim_{\mathrm{H}}\{\theta : \dim_{\mathrm{A}}\mathrm{proj}_\theta E < \min\{1, \dim_{\mathrm{A}} E\}\} = 0$$

Assoud specturm

The Assouad spectrum $\dim_{\mathrm{A}}^{\vartheta} E, (0 < \vartheta < 1)$ of $E \subset \mathbb{R}^2$ is given by

$$\dim_{\mathrm{A}}^{\vartheta} E = \inf \Big\{ \alpha : \text{there exists } c > 0 \text{ s.t. for all } 0 < r < 1 \text{ and } x \in E,\ N_r(B(x, r^{\vartheta}) \cap F) \leq c \Big(\frac{r^{\vartheta}}{r} \Big)^{\alpha} \Big\}.$$

Question Is there a Marstrand-type result for $\dim_{\mathrm{A}}^{\vartheta} E$ for each $0 < \vartheta < 1$ and also for the quasi-Assouad dimension $\dim_{\mathrm{qA}} E = \lim_{\vartheta \to 1} \dim_{\mathrm{A}}^{\vartheta} E$?

Fourier dimension

The Fourier dimension $\dim_F E$ of $E \subset \mathbb{R}^2$ is given by

$$\dim_F E = \sup\Big\{s \le n : \text{ there exists } c > 0 \text{ and } \mu \text{ on } E \text{ such that } |\widehat{\mu}(z)| \le c|z|^{-s/2} \text{ for all } z \in \mathbb{R}^2\Big\}.$$

Then $\dim_F E \le \dim_H E$ by the potential characterisation of $\dim_H E$.
Also, $\widehat{\mu_\theta}(u) = \widehat{\mu}(u\underline{\theta})$ for μ on E and $u \in L_\theta$, so

$$1 \ge \dim_F \text{proj}_\theta E \ge \min\{1, \dim_F E\}.$$

Hence for each θ,

$$\min\{1, \dim_F E\} \le \dim_F \text{proj}_\theta E \le \dim_H \text{proj}_\theta E \le \min\{1, \dim_H E\},$$

In particular, if $\dim_F E = \dim_H E$, that is if E is a Salem set, then

$$\dim_F \text{proj}_\theta E = \dim_H \text{proj}_\theta E = \min\{1, \dim_H E\} \text{ for all } \theta.$$

Question Is there a Marstrand-type result for $\dim_F E$?

Mixed estimates

There are various inequalities that bound the a.s. dimensions of $\dim \text{proj}_\theta E$ in terms of other dimensions E. For example:

Proposition (F, Fraser & Shmerkin, 2021) Let $E \subset \mathbb{R}^2$. Then for almost all θ,

$$\overline{\dim}_{\rm B} \text{proj}_\theta E \geq \overline{\dim}_{\rm B} E - \max\{0, \dim_{\rm qA} E - 1\}$$

In particular, if $\dim_{\rm qA} \leq \max\{1, \overline{\dim}_{\rm B} E\}$ then

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Similarly, good bounds for the dimension of the set of exceptional directions may be obtained by invoking other types of dimension. For example, Fraser & de Orellana (2024) estimate

$$\dim_{\mathrm{H}}\{\theta : \dim_{\mathrm{H}} \mathrm{proj}_\theta E < u\}$$

in terms of the Fourier spectrum $\dim_{\mathrm{F}}^{\alpha} E$.

Specific classes of set

General problem: Find sets or classes of sets where there are no exceptional directions for projections or where the exceptional directions can be identified.

E.g. Salem sets, as above have no exceptional directions.

Finding projection properties of $E \subset \mathbb{R}^2$ is often tied up with finding the dimension of E itself.

Iterated function systems

A family $f_1, \ldots, f_m$ of contractions on $D \subseteq \mathbb{R}^N$, i.e.

$$|f_i(x) - f_i(y)| \leq c_i|x - y| \quad x, y \in D, \quad c_i < 1$$

is called an iterated function system (IFS).

Given an IFS there exists a unique, non-empty compact set E satisfying

$$E = \bigcup_{i=1}^{m} f_i(E),$$

called the attractor of the IFS.

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called the attractor of the IFS.

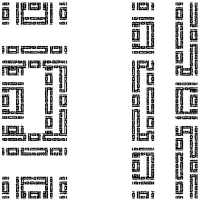
If the f_i are similarities E is called a self-similar set.

If the f_i are conformal maps E is called a self-conformal set.

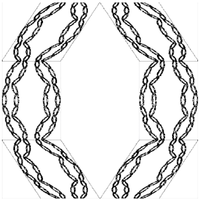
If the $f_i = T_i + t_i$ are affine contractions on $\mathbb{R}^N$, where the T_i are non-singular contracting linear mappings on $\mathbb{R}^n$ and $t_i \in \mathbb{R}^n$ are translation vectors, E is a self-affine set.

We will generally assume that the union $(*)$ is disjoint or perhaps satisfies the open set condition.

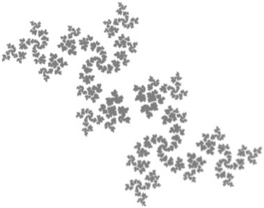
Specific Sets - IFS attractors



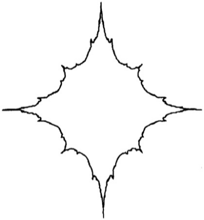
self-similar



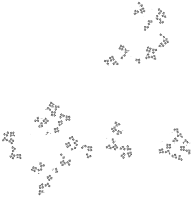
self-affine



self-conformal



nonlinear,
nonconformal



statistically
self-similar

Self-similar sets

Let $f_1, \ldots, f_m : \mathbb{R}^2 \to \mathbb{R}^2$ be an IFS of similarities, so the self-similar set $E \subset \mathbb{R}^2$ satisfies

$$E = \bigcup_{i=1}^{m} f_i(E). \qquad (*)$$

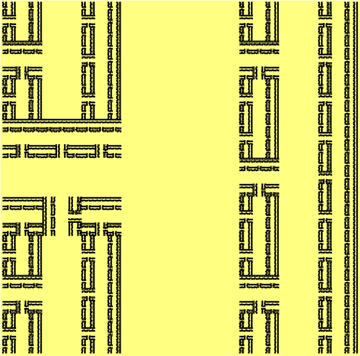
Write the similarities as

$$f_i(x) = r_i O_i(x) + t_i$$

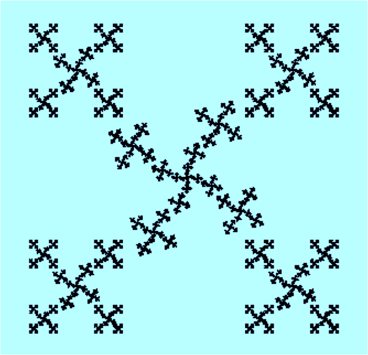
where $0 < r_i < 1$ is the scale factor, O_i is a rotation and t_i is a translation.

The family $\{f_1, \ldots, f_m\}$ has dense rotations if at least one of the O_i is a rotation by an irrational multiple of π, equivalently if group$\{O_1, \ldots, O_m\}$ is dense in $SO(2, \mathbb{R})$.
Otherwise $\{f_1, \ldots, f_m\}$ has finite rotations.

Self-similar sets



finite rotations



dense rotations

Theorem (Farkas 2014) Let $E \subset \mathbb{R}^2$ be a self-similar set defined by a family $\{f_1, \ldots, f_m\}$ of similarities with finite rotations satisfying the open set condition. Then there is at least one value of θ such that $\dim_{\mathrm{H}} \mathrm{proj}_\theta E < \dim_{\mathrm{H}} E$.

Self-similar sets with finite rotations

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Theorem (Hochman 2014) Let $E \subset \mathbb{R}^2$ be a self-similar set defined by a family $\{f_1, \ldots, f_m\}$ of similarities with finite rotations satisfying the strong separation condition. Then

$$\dim_{\mathrm{H}} \mathrm{proj}_\theta E = \min\{\dim_{\mathrm{H}} E, 1\} \text{ for all } \theta \notin B \text{ where } \dim_{\mathrm{H}} B = 0.$$

Self-similar sets with dense rotations

Theorem (Peres & Shmerkin 2009, Hochman & Shmerkin 2012)
Let $E \subset \mathbb{R}^2$ be a self-similar set defined by a family $\{f_1, \ldots, f_m\}$ of similarities with dense rotations. Then

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The proof uses the natural invariant measure on E along with ideas from ergodic scenery flows, CP chains, r-scale entropy, Marstrand's theorem, ...

An alternative proof, using compact group extensions was given by Jin & F (2014).

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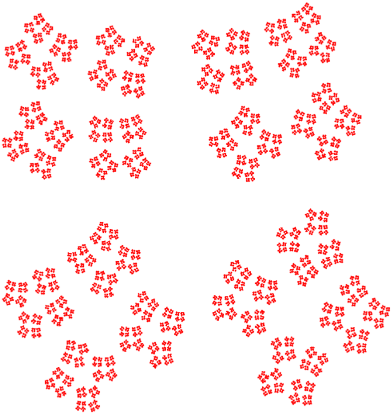
An alternative proof, using compact group extensions was given by Jin & F (2014).

Corollary With $E \subset \mathbb{R}^2$ as above, for all non-singular C^1 functions $h : E \to \mathbb{R}$, where N is a neighbourhood of E,

$$\dim_{\mathrm{H}} h(E) = \min\{\dim_{\mathrm{H}} E, 1\}.$$

This follows using the result for projections locally, noting that at very fine scales h 'looks like' a projection of a small copy of E in some direction.

Statistically self-similar sets - percolation



A self-similar set E has a natural hierarchical construction. This enables us to base a percolation process on a self-similar set. At each stage of the iterated construction we retain each component independently with some probability p.

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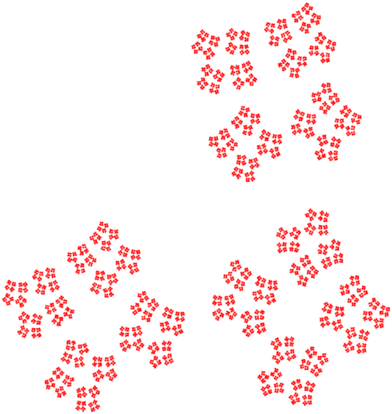

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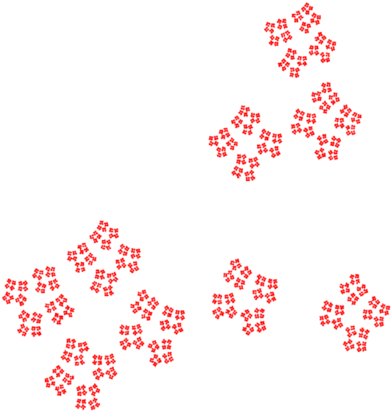

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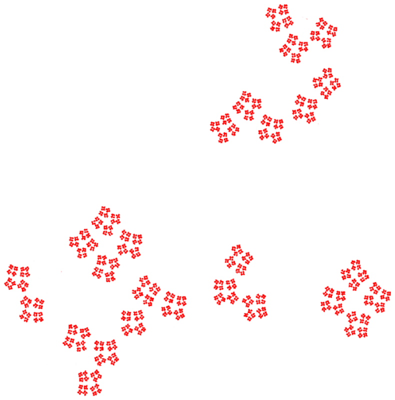

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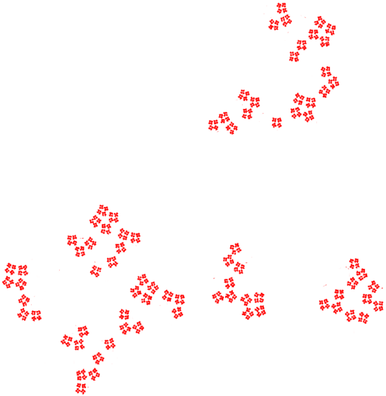

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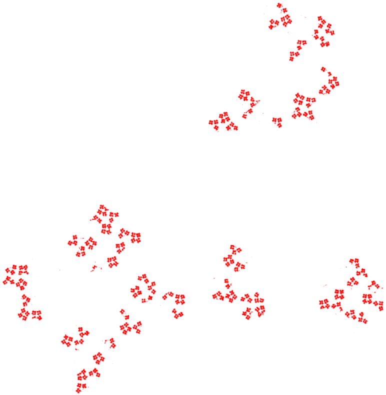

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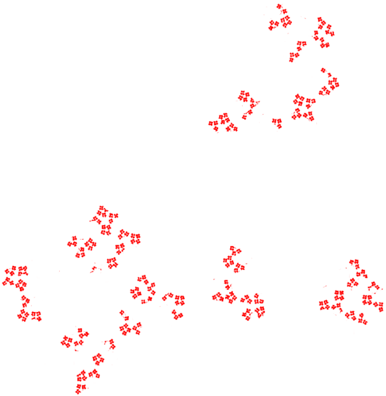


If the underlying self-similar set E is based on m similarities and $p > 1/m$ then the statistically self-similar set $E_p \neq \emptyset$ with positive probability, conditional on which $\dim_{\mathrm{H}} E_p = s$, where $p\sum_{i=1}^{m} r_i^s = 1$.

Projection of statistically self-similar sets

Theorem (F & Jin 2015) Let E be a self-similar set with dense rotations and let E_p be percolation on E where $p > 1/m$. Then, conditional on $E_p \neq \emptyset$, almost surely

$$\dim_{\mathrm{H}} \mathrm{proj}_\theta E_p = \min\{\dim_{\mathrm{H}} E_p, 1\} \text{ for all } \theta.$$

This is a random extension of the deterministic result adding another level of ergodicity to accommodate the randomness of the natural measure supported by the random set E_p.

Self-conformal sets

Theorem (Jin & Bruce 2019) Let $\{f_i\}_{i=1}^m$ be an IFS of $C^{1+\epsilon}$ conformal mappings $f_i : U \to U$ on some convex open set U with OSC holding. Thus $f_i'(x) = r_i(x)O_i(x)$ where $O_i(x)$ are rotations and $0 < r_- \leq r_i(x) \leq r_+ < 1$. The IFS attractor E is called self-conformal.

Then

$$\dim_{\mathrm{H}}\mathrm{proj}_\theta E = \min\{\dim_{\mathrm{H}} E, 1\} \text{ for all } \theta.$$

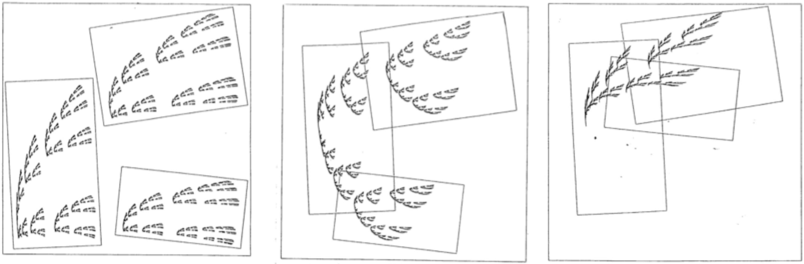
provided that a certain skew product $\sigma_\phi : \{1, 2, \ldots, m\} \times SO(2) \circlearrowleft$ has a dense orbit. (Analogous condition to dense rotations.)

In particular, this holds for totally disconnected Julia sets of $z \mapsto z^2 + c$ if $|c| \geq 2.5$ and $\arg(1 + \sqrt{1-4c})/\pi$ is irrational.

Self-affine sets

Recall that a self-affine set $E \subset \mathbb{R}^2$ is defined by an iterated function system of affine mappings

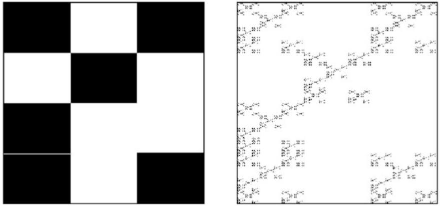
$$f_i(x) = L_i(x) + t_i \text{ where } L_i \text{ is linear and } t_i \text{ is a translation.}$$



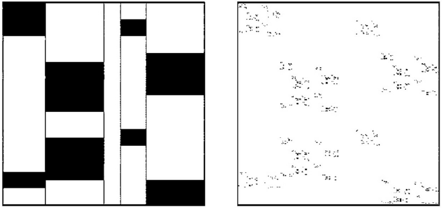
Self-affine sets with the same L_i and different t_i

Self-affine carpets

The attractor E of an IFS of affine mappings $f_i(x) = L_i(x) + t_i$ is a self-affine carpet if the L_i are diagonal matrices. Thus the IFS affine functions are defined by specifying the rectangular images of the unit square.

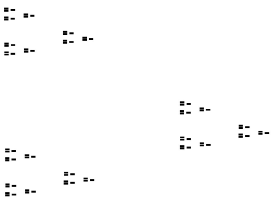


Bedford-McMullen carpet



Lalley-Gatzouras carpet

Projections of self-affine carpets



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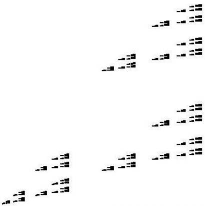

Gatzouras-Lalley

Theorem (Ferguson, Jordan & Shmerkin, 2010) Let E be a Bedford-McMullen (or Gatzouras-Lalley) carpet with $\log m/\log n \notin \mathbb{Q}$. Then

$$\dim_{\mathrm{H}} \mathrm{proj}_\theta E = \min\{\dim_{\mathrm{H}} E, 1\}$$

for all θ apart from possibly when $\theta \in \{0, \pi/2\}$.

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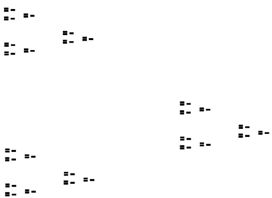

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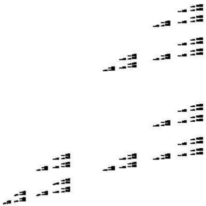

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Theorem (Burrell, F & Fraser, 2021) Let E be a Bedford-McMullen carpet with $\dim_{\mathrm{H}} E < 1 \leq \dim_{\mathrm{B}} E$. Then $\overline{\dim}_{\mathrm{B}} \mathrm{proj}_\theta E < 1$ for almost all θ (regardless of how large $\dim_{\mathrm{B}} E$ is).

Projections of general self-affine sets

Let $f_i(x) = L_i(x) + t_i$ be an IFS where L_i is linear and t_i is a translation, yielding a self-affine set E.

The dimension theory of self-affine sets has recently developed rapidly using techniques from ergodic theory, with projection properties central to this development - Bárány, F, Käenmäki, Kempton, Morris, Hochman, Peres, Rapaport, Shmerkin, ...

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Theorem(Kempton, F, 2017, Bárány, Hochman, Rapaport 2021) If the self-affine IFS $\{f_i\}_{i=1}^m$ on $\mathbb{R}^2$ satisfies the strong open set condition and is totally irreducible (i.e. the L_i do not preserve any finite union of lines) then

$$\dim_{\mathrm{H}} \mathrm{proj}_\theta E = \min\{\dim_{\mathrm{H}} E, 1\} \text{ for all } \theta \in [0, \pi).$$

Strong open set condition can be replaced by exponential separation (Hochman & Rapaport 2022)

3-dimensional analogues ? (Morris, Sert, Rapaport ...)

Other work on dimensions of projections includes

• When do projections of sets have positive Lebesgue measure or non-empty interior? (Numerous authors)

• Projections of measures - almost all results mentioned have measure analogues (Numerous authors)

• Equal dimensions of projections in all / nearly all directions for 'structured' random sets (Jin, F, Shmerkin, Suomala, Käenmäki, Simon, Rams, ...)

• Generalised projections (Peres, Schlag, ...), radial / angular projections

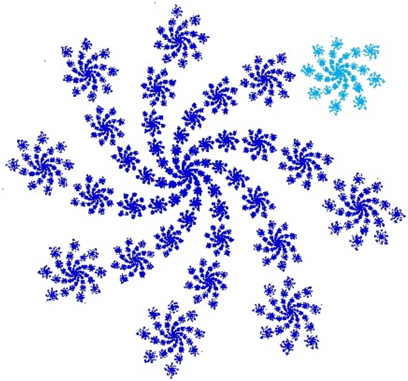
• Projections onto restricted families of subspaces (Fässler, Orponen, Järvenpää^2, Keleti, Leikas, Käenmäki, Venieri, Gan, Guo, Wang, ...)

• Projections in non-Euclidean spaces: Heisenberg groups, normed spaces, ... (Mattila, Balogh, Tyson, Iseli, ...)

• Projections in infinite dimensional spaces (Hunt, Kaloshin, ...)

• Multifractal projection results (Hunt, Kaloshin, F, O'Neil, Olsen, Barral, Bhouri,...)

• ......



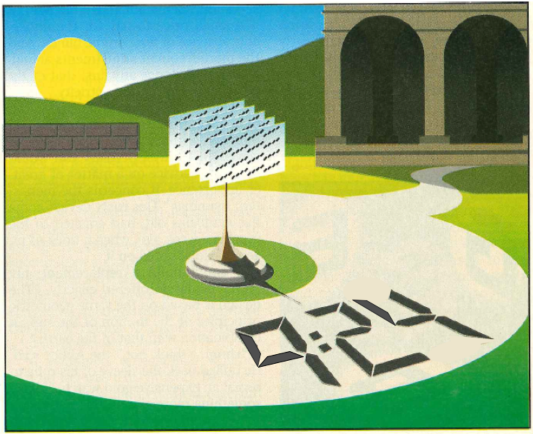
Thank you!

Digital sundials

Digital sundial theorem (F, 1986)

Given a subset E_V of each 2-dimensional subspace V of $\mathbb{R}^3$ (+ measurability condition), there exists a Borel set $E \subset \mathbb{R}^3$ such that for almost all subspaces V

$$\text{Area}(E_V \triangle \text{proj}_V E) = 0.$$



DIGITAL SUNDIAL stands in the courtyard of the Cartesian Monastery, home of Brother Benjamin and the Euclidean monks.

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