Nikodym type sets avoiding lines in many directions

András Máthé
University of Warwick

Fractal Geometry and Stochastics 7
Chemnitz, Germany

26 September 2024

ON THE DIMENSION OF s-NIKODÝM SETS

DAMIAN DĄBROWSKI, MAX GOERING, AND TUOMAS ORPONEN

ABSTRACT. Let $s \in [0,1]$. We show that a Borel set $N \subset \mathbb{R}^2$ whose every point is linearly accessible by an s-dimensional family of lines has Hausdorff dimension at most $2 - s$.

CONTENTS

Otto Nikodym & Stefan Banach

Theorem (Nikodym 1927)

For every $x \in \mathbb{R}^2$ one can find a line ℓ_x going through x such that

$$\bigcup_{x \in \mathbb{R}^2} \ell_x \setminus \{x\}$$

has Lebesgue measure zero.

Theorem (Nikodym 1927)

There is a set $E \subset \mathbb{R}^2$ of full Lebesgue measure such that for every $x \in E$ there is a line ℓ_x that goes through x but no other point of E.

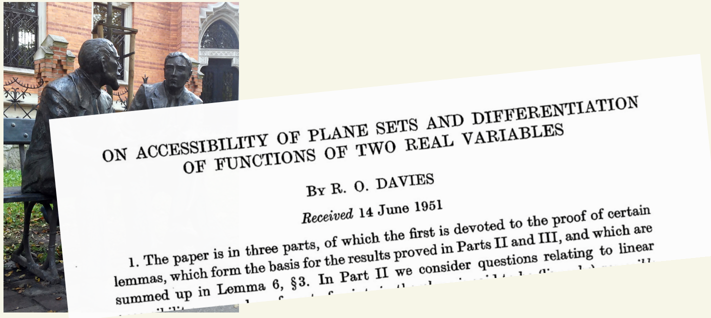
Theorem (Nikodym 1927)

For every $x \in \mathbb{R}^2$ one can find a line ℓ_x going through x such that

$$\bigcup_{x \in \mathbb{R}^2} \ell_x \setminus \{x\}$$

has Lebesgue measure zero.

Can we have more lines through every point?


ON ACCESSIBILITY OF PLANE SETS AND DIFFERENTIATION OF FUNCTIONS OF TWO REAL VARIABLES

By R. O. DAVIES

Received 14 June 1951

1. The paper is in three parts, of which the first is devoted to the proof of certain lemmas, which form the basis for the results proved in Parts II and III, and which are summed up in Lemma 6, §3. In Part II we consider questions relating to linear

Theorem (Nikodym 1927)

There is a set $E \subset \mathbb{R}^2$ of full Lebesgue measure such that for every $x \in E$ there is a line ℓ_x that goes through x but no other point of E.

Theorem (Nikodym 1927)

For every $x \in \mathbb{R}^2$ one can find a line ℓ_x going through x such that

$$\bigcup_{x \in \mathbb{R}^2} \ell_x \setminus \{x\}$$

has Lebesgue measure zero.

We can have continuum many lines in a dense G_δ set of directions.

(Davies 1951)

Can we have lines in directions of positive Hausdorff dimension?

Can we have lines in directions of positive Hausdorff dimension?

No.

Can we have lines in directions of positive Hausdorff dimension?

No.

How big the set E can be to have lines in a t-dimensional set of directions? (Dabrowski, Goering, Orponen 2024)

That is...

Definition

Let $s \in [0,1]$.

$E \subset \mathbb{R}^2$ is an s-Nikodym set

if E is Borel

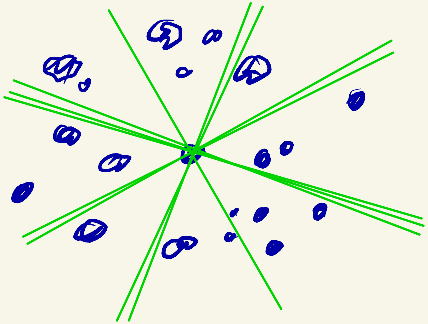
and

for every $x \in E$

there is an s-dimensional

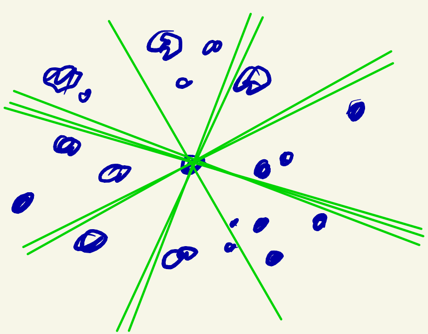
family of lines through x

that only intersect E in x.



Definition

Let $s \in [0,1]$.
$E \subset \mathbb{R}^2$ is an s-Nikodym set
if E is Borel
and
for every $x \in E$
there is an s-dimensional
family of lines through x
that only intersect E in x.



Theorem (D. Dabrowski, M. Goering, T. Orponen 2024)

For any s-Nikodym set E,

$$\dim_H E \leq 2 - s.$$

ON THE DIMENSION OF s-NIKODÝM SETS

DAMIAN DĄBROWSKI, MAX GOERING, AND TUOMAS ORPONEN

ABSTRACT. Let $s \in [0,1]$. We show that a Borel set $N \subset \mathbb{R}^2$ whose every point is linearly accessible by an s-dimensional family of lines has Hausdorff dimension at most $2 - s$.

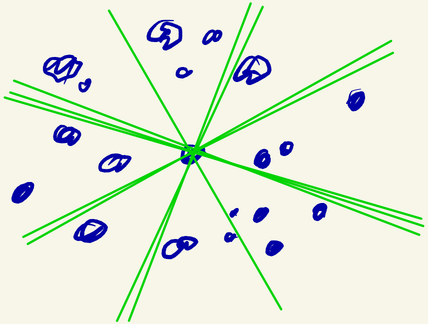
Jun 2024

CONTENTS

Definition

Let $s \in [0,1]$.

$E \subset \mathbb{R}^2$ is an s-Nikodym set if E is Borel and for every $x \in E$ there is an s-dimensional family of lines through x that only intersect E in x.



Theorem (D. Dabrowski, M. Goering, T. Orponen 2024)

For any s-Nikodym set E,

$$\dim_H E \leq 2 - s.$$

Question

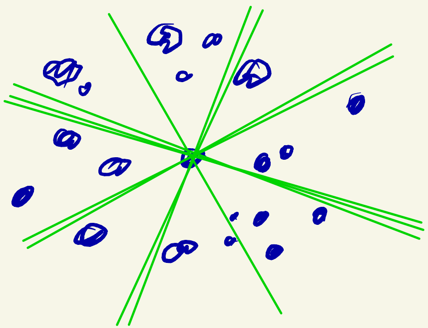
Is this sharp?

Are there s-Nikodym sets of dimension $2-s$?

Definition

Let $s \in [0,1]$.

$E \subset \mathbb{R}^2$ is an s-Nikodym set if E is Borel and for every $x \in E$ there is an s-dimensional family of lines through x that only intersect E in x.



Theorem (D. Dabrowski, M. Goering, T. Orponen 2024)

For any s-Nikodym set E,

$$\dim_H E \leq 2-s.$$

Question

Is this sharp?

Are there s-Nikodym sets of dimension $2-s$?

Answer (A.M. 2024+)

Yes.

In the construction...

Should the lines through each $x \in E$ go in the same s-dimensional set of directions (indpendently of x)?

In the construction...

Should the lines through each $x \in E$ go in the same s-dimensional set of directions (indpendently of x)?

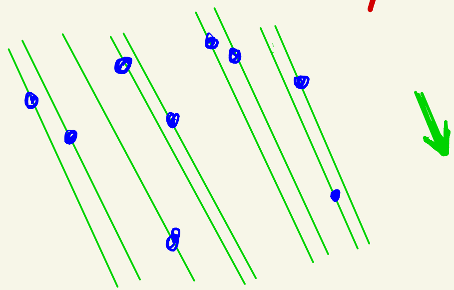
Not necessarily

In the construction...

Should the lines through each $x \in E$ go in the same s-dimensional set of directions (indpendently of x)?

Not necessarily,

but if they do, this is a question about projections.



Theorem (A.M. 2024+) $\forall s \in [0,1]$

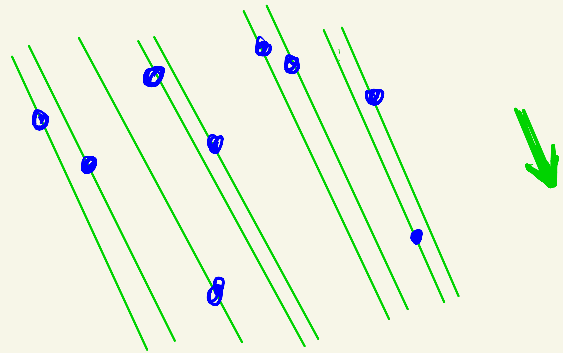
There is a compact set $E \subset \mathbb{R}^2$ of Hausdorff dimension $2-s$ and an s-dimensional set of directions such that each orthogonal projection in these directions is injective.

$\forall s \in [0,1]$

$\exists E \subset \mathbb{R}^2 \quad \dim_H E = 2-s$

$\exists D$ set of directions such that

$\mathrm{proj}_\alpha\big|_E$ is injective $\forall \alpha \in D$.



Theorem (A.M. 2024+) $\forall s \in [0,1]$

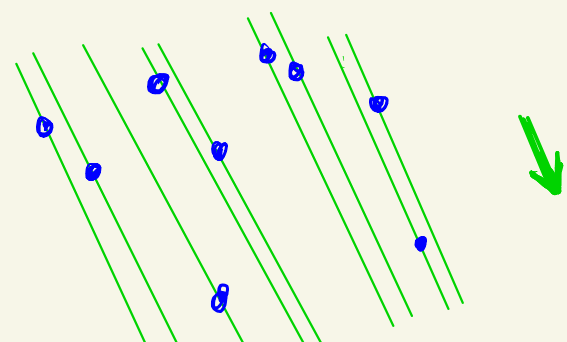
There is a compact set $E \subset \mathbb{R}^2$ of Hausdorff dimension $2-s$ and an s-dimensional set of directions such that each orthogonal projection in these directions is injective.

$\forall s \in [0,1]$

$\exists E \subset \mathbb{R}^2 \quad \dim_H E = 2-s$

$\exists D$ set of directions such that

$\mathrm{proj}_\alpha|_E$ is injective $\forall \alpha \in D$.



$\rightsquigarrow$ Exceptional set of projections.

Theorems about exceptional projections

Theorem (Kaufman)

$E \subset \mathbb{R}^2$ Borel

$\dim_H E = \underline{\underline{s}} \in [0,1]$

Except for a set of directions of dim $\underline{\underline{s}}$,

$\dim_H \mathrm{proj}_\alpha E = s.$

Theorem (Falconer)

$E \subset \mathbb{R}^2$ Borel

$\dim_H E = \underline{\underline{2-s}}$ $\quad$ $s \in [0,1)$

Except for a set of directions of dim $\underline{\underline{s}}$,

$\mathrm{proj}_\alpha E$

has positive Lebesgue measure.

Theorems about exceptional projections

Theorem (Kaufman)

$E \subset \mathbb{R}^2$ Borel

$\dim_H E = \underline{\underline{s}} \in [0,1]$

Except for a set of directions of dim $\underline{\underline{s}}$,

$\dim_H \mathrm{proj}_\alpha E = s.$

Theorem (Falconer)

$E \subset \mathbb{R}^2$ Borel

$\dim_H E = \underline{\underline{2-s}}$ $\quad s \in [0,1)$

Except for a set of directions of dim $\underline{\underline{s}}$,

$\mathrm{proj}_\alpha E$

has positive Lebesgue measure.

Theorem (Kaufman, Mattila)

And these results are sharp.

(Fractal construction based on discrete constructions by Erdős and Elekes.)

sharpness means:

Theorem (Kaufman, Mattila)

$\forall s \in [0,1]$

there is a compact

set $E \subset \mathbb{R}^2$

$\dim_H E = 2 - s - \varepsilon$

such that

$$\dim_H \operatorname{proj}_\alpha E < 1 - \varepsilon'$$

for s-dimensional many α.

Theorem (Kaufman, Mattila)

$\forall s \in [0,1]$

there is a compact set $E \subset \mathbb{R}^2$

$\dim_H E = 2 - s - \varepsilon$

such that

$$\dim_H \mathrm{proj}_\alpha E < 1 - \varepsilon'$$

for s-dimensional many α.

Could we make these projections injective as well?

Theorem (Kaufman, Mattila)

$\forall s \in [0,1]$

there is a compact set $E \subset \mathbb{R}^2$

$\dim_H E = 2 - s - \varepsilon$

such that

$$\dim_H \mathrm{proj}_\alpha E < 1 - \varepsilon'$$

for s-dimensional many α.

Could we make these projections injective as well?

Small projection

= lots of overlaps

Theorem (Kaufman, Mattila)

$\forall s \in [0,1]$

there is a compact set $E \subset \mathbb{R}^2$

$\dim_H E = 2 - s - \varepsilon$

such that

$$\dim_H \operatorname{proj}_\alpha E < 1 - \varepsilon'$$

for s-dimensional many α.

Could we make these projections injective as well?

Small projection

= lots of overlaps

Theorem (Kaufman, Mattila)

$\forall s \in [0,1]$

there is a compact

set $E \subset \mathbb{R}^2$

$\dim_H E = 2 - s - \varepsilon$

such that

$$\dim_H \mathrm{proj}_\alpha E < 1 - \varepsilon'$$

for s-dimensional many α.

Could we make these

projections injective

as well?

Small projection

= lots of overlaps

so probably not?

Theorem (Kaufman, Mattila)

$\forall s \in [0,1]$

there is a compact set $E \subset \mathbb{R}^2$

$\dim_H E = 2 - s - \varepsilon$

such that

$$\dim_H \mathrm{proj}_\alpha E < 1 - \varepsilon'$$

for s-dimensional many α.

Could we make these projections injective as well?

~~Small projection = lots of overlaps~~

~~so probably not?~~

Yes we can.

Theorem (Kaufman, Mattila)

$\forall s \in [0,1]$

there is a compact

set $E \subset \mathbb{R}^2$

$\dim_H E = 2 - s - \varepsilon$

such that

$$\dim_H \mathrm{proj}_\alpha E < 1 - \varepsilon'$$

for s-dimensional many α.

Could we make these projections injective as well?

~~Small projection = lots of overlaps~~

~~so probably not?~~

Yes we can.

Injectivity is not a quantitative property.

Theorem (Kaufman, Mattila)

$\forall s \in [0,1]$

there is a compact

set $E \subset \mathbb{R}^2$

$\dim_H E = 2 - s - \varepsilon$

such that

$$\dim_H \operatorname{proj}_\alpha E < 1 - \varepsilon'$$

for s-dimensional many α.

Theorem (A.M. 2024+)

$\forall s \in [0,1]$

there is a compact

set $E \subset \mathbb{R}^2$

$\dim_H E = 2 - s - \varepsilon$

such that

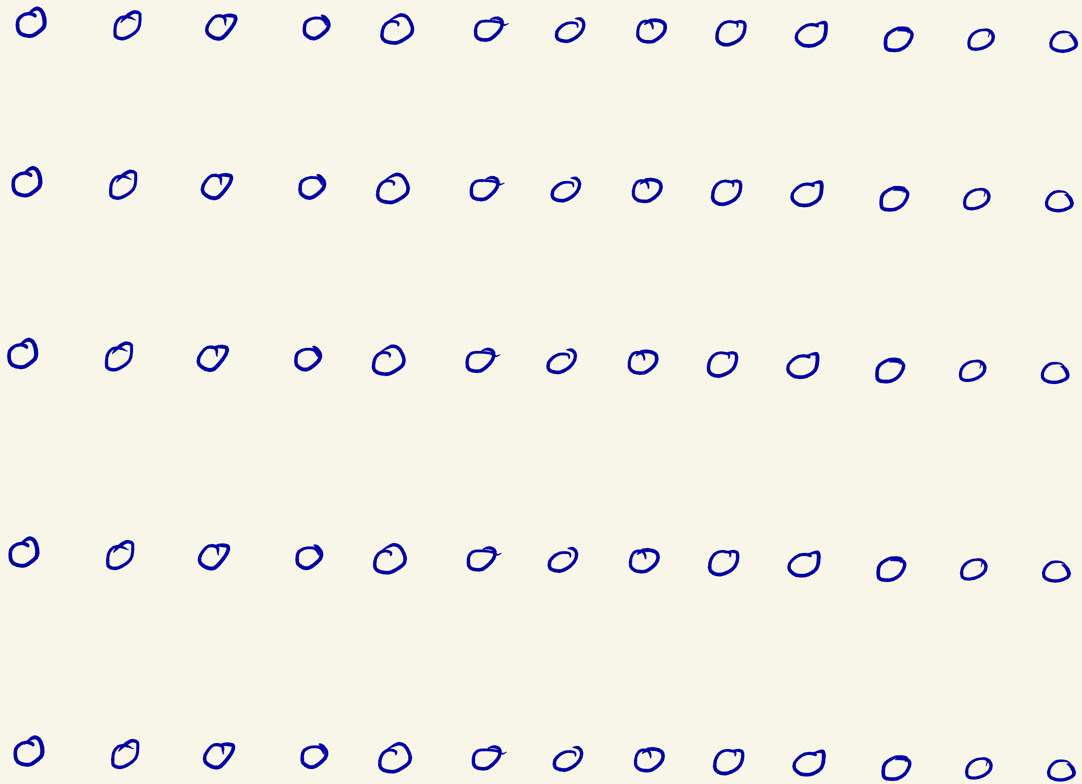
$$\dim_H \operatorname{proj}_\alpha E < 1 - \varepsilon'$$

for s-dimensional many α,

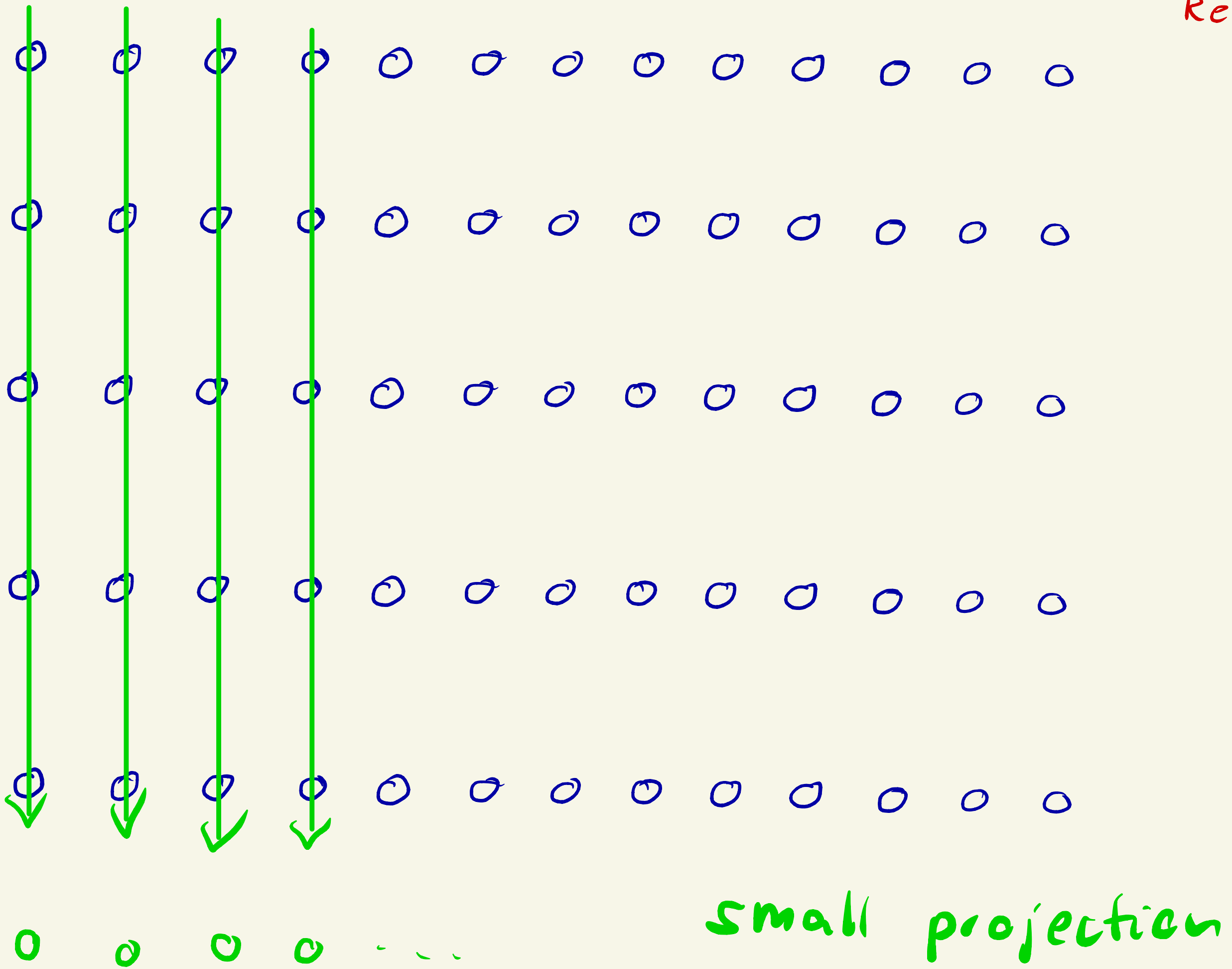
and these projections

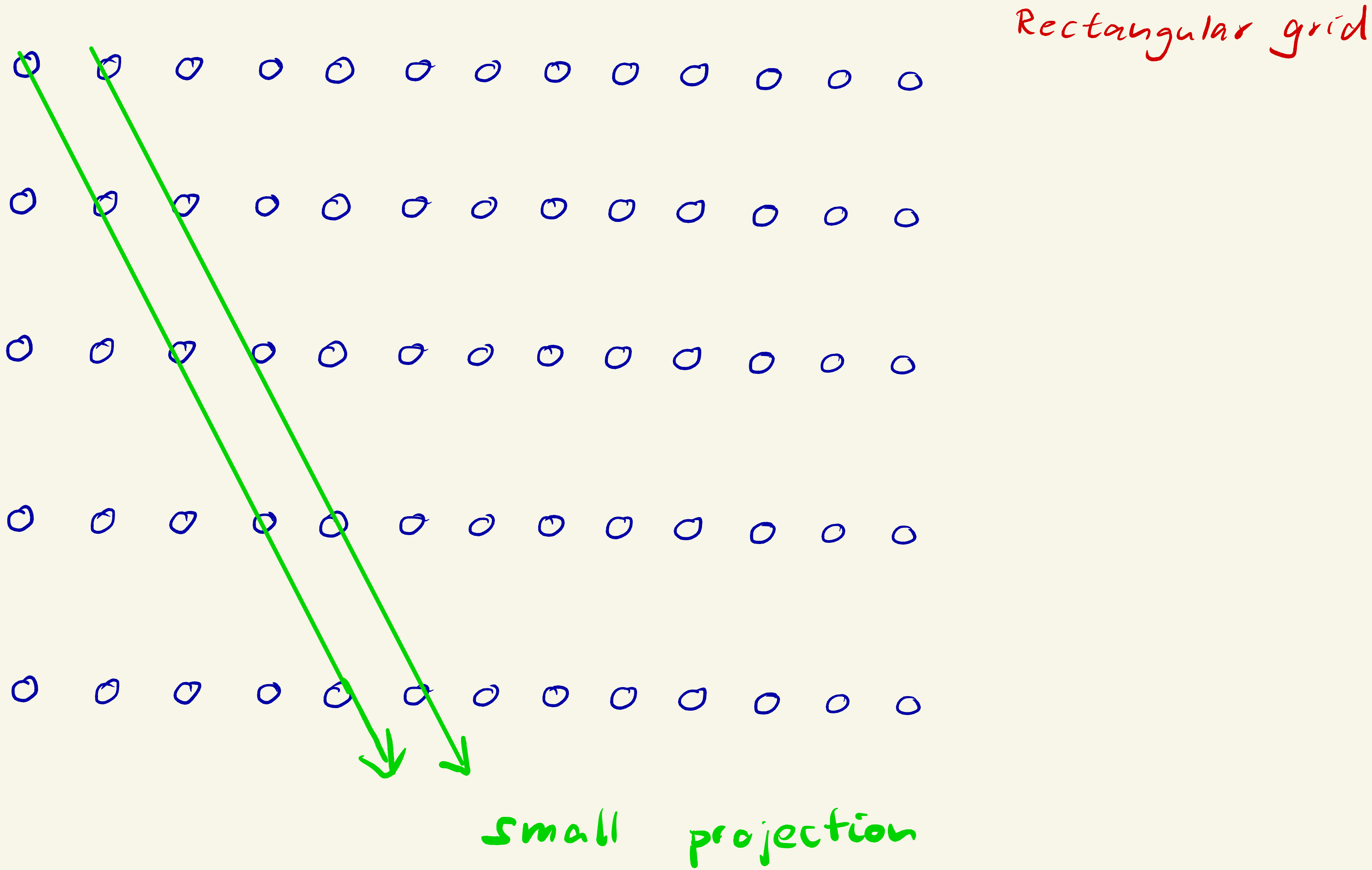
are injective.

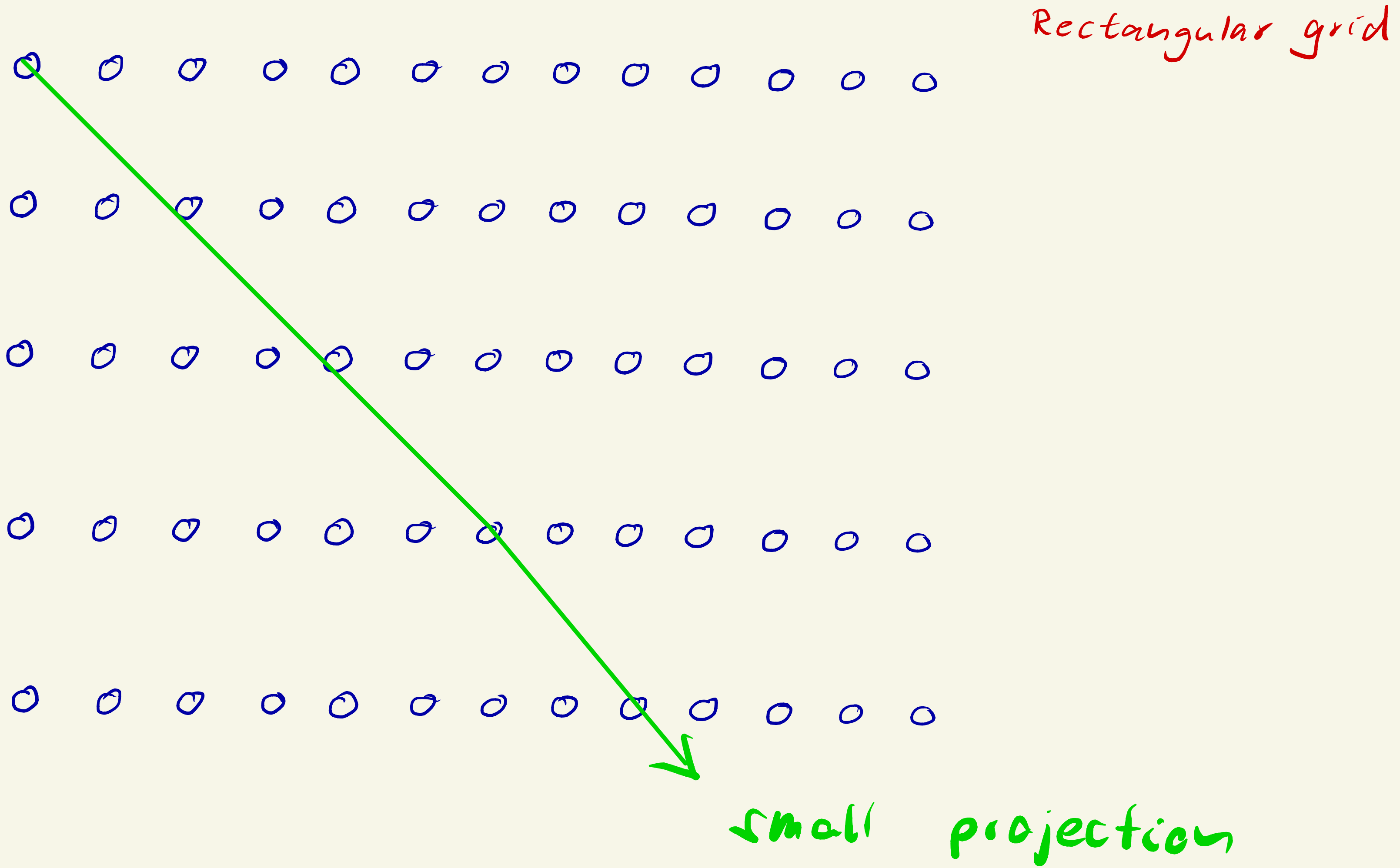
Rectangular grid



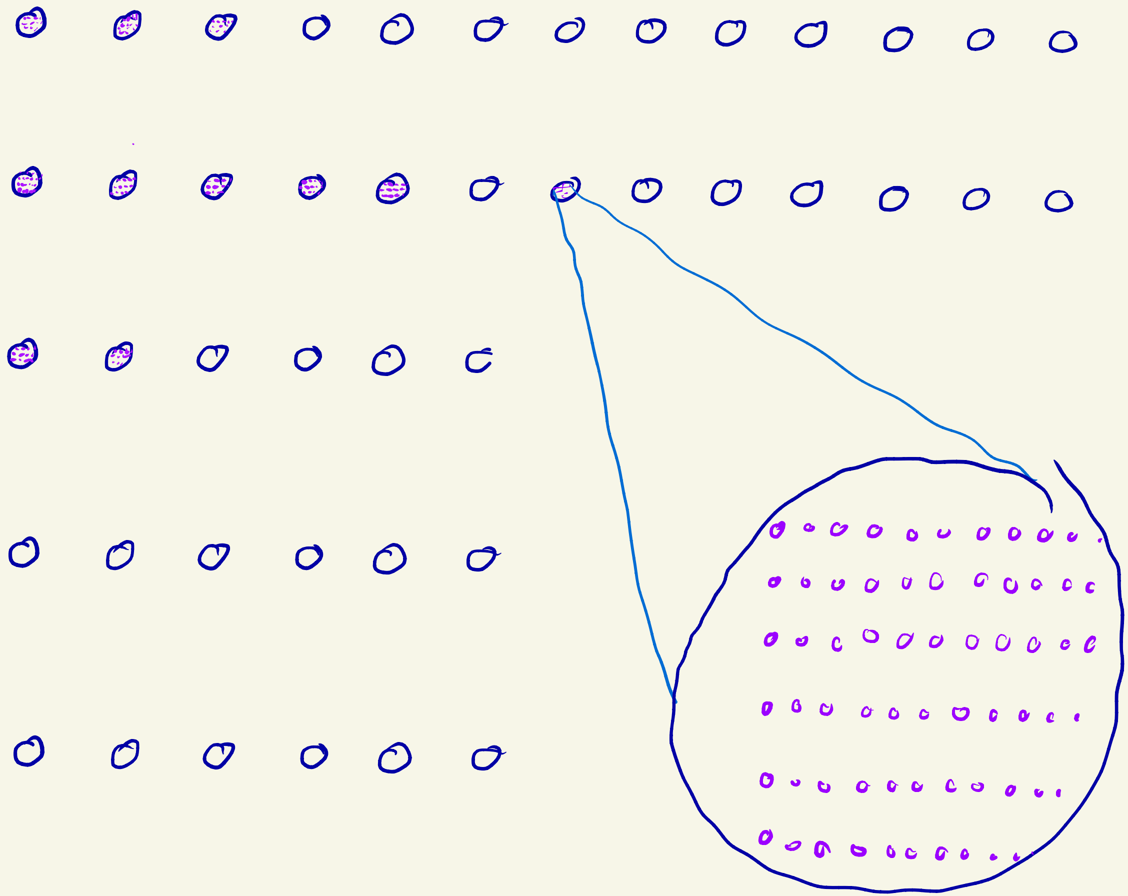
Rectangular grid




Rectangular grid
small projection


Rectangular grid
small projection

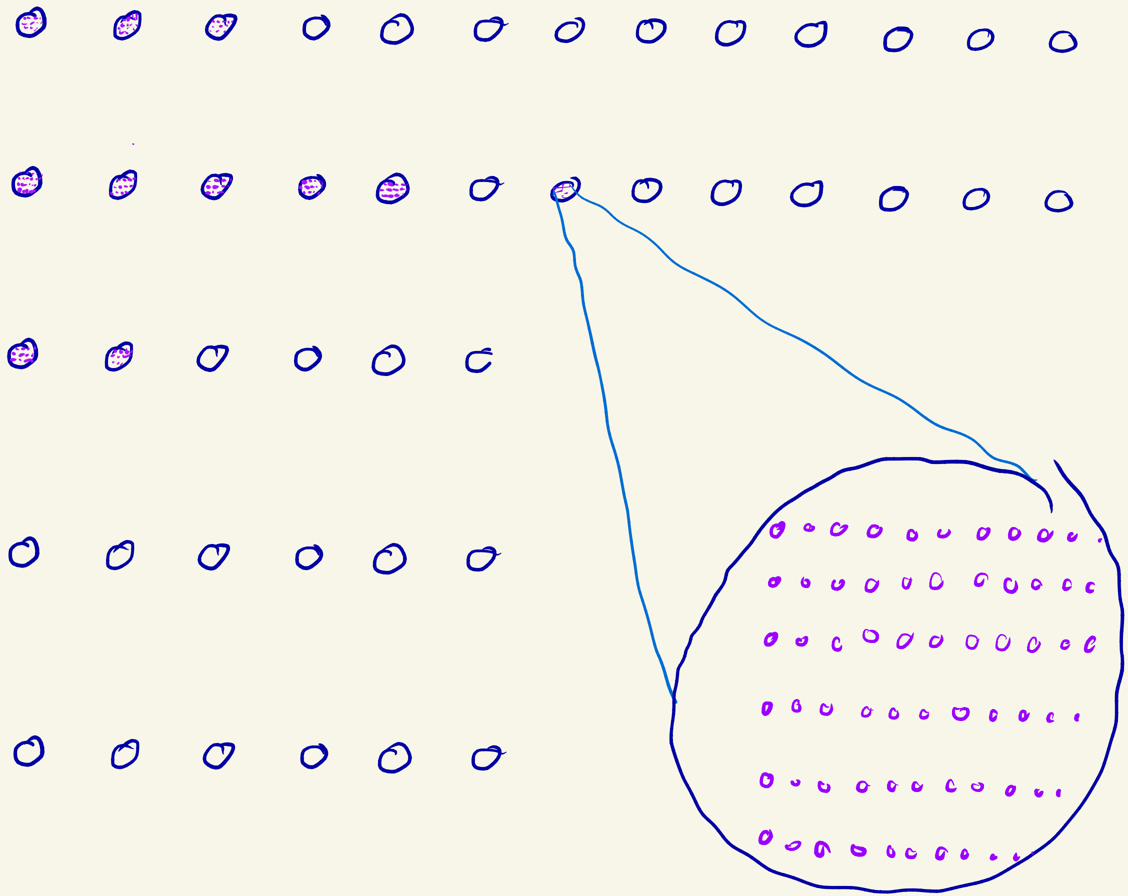
Rectangular grid



Next level:
intersect it
with a
much much
finer
similar*
rectangular
grid

*affine

Rectangular grid

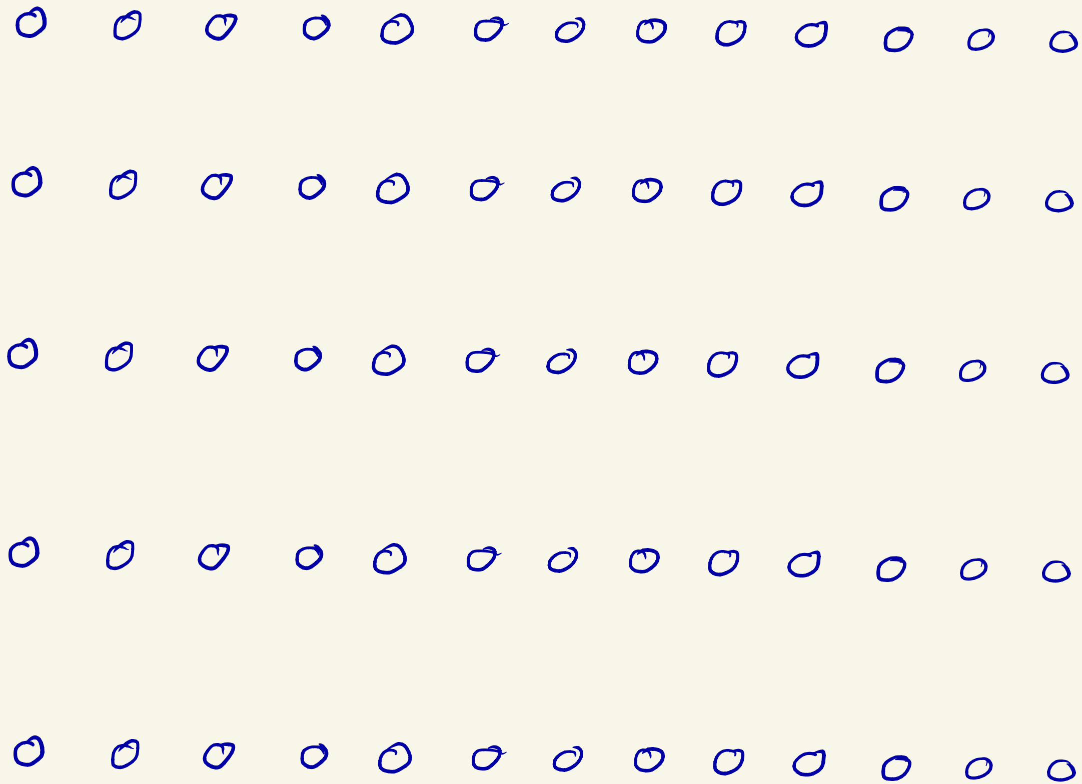


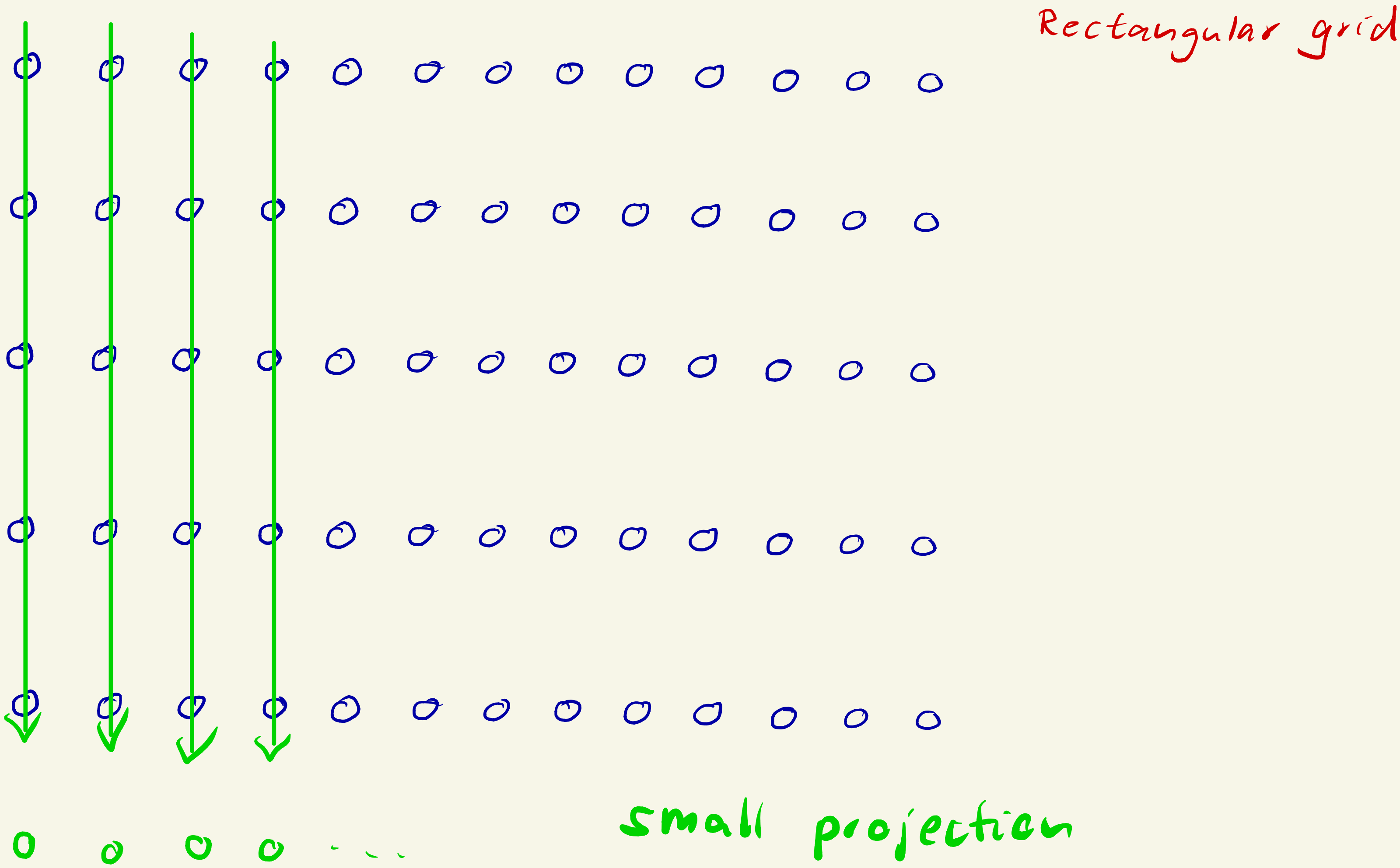
Next level:
intersect it
with a
much much
finer
similar*
rectangular
grid

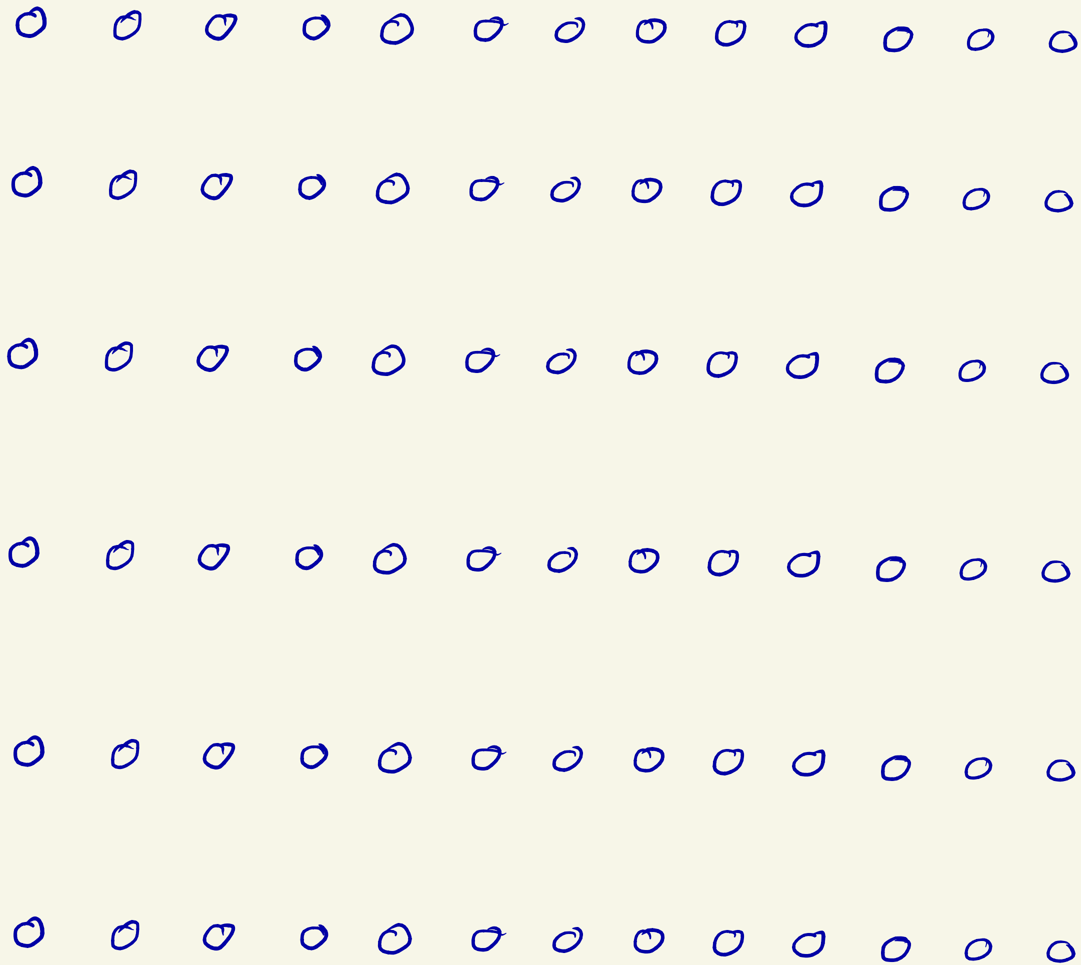
*affine

This how an Elekes-Kaufman-Mattila constr. work

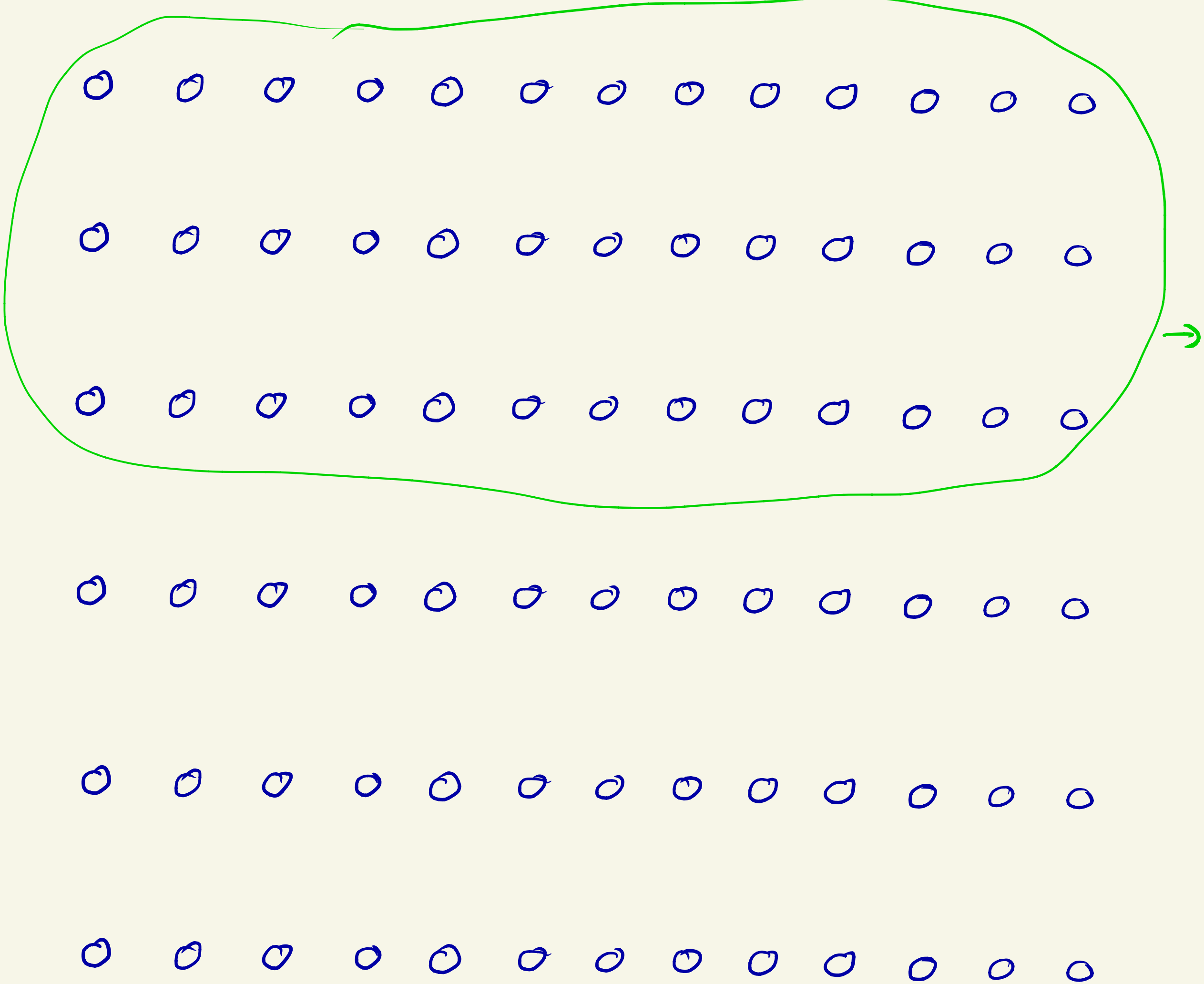
Rectangular grid



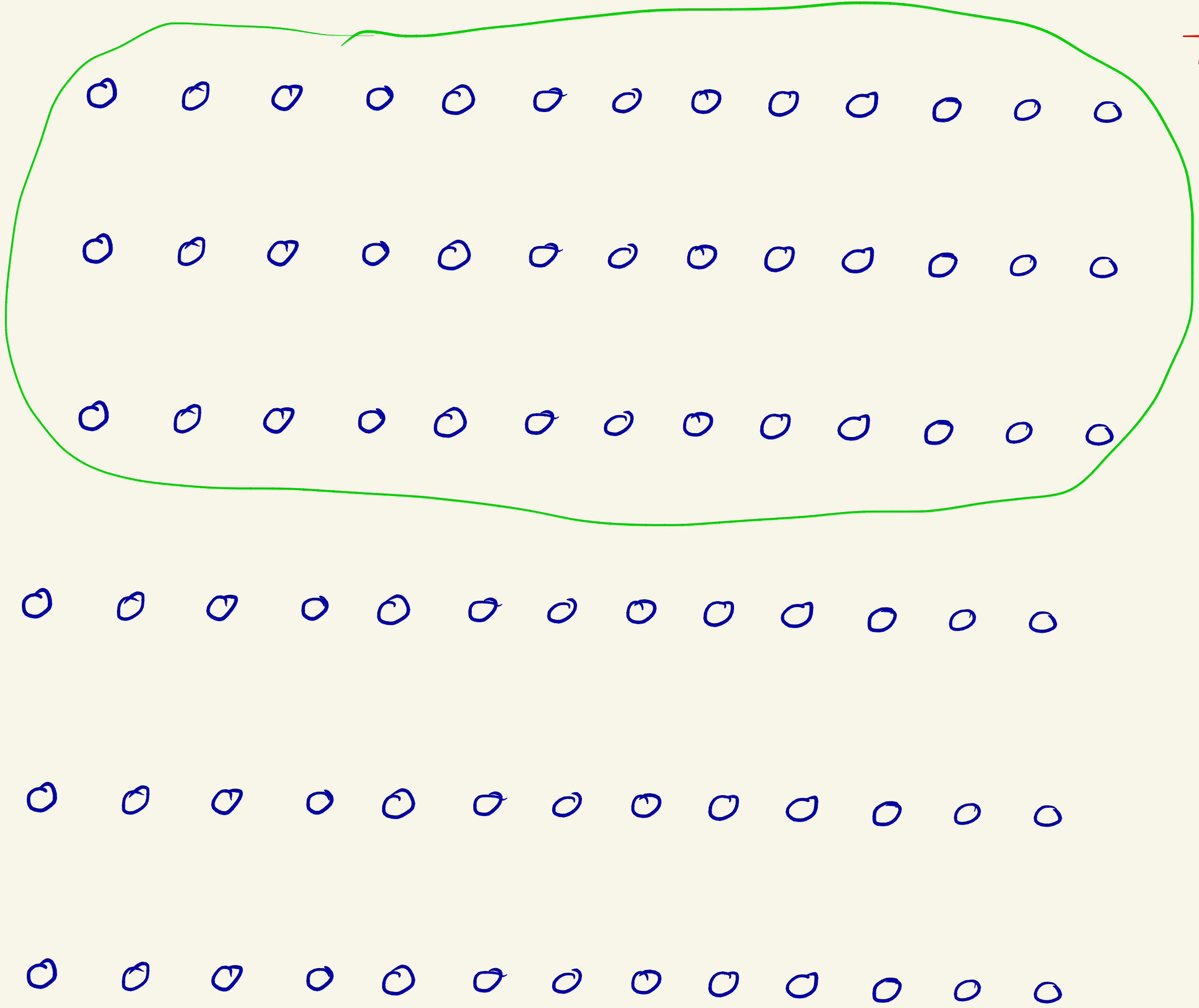

Rectangular grid
small projection



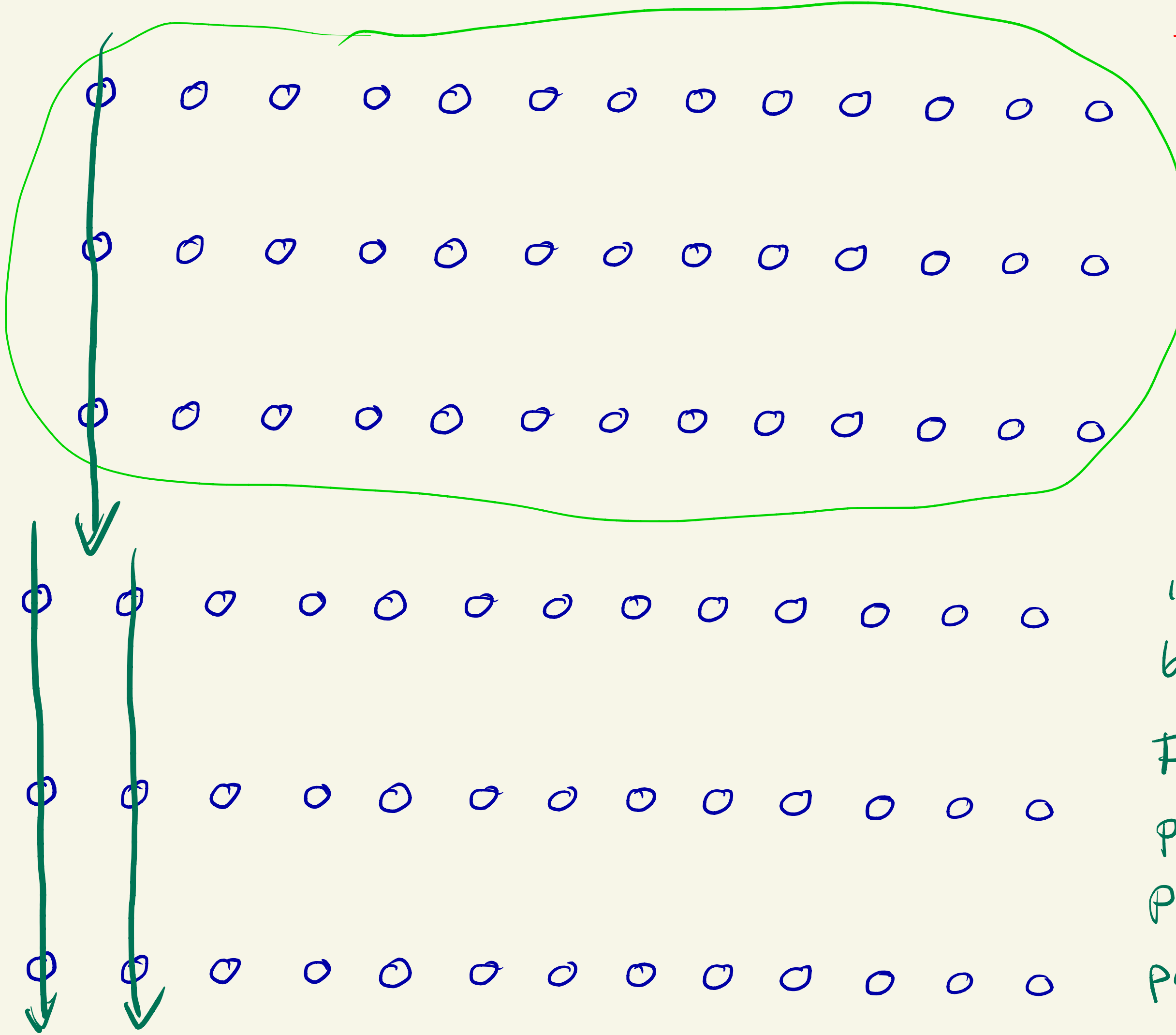
To make these projections injective...



To make these projections injective...



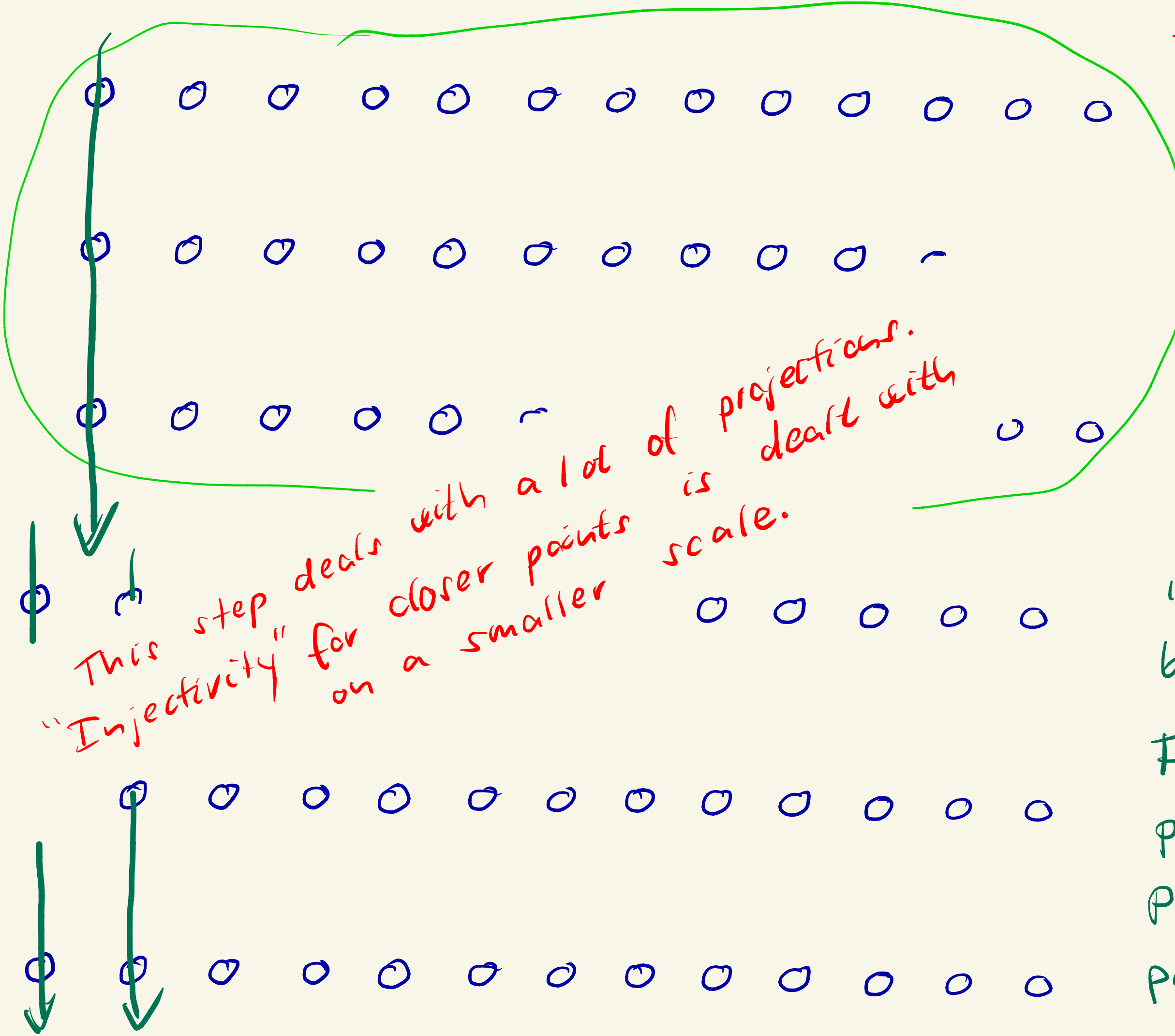
To make these projections injective...



To make these projections injective...

Projection is only "twice" bigger.

Far away points do not project to same point.



To make these projections injective...

This step deals with a lot of projections. "Injectivity" for closer points is dealt with on a smaller scale.

Projection is only "twice" bigger.

Far away points do not project to same point.

"Small projections
can be made injective"

"Small projections can be made injective"

This turns out to be useful for a very different problem as well.

European Journal of Mathematics (2018) 4:687–698
https://doi.org/10.1007/s40879-017-0199-4

RESEARCH ARTICLE

On the center of distances

Wojciech Bielas[1,2] · Szymon Plewik[1] ·
Marta Walczyńska[1,2]

Received: 22 February 2017 / Revised: 20 June 2017 / Accepted: 4 October 2017 /
Published online: 28 November 2017

Abstract We introduce the notion of a center of distances of a metric space

Topological Methods in Nonlinear Analysis
Volume 63, No. 2, 2024, 413–427
DOI: 10.12775/TMNA.2023.023

ON THE OPERATOR OF CENTER OF DISTANCES BETWEEN THE SPACES OF CLOSED SUBSETS OF THE REAL LINE

Artur Bartoszewicz — Małgorzata Filipczak
Grażyna Horbaczewska — Sebastian Lindner
Franciszek Prus-Wiśniowski

Remarks on center of distances

Małgorzata Filipczak

46TH SUMMER SYMPOSIUM IN REAL ANALYSIS
The Promised Land Symposium
June 17–21, 2024

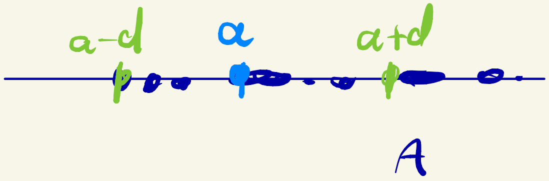
Definition

For a set $A \subset \mathbb{R}$ let

$$S(A) = \{ d \geq 0 : \forall a \in A \quad a+d \text{ or } a-d \text{ is in } A \}$$

"center of distances" of A.

Why? Why not.



European Journal of Mathematics (2018) 4:687–698
https://doi.org/10.1007/s40879-017-0199-4

CrossMark

RESEARCH ARTICLE

On the center of distances

Wojciech Bielas[1,2] · Szymon Plewik[1] ·
Marta Walczyńska[1,2]

Received: 22 February 2017 / Revised: 20 June 2017 / Accepted: 4 October 2017 /

Definition

For a set $A \subset \mathbb{R}$ let

$$S(A) = \{ d \geq 0 : \forall a \in A \ \ a+d \text{ or } a-d \text{ is in } A \}$$

"center of distances" of A.

Why? Why not.

European Journal of Mathematics (2018) 4:687–698
https://doi.org/10.1007/s40879-017-0199-4

CrossMark

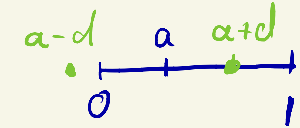
RESEARCH ARTICLE

On the center of distances

Wojciech Bielas[1,2] · Szymon Plewik[1] ·
Marta Walczyńska[1,2]

Received: 22 February 2017 / Revised: 20 June 2017 / Accepted: 4 October 2017 /

Examples



$A = [0, 1]$

$S(A) =$

Definition

For a set $A \subset \mathbb{R}$ let

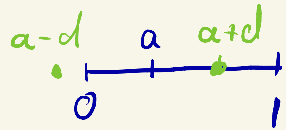
$$S(A) = \{ d \geq 0 : \forall a \in A \quad a+d \text{ or } a-d \text{ is in } A \}$$

"center of distances" of A.

Why? Why not.

Examples

- $A = [0, 1]$



$S(A) = [0, \frac{1}{2}]$

European Journal of Mathematics (2018) 4:687–698
https://doi.org/10.1007/s40879-017-0199-4

CrossMark

RESEARCH ARTICLE

On the center of distances

Wojciech Bielas[1,2] · Szymon Plewik[1] ·
Marta Walczyńska[1,2]

Received: 22 February 2017 / Revised: 20 June 2017 / Accepted: 4 October 2017 /

Definition

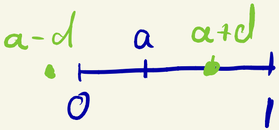
For a set $A \subset \mathbb{R}$ let

$$S(A) = \{ d \geq 0 : \forall a \in A \quad a+d \text{ or } a-d \text{ is in } A \}$$

"center of distances" of A.

Why? Why not.

Examples



- $A = [0, 1]$

  $S(A) = [0, \frac{1}{2}]$

- $A =$ middle-third Cantor set

  $\frac{2}{3} \in S(A)$?

European Journal of Mathematics (2018) 4:687–698
https://doi.org/10.1007/s40879-017-0199-4

CrossMark

RESEARCH ARTICLE

On the center of distances

Wojciech Bielas[1,2] · Szymon Plewik[1] ·
Marta Walczyńska[1,2]

Received: 22 February 2017 / Revised: 20 June 2017 / Accepted: 4 October 2017 /

Definition

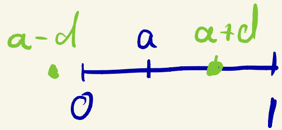
For a set $A \subset \mathbb{R}$ let

$$S(A) = \{ d \geq 0 : \forall a \in A \ \ a+d \text{ or } a-d \text{ is in } A \}$$

"center of distances" of A.

Why? Why not.

Examples



- $A = [0, 1]$

  $S(A) = [0, \frac{1}{2}]$

- $A =$ middle-third Cantor set

  $\frac{2}{3} \in S(A)$? Yes.

European Journal of Mathematics (2018) 4:687–698
https://doi.org/10.1007/s40879-017-0199-4

CrossMark

RESEARCH ARTICLE

On the center of distances

Wojciech Bielas[1,2] · Szymon Plewik[1] ·
Marta Walczyńska[1,2]

Received: 22 February 2017 / Revised: 20 June 2017 / Accepted: 4 October 2017 /

Definition

For a set $A \subset \mathbb{R}$ let

$$S(A) = \{ d \geq 0 : \forall a \in A \quad a+d \text{ or } a-d \text{ is in } A \}$$

"center of distances" of A.

Why? Why not.

European Journal of Mathematics (2018) 4:687–698
https://doi.org/10.1007/s40879-017-0199-4

CrossMark

RESEARCH ARTICLE

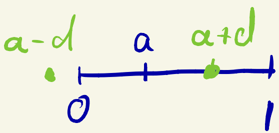
On the center of distances

Wojciech Bielas[1,2] · Szymon Plewik[1] ·
Marta Walczyńska[1,2]

Received: 22 February 2017 / Revised: 20 June 2017 / Accepted: 4 October 2017 /

Examples

- $A = [0, 1]$



$S(A) = [0, \frac{1}{2}]$

- $A =$ middle-third Cantor set

$\frac{2}{3} \in S(A)$? Yes.

$\frac{2}{3^k} \in S(A)$?

Definition

For a set $A \subset \mathbb{R}$ let

$$S(A) = \{ d \geq 0 : \forall a \in A \ \ a+d \text{ or } a-d \text{ is in } A \}$$

"center of distances" of A.

Why? Why not.

European Journal of Mathematics (2018) 4:687–698
https://doi.org/10.1007/s40879-017-0199-4

CrossMark

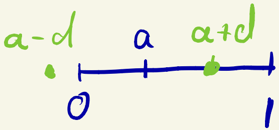
RESEARCH ARTICLE

On the center of distances

Wojciech Bielas[1,2] · **Szymon Plewik**[1] ·
Marta Walczyńska[1,2]

Received: 22 February 2017 / Revised: 20 June 2017 / Accepted: 4 October 2017 /

Examples



- $A = [0, 1]$

  $S(A) = [0, \frac{1}{2}]$

- $A =$ middle-third Cantor set

  $\frac{2}{3} \in S(A)$? Yes.

  $\frac{2}{3^k} \in S(A)$? Yes.

Definition

For a set $A \subset \mathbb{R}$ let

$$S(A) = \{ d \geq 0 : \forall a \in A \ \ a+d \text{ or } a-d \text{ is in } A \}$$

"center of distances" of A.

Why? Why not.

European Journal of Mathematics (2018) 4:687–698
https://doi.org/10.1007/s40879-017-0199-4

CrossMark

RESEARCH ARTICLE

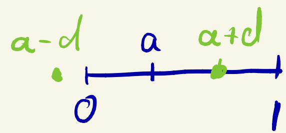
On the center of distances

Wojciech Bielas[1,2] · **Szymon Plewik**[1] · **Marta Walczyńska**[1,2]

Received: 22 February 2017 / Revised: 20 June 2017 / Accepted: 4 October 2017 /

Examples

- $A = [0, 1]$



$S(A) = [0, \frac{1}{2}]$

- $A =$ middle-third Cantor set

$\frac{2}{3} \in S(A)$? Yes.

$\frac{2}{3^k} \in S(A)$? Yes.

$S(A) = \{ \frac{2}{3^k} : k \geq 1 \} \cup \{0\}$?

Definition

For a set $A \subset \mathbb{R}$ let

$$S(A) = \{ d \geq 0 : \forall a \in A \quad a+d \text{ or } a-d \text{ is in } A \}$$

"center of distances" of A.

Why? Why not.

European Journal of Mathematics (2018) 4:687–698
https://doi.org/10.1007/s40879-017-0199-4

CrossMark

RESEARCH ARTICLE

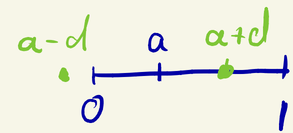
On the center of distances

Wojciech Bielas[1,2] · Szymon Plewik[1] · Marta Walczyńska[1,2]

Received: 22 February 2017 / Revised: 20 June 2017 / Accepted: 4 October 2017 /

Examples

- $A = [0, 1]$



$S(A) = [0, \frac{1}{2}]$

- $A =$ middle-third Cantor set

$\frac{2}{3} \in S(A)$? Yes.

$\frac{2}{3^k} \in S(A)$? Yes.

$S(A) = \{ \frac{2}{3^k} : k \geq 1 \} \cup \{0\}$?

Yes.

Theorem (M. Filipczak et al)

Let $\underline{B} \subset [0, \infty)$ be $\underline{\text{compact}}$,

$0 \in B$.

Then there is a $\underline{\text{closed}}$ set $A \subset [0, \infty)$ such that

$$S(A) = B.$$

Topological **M**ethods in **N**onlinear **A**nalysis
Volume 63, No. 2, 2024, 413–427
DOI: 10.12775/TMNA.2023.023

ON THE OPERATOR OF CENTER OF DISTANCES BETWEEN THE SPACES OF CLOSED SUBSETS OF THE REAL LINE

Artur Bartoszewicz — Małgorzata Filipczak
Grażyna Horbaczewska — Sebastian Lindner
Franciszek Prus-Wiśniowski

Theorem (M. Filipczak et al)

Let $B \subset [0, \infty)$ be compact, $0 \in B$.

Then there is a closed set $A \subset [0, \infty)$ such that

$$S(A) = B.$$

Topological Methods in Nonlinear Analysis
Volume 63, No. 2, 2024, 413–427
DOI: 10.12775/TMNA.2023.023

ON THE OPERATOR OF CENTER OF DISTANCES BETWEEN THE SPACES OF CLOSED SUBSETS OF THE REAL LINE

Artur Bartoszewicz — Małgorzata Filipczak
Grażyna Horbaczewska — Sebastian Lindner
Franciszek Prus-Wiśniowski

About the proof:

A is a locally finite union of intervals, dealing with the complementary intervals of B one by one.

Theorem (M. Filipczak et al)

Let $B \subset [0,\infty)$ be compact, $0 \in B$.

Then there is a closed set $A \subset [0,\infty)$ such that

$$S(A) = B.$$

Topological Methods in Nonlinear Analysis
Volume 63, No. 2, 2024, 413–427
DOI: 10.12775/TMNA.2023.023

ON THE OPERATOR OF CENTER OF DISTANCES BETWEEN THE SPACES OF CLOSED SUBSETS OF THE REAL LINE

Artur Bartoszewicz — Małgorzata Filipczak
Grażyna Horbaczewska — Sebastian Lindner
Franciszek Prus-Wiśniowski

About the proof:

A is a locally finite union of intervals, dealing with the complementary intervals of B one by one.

Question (Filipczak et al)

Is there a compact A for every such compact B? with $S(A) = B$.

The expected answer is "No".

Three questions (Filipczak et al)

Question 1

Is there a compact A for every compact $B \subset [0,\infty)$ with $0 \in B$

such that $S(A) = B$?

Three questions (Filipczak et al)

Question 1

Is there a compact A for every compact $B \subset [0,\infty)$ with $0 \in B$ such that $S(A) = B$?

Question 2

Is there a set A for every set $B \subset [0,\infty)$ with $0 \in B$ such that $S(A) = B$?

Three questions (Filipczak et al)

Question 1

Is there a compact A for every compact $B \subset [0,\infty)$ with $0 \in B$ such that $S(A) = B$?

Question 2

Is there a set A for every set $B \subset [0,\infty)$ with $0 \in B$ such that $S(A) = B$?

Question 3

Is there a Borel set A for every Borel set $B \subset [0,\infty)$ with $0 \in B$ such that $S(A) = B$?

Three questions (Filipczak et al)

Answers? by A.M 2024+

Question 1

Is there a compact A for every compact $B \subset [0,\infty)$ with $0 \in B$ such that $S(A) = B$?

No, as expected.

Question 2

Is there a set A for every set $B \subset [0,\infty)$ with $0 \in B$ such that $S(A) = B$?

Question 3

Is there a Borel set A for every Borel set $B \subset [0,\infty)$ with $0 \in B$ such that $S(A) = B$?

Three questions (Filipczak et al) Answers? by A.M 2024+

Question 1

Is there a compact A for every compact $B \subset [0,\infty)$ with $0 \in B$ such that $S(A) = B$?

No, as expected.

Question 2

Is there a set A for every set $B \subset [0,\infty)$ with $0 \in B$ such that $S(A) = B$?

Question 3

Is there a Borel set A for every Borel set $B \subset [0,\infty)$ with $0 \in B$ such that $S(A) = B$?

Three questions (Filipczak et al) Answers? by A.M 2024+

Question 1

Is there a compact A for every compact $B \subset [0,\infty)$ with $0 \in B$ such that $S(A) = B$?

No, as expected.

Question 2

Is there a set A for every set $B \subset [0,\infty)$ with $0 \in B$ such that $S(A) = B$?

Yes!

Question 3

Is there a Borel set A for every Borel set $B \subset [0,\infty)$ with $0 \in B$ such that $S(A) = B$?

Three questions (Filipczak et al) Answers? by A.M 2024+

Question 1

Is there a compact A for every compact $B \subset [0,\infty)$ with $0 \in B$ such that $S(A) = B$?

No, as expected.

Question 2

Is there a set A for every set $B \subset [0,\infty)$ with $0 \in B$ such that $S(A) = B$?

Yes!

Question 3

Is there a Borel set A for every Borel set $B \subset [0,\infty)$ with $0 \in B$ such that $S(A) = B$?

Yes!

Solution to Q1 is real analysis.

Solution to Q2 is transfinite recursion.

Solution to Q3 is by replacing the Axiom of Choice in the transfinite recursion with "choice" based on a fractal set....

$\forall$ compact B $\exists$ compact A $S(A)=B$

$\forall B \ \exists A \ S(A)=B$

$\forall$ Borel B $\exists$ Borel A $S(A)=B$.

Solution to Q1 is real analysis.

$\forall$ compact B $\exists$ compact A $S(A)=B$

Solution to Q2 is transfinite recursion.

$\forall B \; \exists A \; S(A)=B$

$\forall$ Borel B $\exists$ Borel A $S(A)=B$.

Solution to Q3 is by replacing the Axiom of Choice in the transfinite recursion with "choice" based on a fractal set....

... with lots of injective projections.

Q3 is solved by the following.

Thm (A.M 2024+)

There exists a closed set $E \subset \mathbb{R}^2$ such that

- $\mathrm{proj}_x E = \mathbb{R}$
- all other rational projections are injective
- moreover, "lots of linear independence over $\mathbb{Q}$" the linear equations
$$\sum_{i=1}^{n} \alpha_i x_i + \beta_i y_i = 0 \quad \text{with } \alpha_i \in \mathbb{Q},\ \beta_i \in \mathbb{Q} \setminus \{0\}$$
have NO SOLUTIONS among distinct points $(x_i, y_i) \in E$.

Q3 is solved by the following.

Thm (A.M 2024+)

There exists a closed set $E \subset \mathbb{R}^2$ such that

- $\mathrm{proj}_x E = \mathbb{R}$
- all other rational projections are injective
- moreover, "lots of linear independence over $\mathbb{Q}$"
  the linear equations
  $$\sum_{i=1}^{n} \alpha_i x_i + \beta_i y_i = 0 \quad \text{with } \alpha_i \in \mathbb{Q},\ \beta_i \in \mathbb{Q} \setminus \{0\}$$
have NO SOLUTIONS among distinct points $(x_i, y_i) \in E$.

Using E,

given $B \subset [0, \infty)$

let

$$A = \mathbb{R} \setminus \bigcup \left\{ \begin{array}{l} \text{certain} \\ \text{linear} \\ \text{images} \\ \text{of} \\ E \cap (B^c \times \mathbb{R}) \end{array} \right\}$$