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Estimation of the stochastic leverage effect using the Fourier transform method

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Abstract

We define a non-parametric estimator of the integrated leverage effect as the integrated covariation between the logarithmic asset price and its volatility. In Curato and Sanfelici (2015), a consistent estimator of the leverage effect has been introduced through a pre-estimate of the Fourier coefficients of the volatility. This is a novel approach compared to the ones present in the literature which use a pre-estimate of the spot volatility path. In this paper, we show the asymptotic normality of the Fourier estimator for non-equidistant observations. Moreover, its finite sample properties are analyzed in a simulation study also in the presence of microstructure noise.

MSC: 60F05; 60H05

Keywords: Fourier analysis; Continuous semi-martingales; Leverage effect; Non-parametric estimation techniques

1. Introduction

The leverage effect is defined as the correlation between financial asset returns and the change of their volatilities. Dating back to the seminal papers of Black [9] and Christie [15], this effect has been related to the so called financial leverage: as asset prices decline, companies are more leveraged since the relative value of their debt rises relative to that of their equity causing the assets to become more volatile. Therefore, the financial leverage implies a negative correlation – i.e. leverage effect – between the analyzed asset returns and the change of their own volatility.

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This fact has been incorporated in classical stochastic volatility models as in [6,7,23] where the correlation between the processes driving the logarithmic price and the volatility is modeled by a negative constant parameter. However, the validity of this model assumption has been questioned in several works as well as the dependence of the leverage effect, solely, on the financial leverage—as discussed in [2]. First, it has been documented that the effect is not constant, but itself evolves in time [10,35] and there may be important asymmetries in the way in which the volatility responds to price changes [4], i.e. in the presence of positive shocks (positive return) the volatility may not change or even increase. Moreover, in [13], the authors consider a random correlation parameter between the processes driving the logarithmic price and the volatility in order to model the stochastic skew observed in the currency option data sets. All the above facts motivated the growth of sophisticated mathematical models in which the aforementioned correlation is modeled as a time varying function or, more generally, as a stochastic process itself.

Several authors have proposed non-parametric procedures for estimating the integrated covariation between the logarithmic price and its corresponding volatility, the so called integrated leverage effect, in an Itô semimartingale set-up. The logarithmic price and the volatility are modeled as continuous processes in [8,30] and as semimartingales with jumps, in price and volatility, in [1,4,17]. In [4], the leverage is modeled as a time varying function whereas in subsequent publications, [1,8,17,30], the effect is considered stochastic. The common feature of these estimators is the use of a pre-estimate of the spot volatility in the definition of the integrated covariation by means of different techniques—Fourier transform method [8,17] or local averages of integrated volatility estimators as in [1,4,30]. When high-frequency data are employed, the estimation error due to the latency of the volatility path affects the estimates and bias corrections are typically employed in order to obtain asymptotically unbiased estimators.

In this paper, we propose a different methodology for the estimation of the integrated leverage effect. We model the logarithmic price p and the volatility ν by means of two continuous Itô semimartingales correlated by means of a stochastic process ρ. We do not assume any specific functional form of the volatility, of the variance of the volatility and of the correlation processes. As first shown in Malliavin and Mancino [26,27], the Fourier coefficients of the latent volatility process can be expressed via the Bohr convolution of the Fourier coefficients of the return process. This strategy allows to handle non-equidistant observations of the price and microstructure noise contamination and easily provide an estimator of the spot volatility path by means of the Fourier–Féjer inversion formula. In [8], the Fourier spot volatility estimator is employed in the definition of the integrated leverage effect. However, the asymptotic normality of the latter has not yet been proved. In [19], a new consistent estimator of the integrated leverage effect is introduced modifying the one given in [8]. The Fourier methodology allows to treat the estimation error due to the latency of the volatility process in a novel way by defining an estimation strategy only in the frequency domain. In fact, a pre-estimate of the volatility path is not necessary in order to obtain the estimates. The latency of the volatility, in the estimation of the integrated covariation, can be handled by computing N Fourier coefficients of the volatility process. This is a step that requires the preliminary computation of M Fourier coefficients of the return process. The parameters M and N are called cutting frequencies. In a discrete time framework, let the logarithmic price p be observed on a grid $\mathcal{S}_n := \{0 = t_{0,n} \leq t_{1,n} \leq \cdots \leq t_{k_n,n} = T\}$, for all $i = 0, \ldots, k_n$ and $k_n \leq n$, and define $\tau(n) := \max_{i=0,\ldots,k_n-1} |t_{i+1,n} - t_{i,n}|$. If the asymptotic ratios $N^3/M \to 0$ and $M\tau(n) \to a$, with $a > 0$ as $N, M, n \to \infty$ and $\tau(n) \to 0$ are satisfied, we can prove a central limit theorem for the integrated leverage effect estimator initially presented in [19]. The above ratios play a fundamental role in the theorem and in the finite sample properties of the estimator, as discussed in detail in Sections 3 and 5.

Due to the different modeling set-ups assumed by the authors in [1,4,8,17,30], comparing different estimators of the integrated leverage effect is difficult. The estimators in [1,30] are the most similar to the one defined in this work. However, they do not allow to consider a general specification of the stochastic correlation between the Itô semimartingales p and ν, respectively, the logarithmic price and the volatility process, as the model set-up presented in Section 2 does. In the case of equidistant observations, a rate of convergence of $n^{\frac{1}{4}}$ for the estimation error, can be reached in [1,30]. Moreover in [30], when non-equidistant observations of the price process are considered or microstructure noise contamination is present, a central limit theorem is obtained respectively under specific assumptions on $\mathcal{S}_n$ – the latter is a typical framework under which central limit theorems based on realized volatility estimates hold [5] – or additive microstructure noise. Although, there is no formal proof for this so far, the rate $n^{\frac{1}{4}}$ is probably the optimal rate of convergence for the integrated leverage effect, as discussed in [3, Chapter 8, Theorem 8.14.]. This said, the central limit theorem here presented achieves a rate less than $1/6$ when the ratio between the cutting frequencies N and M is optimally chosen. We show that this is independent of the structure of the time grid $\mathcal{S}_n$, the parameter a and depends on the L_2-norm of the Dirichlet kernel involved in the determination of the Fourier coefficients of the return and the volatility processes.

The Fourier estimator can be used to estimate the integrated leverage effect under very general model specification as for example the Generalized Heston model, presented in [34]. This is a continuous stochastic volatility model where three independent Brownian motions drive the dynamics of the logarithmic price and the volatility processes. To conclude, we then test the finite sample performances of the Fourier estimator on logarithmic price and volatility paths, drawn by the above model and the classical Heston model [23], on not-equidistant time grids and in the presence of additive microstructure noise contamination. An optimal selection rule for the cutting frequencies M and N is given for Monte Carlo data based on the minimization of the mean squared error of the estimate. In fact, being the mean squared error the sum of the squared bias and the variance, it constitutes a suitable criterion to select the cutting frequencies in the finite sample.

The paper is organized as follows. The model setting is carefully described in Section 2. In Section 3, we define the Fourier estimator of the integrated leverage effect and its asymptotic properties. In Section 4, the detailed proof of the central limit theorem is given. In Section 5, we present a simulation analysis to test the finite sample properties of the estimator and a selection rule for the cutting frequencies. Section 6 concludes.

2. Model setting

Suppose that $W(t)$, $t \geq 0$, and $Z(t)$, $t \geq 0$, are two correlated standard Brownian motions defined on the complete probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$. That is, a canonical probability space where $\Omega = \mathcal{C}_0(\mathbb{R}_+)$, and $\mathcal{F} = (\mathcal{F}_t)$ is the usual augmentation of the natural filtration generated by W and Z. The correlation process is defined as $\rho(t)$ with values in $[-1, 1]$ such that $\langle dW(t), dZ(t)\rangle = \rho(t)dt$. The temporal window, in which our analysis is performed, is $[0, 2\pi]$ in order to lighten the notations in what follows. However, by rescaling the unit of time all the results apply on a general interval $[0, T]$.

The logarithmic price and the volatility processes are defined as solutions of the system of equations

$$\begin{cases} dp(t) &= a(t)\,dt + \sigma(t)\,dW(t) \\ d\nu(t) &= b(t)\,dt + \gamma(t)\,dZ(t) \end{cases} \tag{1}$$

where $\nu(t) = \sigma^2(t)$ is the process we call volatility throughout the paper. The processes that appear in (1) satisfy the following assumptions:

H1. $a(t), b(t), \sigma(t), \gamma(t)$ and $\rho(t)$ are $\mathbb{R}$-valued processes, almost surely continuous on $[0, 2\pi]$ and adapted to the filtration $\mathcal{F}$ such that

$$\mathbb{E}\Big[\sup_{t\in[0,2\pi]} |a(t)|^4\Big] < \infty, \quad \mathbb{E}\Big[\sup_{t\in[0,2\pi]} |b(t)|^4\Big] < \infty,$$

$$\mathbb{E}\Big[\sup_{t\in[0,2\pi]} |\sigma(t)|^4\Big] < \infty, \quad \mathbb{E}\Big[\sup_{t\in[0,2\pi]} |\gamma(t)|^4\Big] < \infty,$$

$$\mathbb{E}\Big[\sup_{t\in[0,2\pi]} |\rho(t)|^4\Big] < \infty.$$

H2. Let $\mathbb{D}^{1,p}$ be the space of $\mathbb{R}$-valued measurable and adapted processes admitting a first order Malliavin derivative $\mathcal{D}$ that is p-integrable. We define $\mathbb{D}^{1,\infty} = \bigcap_{p\geq 1}\mathbb{D}^{1,p}$. Then, the processes $a(t), b(t), \sigma(t), \gamma(t) \in \mathbb{D}^{1,\infty}$ and $\forall p \geq 1$

$$\mathbb{E}\Big[\sup_{s,t\in[0,2\pi]} \Big|\mathcal{D}_s a(t)\Big|^p\Big] < \infty, \quad \mathbb{E}\Big[\sup_{s,t\in[0,2\pi]} \Big|\mathcal{D}_s b(t)\Big|^p\Big] < \infty,$$

$$\mathbb{E}\Big[\sup_{s,t\in[0,2\pi]} \Big|\mathcal{D}_s \sigma(t)\Big|^p\Big] < \infty, \quad \mathbb{E}\Big[\sup_{s,t\in[0,2\pi]} \Big|\mathcal{D}_s \gamma(t)\Big|^p\Big] < \infty.$$

We refer the reader to [31, Section 1.5] for further details regarding the construction of the space $\mathbb{D}^{1,\infty}$ and to [31] for the basic theory of Malliavin calculus.

We denote by (H) the ensemble of all the above assumptions. Model (1) describes the dynamics of an underlying efficient price process in the absence of market frictions. The parametric models, e.g. Heston, CEV, and the Generalized Heston model defined in [34] satisfy our assumptions.

Remark 1. Assumption (H2) is clearly satisfied for diffusion processes having globally Lipschitz coefficients with linear growth [31, Theorem 2.2.1]. Moreover, Feller diffusions – as in the Heston model – satisfy Assumption (H2), [21]. For example,

$$d\nu(t) = \kappa(\beta - \nu(t))dt + \chi\sqrt{\nu(t)}dW(t),$$

assuming that $2\kappa\beta \geq \chi^2$, admits a solution $\nu(t) \in \mathbb{D}^{1,\infty}$.

The leverage process $\eta(t)$ is defined by means of the covariation between the returns and the increments of the volatility process as

$$\langle dp(t), d\nu(t)\rangle = \eta(t)dt. \tag{2}$$

We are interested in estimating the integrated covariation between the logarithmic price p and the volatility process ν

$$\hat{\eta} = \int_0^{2\pi} \eta(t)dt. \tag{3}$$

Remark 2. In [1], the authors work on an underlying model that admits jumps in the logarithmic price and the volatility dynamics, see [1, Assumption (H)]. In the continuous case, the estimator in [1] can still be used but at the cost of more restrictive assumptions on the volatility process than in our Assumptions (H). A more careful comparison can be made with the results in [30]. Here, $a(t)$, $b(t)$ and $\gamma(t)$ are assumed to be locally bounded in absolute value and $\sigma(t)$, in particular, locally bounded away from zero. However, a stochastic correlation process $\rho(t)$, as for example in the Generalized Heston model [34], cannot be defined in the model set-up described in [30], see [30, Appendix A] therein for more details on the filtration on which the processes are considered adapted. To conclude, it is important to notice that a different underlying filtration, as long as it contains the driving processes W and Z, can be chosen to include the model set-up we are interested in studying. However, we choose this specific filtration in view of the use of Malliavin calculus that is pivotal in the proof of Theorem 3.

3. The Fourier estimator of the integrated leverage effect

3.1. *Definition of the Fourier coefficients*

In this section, we define the estimators of the Fourier coefficients of the leverage process and their statistical properties.

Following [26], we define the Fourier coefficients of the returns and of the increments of the volatility process as

$$c(l;dp)=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}lt}\,dp(t), \tag{4}$$

and

$$c(l;dv)=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}lt}\,dv(t), \tag{5}$$

for each $l\in\mathbb{Z}$.

Given two functions Φ and Ψ on the integers $\mathbb{Z}$, we say that the Bohr convolution product exists if the following limit exists for all integers h

$$(\Phi*\Psi)(h):=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{|l|\le N}\Phi(l)\Psi(h-l).$$

Under Assumptions (H), let $(p(t),\nu(t))$ be a solution of system (1). For a fixed h, defining $\Phi(l):=c(l;dv)$ and $\Psi(h-l):=c(h-l,dp)$, the limit in probability of the Bohr convolution product exists and converges to the hth Fourier coefficient of the leverage process. This result is shown in [27, Theorem 2.1] in the case of the covariance process. The hth Fourier coefficient of $\eta(t)$ is

$$c(h;\eta)=\lim_{N\to\infty}\frac{2\pi}{2N+1}\sum_{|l|\le N}c(l;dv)c(h-l;dp)=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}ht}\eta(t)dt. \tag{6}$$

The above identity has the obvious drawback to be feasible only when continuous observations of the logarithmic price and the volatility process are available. Therefore, we have two bottlenecks to overcome, namely, the latency of the volatility process and the availability of discrete observations of the logarithmic price. We present below an estimation procedure in which two errors – that respectively allow to measure the impact of the two bottlenecks – arise. We call them, respectively, truncation error and discretization error.

Let us assume, first, that we can observe continuously the logarithmic price and that the volatility process is latent.

Let $D_N(t)$ denote the normalized Dirichlet kernel defined by

$$D_N(t) = \frac{1}{2N+1} \sum_{|l| \le N} \mathrm{e}^{\mathrm{i}lt} \tag{7}$$

and $D'_N(t)$ its first derivative

$$D'_N(t) = \frac{1}{2N+1} \sum_{|l| \le N} il\mathrm{e}^{\mathrm{i}lt}. \tag{8}$$

Remark 3. In the Fourier analysis, the properties of the first derivative of the Dirichlet kernel are of interest when the summability of the first derivative of a classical Fourier series expansion – for deterministic and periodic functions – is investigated, for an overview on the topic see [33]. When the Bohr convolution product is used to define the Fourier coefficients (6), the first derivative of the Dirichlet kernel implicitly appears in their definition.

We recall some useful properties of the normalized Dirichlet kernel. The proof of the results below is straightforward and we omit it.

Proposition 1. *Let $D_N(t)$ be the normalized Dirichlet kernel defined in (7), then the following properties are satisfied.*

1. $\int_0^{2\pi} |D_N(u)|^2 \, du = \frac{2\pi}{2N+1}$,
2. $\forall p > 1$, *there exists a constant $\mathcal{C}_p$ such that* $\int_0^{2\pi} |D_N(u)|^p \, du = \frac{\mathcal{C}_p}{2N+1}$.

For all $l \neq 0$, by means of the use of the integration by parts formula, we have that

$$c(l; d\nu) = \mathrm{i}lc(l; \nu) + \frac{1}{2\pi}(\nu(2\pi) - \nu(0)), \tag{9}$$

where

$$c(l; \nu) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}lt} \nu(t) dt.$$

Therefore, the limit (6) becomes

$$\begin{aligned} c(h; \eta) &= \lim_{N \to \infty} \frac{2\pi}{2N+1} \sum_{|l| \le N} (\mathrm{i}lc(l; \nu) + \frac{1}{2\pi}(\nu(2\pi) - \nu(0)))c(h-l; dp) \qquad (10) \\ &= \lim_{N \to \infty} \frac{2\pi}{2N+1} \sum_{|l| \le N} \mathrm{i}lc(l; \nu)c(h-l; dp) \\ &\quad + \frac{2\pi}{2N+1} \sum_{|l| \le N} \frac{1}{2\pi}(\nu(2\pi) - \nu(0))c(h-l; dp) \\ &= \lim_{N \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}ht} D'_N(t-s)\nu(s) \, ds \, dp(t) \\ &\quad + \frac{1}{2\pi} \int_0^{2\pi} (\nu(2\pi) - \nu(0))\mathrm{e}^{-\mathrm{i}ht} D_N(t) \, dp(t). \end{aligned}$$

The second summand converges to 0 in probability as N converges to infinity. In fact, by applying the Itô isometry and the Cauchy–Schwarz inequality

$$\mathbb{E}\Big[\Big|\frac{1}{2\pi}\int_0^{2\pi}(\nu(2\pi)-\nu(0))\mathrm{e}^{-\mathrm{i}ht}D_N(t)\,dp(t)\Big|^2\Big] \leq C\frac{2\pi}{2N+1}$$

because of Proposition 1 and Assumption (H1), where C is a constant independent of N.

Thus, when the volatility is a latent process

$$c(h;\eta) = \lim_{N\to\infty}\frac{2\pi}{2N+1}\sum_{|l|\leq N}\mathrm{i}lc(l;\nu)c(h-l;dp) \tag{11}$$

Formula (11) can also be interpreted as subtracting $c(0,\nu)$ – the 0th Fourier coefficient of the volatility – from the Fourier coefficients $c(l;d\nu)$ defined in (9) for each $l\neq 0$. In order to construct a feasible estimation procedure for the hth Fourier coefficient of the leverage process, we consider the truncation of the limit in (11). Thus,

$$c_N(h;\eta) = \frac{2\pi}{2N+1}\sum_{|l|\leq N}\mathrm{i}lc(l;\nu)c(h-l;dp) \tag{12}$$

in which only the Fourier coefficients of the return and volatility process appear. In [8], the estimation procedure is described starting from Eq. (10). With respect to this approach, we have the fundamental advantage to require only the knowledge of the Fourier coefficients of the volatility process. Therefore, the error due to the estimation of a spot volatility path can be overcome defining an estimation strategy only in the frequency domain.

We can now assume to observe $p(t)$ on a discrete non-equidistant time grid, a step that adds to the procedure a discretization error. Let

$$\mathcal{S}_n := \{0 = t_{0,n} \leq t_{1,n} \leq \cdots \leq t_{k_n,n} = 2\pi\}, \quad \text{for all } i = 0,\ldots,k_n \text{ and } k_n \leq n.$$

We define $\tau(n) := \max_{i=0,\ldots,k_n-1}|t_{i+1,n}-t_{i,n}|$ and the discrete observed return as $\delta_{i,n}(p) = p(t_{i+1,n})-p(t_{i,n})$ for all $i = 0,\ldots,k_n-1$. By means of the classical definition of the discrete Fourier transform, we estimate $c(s;dp)$ as

$$c_n(s;dp) = \frac{1}{2\pi}\sum_{i=0}^{k_n-1}\mathrm{e}^{-\mathrm{i}st_{i,n}}\delta_{i,n}(p) \tag{13}$$

for any integer s such that $|s|\leq M+N$. We define the estimators of the Fourier coefficients of the volatility process as in [26]

$$c_{n,M}(l;\nu) = \frac{2\pi}{2M+1}\sum_{|s|\leq M}c_n(s;dp)c_n(l-s;dp) \tag{14}$$

for any integer l such that $|l|\leq N$. Finally, using the definition (12), (13) and (14), we get the estimators of the Fourier coefficients of the leverage process for any integer h such that $|h|\leq N$.

$$c_{n,M,N}(h;\eta) = \frac{2\pi}{2N+1}\sum_{|l|\leq N}\mathrm{i}lc_{n,M}(l;\nu)c_n(h-l;dp). \tag{15}$$

The above estimators are written as functions of the parameters n, M and N, respectively, the number of observations available, the number of the Fourier coefficients of the discrete observed return and of the latent volatility. The parameters M and N are called cutting frequencies.

Remark 4. The Fourier methodology differs from the one used in the papers [1,30], not just because it is an analysis in the frequency domain, but also as regards the use of the observed values of $p(t)$ in the computation of the integrated leverage effect. In fact, from (13) and (14), it can be observed that each Fourier coefficient of the return and the volatility processes is computed using all available observations in $\mathcal{S}_n$. In contrast in the papers [1,30], a pre-estimate of the volatility path is used, specifically, local average of integrated realized volatility estimators are employed. The observations are then divided into blocks. The observations in each block are used to determine a point estimation of the volatility path therein.

Therefore, we have the estimation error decomposition

$$c_{n,M,N}(h;\eta) - \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}ht}\eta(t)dt$$
$$= \frac{2\pi}{2N+1}\sum_{|l|\le N}(\mathrm{i}l c_{n,M}(l;\nu)c_n(h-l;dp) - \mathrm{i}l c(l;\nu)c(h-l;dp)) \tag{16}$$
$$+ \frac{2\pi}{2N+1}\sum_{|l|\le N}\mathrm{i}l c(l;\nu)c(h-l;dp) - \frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}ht}\eta(t)dt \tag{17}$$

where the two summands are respectively called discretization and truncation error. The estimation of their orders of magnitude in L_1-norm is presented in detail in the case of the 0th Fourier coefficient in Section 4.2 and can be found for each hth Fourier coefficient in [19]. This result is crucial in order to show the following consistency result.

Theorem 2. *For all $|h| \le N$, let $c_{n,M,N}(h;\eta)$ be the estimators of the Fourier coefficients of the leverage process defined in (15). We assume that Assumptions* (H),

$$\frac{N^2}{M} \to 0 \quad \textit{and} \quad M\tau(n) \to a \tag{18}$$

with $a > 0$ hold true as $n, N, M \to \infty$ and $\tau(n) \to 0$. Then

$$c_{n,M,N}(h;\eta) \xrightarrow{\mathbb{P}} \frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}ht}\eta(t)dt. \tag{19}$$

We refer the reader to [19] for the proof.

Remark 5. The parameter a greater than zero is a necessary assumption for showing an inequality needed to compute the order of magnitude of the L_1-norm of the discretization error (16). It was first shown in [16, Lemma 3], and we recall the result in Section 4.1.

3.2. Integrated estimator

An estimator of the integrated covariation (3) can be simply obtained by means of definition (15) for $h = 0$

$$\hat{\eta}_{n,M,N} = 2\pi c_{n,M,N}(0;\eta). \tag{20}$$

The consistency of $\hat{\eta}_{n,M,N}$ follows by the consistency of the estimator of the 0th Fourier coefficient already proved in Theorem 2. We call the estimator (20) the Fourier estimator of the integrated leverage effect. The explicit form of the estimator (20) is

$$\hat{\eta}_{n,M,N} = \sum_{j=0}^{k_n-1}\sum_{j'=0}^{k_n-1}\sum_{j''=0}^{k_n-1} D_M(t_{j,n}-t_{j',n})D'_N(t_{j',n}-t_{j'',n})\delta_{j,n}(p)\delta_{j',n}(p)\delta_{j'',n}(p)$$

$$= \sum_{j,j',j'':j\neq j'} D_M(t_{j,n}-t_{j',n})D'_N(t_{j',n}-t_{j'',n})\delta_{j,n}(p)\delta_{j',n}(p)\delta_{j'',n}(p) \tag{21}$$

$$+ \sum_{j,j',j'':j=j'} D'_N(t_{j',n}-t_{j'',n})\delta^2_{j,n}(p)\delta_{j'',n}(p), \tag{22}$$

where D_M and D'_N are respectively defined in (7) and (8).

The term (21) depends on products of odd Itô semimartingale increments, whereas (22) depends on the product between simple and square returns at each possible lag in the considered time window. The contribution of different lags of squared returns in the definition of the leverage effect has also been considered in [1].

Under the modeling assumptions in (1), we are able to prove that the estimation error converges stably in law[1] to a mixed normal distribution.

Theorem 3. *We assume that Assumptions* (H) *and the following relations*

$$\frac{N^3}{M} \to 0 \quad \text{and} \quad M\tau(n) \to a \tag{23}$$

with $a > 0$ hold true as $n, N, M \to \infty$ and $\tau(n) \to 0$. Then

$$\sqrt{N}\left(\hat{\eta}_{n,M,N} - \hat{\eta}\right) \xrightarrow{st} \int_0^{2\pi} \sqrt{\varphi(s)}dW'(s) \tag{24}$$

where

$$\varphi(s) = \pi(\nu(s)\gamma^2(s) + \eta^2(s)) + \frac{\pi}{2}\nu(0)\gamma^2(s) + \frac{\pi}{2}(\nu(2\pi) - \nu(0))^2\nu(2\pi) \tag{25}$$

and W' is a Brownian motion defined on an extension of the original probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and independent of the original σ-algebra $\mathbb{F}$.

Remark 6. The asymptotic rate of convergence of the estimation error is less than 1/6. This depends on the L_2-norm of the Dirichlet kernel, see Proposition 1, appearing in the truncation error (17) which also has the leading order of magnitude in the error decomposition. The parameter a, consequently, does not appear in the asymptotic conditional variance, as shown in detail in Section 4.2, because the discretization error (16) is negligible in probability. Therefore, the asymptotic properties of the estimator (20) do not depend on the structure of the time grid on which the logarithmic price is observed.

The main martingale representation of the estimation error and the complete proof of Theorem 3 are presented in Section 4.

The integrated asymptotic variance

$$\int_0^{2\pi} \varphi(s)ds \tag{26}$$

[1] Let Y_n be a sequence of $\mathbb{R}$-valued measurable random variables on $(\Omega, \mathbb{F}, \mathbb{P})$. We say that Y_n converges stably in law with limit Y, written $Y_n \xrightarrow{st} Y$, where Y is defined on an extension of the original probability space $(\Omega', \mathbb{F}', \mathbb{P}')$, if for any bounded, continuous function g and any bounded $\mathbb{F}$-measurable random variable Z it holds that $\mathbb{E}[g(Y_n)Z] \to \mathbb{E}'[g(Y)Z]$ as $n \to \infty$.

can be estimated in the Fourier framework. In fact, we can define an estimator for the 0th Fourier coefficient of the stochastic function $\varphi(s)$. First, we introduce the estimators of the Fourier coefficients of the volatility of volatility process $\gamma^2(s)$, for each $|h| \leq P$

$$c_{n,M,N}(h;\gamma^2) = \frac{2\pi}{2N+1} \sum_{|l|\leq N} l(l-h) c_{n,M}(l;\nu) c_{n,M}(h-l;\nu), \tag{27}$$

and the Laplace estimator of the spot volatility for $t \in (0, 2\pi)$

$$\hat{\nu}(t) = \sum_{i,j>0} \delta_{i,n}(p)\delta_{j,n}(p) \frac{\sin(M(t_{i,n}-t_{j,n}))}{M(t_{i,n}-t_{j,n})} \frac{1}{h}(1-\frac{|t-t_{j,n}|}{h}) 1_{[-1,1]}(\frac{t-t_{j,n}}{h}), \tag{28}$$

The estimators (27) and (28) have been introduced respectively in [32] and [18]. Therefore, estimating preliminarily $|s| \leq M+N+P$ Fourier coefficients $c_n(s;dp)$,

$$\begin{aligned} c_{n,N,M,P}(0,\varphi) = \frac{1}{2} \sum_{|j|\leq P} & c_{n,M,N}(j,\gamma^2) c_{n,M}(-j,\nu) + c_{n,M,N}(j,\eta) c_{n,M,N}(-j,\eta) \\ & + \frac{1}{4}\hat{\nu}(t_1) c_{n,M,N}(0;\gamma^2) + \frac{1}{4}(\hat{\nu}(t_2)-\hat{\nu}(t_1))^2 \hat{\nu}(t_2) \end{aligned} \tag{29}$$

where P is a constant less than N and $t_1 = 0+\epsilon$ and $t_2 = 2\pi - \epsilon$, for an $\epsilon > 0$ and small, are the points in which we estimate the volatility path. The first and second summands of (25) are defined following the estimation methodology described in [29]. Here, the estimator of the 0th Fourier coefficient of products of even power of latent variables is introduced for the quarticity. The asymptotic ratios between the parameters n, N, M, P and h need to be carefully studied in order to show that $2\pi c_{n,N,M,P}(0,\varphi)$ is a consistent estimator of (26). This problem is outside the scope of the present paper. However, estimators based on realized covariances, see for a review [3], can also be employed.

3.3. Path estimator and multivariate set-up

We now illustrate some applications and open problems connected to the Fourier methodology developed in this section.

First of all, it can be applied to the estimation of a leverage process path. In fact, the estimators (15) can be used to define for all $t \in (0, 2\pi)$,

$$\hat{\eta}_{n,M,N}(t) = \sum_{|h|\leq N} \left(1-\frac{|h|}{N}\right) e^{iht} c_{n,M,N}(h;\eta). \tag{30}$$

This is a consistent estimator of $\eta(t)$, a result that can be readily proved by means of the use of the Féjer Theorem for continuous functions and Theorem 2. However, as of today, there is no proof of a central limit theorem for the estimator (30).

Starting by the estimation strategy that leads to the estimators (15), it is also easy to approach the estimation of the leverage effect in a multivariate set-up. Furthermore, when prices of different financial assets are recorded on non-synchronous time grid, as when working with tick data, the Fourier estimators overcome easily this challenging problem.

For simplicity, in the following, let us consider a 2-dimensional logarithmic price process. We assume that $p(t) = (p_1(t), p_2(t))$ and its covariance matrix process Σ are solutions of the system of equations

$$\begin{cases} dp_i(t) & = a_i(t)\,dt + \sum_{r=1}^{d} \sigma_i^r(t) dW_r(t) \\ d\Sigma_{i,j}(t) & = b_{i,j}(t)\,dt + \sum_{k=1}^{p} \Lambda_{i,j}^k(t)\,dZ_k(t) \quad \text{for all } i,j = 1,2, \end{cases} \tag{31}$$

where $\Sigma_{i,j} = \sum_{r=1}^{d} \sigma_i^r(t)\sigma_j^r(t)$ with values in $\mathbb{S}_2^+$, the set of the positive semidefinite matrices, W and Z are respectively a d-dimensional and a p-dimensional Brownian motion such that $\langle dW_r(t), dZ_k(t)\rangle = \rho_{r,k}(t)dt$. The processes $a_i(t)$, $b_{i,j}(t)$, $\sigma_i^r(t)$, $\Lambda_{i,j}^k(t)$, $\rho_{r,k}(t)$ satisfy Assumptions (H) for each $i, j = 1, 2$, $r = 1, \dots, d$ and $k = 1, \dots, p$. Thus, the covariation between the logarithmic prices, for $l = 1, 2$, and the elements of the covariance matrix, for $i, j = 1, 2$, are defined by

$$\langle dp_l(t), d\Sigma_{i,j}\rangle = \eta_{i,j}^l(t)dt,$$

and, their respective integrated covariation as

$$\int_0^{2\pi} \eta_{i,j}^l(t)dt. \tag{32}$$

When $p_1(t)$ and $p_2(t)$ are observed on two time grids $\mathcal{S}_n$ and $\mathcal{S}_{n'}$, not necessarily the same, we can then compute the Fourier coefficients of their returns by using the discrete Fourier transform as in (13) and of the elements of their covariance matrix $\Sigma_{i,j}$ using the estimators defined in [27]. Estimators of (32), that determine the response of variances and covariance to the returns of the asset price $p_1(t)$, for $i, j = 1, 2$, are defined by

$$\hat{\eta}_{n,M,N,i,j} = \frac{4\pi^2}{2N+1} \sum_{|l|\le N} \mathrm{i}l c_{n,n',M}(l; \Sigma_{i,j}) c_n(-l; dp_1). \tag{33}$$

Similar estimators can be defined with respect to the asset price $p_2(t)$. We use the subscript n' in the above definition to indicate the presence of non-synchronicity in the time grids $\mathcal{S}_n$ and $\mathcal{S}_{n'}$.

The estimators (33) constitute interesting instruments for the analysis of stochastic volatility models, e.g. for equity indices or aggregate market portfolio, when the asymmetric joint dynamic dependencies between financial assets are of interest. A different approach to model the leverage effect in a multivariate set-up, using high frequency data, can be found in [12] in the context of a GARCH model. The model exploits estimates of variances and covariances based on the signs of high frequency returns, measure known as realized semivariances [11], into the modeling of the conditional variance matrix.

One final remark on the asymptotic properties of the estimators (33). The non-synchronicity of the observed logarithmic prices affects the asymptotical unbiasedness of the Fourier estimator, see [16] in the case of the covariance matrix estimation, then the asymptotic properties in the univariate set-up are not straightforwardly extendable in this scenario and are open problems in literature.

4. Proof of Theorem 3

4.1. Notations and preliminary results

First, we introduce some preliminary definitions.

Definition 1. A summability kernel is a sequence $\{k_n\}$ of continuous t-periodic functions satisfying:

1. $\frac{1}{t}\int_0^t k_n(s)\,ds = 1$.
2. $\frac{1}{t}\int_0^t |k_n(s)|\,ds \leq A$ where $A \in \mathbb{N}$.
3. For all $0 < \delta < \frac{t}{2}$, $\lim_{n\to\infty}\int_\delta^{t-\delta}|k_n(t)|\,dt = 0$.

A positive summability kernel is one such that $k_n(s) \geq 0$ for all s and n.

Definition 2. In the interval $[-t/2, t/2]$ for each $t \in (0, 2\pi]$, we call

$$D_N(s,t) = \frac{1}{2N+1}\sum_{|k|\leq N} e^{is\frac{2\pi}{t}k} = \frac{1}{2N+1}\frac{\sin((2N+1)(2\pi/t)s/2)}{\sin((2\pi/t)s/2)}, \tag{34}$$

the rescaled Dirichlet kernel.

The above kernel appears when we perform the Fourier estimation methodology in a time window $[0, t]$.

Lemma 4. *Let $D_N(s,t)$ be the rescaled Dirichlet kernel defined in the interval $[-t/2, t/2]$ for each $t \in (0, 2\pi]$. Then*

1. $$\int_0^t |D_N(s,t)|^2 ds = \frac{t}{2N+1}, \quad \forall N \in \mathbb{N}. \tag{35}$$
2. *For each $p > 1$, there exists a constant C_p such that*
$$\int_0^t |D_N(s,t)|^p ds \leq t\frac{C_p}{2N+1}, \quad \forall N \in \mathbb{N}. \tag{36}$$
3. *The sequence of continuous t-periodic functions*
$$K_N(s,t) = \frac{1}{N+1}\frac{\sin^2((N+1)(2\pi/t)s/2)}{\sin^2((2\pi/t)s/2)} \tag{37}$$
is a positive summability kernel respect to the argument s.
4. *Let us consider the sequence of continuous t-periodic functions*
$$ND_N^2(s,t) = \frac{N}{2N+1}K_{2N}(s,t),$$
and a continuous function $g : [0,t) \to \mathbb{R}$ such that the left limit $g(t^-) = \lim_{s\to t^-} g(s)$ exists and is finite. Then, for all $\epsilon \in (0, t/2)$
$$\frac{1}{t}\int_0^\epsilon + \int_{t-\epsilon}^t ND_N^2(s,t)\,g(s)ds \to \frac{g(0)+g(t^-)}{4} \tag{38}$$
as $N \to \infty$.

Proof. Using classical trigonometric tools it is easy to show that (35)–(36)–(37) hold. The Féjer theorem [25, Theorem 3.1, Chapter 1] has to be employed to show (38).

Another important tool used in our proof is Lemma 5, that gives an estimation of the L_p-norm of the rescaled Dirichlet kernel in discrete time.

We consider the following notations. Consider a discrete non-equidistant time grid $\mathcal{S}_n$ for any $n \geq 1$, we define $\phi_n(s) := \sup_{i=0,\dots,k_n}\{s_{i,n} : s_{i,n} \leq s\}$, thus the rescaled Dirichlet kernel

$$D_N(\phi_n(s), t) = \frac{1}{2N+1} \sum_{|k|\leq N} \mathrm{e}^{\mathrm{i}\phi_n(s)\frac{2\pi}{t}k}$$

satisfies the following property.

Lemma 5. *We assume that $\tau(n) \to 0$ as $n \to +\infty$ and that $N\tau(n) \to a$, where $a > 0$ as $N, n \to +\infty$. Then for all $t \in (0, 2\pi]$*

$$\forall p > 1, \exists C_p, \quad \limsup_{n,N\to+\infty} \tau(n)^{-1} \sup_{s\in[0,t]} \int_0^t |D_N(\phi_n(s) - \phi_n(u), t)|^p du \leq C_p. \tag{39}$$

Proof. For $t = 2\pi$, Lemma 5 is proved in [16, Lemma 3]. For $t \in (0, 2\pi)$, the proof is a straightforward extension of the latter.

Hereafter, we will use the following equivalent integral definition for the Fourier coefficients (13)

$$c_n(s; dp) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}s\phi_n(u)} dp(u). \tag{40}$$

4.2. *Martingale representation of the estimation error*

Along the proof, C will denote a positive constant, not necessarily the same at different occurrences.

Let us decompose

$$\sqrt{N}\Big(\hat{\eta}_{n,M,N} - \int_0^{2\pi} \eta(t)dt\Big) \tag{41}$$

$$= \sqrt{N}\Big(\frac{4\pi^2}{2N+1} \sum_{|l|\leq N} \mathrm{i}l c_{n,M}(l; \nu) c_n(-l; dp) - \mathrm{i}l c(l; \nu) c(-l; dp)\Big) \tag{42}$$

$$+ \sqrt{N}\Big(\frac{4\pi^2}{2N+1} \sum_{|l|\leq N} \mathrm{i}l c(l; \nu) c(-l; dp) - \int_0^{2\pi} \eta(t)dt\Big). \tag{43}$$

The summand (42) represents the discretization error of the estimate whereas the summand (43) represents the truncation error.

Applying the Cauchy–Schwarz inequality to the summand (42)

$$\mathbb{E}\Big[\Big|\frac{4\pi^2}{2N+1} \sum_{|l|\leq N} \mathrm{i}l c_{n,M}(l; \nu) c_n(-l; dp) - \mathrm{i}l c(l; \nu) c(-l; dp)\Big|\Big]$$
$$\leq \frac{4\pi^2}{2N+1} \sum_{|l|\leq N} |l| \Big(\mathbb{E}[c_{n,M}(l; \nu)^2]^{\frac{1}{2}} \mathbb{E}[(c_n(-l; dp) - c(-l; dp))^2]^{\frac{1}{2}}$$
$$+ \mathbb{E}[c(-l; dp)^2]^{\frac{1}{2}} \mathbb{E}[(c_{n,M}(l; \nu) - c(l; \nu))^2]^{\frac{1}{2}}\Big).$$

The L_2-norm of the Fourier coefficients $c(l; dp)$ defined in (4) is bounded under Assumptions (H), whereas for each $|l| \leq M + N$

$$\mathbb{E}\Big[(c_n(l; dp) - c(l; dp))^2\Big] \tag{44}$$
$$\leq \mathbb{E}\Big[\Big(\frac{1}{2\pi}\int_0^{2\pi}(\mathrm{e}^{-\mathrm{i}l\phi_n(t)} - \mathrm{e}^{-\mathrm{i}lt})\sigma(t)\,dW(t) + \frac{1}{2\pi}\int_0^{2\pi}(\mathrm{e}^{-\mathrm{i}l\phi_n(t)} - \mathrm{e}^{-\mathrm{i}lt})\,a(t)\,dt\Big)^2\Big].$$

After using the Itô isometry (44) is less than or equal to

$$C\mathbb{E}\Big[\frac{1}{4\pi^2}\int_0^{2\pi}(\mathrm{e}^{-\mathrm{i}l(\phi_n(t)-t)} - 1)(\mathrm{e}^{\mathrm{i}l(\phi_n(t)-t)} - 1)\sigma^2(t)\,dt\Big]$$
$$+ C\mathbb{E}\Big[\frac{1}{4\pi^2}\Big(\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}lt}(\mathrm{e}^{-\mathrm{i}l(\phi_n(t)-t)} - 1)\,a(t)\,dt\Big)^2\Big]$$

By means of the Hölder inequality with $p = \infty$ and $p = 1$ and Taylor's formula

$$\mathbb{E}\Big[\frac{1}{4\pi^2}\int_0^{2\pi}(\mathrm{e}^{-\mathrm{i}l(\phi_n(t)-t)} - 1)(\mathrm{e}^{\mathrm{i}l(\phi_n(t)-t)} - 1)\sigma^2(t)\,dt\Big]$$
$$\leq C\int_0^{2\pi}(|l|\,|\phi_n(t) - t| + l^2 o(|\phi_n(t) - t|^2))^2 dt \leq CN^2\tau^2(n) + o(1),$$

and

$$\mathbb{E}\Big[\frac{1}{4\pi^2}\Big(\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}lt}(\mathrm{e}^{-\mathrm{i}l(\phi_n(t)-t)} - 1)\,a(t)\,dt\Big)^2\Big]$$
$$\leq C\int_{[0,2\pi]^2}(|l|\,|\phi_n(t) - t| + l^2 o(|\phi_n(t) - t|^2))(|l|\,|\phi_n(s) - s| + l^2 o(|\phi_n(s) - s|^2))dtds$$
$$\leq CN^2\tau^2(n) + o(1)$$

By definition (14) and applying the product rule to the term $c_n(s; dp)c_n(l - s; dp)$, we obtain the following decomposition

$$c_{n,M}(l; \nu) = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}l\phi_n(t)}\nu(t)dt$$
$$+ I_{M,n} + \tilde{I}_{M,n} + H^1_{M,n} + H^2_{M,n} + H^3_{M,n} + \tilde{H}^1_{M,n} + \tilde{H}^2_{M,n} + \tilde{H}^3_{M,n}.$$

Referring to $D_M(\phi_n(s))$ as the normalized Dirichlet kernel for $t = 2\pi$ and $s \in [0, 2\pi]$

$$I_{M,n} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^t \mathrm{e}^{-\mathrm{i}l\phi_n(u)}D_M(\phi_n(t) - \phi_n(u))\sigma(u)\,dW(u)\,\sigma(t)\,dW(t),$$
$$H^1_{M,n} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^t \mathrm{e}^{-\mathrm{i}l\phi_n(u)}D_M(\phi_n(t) - \phi_n(u))\,a(u)\,du\,\sigma(t)\,dW(t),$$
$$H^2_{M,n} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^t \mathrm{e}^{-\mathrm{i}l\phi_n(u)}D_M(\phi_n(t) - \phi_n(u))\sigma(u)\,dW(u)\,a(t)\,dt,$$
$$H^3_{M,n} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^t \mathrm{e}^{-\mathrm{i}l\phi_n(u)}D_M(\phi_n(t) - \phi_n(u))\,a(u)\,du\,a(t)\,dt,$$
$$\tilde{I}_{M,n} = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}l\phi_n(t)}\int_0^t D_M(\phi_n(t) - \phi_n(u))\sigma(u)\,dW(u)\,\sigma(t)\,dW(t),$$
$$\tilde{H}^1_{M,n} = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}l\phi_n(t)}\int_0^t D_M(\phi_n(t) - \phi_n(u))\,a(u)\,du\,\sigma(t)\,dW(t),$$

$$\tilde{H}^2_{M,n} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}l\phi_n(t)} \int_0^t D_M(\phi_n(t)-\phi_n(u))\sigma(u)\,dW(u)\,a(t)\,dt,$$
$$\tilde{H}^3_{M,n} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}l\phi_n(t)} \int_0^t D_M(\phi_n(t)-\phi_n(u))\,a(u)\,du\,a(t)\,dt.$$

The L_2-norm of $\tilde{I}_{M,n}$, $\tilde{H}^1_{M,n}$, $\tilde{H}^2_{M,n}$ $\tilde{H}^3_{M,n}$, respectively, have the same order of magnitude as $I_{M,n}$, $H^1_{M,n}$, $H^2_{M,n}$ and $H^3_{M,n}$. Then, we just present the estimation of the L_2-norm of the latter quantities.

$$\mathbb{E}[(I_{M,n})^2] = \mathbb{E}\Big[\frac{1}{4\pi^2}\int_0^{2\pi}\Big(\int_0^t \mathrm{e}^{-\mathrm{i}l\phi_n(u)} D_M(\phi_n(t)-\phi_n(u))\sigma(u)dW(u)\Big)^2 \nu(t)dt\Big]$$
$$\leq C\mathbb{E}[\sup_{t\in[0,2\pi]} \nu^2(t)]\int_0^{2\pi}\int_0^t D^2_M(\phi_n(t)-\phi_n(u))dudt \leq \frac{C}{M}$$

by means of Lemma 5. Using similar tools, it can be shown that

$$\mathbb{E}[(H^1_{M,n})^2] \leq \frac{C}{M^{\frac{2}{p}}} \quad and \quad \mathbb{E}[(H^3_{M,n})^2] \leq \frac{C}{M^{\frac{2}{p}}},$$

for $p \in (1, 2)$. Regarding the estimation of the term $H^2_{M,n}$, the duality property for the stochastic integrals, [31, Formula 1.42], has to be used in order to obtain

$$\mathbb{E}[(H^2_{M,n})^2] \leq \frac{C}{M^{\frac{2+p}{2p}}},$$

for $p \in (1, 2)$. The complete computations of the above estimations can be found in [19, Theorem 3.1]. Then, it can be proved

$$\mathbb{E}[(c_{n,M}(l;\nu) - c(l;\nu))^2] \leq C\mathbb{E}\Big[\Big(\frac{1}{2\pi}\int_0^{2\pi}\Big(\mathrm{e}^{-\mathrm{i}l\phi_n(t)} - \mathrm{e}^{-\mathrm{i}lt}\Big)\nu(t)dt$$
$$+ I_{M,n} + \tilde{I}_{M,n} + H^1_{M,n} + H^2_{M,n} + H^3_{M,n} + \tilde{H}^1_{M,n} + \tilde{H}^2_{M,n} + \tilde{H}^3_{M,n}\Big)^2\Big]$$
$$\leq CN^2\tau^2(n) + \frac{C}{M} + o(1)$$

by means of the use of Hölder inequality and Taylor's formula as in (44) and that the L_1-norm of the discretization error

$$\mathbb{E}\Big[\Big|\sqrt{N}\frac{4\pi^2}{2N+1}\sum_{|l|\leq N} ilc_{n,M}(l;\nu)c_n(l;dp) - ilc(l;\nu)c(l;dp)\Big|\Big]$$
$$\leq CN^{\frac{5}{2}}\tau(n) + C\frac{N^{\frac{3}{2}}}{\sqrt{M}} + o(1).$$

Due to Assumptions (23) the above terms go to zero as $n, N, M \to \infty$ and $\tau(n) \to 0$.

Using the product rule we obtain that the truncation error (43) – because of formula (9) – can be decomposed as

$$\sqrt{N}\Big(\frac{4\pi^2}{2N+1}\sum_{|l|\leq N} \mathrm{i}lc(l;\nu)c(-l;dp) - \int_0^{2\pi}\eta(t)dt\Big)$$
$$= \underbrace{\sqrt{N}\int_0^{2\pi}\int_0^s D_N(s-u)dp(u)d\nu(s)}_{(M_{1,N}(2\pi))} + \underbrace{\sqrt{N}\int_0^{2\pi}\int_0^s D_N(s-u)d\nu(u)dp(s)}_{(M_{2,N}(2\pi))}$$

$$
-\underbrace{\sqrt{N}\int_0^{2\pi}\int_0^s D_N(u)dp(u)d\nu(s)}_{(M_{3,N}(2\pi))}-\underbrace{\sqrt{N}\int_0^{2\pi}\int_0^s D_N(s)d\nu(u)dp(s)}_{(M_{4,N}(2\pi))}
$$
$$
-\underbrace{\sqrt{N}\int_0^{2\pi} D_N(u)\eta(u)du}_{(M_{5,N}(2\pi))}\,.
$$

Let us analyze the first double integral

$$
\mathbb{E}\Big[\Big|\int_0^{2\pi}\int_0^s D_N(s-u)dp(u)d\nu(s)\Big|\Big]=\mathbb{E}\Big[\Big|\int_0^{2\pi}\int_0^s D_N(s-u)\sigma(u)dW(u)\gamma(s)dZ(s)
$$
$$
+\int_0^{2\pi}\int_0^s D_N(s-u)\sigma(u)dW(u)b(s)ds+\int_0^{2\pi}\int_0^s D_N(s-u)a(u)d(u)\gamma(s)dZ(s)
$$
$$
+\int_0^{2\pi}\int_0^s D_N(s-u)a(u)dub(s)ds\Big|\Big]
$$

The first two summands of the above decomposition have a L_1-norm respectively of order $O(N^{-\frac{1}{2}})$ and $O(N^{-\frac{2+p}{4p}})$ and the third and the fourth one are of order $O(N^{-\frac{1}{p}})$, where $p\in(1,2)$. These estimations are performed by means of the use of Proposition 1, the Hölder inequality and the duality property for the stochastic integrals, [31, Formula 1.42], in the case of the second summand. In the next section, we work extensively with similar estimations and present the exact calculations. The above calculations show that the drift components of the logarithmic price and the volatility process are negligible in probability with respect to the diffusive components under Assumptions (23). The L_1-norm of the summands $M_{1,N}(2\pi)$, $M_{2,N}(2\pi)$, $M_{3,N}(2\pi)$, $M_{4,N}(2\pi)$ has evidently the same order of magnitude.

By means of Proposition 1,

$$
\mathbb{E}[|M_{5,N}(2\pi)|]\le C\mathbb{E}\Big[\sup_{t\in[0,2\pi]}|\eta(t)|\Big]\sqrt{N}\Big(\int_0^{2\pi}|D_N(u)|^p du\Big)^{\frac{1}{p}}\le\frac{C}{N^{\frac{2-p}{2p}}}.
$$

Choosing $p\in(1,2)$ we obtain that the term $M_{5,N}(2\pi)$ converges to zero in L_1-norm as $N\to\infty$. Thus,

$$
\mathbb{E}\Big[\Big|\sqrt{N}\frac{4\pi^2}{2N+1}\sum_{|l|\le N} \mathrm{i}lc(l;\nu)c(-l;dp)-\int_0^{2\pi}\eta(t)dt\Big|\Big]
$$
$$
\le\mathbb{E}[|M_{1,N}(2\pi)+M_{2,N}(2\pi)+M_{3,N}(2\pi)+M_{4,N}(2\pi)|]+o_p(1).
$$

In the error decomposition (41), the truncation error (43) has the leading order of magnitude and the discretization error (42) is negligible in probability. Therefore, its asymptotic distribution just depends on $M_{i,N}(2\pi)$ for $i=1,\dots,4$.

To simplify notation we consider the process $p(t)$ and $\nu(t)$ as martingales in the following

$$
\begin{cases} dp(t) &=\sqrt{\nu(t)}dW(t)\\ d\nu(t) &=\gamma(t)dZ(t)\end{cases} \tag{45}
$$

In order to show the asymptotic result in (24), we need another preliminary step. We define

$$M_N(t) = \underbrace{\sqrt{N}\int_0^t\int_0^s D_N(s-u,t)dp(u)d\nu(s)}_{(M_{1,N}(t))} + \underbrace{\sqrt{N}\int_0^t\int_0^s D_N(s-u,t)d\nu(u)dp(s)}_{(M_{2,N}(t))}$$

$$-\underbrace{\sqrt{N}\int_0^t\int_0^s D_N(u,t)dp(u)d\nu(s)}_{(M_{3,N}(t))} - \underbrace{\sqrt{N}\int_0^t\int_0^s D_N(s,t)d\nu(u)dp(s)}_{(M_{4,N}(t))}$$

where we remind the reader that $D_N(s-u,t)$ is the rescaled Dirichlet kernel defined in (34).

Remark 7. The intuition behind the definition of the process $M_N(t)$ is the following. In order to estimate

$$\int_0^t \eta(s)ds,$$

we should employ an estimator of the 0th Fourier coefficient of the leverage process using observations of the logarithmic price in $[0,t]$. By means of Lemmas 4 and 5, we can show that the asymptotic properties of the error distribution depend exactly on the sequence $M_N(t)$ for each t.

The definition of $M_N(t)$ makes it now possible to use Jacod's stable limit Theorem, [24, Theorem 2.1], and then for $t = 2\pi$ to determine the asymptotic error distribution of the Fourier estimator of the integrated leverage effect.

4.3. Asymptotic error distribution

Jacod's stable limit theorem implies that it suffices to study for all $t \in [0,2\pi]$ the probability limit of the brackets $\langle M_N(t), W(t)\rangle$, $\langle M_N(t), Z(t)\rangle$, $\langle M_N(t), M_N(t)\rangle$ and $\langle M_N(t), N(t)\rangle$ where $N(t)$ belongs to the set of the bounded martingales adapted to the filtration $\mathcal{F}$ with $N(0) = 0$ such that $\langle W(t), N(t)\rangle = 0$ and $\langle Z(t), N(t)\rangle = 0$. The limit $\langle M_N(t), N(t)\rangle$ is obviously equal to zero for all $N(t)$. Thus, we divide the proof into three steps. In the first two steps we prove the asymptotic orthogonality of the sequence with respect to the Brownian motions W and Z. In the last we focus our attention on the limit in probability of the quadratic variation process.

In what follows, we indicate

$$Y_{1,N}(z,s) = \int_0^s D_N(z-u,t)dp(u) \tag{46}$$

$$Y_{2,N}(z,s) = \int_0^s D_N(z-u,t)d\nu(u) \tag{47}$$

$$Y_{3,N}(s) = \int_0^s D_N(u,t)dp(u) \tag{48}$$

First step: we prove that for each $t \in [0,2\pi]$, $\langle M_N(t), W(t)\rangle$ converges to zero in the L_2-norm.

$$\mathbb{E}[\langle M_N(t), W(t)\rangle^2]$$
$$= \mathbb{E}[|\langle M_{1,N}(t), W(t)\rangle + \langle M_{2,N}(t), W(t)\rangle + \langle M_{3,N}(t), W(t)\rangle + \langle M_{4,N}(t), W(t)\rangle|^2]$$

$$
\begin{aligned}
&= \mathbb{E}\Big[N\Big(\int_0^t Y_{1,N}(s,s)\gamma(s)\rho(s)ds + \int_0^t Y_{2,N}(s,s)\sqrt{\nu(s)}ds \\
&\quad - \int_0^t Y_{3,N}(s)\gamma(s)\rho(s)ds - \int_0^t \int_0^s D_N(s,t)d\nu(u)\sqrt{\nu(s)}ds\Big)^2\Big] \\
&\le \underbrace{CN\int_{[0,t]^2} \mathbb{E}[Y_{1,N}(s,s)Y_{1,N}(s',s')\gamma(s)\rho(s)\gamma(s')\rho(s')]dsds'}_{(I_{1,1})} \\
&\quad + \underbrace{CN\int_{[0,t]^2} \mathbb{E}[Y_{2,N}(s,s)Y_{2,N}(s',s')\sqrt{\nu(s)}\sqrt{\nu(s')}]dsds'}_{(I_{1,2})} \\
&\quad + \underbrace{CN\int_{[0,t]^2} \mathbb{E}[Y_{3,N}(s)Y_{3,N}(s')\gamma(s)\rho(s)\gamma(s')\rho(s')]dsds'}_{(I_{1,3})} \\
&\quad + \underbrace{CN\int_{[0,t]^2} |D_N(s,t)D_N(s',t)|\mathbb{E}[(\nu(s)-\nu(0))(\nu(s')-\nu(0))\sqrt{\nu(s)}\sqrt{\nu(s')}]dsds'}_{(I_{1,4})}
\end{aligned}
$$

The integrals $I_{1,1}$, $I_{1,2}$ and $I_{1,3}$ can be treated using the same procedure. We will only evaluate $I_{1,1}$. We show the calculation in the set of integration $\{s \le s' : (s,s') \in [0,t]^2\}$. In its complementary set, the duality property for the stochastic integrals leads to the terms discussed below where s' is in place of the variable s and vice versa. By means of the product rule [31, Lemma 1.2.2] and [31, Formula (1.65)]

$$
\begin{aligned}
&N\int_{0\le s\le s'\le t} \mathbb{E}[Y_{1,N}(s,s)Y_{1,N}(s',s')\gamma(s)\rho(s)\gamma(s')\rho(s')]dsds' \\
&= N\int_{0\le s\le s'\le t} \mathbb{E}[\gamma(s)\rho(s)\gamma(s')\rho(s')\int_0^s D_N(s-u,t)D_N(s'-u,t)1_{\{u\le s'\}}\nu(u)du]dsds' \\
&\quad + N\int_{0\le s\le s'\le t} \mathbb{E}[\gamma(s)\rho(s)\gamma(s')\rho(s')\int_0^s D_N(s-u,t)\sqrt{\nu(u)} \\
&\quad \times \Big(\int_u^{s'} D_N(s'-v,t)\mathcal{D}_u(\sqrt{\nu(v)})dW(v)\Big)du]dsds' \\
&\quad + N\int_{0\le s\le s'\le t} \mathbb{E}[Y_{1,N}(s',s')\int_0^s D_N(s-u,t)\sqrt{\nu(u)} \\
&\quad \times \mathcal{D}_u(\gamma(s)\rho(s)\gamma(s')\rho(s'))du]dsds'.
\end{aligned}
$$

We have that using Fubini's theorem and Assumption (H2)

$$
\begin{aligned}
&N\int_{0\le s\le s'\le t} \mathbb{E}[\gamma(s)\rho(s)\gamma(s')\rho(s')\int_0^s D_N(s-u,t)D_N(s'-u,t)1_{\{u\le s'\}}\nu(u)du]dsds' \\
&\le CN\int_0^t\Big(\int_u^t |D_N(s-u,t)|ds\int_u^t |D_N(s'-u,t)|ds'\Big)du \\
&\le CN\Big(\int_0^t |D_N(s-u,t)|^p du\Big)^{\frac{2}{p}} \le \frac{C}{N^{\frac{2-p}{p}}}.
\end{aligned}
$$

The Cauchy–Schwarz inequality and the property of the rescaled Dirichlet kernel lead to

$$
\Big|N\int_{0\le s\le s'\le t} \mathbb{E}[\gamma(s)\rho(s)\gamma(s')\rho(s')\int_0^s D_N(s-u,t)\sqrt{\nu(u)}
$$

$$\times\Big(\int_u^{s'} D_N(s'-v,t)\mathcal{D}_u(\sqrt{\nu(v)})dW(v)\Big)du]dsds'\Big|$$
$$\le CN\int_{0\le s\le s'\le t}\int_0^s |D_N(s-u,t)|du\Big(\int_0^{s'} D_N^2(s'-v,t)dv\Big)^{\frac{1}{2}}dsds' \le \frac{C}{N^{\frac{2-p}{2p}}}.$$

We proceed similarly for the third summand

$$N\int_{0\le s\le s'\le t}\mathbb{E}[Y_{1,N}(s',s')\int_0^s D_N(s-u,t)\sqrt{\nu(u)}\mathcal{D}_u(\gamma(s)\rho(s)\gamma(s')\rho(s'))du]dsds'$$
$$\le CN\int_{0\le s\le s'\le t}\Big(\int_0^{s'} D_N^2(s'-u,t)du\Big)^{\frac{1}{2}}\Big(\int_0^s |D_N(s-u,t)|^p du\Big)^{\frac{1}{p}}dsds' \le \frac{C}{N^{\frac{2-p}{2p}}}.$$

It remains to evaluate the integral $I_{1,4}$

$$N\int_{[0,t]^2}|D_N(s,t)D_N(s',t)|\mathbb{E}[(\nu(s)-\nu(0))(\nu(s')-\nu(0))\sqrt{\nu(s)}\sqrt{\nu(s')}]dsds'$$
$$\le CN\Big(\int_0^t |D_N(s,t)|^p ds\Big)^{\frac{2}{p}} \le \frac{C}{N^{\frac{2-p}{p}}}.$$

Therefore, choosing $p\in(1,2)$ all the above estimates go to zero as $N\to\infty$.

Second step: proceeding as in the computation of the *First step*, the bracket $\langle M_N(t), Z(t)\rangle$ converges to zero in L_2-norm for all $t\in[0,2\pi]$.

Third step: we have that

$$\begin{aligned}
&\langle M_N(t), M_N(t)\rangle\\
&\quad=\langle M_{1,N}(t), M_{1,N}(t)\rangle+2\langle M_{1,N}(t), M_{2,N}(t)\rangle+2\langle M_{1,N}(t), M_{3,N}(t)\rangle\\
&\qquad+2\langle M_{1,N}(t), M_{4,N}(t)\rangle\\
&\qquad+\langle M_{2,N}(t), M_{2,N}(t)\rangle+2\langle M_{2,N}(t), M_{3,N}(t)\rangle+2\langle M_{2,N}(t), M_{4,N}(t)\rangle\\
&\qquad+\langle M_{3,N}(t), M_{3,N}(t)\rangle\\
&\qquad+2\langle M_{3,N}(t), M_{4,N}(t)\rangle+\langle M_{4,N}(t), M_{4,N}(t)\rangle.
\end{aligned}\tag{49}$$

The first bracket

$$\begin{aligned}
\langle M_{1,N}(t), M_{1,N}(t)\rangle &= N\int_0^t\Big(\int_0^s D_N(s-u,t)dp(u)\Big)^2\gamma^2(s)\,ds\\
&=\underbrace{N\int_0^t\int_0^s D_N^2(s-u,t)\,\nu(u)\,du\,\gamma^2(s)\,ds}_{(I_{3,1})}\\
&\quad+\underbrace{2N\int_0^t\int_0^s Y_{1,N}(s,u)\,D_N(s-u,t)\,dp(u)\,\gamma^2(s)\,ds}_{(I_{3,2})}.
\end{aligned}$$

where the Itô isometry is applied to $Y_{1,N}^2(s,s)$.

We will use the same procedure to compute all the brackets in (49) labeling each integral for the convenience of the reader.

$$\langle M_{2,N}(t), M_{2,N}(t)\rangle = \underbrace{N\int_0^t\int_0^s D_N^2(s-u,t)\,\gamma^2(u)\,du\,\nu(s)\,ds}_{(I_{3,3})}$$

$$
\begin{aligned}
&+ \underbrace{2N \int_0^t \int_0^s Y_{2,N}(s,u)\, D_N(s-u,t)\, d\nu(u)\, \nu(s)\, ds}_{(I_{3,4})} \\
\langle M_{3,N}(t), M_{3,N}(t)\rangle &= \underbrace{N \int_0^t \int_0^s D_N^2(u,t)\, \nu(u)\, du\, \gamma^2(s)\, ds}_{(I_{3,5})} \\
&+ \underbrace{2N \int_0^t \int_0^s Y_{3,N}(u)\, D_N(u,t)\, dp(u)\, \gamma^2(s)\, ds}_{(I_{3,6})} \\
\langle M_{4,N}(t), M_{4,N}(t)\rangle &= \underbrace{N \int_0^t D_N^2(s,t)\Big(\int_0^s d\nu(u)\Big)^2 \nu(s) ds}_{(I_{3,7})} \\
2\langle M_{1,N}(t), M_{2,N}(t)\rangle &= \underbrace{2N \int_0^t \int_0^s D_N^2(s-u,t)\, \eta(u)\, du\, \eta(s)\, ds}_{(I_{3,8})} \\
&+ \underbrace{2N \int_0^t \int_0^s Y_{1,N}(s,u)\, D_N(s-u,t)\, d\nu(u)\, \eta(s)\, ds}_{(I_{3,9})} \\
&+ \underbrace{2N \int_0^t \int_0^s Y_{2,N}(s,u)\, D_N(s-u,t)\, dp(u)\, \eta(s) ds}_{(I_{3,10})} \\
2\langle M_{1,N}(t), M_{3,N}(t)\rangle &= \underbrace{-2N \int_0^t \int_0^s D_N(s-u,t)\, D_N(u,t)\, \nu(u)\, du \gamma^2(s) ds}_{(I_{3,11})} \\
&- \underbrace{2N \int_0^t \int_0^s Y_{1,N}(s,u)\, D_N(u,t)\, dp(u)\, \gamma^2(s) ds}_{(I_{3,12})} \\
&- \underbrace{2N \int_0^t \int_0^s Y_{3,N}(u)\, D_N(s-u,t)\, dp(u)\, \gamma^2(s)\, ds}_{(I_{3,13})}
\end{aligned}
$$

$$
\begin{aligned}
2\langle M_{1,N}(t), M_{4,N}(t)\rangle &= \underbrace{-2N \int_0^t D_N(s,t) \int_0^s D_N(s-u,t) \eta(u) du \eta(s)\, ds}_{(I_{3,14})} \\
&- \underbrace{2N \int_0^t D_N(s,t) \int_0^s Y_{1,N}(s,u)\, d\nu(u) \eta(s) ds}_{(I_{3,15})} \\
&- \underbrace{2N \int_0^t D_N(s,t) \int_0^s (\nu(u) - \nu(0)) D_N(s-u,t)\, dp(u) \eta(s)\, ds}_{(I_{3,16})} \\
2\langle M_{2,N}(t), M_{3,N}(t)\rangle &= \underbrace{-2N \int_0^t \int_0^s D_N(s-u,t) D_N(u,t) \eta(u)\, du\, \eta(s)\, ds}_{(I_{3,17})} \\
&- \underbrace{2N \int_0^t \int_0^s Y_{2,N}(s,u) D_N(u,t)\, dp(u)\, \eta(s)\, ds}_{(I_{3,18})}
\end{aligned}
$$

$$- \underbrace{2N\int_0^t\int_0^s Y_{3,N}(u)D_N(s-u,t)\,d\nu(u)\,\eta(s)\,ds}_{(I_{3,19})}$$

$$2\langle M_{2,N}(t), M_{4,N}(t)\rangle = \underbrace{-2N\int_0^t D_N(s,t)\int_0^s D_N(s-u,t)\gamma^2(u)\,du\,\nu(s)ds}_{(I_{3,20})}$$

$$- \underbrace{2N\int_0^t D_N(s,t)\int_0^s Y_{2,N}(s,u)d\nu(u)\,\nu(s)ds}_{(I_{3,21})}$$

$$- \underbrace{-2N\int_0^t D_N(s,t)\int_0^s (\nu(u)-\nu(0))D_N(s-u,t)d\nu(u)\nu(s)ds}_{(I_{3,22})}$$

$$2\langle M_{3,N}(t), M_{4,N}(t)\rangle = \underbrace{2N\int_0^t D_N(s,t)\int_0^s D_N(u,t)\eta(u)du\eta(s)ds}_{(I_{3,23})}$$

$$+ \underbrace{2N\int_0^t D_N(s,t)\int_0^s Y_{3,N}(u)d\nu(u)\eta(s)ds}_{(I_{3,24})}$$

$$+ \underbrace{2N\int_0^t D_N(s,t)\int_0^s (\nu(u)-\nu(0))D_N(u,t)dp(u)\eta(s)ds}_{(I_{3,25})}.$$

The integrals

$$I_{3,1}+I_{3,3}+I_{3,5}+I_{3,7}+I_{3,8} \to \frac{t}{2}\int_0^t(\nu(s)\gamma^2(s)+\eta^2(s))ds + \frac{t}{4}\nu(0)\int_0^t\gamma^2(s)ds + \frac{t}{4}(\nu(t)-\nu(0))^2\nu(t)$$

as $N\to\infty$ a.s. and for all $t\in[0,2\pi]$.

In the integrals $I_{3,1}$, $I_{3,3}$ and $I_{3,8}$, the sequence ND_N^2 is centered in s for each $s\in(0,t)$, we then compute the limit using the result (38) and considering the sequence just integrated with respect to the interval $[s-\epsilon,s)$ for $\epsilon\in(0,s/2)$. For the integral $I_{3,5}$, the sequence ND_N^2 is centered in 0 and the same applies but considering the sequence integrated with respect to the interval $[0,\epsilon)$ for $\epsilon\in(0,s/2)$ and $s\in(0,t)$. Just for the computation of the limit involving the integral $I_{3,7}$, the full result in (38) is used.

The integrals $I_{3,11}$, $I_{3,14}$, $I_{3,17}$, $I_{3,20}$ and $I_{3,23}$ converge to zero in L_1-norm for all $t\in[0,2\pi]$. The computation for each integral is similar. It is based on the use of the property of the rescaled Dirichlet kernel. We evaluate just the integral $I_{3,11}$.

$$\begin{aligned}
&\mathbb{E}\Big[\Big|-2N\int_0^t\int_0^s D_N(s-u,t)\,D_N(u,t)\,\nu(u)\,du\gamma^2(s)ds\Big|\Big]\\
&\le C\mathbb{E}[\sup_{t\in[0,2\pi]}|\nu(t)\gamma^2(t)|]2N\int_0^t\Big(\int_0^s|D_N(s-u,t)D_N(u,t)|^r\,du\Big)^{\frac{1}{r}}ds\\
&\le C2N\int_0^t\Big(\int_0^s|D_N(s-u,t)|^p du\Big)^{\frac{1}{p}}\Big(\int_0^s|D_N(u,t)|^{p'}du\Big)^{\frac{1}{p'}}ds\\
&\le C\frac{1}{N^{\frac{2-p}{2p}}}\frac{1}{N^{\frac{2-p'}{2p'}}}
\end{aligned}$$

applying the Young's inequality for convolutions with $p, p' \in (1,2)$ and $r > 1$ such that $\frac{1}{p} + \frac{1}{p'} = 1 + \frac{1}{r}$. The above term goes to zero as $N \to \infty$.

It can be proved that $I_{3,15}$, $I_{3,16}$, $I_{3,21}$, $I_{3,22}$, $I_{3,24}$ and $I_{3,25}$ converge to zero in L_2-norm using the same procedure. We present the computation for the integral $I_{3,15}$. We call

$$P_N(s) = \int_0^s Y_{1,N}(s,u)d\nu(u),$$

then the L_2-norm of the integral $I_{3,15}$

$$\mathbb{E}\Big[\Big(-2N\int_0^t D_N(s,t)P_N(s)\eta(s)ds\Big)^2\Big]$$
$$= 4N^2 \int_{[0,t]^2} D_N(s,t)D_N(s',t)\mathbb{E}\Big[P_N(s)P_N(s')\eta(s)\eta(s')\Big]dsds'. \tag{50}$$

Using several times the Cauchy–Schwarz and the Burkholder–Gundy inequalities

$$\begin{aligned}\mathbb{E}[P_N(s)P_N(s')\eta(s)\eta(s')] &\leq C\mathbb{E}\Big[\Big(\int_0^s Y_{1,N}(s,u)\,d\nu(u)\Big)^2\Big(\int_0^{s'} Y_{1,N}(s',v)d\nu(v)\Big)^2\Big]^{\frac{1}{2}}\\ &\leq C\mathbb{E}\Big[\int_0^s Y_{1,N}^4(s,u)\gamma^4(u)\,du\Big]^{\frac{1}{2}}\\ &\leq C\Big(\int_0^s\Big(\int_0^u D_N^2(s-w,t)dw\Big)^2du\Big)^{\frac{1}{2}} \leq \frac{C}{N}.\end{aligned}$$

Then, for all $t \in (0, 2\pi]$, (50) is less than or equal to

$$CN\int_{[0,t]^2} |D_N(s,t)D_N(s',t)|dsds' \leq \frac{C}{N^{\frac{2-p}{p}}}$$

which goes to zero choosing $p \in (1,2)$ as $N \to \infty$.

We evaluate the integrals $I_{3,2}$, $I_{3,4}$, $I_{3,6}$, $I_{3,9}$, $I_{3,10}$, $I_{3,12}$, $I_{3,13}$, $I_{3,18}$ and $I_{3,19}$ and show that they converge to zero in L_2-norm. In this case, the computation in L_2-norm is more technical and involves the use of Malliavin calculus. We explain the procedure in the case of the integral $I_{3,2}$. Let us call

$$Z_N(s) = \int_0^s Y_{1,N}(s,u)\,D_N(s-u,t)\,dp(u),$$

then the L_2-norm of the integral $I_{3,2}$

$$\mathbb{E}\Big[\Big|2N\int_0^t\int_0^s Y_{1,N}(s,u)\,D_N(s-u,t)\,dp(u)\,\gamma^2(s)\,ds\Big|^2\Big]$$
$$= 4N^2\int_{[0,t]^2} \mathbb{E}[Z_N(s)Z_N(s')\gamma^2(s)\gamma^2(s')]dsds'. \tag{51}$$

We show the calculation in the set of integration $\{s \leq s' : (s,s') \in [0,t]^2\}$. In the complementary set we obtain equal terms, substituting the variable s' with s and vice versa, with the same asymptotic behavior. By means of the product rule [31, Lemma 1.2.2] and [31, Formula (1.65)], we obtain

$$\mathbb{E}[Z_N(s)Z_N(s')\gamma^2(s)\gamma^2(s')] = \mathbb{G}_N^1(s,s') + \mathbb{G}_N^2(s,s') + \mathbb{G}_N^3(s,s'),$$

where

$$\mathbb{G}_N^1(s,s') = \mathbb{E}\Big[\gamma^2(s)\gamma^2(s')\int_0^s Y_{1,N}(s,u)D_N(s-u,t)\nu(u)Y_{1,N}(s',u)$$

$$\times\ D_N(s'-u,t)1_{\{u\le s'\}}du\Big],$$

$$\mathbb{G}_N^2(s,s') = \mathbb{E}\Big[\gamma^2(s)\gamma^2(s')\int_0^s Y_{1,N}(s,u)D_N(s-u,t)\sqrt{\nu(u)}\times\Big(\int_u^{s'} D_N(s'-v,t)\mathcal{D}_u(Y_{1,N}(s',v)\sqrt{\nu(v)})dW(v)\Big)du\Big],$$

$$\mathbb{G}_N^3(s,s') = \mathbb{E}[Z_N(s')\int_0^s Y_{1,N}(s,u)D_N(s-u,t)\sqrt{\nu(u)}\mathcal{D}_u(\gamma^2(s)\gamma^2(s'))du].$$

Then, we study

$$4N^2\int_{0\le s\le s'\le t}(\mathbb{G}_N^1(s,s')+\mathbb{G}_N^2(s,s')+\mathbb{G}_N^3(s,s'))dsds'. \tag{52}$$

Let us turn our attention to the first summand.

$$\begin{aligned}4N^2&\int_{0\le s\le s'\le t}\mathbb{G}_N^1(s,s')dsds'\\ &\le CN^2\int_{0\le s\le s'\le t}\int_0^s \mathbb{E}[Y_{1,N}^2(s,u)Y_{1,N}^2(s',u)]^{\frac{1}{2}}\\ &\quad\times|D_N(s-u,t)D_N(s'-u,t)|1_{\{u\le s'\}}dudsds'.\end{aligned} \tag{53}$$

We observe that, using successively the Cauchy–Schwarz and the Burkholder–Gundy inequalities

$$\begin{aligned}&\mathbb{E}[Y_{1,N}^2(s,u)Y_{1,N}^2(s',u)]^{\frac{1}{2}}\\ &\le \mathbb{E}\Big[\Big(\int_0^u D_N(s-v,t)\sqrt{\nu(v)}dW(v)\Big)^4\Big]^{\frac{1}{4}}\mathbb{E}\Big[\Big(\int_0^u D_N(s'-v,t)\sqrt{\nu(v)}dW(v)\Big)^4\Big]^{\frac{1}{4}}\\ &\le C\mathbb{E}\Big[\Big(\int_0^t D_N^2(s-v,t)dv\Big)^2\Big]^{\frac{1}{2}}\le\frac{C}{N}.\end{aligned} \tag{54}$$

Coming back to the estimation of the first summand, we have that (53) is less than or equal to

$$\begin{aligned}&CN\int_{0\le s\le s'\le t}\int_0^s|D_N(s-u,t)D_N(s'-u,t)|1_{\{u\le s'\}}dudsds'\\ &\le CN\int_0^t\Big(\int_u^t|D_N(s-u,t)|ds\Big)^2du\le CN\int_0^t\Big(\int_0^t|D_N(s-u,t)|^pds\Big)^{\frac{2}{p}}du\\ &\le\frac{C}{N^{\frac{2}{p}-1}}.\end{aligned}$$

We used Fubini's theorem and the property of the rescaled Dirichlet kernel to attain this result. Therefore, choosing $p\in(1,2)$ this term goes to zero.

Regarding the second summand: first of all we observe that the Malliavin derivative

$$\begin{aligned}\mathcal{D}_u(Y_{1,N}(s',v)\sqrt{\nu(v)}) &= Y_{1,N}(s',v)\mathcal{D}_u(\sqrt{\nu(v)})+D_N(s'-u,t)\sqrt{\nu(u)}1_{\{u\le v\}}\sqrt{\nu(v)}\\ &\quad+\int_u^v D_N(s'-v',t)\mathcal{D}_u(\sqrt{\nu(v')})dW(v')\sqrt{\nu(v)}\end{aligned}$$

and we can decompose the summand as follows

$$4N^2\int_{0\le s\le s'\le t}\mathbb{G}_N^2(s,s')dsds' =$$

$$
\begin{aligned}
&4N^2\int_{0\le s\le s'\le t}\mathbb{E}\Big[\gamma^2(s)\gamma^2(s')\int_0^s Y_{1,N}(s,u)D_N(s-u,t)\sqrt{\nu(u)}\\
&\quad\times\Big(\underbrace{\int_u^{s'} D_N(s'-v,t)Y_{1,N}(s',v)\mathcal{D}_u(\sqrt{\nu(v)})dW(v)}_{(I_{3,26})}\Big)du\Big]dsds'\\
&\quad+4N^2\int_{0\le s\le s'\le t}\mathbb{E}\Big[\gamma^2(s)\gamma^2(s')\int_0^s Y_{1,N}(s,u)D_N(s-u,t)\sqrt{\nu(u)}\\
&\quad\times\Big(\underbrace{\int_u^{s'} D_N(s'-v,t)D_N(s'-u,t)\sqrt{\nu(u)}1_{\{u\le v\}}\sqrt{\nu(v)}dW(v)}_{(I_{3,27})}\Big)du\Big]dsds'\\
&\quad+4N^2\int_{0\le s\le s'\le t}\mathbb{E}\Big[\gamma^2(s)\gamma^2(s')\int_0^s Y_{1,N}(s,u)D_N(s-u,t)\sqrt{\nu(u)}\\
&\quad\times\Big(\underbrace{\int_u^{s'} D_N(s'-v,t)\sqrt{\nu(v)}\int_u^v D_N(s'-v',t)\mathcal{D}_u(\sqrt{\nu(v')})dW(v')dW(v)}_{(I_{3,28})}\Big)du\Big]dsds'.
\end{aligned}
$$

Throughout the below estimate we use the Cauchy–Schwarz inequality. The integral $I_{3,26}$ is less than or equal to

$$
\begin{aligned}
&CN^2\int_{0\le s\le s'\le t}\int_0^s|D_N(s-u,t)|\mathbb{E}\Big[\Big|Y_{1,N}(s,u)\int_u^{s'} D_N(s'-v,t)Y_{1,N}(s',v)\\
&\quad\times\mathcal{D}_u(\sqrt{\nu(v)})dW(v)\Big|^2\Big]^{\frac{1}{2}}dudsds'\\
&\le CN^2\int_{0\le s\le s'\le t}\int_0^s|D_N(s-u,t)|\mathbb{E}\Big[\int_0^{s'} Y^2_{1,N}(s,u)D^2_N(s'-v,t)Y^2_{1,N}(s',v)\\
&\quad\times\mathcal{D}_u(\sqrt{\nu(v)})^2dv\Big]^{\frac{1}{2}}dudsds'\\
&\le CN^2\int_{0\le s\le s'\le t}\int_0^s|D_N(s-u,t)|\mathbb{E}[\sup_{u,v\in[0,t]}\mathcal{D}_u(\sqrt{\nu(v)})^4]^{\frac{1}{4}}\\
&\quad\times\Big[\int_0^{s'}\mathbb{E}[|Y^2_{1,N}(s,u)Y^2_{1,N}(s',v)|^2]^{\frac{1}{2}}D^2_N(s'-v,t)dv\Big]^{\frac{1}{2}}dudsds',
\end{aligned}
$$

we observe that, using successively Cauchy–Schwarz and Burkholder–Gundy inequalities

$$
\begin{aligned}
&\mathbb{E}[|Y^2_{1,N}(s,u)Y^2_{1,N}(s',v)|^2]^{\frac{1}{2}}\le\mathbb{E}[Y^8_{1,N}(s,u)]^{\frac{1}{4}}\mathbb{E}[Y^8_{1,N}(s',v)]^{\frac{1}{4}}\\
&\le C\mathbb{E}[\Big(\int_0^t D^2_N(w,t)dw\Big)^4]^{\frac{1}{2}}\le\frac{C}{N^2}.
\end{aligned}
$$

Therefore $I_{3,26}$ is less than or equal to

$$
\begin{aligned}
&\le CN\int_{0\le s\le s'\le t}\int_0^s|D_N(s-u,t)|\Big[\int_0^{s'} D^2_N(s'-v,t)dv\Big]^{\frac{1}{2}}dudsds'\\
&=CN\int_{0\le s\le s'\le t}\Big(\int_0^s|D_N(s-u,t)|du\Big)\Big(\int_0^{s'} D^2_N(s'-v,t)dv\Big)^{\frac{1}{2}}dsds'\le\frac{C}{N^{\frac{2-p}{2p}}}.
\end{aligned}
$$

Therefore, choosing $p\in(1,2)$, the above term goes to zero.

Using several times the Cauchy–Schwarz inequality, we have that $I_{3,27}$ is less than or equal to

$$
\begin{aligned}
&CN^2 \int_{0\le s\le s'\le t} \int_0^s |D_N(s-u,t)| \\
&\quad \times \mathbb{E}\Big[\Big| \int_u^{s'} Y_{1,N}(s,u) D_N(s'-v,t) D_N(s'-u,t) 1_{\{u\le v\}} \\
&\quad \times \sqrt{\nu(u)}\sqrt{\nu(v)} dW(v) \Big|^2\Big]^{\frac{1}{2}} du ds ds' \\
&= CN^2 \int_{0\le s\le s'\le t} \int_0^s |D_N(s-u,t)| \mathbb{E}\Big[\int_u^{s'} Y_{1,N}^2(s,u) D_N^2(s'-v,t) \\
&\quad \times D_N^2(s'-u,t) 1_{\{u\le v\}} \nu(u)\nu(v) dv \Big]^{\frac{1}{2}} du ds ds' \\
&\le CN^2 \int_{0\le s\le s'\le t} \int_0^s |D_N(s-u,t)| \Big[\int_u^{s'} D_N^2(s'-v,t) D_N^2(s'-u,t) \\
&\quad \times \mathbb{E}[Y_{1,N}^4(s,u)]^{\frac{1}{2}} 1_{\{u\le v\}} dv \Big]^{\frac{1}{2}} du ds ds'
\end{aligned}
$$

by means of the estimate (54)

$$
\begin{aligned}
&\le CN^{\frac{3}{2}} \int_{0\le s\le s'\le t} \int_0^s |D_N(s-u,t) D_N(s'-u,t)| \\
&\quad \times \Big[\int_0^t D_N^2(s'-v,t) 1_{\{u\le v\}} dv \Big]^{\frac{1}{2}} du ds ds' \\
&\le CN \int_{0\le s\le s'\le t} \int_0^s |D_N(s-u,t)||D_N(s'-u,t)| du ds ds' \\
&\le CN \int_0^t \Big(\int_u^t |D_N(s-u,t)| ds \Big) \Big(\int_u^t |D_N(s'-u,t)| ds' \Big) du \le \frac{C}{N^{\frac{2}{p}-1}}.
\end{aligned}
$$

We obtain the last estimate using Fubini's theorem and the properties of the Dirichlet kernel. Also in this case, if we choose $p \in (1,2)$, the above term goes to zero.

Proceeding as in the previous cases, $I_{3,28}$ is less than or equal to

$$
\begin{aligned}
&CN^2 \int_{0\le s\le s'\le t} \int_0^s |D_N(s-u,t)| \\
&\quad \times \mathbb{E}\Big[\Big| \int_u^{s'} Y_{1,N}(s,u) D_N(s'-v,t) \int_u^v D_N(s'-v',t) \mathcal{D}_u(\sqrt{\nu(v')}) dW(v') \\
&\quad \times \sqrt{\nu(v)} dW(v) \Big|^2\Big]^{\frac{1}{2}} du ds ds' \\
&\le CN^2 \int_{0\le s\le s'\le t} \int_0^s |D_N(s-u,t)| \mathbb{E}\Big[\int_u^{s'} Y_{1,N}^2(s,u) D_N^2(s'-v,t) \\
&\quad \times \Big(\int_u^v D_N(s'-v',t) \mathcal{D}_u(\sqrt{\nu(v')}) dW(v') \Big)^2 \nu(v) dv \Big]^{\frac{1}{2}} du ds ds' \\
&\le CN^2 \int_{0\le s\le s'\le t} \int_0^s |D_N(s-u,t)| \Big[\int_0^{s'} \mathbb{E}[Y_{1,N}^8(s,u)]^{\frac{1}{2}} D_N^4(s'-v,t) dv \Big]^{\frac{1}{4}}
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times \Big[\int_0^{s'}\Big(\int_0^{v} D_N^2(s'-v',t)dv'\Big)^2 dv\Big]^{\frac{1}{4}} dudsds' \\
&\le CN^2\int_{0\le s\le s'\le t}\int_0^s |D_N(s-u,t)|\frac{1}{N^{\frac{1}{2}}}\Big[\int_0^t D_N^4(s'-v,t)dv\Big]^{\frac{1}{4}} \\
&\quad \times \Big[\Big(\int_0^{t} D_N^2(s'-v',t)dv'\Big)^2\Big]^{\frac{1}{4}} dudsds' \\
&\le \frac{C}{N^{\frac{4-3p}{4p}}},
\end{aligned}
$$

choosing $p \in (1, 4/3)$ the above term goes to zero.

Concerning the third summand of (52), we have that

$$
\begin{aligned}
&4N^2\int_{0\le s\le s'\le t}\mathbb{G}_N^3(s,s')dsds' \\
&\le CN^2\int_{0\le s\le s'\le t}\int_0^s \mathbb{E}[Z_N^2(s')]^{\frac{1}{2}}\mathbb{E}[Y_{1,N}^4(s,u)]^{\frac{1}{4}}|D_N(s-u,t)|dudsds' \\
&\le CN^2\int_{0\le s\le s'\le t}\int_0^s |D_N(s-u,t)|\mathbb{E}[Z_N^2(s')]^{\frac{1}{2}}\frac{1}{N^{\frac{1}{2}}}dudsds' \\
&\le \frac{C}{N^{\frac{2-p}{2p}}}
\end{aligned}
$$

because of the following estimation

$$
\begin{aligned}
\mathbb{E}[Z_N^2(s')]^{\frac{1}{2}} &= \mathbb{E}\Big[\Big(\int_0^{s'} Y_{1,N}(s',u)D_N(s'-u,t)\sqrt{v(u)}dW(u)\Big)^2\Big]^{\frac{1}{2}} \\
&\le C\Big[\int_0^{s'}\mathbb{E}[Y_{1,N}^4(s'-u)]^{\frac{1}{2}}D_N^2(s'-u,t)du\Big]^{\frac{1}{2}} \le \frac{C}{N}.
\end{aligned}
$$

Therefore, choosing $p \in (1, 2)$ the third summand goes to zero in L_2-norm and we conclude.

5. Simulation analysis

In this section, we test the finite sample properties of the Fourier estimator as function of the cutting frequencies M and N. These parameters correspond to the highest frequency coefficients (13), (14) which have to be included in the estimator $\hat{\eta}_{n,M,N}$ and selected with respect to an optimality criterion in the finite sample. We choose to minimize the real mean squared error of the estimate (MSE in what follows) and to call optimal the cutting frequencies for which the minimum is reached. We then perform a sensitivity study on the MSE as function of the parameters M and N, a study on the behavior of the asymptotic statistic determined in Theorem 3 and we use the optimal selection rule to estimate the leverage effect in data affected by additive microstructure noise contaminations.

Let us start by describing the data sets used in the simulation analysis. The estimation procedure developed in the paper is not sensitive to the type of grid, equidistant or non-equidistant, on which the data are recorded, see Remark 6. Thus, we define the time grid $\mathcal{S}_n := \{0 = t_0 \le t_1 \le \cdots \le t_n = T\}$ as generated by a Beta distribution $B(a, b)$ with parameters $a = 2$ and $b = 5$ that models the trade duration between subsequent transactions. Our time grid shall resemble the times corresponding to the trade of a liquid stock – see [22, Chapter 3.3] – in a day corresponding to $T = 6$ hours. Referring to the notations of Section 3, for simplicity, we assume $k_n = n$ and $t_{i,n} = t_i$ for $i = 1, \ldots, n$. We construct two different time grids on a day: the first one is generated by a Beta distribution rescaled to the interval [0, 10.8] seconds where a trade occurs on average every 3.8 s and the second one by a Beta distribution rescaled to the interval [0, 3.09] seconds where a trade occurs on average every 0.88 s. We then obtain time grids with $n = 6962$ and $n = 24\,405$ time points, respectively. We simulate 250 returns and volatility paths drawn from the Heston [23] and the Generalized Heston model [34] on these two time grids. The Heston model used in the simulation is

$$\begin{cases} dp(t) &= \sqrt{\nu(t)}dW_1(t) \\ d\nu(t) &= \kappa(\beta - \nu(t))dt + \chi\sqrt{\nu}(t)dW_2(t), \end{cases} \tag{55}$$

where W_1 and W_2 are correlated Brownian motions. The parameter values are $\kappa = 0.01$, $\beta = 1.0$, $\chi = 0.05$ and the correlation parameter is chosen as $\rho = -0.2$.

Regarding the Generalized Heston model framework, which introduces stochastic correlation by adding a further source of randomness in the Heston model, we assume that

$$\begin{cases} dp(t) &= \sqrt{\nu(t)}dX(t) \\ dX(t) &= \rho(t)dW_1(t) + \sqrt{1-\rho^2(t)}dW_2(t) \\ d\nu(t) &= \kappa(\beta - \nu(t))dt + \chi\sqrt{\nu(t)}dW_1(t), \end{cases} \tag{56}$$

where $\rho(t)$ satisfies the stochastic differential equation

$$d\rho(t) = ((2\xi - \iota) - \iota\rho(t))dt + \theta\sqrt{(1+\rho(t))(1-\rho(t))}dW_0,$$

ι, ξ and θ are positive constants and W_0 is a Brownian motion. The processes $W_0(t)$, $W_1(t)$ and $W_2(t)$ are assumed to be independent. The parameter values used in the simulation are $\kappa = 0.01$, $\beta = 1.0$, $\chi = 0.05$ and $\xi = 0.2$, $\iota = 0.5$, $\theta = 0.5$, where the last three parameters are chosen in the range prescribed in [34] such that $\rho(t) \in [-1, 1]$. The initial values used are $\nu(0) = 1$, $p(0) = \log(100)$ and $\rho(0) = 0.04$.

We compute on each time grid and path $\hat{\eta}_{n,M,N}$ where $M \in \{1, \ldots, 3480\}$ and $N \in \{1, \ldots, 20\}$, for the data sets with $n = 6962$, and $M \in \{1, \ldots, 12\,202\}$ and $N \in \{1, \ldots, 20\}$ for the ones with $n = 24\,405$. The upper bound of the parameter M is consistent with the Nyquist frequency, i.e. the ratio $\frac{M}{n} = \frac{1}{2}$. This is a typical bound used in the Fourier framework to avoid aliasing effect. Moreover, we respect this bound also due to the asymptotic properties of the Fourier coefficients of the volatility process, as addressed in [28, Remark 3.2]. For the same selection of frequencies M and N we compute

$$\text{BIAS} = \mathbb{E}[\hat{\eta}_{n,M,N} - \hat{\eta}], \quad \text{and} \quad \text{MSE} = \mathbb{E}[(\hat{\eta}_{n,M,N} - \hat{\eta})^2],$$

where $\hat{\eta}$ is the real value of the integrated leverage effect as defined in (3). In the set-up described in this section, the value of $\hat{\eta}$ can be computed by using Riemann sums. We can then estimate an average of the real integrated leverage over the 250 replications in our data sets. We call it $\overline{\eta}$. The optimal couple (M^*, N^*) is then defined as

$$(M^*, N^*) = argmin\, \text{MSE}(M, N).$$

The results of the optimization are presented in Table 1. It is important to emphasize that whereas the integrated leverage effect is always negative in the Heston model framework, in the case of the Generalized Heston model it can be either positive or negative for different trajectories. The optimal cutting frequency M^* selected in the case of the Heston model is always lower than the Generalized Heston model scenarios, whereas the contrary happens for the selection of the cutting frequency N^*. Moreover, M^* is always bigger than $(N^*)^3$ in line with the assumptions

Table 1
Selected cutting frequencies M^* and N^* with respect to the MSE (computed over 250 replications of the return and volatility paths) for the Heston and Generalized Heston model data sets. $\overline{\eta}$ represents the average real integrated leverage for each data set.

$\overline{\eta}$	n	M^*	N^*	MSE	BIAS
Heston model					
$-2.499e-03$	6962	1840	3	$5.87e-06$	$2.37e-03$
$-2.504e-03$	244 05	5923	6	$5.77e-06$	$2.28e-04$
G-Heston model					
$-1.075e-03$	6962	1940	3	$4.02e-06$	$1.10e-03$
$-1.082e-03$	24 405	9128	5	$3.40e-06$	$9.35e-04$

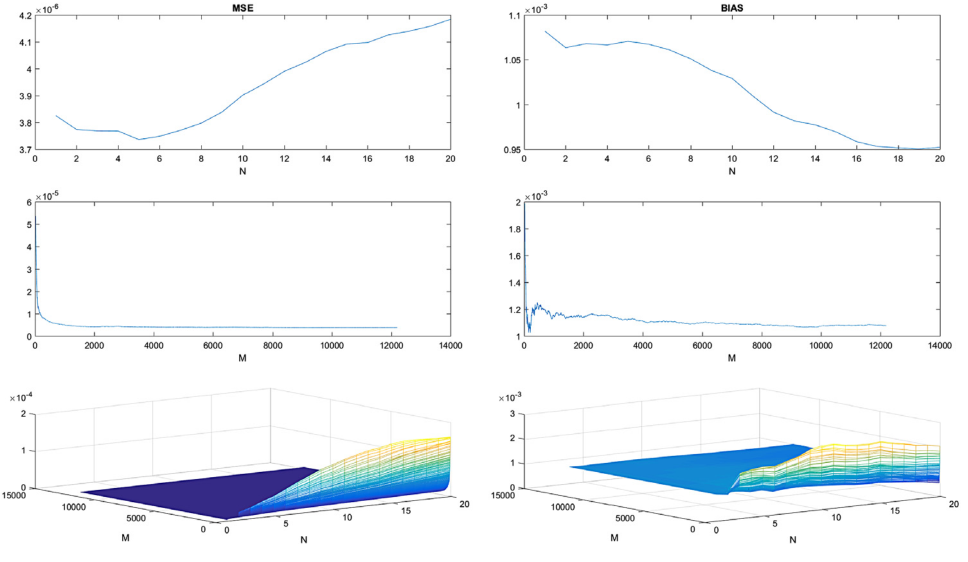

Fig. 1. Real MSE and BIAS of $\hat{\eta}_{n,M,M}$ averaged on the whole data set (lower panels) and their sections relative to the optimal selected cutting frequencies $N^* = 5$ (middle panels) and $M^* = 9128$ (upper panels).

of Theorem 3. Overall, across the scenarios, when n, M^*, N^* increase the performances of the Fourier estimator improve. In particular the BIAS decreases even though the estimator has a positive BIAS on average.

We now study in detail the Generalized Heston model data set with $n = 24\,405$. The sensitivity analysis in the case of the Heston model is similar although the minimum of the MSE is reached for another selection of parameters. In Fig. 1, we plot the MSE, the BIAS and their sections relative to the optimal cutting frequencies ($M^* = 9128$, $N^* = 5$). Once N^* has been selected, the MSE is quite robust to the choice of the cutting frequency M for $M \geq M^*$ whereas the BIAS has a minimum point that does not correspond to M^*. On the contrary, when M^* is selected, there is just one possible N that minimizes either the MSE or the BIAS. Thus, (M^*, N^*) do not correspond to a minimum of the BIAS. This means that to improve the performances of the estimator a bias correction has to be made. This problem is tackled in [20].

Table 2
Selection of the cutting frequencies (M, N). Generalized Heston model data sets with $n = 24\,405$.

	M	N
(a)	9 128	5
(b)	10 953	6
(c)	11 866	7
(d)	12 202	7

The presence of a positive BIAS is also evident if we focus on the behavior of the statistic

$$\sqrt{N}\frac{(\hat{\eta}_{n,M,N} - \eta_{\text{real}})}{\sqrt{\int_0^{2\pi} \varphi(s)\,ds}}, \tag{57}$$

where φ is the asymptotic conditional variance defined in (25), that converges in distribution to a Standard Normal random variable due to Theorem 3 for n, M, N that goes to infinity. We here estimate the true integrated asymptotic variance by means of Riemann sums. Given $n = 24\,405$, we select the frequencies N and M as in Table 2 such that in (a), (b) and (c) they correspond to $\lceil kN^* \rceil$ and $\lfloor kM^* \rfloor$ for $k = 1.2, 1.3$ and in (d) to the choice of $\lfloor n/2 \rfloor$ for the parameter M and of $\lceil N^* \lfloor n/2 \rfloor / M^* \rceil$ for the parameter N. The (a) selection corresponds to (M^*, N^*) and the selection (d) is the biggest possible in the considered scenario, i.e. the cutting frequency M is the biggest integer such that $M/n \le 1/2$, the Nyquist frequency. The QQ-plots relative to the statistic (57) are shown in Fig. 2. The cutting frequency N that we are allowed to use is not greater than 7 and evidently a BIAS is present in the estimates due to the finite sample. Nevertheless, the statistic (57) gives a good guide to the behavior of the error distribution as the sensitivity analysis does in Fig. 1. The QQ-plot in the upper left panel has the best coverage between the four selection of cutting frequencies in Table 2.

Finally, even if the results presented in this paper do not take into account the microstructure noise contamination effects, we apply the Fourier methodology to data affected by additive noise. We add microstructure noise contamination to the efficient logarithmic price in equilibrium, $p(t)$, defined in (55) and (56). Thus, the logarithm of the observed price is

$$\widetilde{p}(t_{i,n}) = p(t_{i,n}) + \zeta(t_{i,n}) \tag{58}$$

where $\zeta(t)$ is the microstructure noise. The random shocks ζ are considered i.i.d. Gaussian and independent of p. This is typical of the bid–ask bounce effects in the case of exchange rates and, to a lesser extent, in the case of equities. We consider noise-to-signal ratio $\lambda = std(\zeta)/std(r) = 3$, where r is the series of the discrete simulated return $r_i = \delta_{i,n}(p)$ for $i = 0, \ldots, n-1$, not affected by noise, and ζ the series of the random shocks on the same time grid. The results of the optimization for the selection of the cutting frequencies (M^*, N^*) are presented in Table 3 for the Heston and the Generalized Heston model data sets. The performances of the estimator in the presence of noise are similar to the ones in Table 1 as well as the presence of a BIAS that, however, it is positive just in the case of the Generalized Heston model. We notice that the selected (M^*, N^*) are smaller than those in the no-noise case. This is a typical behavior observed in the class of the Fourier estimators in the presence of microstructure noise contamination effects, see [28, Chapter 5] for a review.

To conclude, a note on the practical use of the estimator. The underlying model (1), can be suitable to describe tick data. In fact, the presence of jumps in the dynamics of ultra high-frequency data (millisecond precision) is questionable, see [14] for an empirical study on this

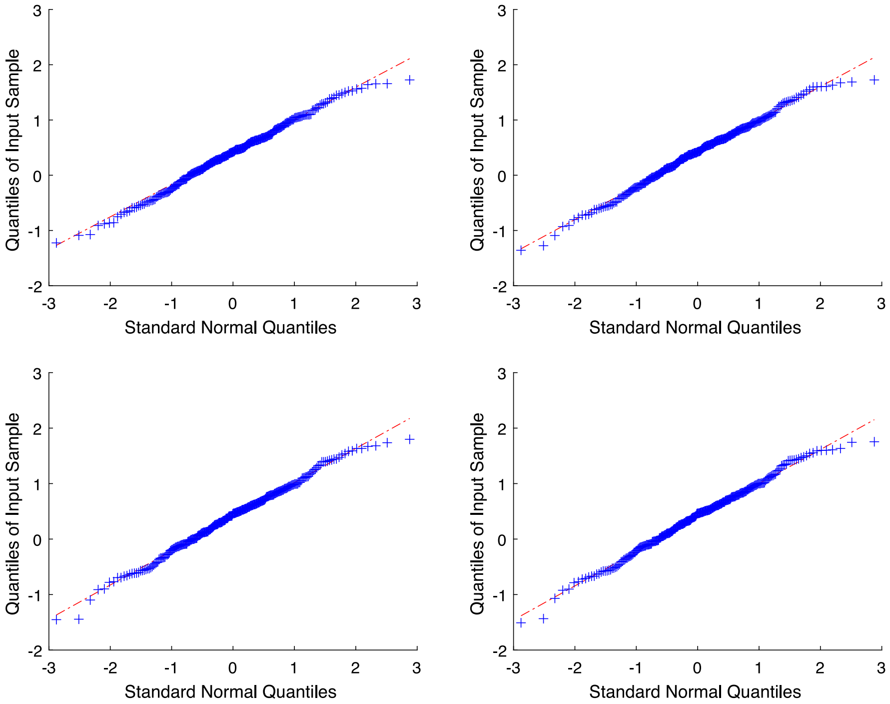

Fig. 2. Quantile–quantile plot of the sample quantiles of the standardized estimation errors versus the theoretical quantiles of a standardized normal distribution for the Generalized Heston model data set (250 replications of the return and volatility paths) for the selected frequencies in Table 2: (a) upper left, (b) upper right, (c) lower left and (d) lower right panel.

Table 3
Selected cutting frequencies M^* and N^* with respect to the MSE (computed over 250 replications of the return and volatility paths) for the Heston and Generalized Heston model data sets with $n = 24\,405$ in the case of microstructure noise contamination. The noise to signal ratio $\lambda = 3$. $\overline{\eta}$ represents the average real integrated leverage for each data set.

$\overline{\eta}$	M^*	N^*	MSE	BIAS
Heston model				
$-2.504e-03$	3014	2	$6.03e-06$	$-9.86e-05$
G-Heston model				
$-1.082e-03$	3497	1	$3.58e-06$	$1.05e-04$

issue. However, for tick data the microstructure contamination effects are not negligible. It is then worth developing directly a feasible selection rule for the optimal couple (M^*, N^*) under the model specification (58). This issue is addressed in [20]. The methodology here developed has similarities with the feasible selection rules defined for the estimator of the quarticity [29] and the volatility of volatility [32] in the presence of noise.

6. Conclusion

In a continuous semimartingale set up, the asymptotic normality of the Fourier estimator of the integrated leverage effect defined in [19] is attained in the presence of non-equidistant observations of the logarithmic price process. The proof is conducted using Malliavin calculus and is a stable limit result. The asymptotic rate of the central limit theorem as well as the finite sample performances of the estimator depends on the parameters M and N, respectively, the number of the Fourier coefficients of the return and the volatility process to include in the estimation procedure. In the finite sample, a mean squared error based optimal selection rule for the parameters M and N is addressed for Monte Carlo data. Moreover, the asymptotic theory and the finite sample performances of the Fourier estimator constitute the foundation for the definition of spot and multivariate estimators that aim to analyze the asymmetry in the dynamics of the logarithmic prices and their volatilities.

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