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A generalization of the Funk–Radon transform to circles passing through a fixed point

Michael Quellmalz*

The Funk–Radon transform assigns to a function on the two-sphere its mean values along all great circles. We consider the following generalization: we replace the great circles by the small circles being the intersection of the sphere with planes containing a common point $\boldsymbol{\zeta}$ inside the sphere. If $\boldsymbol{\zeta}$ is the origin, this is just the classical Funk–Radon transform. We find two mappings from the sphere to itself that enable us to represent the generalized Radon transform in terms of the Funk–Radon transform. This representation is utilized to characterize the nullspace and range as well as to prove an inversion formula of the generalized Radon transform.

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1 Background

On the two-dimensional sphere $\mathbb{S}^2$, every circle can be described as the intersection of the sphere with a plane,

$$\mathscr{C}(\boldsymbol{\xi}, x) = \left\{ \boldsymbol{\eta} \in \mathbb{S}^2 \mid \langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = x \right\},$$

where $\boldsymbol{\xi} \in \mathbb{S}^2$ is the normal vector of the plane and $x \in [-1, 1]$ is the signed distance of the plane to the origin. For $x = \pm 1$, the circle $\mathscr{C}(\boldsymbol{\xi}, x)$ consists of only the singleton $\pm\boldsymbol{\xi}$. The spherical mean operator $\mathcal{S} \colon C(\mathbb{S}^2) \to C(\mathbb{S}^2 \times [-1, 1])$ assigns to a continuous function f defined on $\mathbb{S}^2$ its mean values along all circles of the sphere, i.e.

$$\mathcal{S}f(\boldsymbol{\xi}, x) = \int_{\mathscr{C}(\boldsymbol{\xi}, x)} f(\boldsymbol{\eta}) \, \mathrm{d}\mu(\boldsymbol{\eta}),$$

where μ denotes the Lebesgue measure on the circle $\mathscr{C}(\boldsymbol{\xi}, x)$ normalized such that $\mu(\mathscr{C}(\boldsymbol{\xi}, x)) = 1$. The inversion of the spherical mean operator $\mathcal{S}$ is an overdetermined problem, e.g. $\mathcal{S}f(\boldsymbol{\xi}, 1) = f(\boldsymbol{\xi})$ for all $\boldsymbol{\xi} \in \mathbb{S}^2$. However, in practical applications $\mathcal{S}f$ is often known only on a two-dimensional sub-manifolds of $\mathbb{S}^2 \times [-1, 1]$.

An important example of such restriction is the Funk–Radon transform $\mathcal{F}$, namely the restriction of $\mathcal{S}$ to $x = 0$. It computes the averages along all great circles $\mathscr{C}(\boldsymbol{\xi}, 0)$ of the sphere. Based on the work of Minkowski [13], Funk [6] showed that every even, continuous function can be reconstructed from its Funk–Radon transform. There are several reconstruction formulas, a famous one is due to

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Helgason [8, Sec. III.1.C]. The range of the Funk–Radon transform in terms of Sobolev spaces was characterized by Strichartz [23].

A similar problem is the restriction of $\mathcal{S}$ to a fixed value $x = x_0 \in [-1, 1]$, which corresponds to the family of circles with fixed diameter. Schneider [22] proved a so-called "freak theorem", which says that the set of values x_0 for which $\mathcal{S}f(\boldsymbol{\xi}, x_0) = 0$ for all $\boldsymbol{\xi} \in \mathbb{S}^2$ does not imply $f = 0$ is countable and dense in $[-1, 1]$. These possible values of x_0 were further investigated by Rubin [19]. Similar results were obtained for circles whose radius is one of two fixed values by Volchkov and Volchkov [25].

Abouelaz and Daher [1] considered the restriction to the family of circles containing the north pole. An inversion formula was found by Gindikin et al. [7]. Helgason [8, Sec. III.1.D] gave this restriction the name spherical slice transform and showed that it is injective for continuously differentiable functions vanishing at the north pole. Injectivity has also been shown to hold for square-integrable functions vanishing in a neighborhood of the north pole [5], and for bounded functions [20, Sec. 7].

Restricted to $\xi_3 = 0$, which corresponds to the family of circles perpendicular to the equator, the mean operator is injective for all functions f that are even with respect to the north–south direction, i.e. $f(\xi_1, \xi_2, \xi_3) = f(\xi_1, \xi_2, -\xi_3)$. Different reconstruction schemes were proposed in [7, 26, 10]. More generally, it was shown in [3] (see also [2]) that the restriction of the mean operator to the set $A \times [-1, 1]$, where A is some subset of $\mathbb{S}^2$, is injective if and only if A is not contained in the zero set of any harmonic polynomial.

In this article, we are going to look at circles that are the intersections of the sphere with planes containing a fixed point $(0, 0, z)^\top$ located on the north–south axis inside the unit sphere, where $z \in [0, 1)$. The spherical transform

$$\mathcal{U}_z f(\boldsymbol{\xi}) = \mathcal{S}f(\boldsymbol{\xi}, z\xi_3), \qquad \boldsymbol{\xi} \in \mathbb{S}^2,$$

computes the mean values of a continuous function $f \colon \mathbb{S}^2 \to \mathbb{C}$ along all such circles. The spherical transform $\mathcal{U}_z$ was first investigated by Salman [21] in 2015. He proved the injectivity of the spherical transform for smooth functions supported inside the spherical cap $\{\boldsymbol{\xi} \in \mathbb{S}^2 \mid \xi_3 < z\}$. Furthermore, he showed an inversion formula (see Proposition 6.1) using stereographic projection combined with an inversion formula of a Radon-like transform in the plane, which integrates along all circles that intersect the unit circle in two antipodal points.

The central result of this paper is Theorem 3.1, where we prove the factorization of the spherical transform

$$\mathcal{U}_z = \mathcal{N}_z \mathcal{F} \mathcal{M}_z,$$

into the Funk–Radon transform $\mathcal{F}$ and the two operators $\mathcal{M}_z$ and $\mathcal{N}_z$, which are defined in (3.1) and (3.2), respectively. Both $\mathcal{M}_z$ and $\mathcal{N}_z$ consist of a dilation from the sphere to itself composed with the multiplication of some weight.

Based on this factorization, we show in Theorem 4.4 that the nullspace of the spherical transform $\mathcal{U}_z$ consists of all functions that are, multiplied with some weight, odd with respect to the point reflection of the sphere about the point $(0, 0, z)^\top$. Moreover, it turns out that the ranges of the spherical transform and the Funk–Radon transform coincide, considered they are both defined on square-integrable functions on the sphere, see Theorem 4.7. The relation with the Funk–Radon transform also allows us to state an inversion formula of the spherical transform in Theorem 5.1. We close the paper in Section 6 by reviewing the proof of Theorem 3.1 from a different perspective that is connected with Salman's approach.

2 Definitions

We denote with $\mathbb{R}$ and $\mathbb{C}$ the fields of real and complex numbers, respectively. We define the two-dimensional sphere $\mathbb{S}^2 = \{\boldsymbol{\xi} \in \mathbb{R}^3 \mid \|\boldsymbol{\xi}\| = 1\}$ as the set of unit vectors $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^\top \in \mathbb{R}^3$ in the three-dimensional Euclidean space $\mathbb{R}^3$ equipped with the scalar product $\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3$ and the norm $\|\boldsymbol{\xi}\| = \langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle^{1/2}$. We make use of the sphere's parametrization in terms of cylinder coordinates,

$$\boldsymbol{\xi}(\varphi, t) = \left(\cos\varphi\sqrt{1-t^2}, \sin\varphi\sqrt{1-t^2}, t\right)^\top, \qquad \varphi \in [0, 2\pi),\ t \in [-1, 1], \tag{2.1}$$

where we assume that the longitude φ is 2π-periodic. Let $f\colon \mathbb{S}^2 \to \mathbb{C}$ be some measurable function. With respect to cylinder coordinates, the surface measure $\mathrm{d}\boldsymbol{\xi}$ on the sphere reads

$$\int_{\mathbb{S}^2} f(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = \int_{-1}^{1}\int_{0}^{2\pi} f(\boldsymbol{\xi}(\varphi, t))\,\mathrm{d}\varphi\,\mathrm{d}t.$$

The Hilbert space $L^2(\mathbb{S}^2)$ is defined as the space of all measurable functions $f\colon \mathbb{S}^2 \to \mathbb{C}$, whose norm $\|f\|_{L^2(\mathbb{S}^2)} = \langle f, f \rangle^{1/2}$ is finite, where

$$\langle f, g \rangle = \int_{\mathbb{S}^2} f(\boldsymbol{\xi})\overline{g(\boldsymbol{\xi})}\,\mathrm{d}\boldsymbol{\xi}$$

is the usual L^2-inner product. Furthermore, we denote with $C(\mathbb{S}^2)$ the set of continuous, complex-valued functions defined on the sphere.

Let $\boldsymbol{\gamma}\colon [0, 1] \to \mathbb{S}^2$, $s \mapsto \boldsymbol{\gamma}(\varphi(s), t(s))$ be a regular path on the sphere parameterized in cylinder coordinates. The line integral of a function $f \in C(\mathbb{S}^2)$ along the path $\boldsymbol{\gamma}$ with respect to the arc-length $\mathrm{d}\ell$ is given by

$$\int_{\boldsymbol{\gamma}} f\,\mathrm{d}\ell = \int_0^1 f(\boldsymbol{\gamma}(\varphi(s), t(s)))\sqrt{(1 - t(s)^2)\left(\frac{\mathrm{d}\varphi(s)}{\mathrm{d}s}\right)^2 + \frac{1}{1 - t(s)^2}\left(\frac{\mathrm{d}t(s)}{\mathrm{d}s}\right)^2}\,\mathrm{d}s. \tag{2.2}$$

The spherical transform. Every circle on the sphere can be described as the intersection of the sphere with a plane, i.e.

$$\mathscr{C}(\boldsymbol{\xi}, x) = \left\{\boldsymbol{\eta} \in \mathbb{S}^2 \mid \langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = x\right\},$$

where $\boldsymbol{\xi} \in \mathbb{S}^2$ is the normal vector of the plane and $x \in [-1, 1]$ is the signed distance of the plane to the origin. We consider circles whose planes have a common point $\boldsymbol{\zeta} \in \mathbb{R}^3$ located in the interior of the unit ball, i.e. $\|\boldsymbol{\zeta}\| < 1$. We say that a circle passes through $\boldsymbol{\zeta}$ if its respective plane contains $\boldsymbol{\zeta}$. By rotational symmetry, we can assume that the point $\boldsymbol{\zeta}$ lies on the positive ξ_3 axis. For $z \in [0, 1)$, we set

$$\boldsymbol{\zeta}_z = (0, 0, z)^\top.$$

The circles passing through $\boldsymbol{\zeta}_z$ can be described by $\mathscr{C}(\boldsymbol{\xi}, x)$ with $\boldsymbol{\xi} \in \mathbb{S}^2$ and $x = \langle \boldsymbol{\xi}, \boldsymbol{\zeta}_z \rangle = z\xi_3$. For a function $f \in C(\mathbb{S}^2)$, we define the spherical transform

$$\mathcal{U}_z f(\boldsymbol{\xi}) = \frac{1}{2\pi\sqrt{1 - z^2\xi_3^2}} \int_{\mathscr{C}(\boldsymbol{\xi}, z\xi_3)} f(\boldsymbol{\eta})\,\mathrm{d}\ell(\boldsymbol{\eta}), \qquad \boldsymbol{\xi} \in \mathbb{S}^2, \tag{2.3}$$

which computes the mean values of f along all circles passing through $\boldsymbol{\zeta}_z$. Note that the denominator in (2.3) is equal to the circumference of the circle $\mathscr{C}(\boldsymbol{\xi}, z\xi_3)$.

The Funk–Radon transform. Setting the parameter $z = 0$, the point $\boldsymbol{\zeta}_0 = (0,0,0)^\top$ is the center of the sphere. Hence, the spherical transform $\mathcal{U}_0$ integrates along all great circles of the sphere. This special case is the Funk–Radon transform

$$\mathcal{F}f(\boldsymbol{\xi}) = \mathcal{U}_0 f(\boldsymbol{\xi}) = \frac{1}{2\pi} \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = 0} f(\boldsymbol{\eta}) \, \mathrm{d}\ell(\boldsymbol{\eta}), \qquad \boldsymbol{\xi} \in \mathbb{S}^2, \tag{2.4}$$

which is also known by the terms Funk transform, Minkowski–Funk transform or spherical Radon transform, where the latter term is occasionally also refers to means over spheres in $\mathbb{R}^3$, cf. [16].

3 Relation with the Funk–Radon transform

Let $z \in [0,1)$ and $f \in C(\mathbb{S}^2)$. We define the two transformations $\mathcal{M}_z, \mathcal{N}_z \colon C(\mathbb{S}^2) \to C(\mathbb{S}^2)$ by

$$\mathcal{M}_z f(\boldsymbol{\xi}(\varphi, t)) = \frac{\sqrt{1-z^2}}{1+zt} f\left(\boldsymbol{\xi}\left(\varphi, \frac{t+z}{1+zt}\right)\right), \qquad \boldsymbol{\xi} \in \mathbb{S}^2 \tag{3.1}$$

and

$$\mathcal{N}_z f(\boldsymbol{\xi}(\varphi, t)) = \frac{1}{\sqrt{1-z^2t^2}} f\left(\boldsymbol{\xi}\left(\varphi, t\sqrt{\frac{1-z^2}{1-z^2t^2}}\right)\right), \qquad \boldsymbol{\xi} \in \mathbb{S}^2. \tag{3.2}$$

Theorem 3.1. *Let $z \in [0,1)$. Then the factorization of the spherical transform*

$$\mathcal{U}_z = \mathcal{N}_z \mathcal{F} \mathcal{M}_z \tag{3.3}$$

holds, where $\mathcal{F}$ is the Funk–Radon transform (2.4).

Proof. Let $f \in C(\mathbb{S}^2)$ and $\boldsymbol{\xi} \in \mathbb{S}^2$. By the definition of $\mathcal{U}_z$ in (2.3), we have

$$2\pi \sqrt{1 - z^2 \xi_3^2}\, \mathcal{U}_z f(\boldsymbol{\xi}) = \int_{\mathscr{C}(\boldsymbol{\xi}, z\xi_3)} f(\boldsymbol{\eta}) \, \mathrm{d}\ell(\boldsymbol{\eta}), \tag{3.4}$$

where $\mathrm{d}\ell$ is the arc-length. We are going to use cylinder coordinates $\boldsymbol{\eta}(\psi, u) \in \mathbb{S}^2$, see (2.1). Let

$$[0,1] \to \mathscr{C}(\boldsymbol{\xi}, z\xi_3) \subset \mathbb{S}^2, \quad s \mapsto \boldsymbol{\eta}(\psi(s), u(s))$$

be some parameterization of the circle $\mathscr{C}(\boldsymbol{\xi}, z\xi_3)$, which acts as domain of integration in (3.4). Then we have by (2.2)

$$2\pi \sqrt{1 - z^2 \xi_3^2}\, \mathcal{U}_z f(\boldsymbol{\xi}) = \int_0^1 f(\boldsymbol{\eta}(\psi, u)) \sqrt{(1-u^2)\left(\frac{\mathrm{d}\psi}{\mathrm{d}s}\right)^2 + \frac{1}{1-u^2}\left(\frac{\mathrm{d}u}{\mathrm{d}s}\right)^2} \, \mathrm{d}s.$$

We perform the substitution $u(s) \mapsto v(s)$, where

$$u = \frac{v+z}{1+zv}. \tag{3.5}$$

By the chain rule,

$$\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{\mathrm{d}u}{\mathrm{d}v}\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{1+zv-z(z+v)}{(1+zv)^2}\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{1-z^2}{(1+zv)^2}\frac{\mathrm{d}v}{\mathrm{d}s}.$$

Thus, we have

$$
\begin{aligned}
&2\pi\sqrt{1-z^2\xi_3^2}\,\mathcal{U}_z f(\boldsymbol{\xi})\\
&=\int_0^1 f\left(\boldsymbol{\eta}\left(\psi,\frac{v+z}{1+zv}\right)\right)\sqrt{\frac{1+z^2v^2-z^2-v^2}{(1+zv)^2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}s}\right)^2+\frac{(1+zv)^2}{1+z^2v^2-z^2-v^2}\frac{(1-z^2)^2}{(1+zv)^4}\left(\frac{\mathrm{d}v}{\mathrm{d}s}\right)^2}\,\mathrm{d}s\\
&=\int_0^1 f\left(\boldsymbol{\eta}\left(\psi,\frac{v+z}{1+zv}\right)\right)\sqrt{\frac{(1-v^2)(1-z^2)}{(1+zv)^2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}s}\right)^2+\frac{1-z^2}{(1-v^2)(1+zv)^2}\left(\frac{\mathrm{d}v}{\mathrm{d}s}\right)^2}\,\mathrm{d}s\\
&=\int_0^1 f\left(\boldsymbol{\eta}\left(\psi,\frac{v+z}{1+zv}\right)\right)\frac{\sqrt{1-z^2}}{1+zv}\sqrt{(1-v^2)\left(\frac{\mathrm{d}\psi}{\mathrm{d}s}\right)^2+\frac{1}{1-v^2}\left(\frac{\mathrm{d}v}{\mathrm{d}s}\right)^2}\,\mathrm{d}s. \qquad (3.6)
\end{aligned}
$$

Plugging (3.2) into the last equation, we obtain by (2.2)

$$
2\pi\sqrt{1-z^2\xi_3^2}\,\mathcal{U}_z(\boldsymbol{\xi})=\int_{\mathscr{D}_z(\boldsymbol{\xi})}\mathcal{M}_z f(\boldsymbol{\eta})\,\mathrm{d}\ell(\boldsymbol{\eta}), \qquad (3.7)
$$

where

$$
\mathscr{D}_z(\boldsymbol{\xi})=\left\{\boldsymbol{\eta}(\psi,v)\in\mathbb{S}^2:\boldsymbol{\eta}\left(\psi,\frac{v+z}{1+vz}\right)\in\mathscr{C}(\boldsymbol{\xi},z\xi_3)\right\}.
$$

In the second part of the proof, we are going to show that

$$
\mathscr{D}_z(\boldsymbol{\xi}(\varphi,t))=\mathscr{C}\left(\boldsymbol{\xi}\left(\varphi,t\sqrt{\frac{1-z^2}{1-z^2t^2}}\right),0\right), \qquad (3.8)
$$

which is a great circle on the sphere. The point $\boldsymbol{\eta}(\psi,v)\in\mathbb{S}^2$ lies in the set $\mathscr{D}_z(\boldsymbol{\xi}(\varphi,t))$ if and only if

$$
\left\langle\boldsymbol{\eta}\left(\psi,\frac{v+z}{1+vz}\right),\boldsymbol{\xi}(\varphi,t)\right\rangle=zt.
$$

By the definition of the cylinder coordinates (2.1), this equation can be rewritten as

$$
(\cos\psi\cos\varphi+\sin\psi\sin\varphi)\sqrt{1-\left(\frac{v+z}{1+vz}\right)^2}\sqrt{1-t^2}+t\frac{v+z}{1+vz}=zt.
$$

By the addition formula for the cosine, this is equivalent to

$$
\cos(\varphi-\psi)\frac{\sqrt{1-v^2}\sqrt{1-z^2}}{1+vz}\sqrt{1-t^2}+t\frac{v-vz^2}{1+vz}=0.
$$

Now we multiply the last equation with $(1+vz)(1-z^2)^{-1/2}(1-z^2t^2)^{-1/2}$ and obtain

$$
\begin{aligned}
0&=\cos(\varphi-\psi)\sqrt{1-v^2}\frac{\sqrt{1-t^2}}{\sqrt{1-z^2t^2}}+tv\frac{\sqrt{1-z^2}}{\sqrt{1-z^2t^2}}\\
&=\cos(\varphi-\psi)\sqrt{1-v^2}\sqrt{1-t^2\frac{1-z^2}{1-z^2t^2}}+vt\sqrt{\frac{1-z^2}{1-z^2t^2}},
\end{aligned}
$$

which is exactly the equation of the great circle

$$
\mathscr{C}\left(\boldsymbol{\xi}\left(\varphi,t\sqrt{\frac{1-z^2}{1-z^2t^2}}\right),0\right).
$$

This shows (3.8). Combining (3.7) and (3.8), we obtain

$$
\mathcal{U}_z(\boldsymbol{\xi}(\varphi,t))=\frac{1}{\sqrt{1-z^2\xi_3^2}}\mathcal{F}\mathcal{M}_z f\left(\boldsymbol{\xi}\left(\varphi,t\sqrt{\frac{1-z^2}{1-z^2t^2}}\right)\right). \qquad \blacksquare
$$

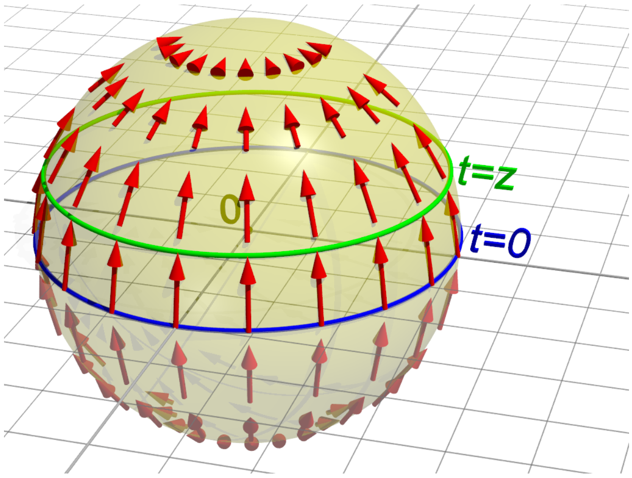

Figure 3.1: The red arrows indicate the transformation $\mathbf{h}_z\colon \mathbb{S}^2 \to \mathbb{S}^2$, which was defined in (3.9) and maps the equator (blue) to the circle of latitude z (green), for $z = 0.33$.

The proof of the decomposition of the spherical transform $\mathcal{U}_z$ in Theorem 3.1 is based on the substitution (3.5), which can be expressed as the transformation

$$\mathbf{h}_z\colon \mathbb{S}^2 \to \mathbb{S}^2, \quad \mathbf{h}_z(\boldsymbol{\xi}(\varphi,t)) = \boldsymbol{\xi}\left(\varphi, \frac{t+z}{1+zt}\right) \tag{3.9}$$

where $z \in [0,1)$. Then $\mathcal{M}_z f(\boldsymbol{\xi}) = f \circ \mathbf{h}_z(\boldsymbol{\xi}) \cdot \sqrt{1-z^2}/(1+z\xi_3)$. By (3.6), the map $\mathbf{h}_z$ is conformal, i.e., it preserves angles. The transformation $\mathbf{h}_z$ moves the points on the sphere northwards while leaving the north and south pole unchanged. It maps the equator $t=0$ to the circle of latitude $t = z$, see Figure 3.1. Moreover, $\mathbf{h}_z$ maps all great circles to circles passing through $\boldsymbol{\zeta}_z$. An interpretation of $\mathbf{h}_z$ in terms of the stereographic projection will be given in Section 6.

4 Properties of the spherical transform

4.1 The operators $\mathcal{M}_z$ and $\mathcal{N}_z$

In the following two lemmas, we investigate the two transformations $\mathcal{M}_z$ and $\mathcal{N}_z$ from Theorem 3.1 as operators $L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ and compute their inverses.

Lemma 4.1. *The operator $\mathcal{M}_z$ given in* (3.1) *can be extended to a unitary operator $\mathcal{M}_z\colon L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$. Its inverse is given by*

$$\mathcal{M}_z^{-1} g(\boldsymbol{\xi}(\varphi,u)) = g\left(\boldsymbol{\xi}\left(\varphi, \frac{u-z}{1-zu}\right)\right) \frac{\sqrt{1-z^2}}{1-zu}, \qquad \boldsymbol{\xi}(\varphi,u) \in \mathbb{S}^2. \tag{4.1}$$

Proof. Let $f \in C(\mathbb{S}^2)$. In order to proof that $\mathcal{M}_z$ is unitary, we are going to show first that $\|\mathcal{M}_z f\|_{L^2(\mathbb{S}^2)} = \|f\|_{L^2(\mathbb{S}^2)}$, which implies that $\mathcal{M}_z$ is an isometry on $L^2(\mathbb{S}^2)$ since the continuous functions $C(\mathbb{S}^2)$ are dense in $L^2(\mathbb{S}^2)$. In the integral

$$\|\mathcal{M}_z f\|_{L^2(\mathbb{S}^2)}^2 = \int_0^{2\pi} \int_{-1}^1 \left| f\left(\boldsymbol{\xi}\left(\varphi, \frac{t+z}{1+zt}\right)\right) \frac{\sqrt{1-z^2}}{1+zt} \right|^2 \mathrm{d}t\, \mathrm{d}\varphi,$$

we substitute

$$t = \frac{u-z}{1-zu}, \qquad \mathrm{d}t = \frac{1-z^2}{(1-zu)^2} \mathrm{d}u \tag{4.2}$$

and obtain

$$\begin{aligned}\|\mathcal{M}_z f\|^2_{L^2(\mathbb{S}^2)} &= \int_0^{2\pi}\int_{-1}^1 \left| f\left(\boldsymbol{\xi}\left(\varphi, \frac{u-z+z(1-zu)}{1-zu+z(u-z)}\right)\right)\right|^2 \frac{(1-z^2)(1-zu)^2}{(1-zu+z(u-z))^2}\frac{1-z^2}{(1-zu)^2}\,\mathrm{d}u\,\mathrm{d}\varphi \\ &= \int_0^{2\pi}\int_{-1}^1 |f(\boldsymbol{\xi}(\varphi,u))|^2\,\mathrm{d}u\,\mathrm{d}\varphi = \|f\|^2_{L^2(\mathbb{S}^2)}.\end{aligned}$$

For the inversion formula (4.1), we apply the substitution (4.2) to (3.1) and obtain

$$\mathcal{M}_z f\left(\boldsymbol{\xi}\left(\varphi, \frac{u-z}{1-zu}\right)\right)\frac{1+z\frac{u-z}{1-zu}}{\sqrt{1-z^2}} = f(\boldsymbol{\xi}(\varphi,u)), \qquad \boldsymbol{\xi}(\varphi,u)\in\mathbb{S}^2.$$

This equality implies that $\mathcal{M}_z$ is surjective and hence unitary. ■

Lemma 4.2. *The operator $\mathcal{N}_z$ given in* (3.2) *can be extended to a bijective and continuous operator $\mathcal{N}_z\colon L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)$ satisfying*

$$\|f\|_{L^2(\mathbb{S}^2)} \le \|\mathcal{N}_z f\|_{L^2(\mathbb{S}^2)} \le (1-z^2)^{-1/4}\,\|f\|_{L^2(\mathbb{S}^2)} \tag{4.3}$$

for all $f\in L^2(\mathbb{S}^2)$. Its inverse is given by

$$\mathcal{N}_z^{-1} g(\boldsymbol{\xi}(\varphi,u)) = g\left(\boldsymbol{\xi}\left(\varphi, \frac{u}{\sqrt{1-z^2+z^2u^2}}\right)\right)\sqrt{\frac{1-z^2}{1-z^2+z^2u^2}}. \tag{4.4}$$

Proof. Let $f\in C(\mathbb{S}^2)$. In the integral

$$\|\mathcal{N}_z f\|^2_{L^2(\mathbb{S}^2)} = \int_0^{2\pi}\int_{-1}^1 \left|\frac{1}{\sqrt{1-z^2t^2}} f\left(\boldsymbol{\xi}\left(\varphi, t\sqrt{\frac{1-z^2}{1-z^2t^2}}\right)\right)\right|^2 \mathrm{d}t\,\mathrm{d}\varphi,$$

we substitute

$$t = \frac{u}{\sqrt{1-z^2+u^2z^2}}$$

with the derivative

$$\frac{\mathrm{d}t}{\mathrm{d}u} = \frac{\sqrt{1-z^2+u^2z^2} - \frac{2u^2z^2}{2\sqrt{1-z^2+u^2z^2}}}{1-z^2+u^2z^2} = \frac{1-z^2}{(1-z^2+u^2z^2)^{3/2}}.$$

Hence, we have

$$\begin{aligned}\|\mathcal{N}_z f\|^2_{L^2(\mathbb{S}^2)} &= \int_0^{2\pi}\int_{-1}^1 |f(\boldsymbol{\xi}(\varphi,u))|^2 \frac{1}{1-z^2\frac{u^2}{1-z^2+u^2z^2}}\frac{1-z^2}{(1-z^2+u^2z^2)^{3/2}}\,\mathrm{d}u\,\mathrm{d}\varphi \\ &= \int_0^{2\pi}\int_{-1}^1 |f(\boldsymbol{\xi}(\varphi,u))|^2 \frac{1}{\sqrt{1-z^2+u^2z^2}}\,\mathrm{d}u\,\mathrm{d}\varphi. \end{aligned} \tag{4.5}$$

Since the weight $(1-z^2+u^2z^2)^{-1/2}$ in the integrand of (4.5) for $u\in[-1,1]$ attains its maximum value of $(1-z^2)^{-1/2}$ at $u=0$ and its minimum 1 at $u=\pm1$, we can conclude (4.3). For the inversion formula (4.4), we apply the substitution from the first part of the proof to (3.2) and obtain

$$f(\boldsymbol{\xi}(\varphi,u)) = \sqrt{1-\frac{z^2u^2}{1-z^2+z^2u^2}}\,\mathcal{N}_z f\left(\boldsymbol{\xi}\left(\varphi, \frac{u}{\sqrt{1-z^2+z^2u^2}}\right)\right). \qquad ■$$

4.2 Nullspace

Lemma 4.3. *Let $z \in [0,1)$. We define*

$$\mathbf{R}_z\colon \mathbb{S}^2 \to \mathbb{S}^2, \quad \boldsymbol{\xi}(\varphi,t) \mapsto \boldsymbol{\xi}\left(\varphi+\pi, \frac{2z-t-tz^2}{1-2tz+z^2}\right).$$

Then $\mathbf{R}_z$ is the point reflection of the sphere across the point $\boldsymbol{\zeta}_z = (0,0,z)^\top$, i.e., for every $\boldsymbol{\xi} \in \mathbb{S}^2$ the three points $\boldsymbol{\xi}$, $\mathbf{R}_z\boldsymbol{\xi}$ and $\boldsymbol{\zeta}_z$ are located on one line.

Proof. We are going to show that $\mathbf{R}_z\boldsymbol{\xi}$ can be written as an affine combination of $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}_z$. We assume that $\varphi = 0$, the general case then follows by rotation about the north–south axis. We have

$$\boldsymbol{\xi}(0,t) = (\sqrt{1-t^2},\, 0,\, t)^\top.$$

Setting

$$\alpha = \frac{z^2-1}{1-2tz+z^2} < 0,$$

we obtain in 3D coordinates

$$\begin{aligned}
\alpha\boldsymbol{\xi} + (1-\alpha)\boldsymbol{\zeta}_z &= \left(\frac{z^2-1}{1-2tz+z^2}\sqrt{1-t^2}, 0, \frac{z^2-1}{1-2tz+z^2}(t-z)+z\right)^\top \\
&= \left(\sqrt{1-\frac{(1-2tz+z^2)^2-(z^2-1)^2(1-t^2)}{(1-2tz+z^2)^2}}, 0, \frac{tz^2-t-z^3+2z-2tz^2+z^3}{1-2tz+z^2}\right)^\top \\
&= \left(\sqrt{1-\left(\frac{2z-t-tz^2}{1-2tz+z^2}\right)^2}, 0, \frac{2z-t-tz^2}{1-2tz+z^2}\right)^\top = \mathbf{R}_z\boldsymbol{\xi}.
\end{aligned}$$

■

The following theorem shows that the functions in the nullspace of the spherical transform $\mathcal{U}_z$ can be imagined as the set of functions that are odd with respect to the point reflection $\mathbf{R}_z$ and the multiplication with some weight.

Theorem 4.4. *Let $z \in [0,1)$. The nullspace of the spherical transform $\mathcal{U}_z$ consists of all functions $f \in L^2(\mathbb{S}^2)$ for which $\mathcal{M}_z f$ is odd. The latter is equivalent to the condition that for almost every $\boldsymbol{\xi} \in \mathbb{S}^2$*

$$f(\boldsymbol{\xi}) = f(\mathbf{R}_z(\boldsymbol{\xi}))\frac{1-z^2}{2z\xi_3-1-z^2}. \tag{4.6}$$

Proof. Let $f \in L^2(\mathbb{S}^2)$ with $\mathcal{U}_z f = 0$. By the factorization (3.3), we have $\mathcal{N}_z\mathcal{F}\mathcal{M}_z f = 0$. Since $\mathcal{N}_z$ is injective by Lemma 4.2, we conclude that $\mathcal{F}\mathcal{M}_z f = 0$. It is well-known that the nullspace of the Funk–Radon transform $\mathcal{F}$ consists of the odd functions, cf. [6]. It follows that $\mathcal{U}_z f = 0$ if and only if $\mathcal{M}_z f$ is odd. That is, for almost every $\boldsymbol{\xi} \in \mathbb{S}^2$, we have

$$\mathcal{M}_z f(\boldsymbol{\xi}) = -\mathcal{M}_z f(-\boldsymbol{\xi})$$

and hence in cylinder coordinates

$$f\left(\boldsymbol{\xi}\left(\varphi, \frac{t+z}{1+tz}\right)\right)\frac{\sqrt{1-z^2}}{1+zt} = -f\left(\boldsymbol{\xi}\left(\varphi+\pi, \frac{-t+z}{1-tz}\right)\right)\frac{\sqrt{1-z^2}}{1-zt}. \tag{4.7}$$

By setting

$$t = \frac{z-u}{uz-1},$$

equation (4.7) becomes

$$f\left(\boldsymbol{\xi}\left(\varphi, \frac{z-u+z(uz-1)}{uz-1+(z-u)z}\right)\right) = -f\left(\boldsymbol{\xi}\left(\varphi+\pi, \frac{-z+u+z(uz-1)}{uz-1-z(z-u)}\right)\right)\frac{uz-1+z(z-u)}{uz-1-z(z-u)},$$

which is equivalent to

$$f(\boldsymbol{\xi}(\varphi,u)) = f\left(\boldsymbol{\xi}\left(\varphi+\pi, \frac{u-2z+uz^2}{2uz-1-z^2}\right)\right)\frac{1-z^2}{2uz-z^2-1}. \qquad \blacksquare$$

Remark 4.5. In our considerations, we have left out the case $z = 1$, in which the spherical transform $\mathcal{U}_1$ computes the mean values along all circles passing through the north pole $(0,0,1)^\top$. This case $\mathcal{U}_1$ is also known as the spherical slice transform. The spherical transform $\mathcal{U}_z$ for $z < 1$ has a nonempty nullspace according to the previous theorem, whereas the spherical slice transform is injective for all bounded functions, which was shown in [20].

4.3 Range

In order to obtain a description of the range of the spherical transform $\mathcal{U}_z$, we introduce Sobolev spaces on the sphere. For more details on such Sobolev spaces, we refer the reader to [12]. We start by defining the associated Legendre polynomials

$$P_n^k(t) = \frac{(-1)^k}{2^k k!}\left(1-t^2\right)^{k/2}\frac{\mathrm{d}^{n+k}}{\mathrm{d}t^{n+k}}\left(t^2-1\right)^k, \qquad t\in[-1,1],$$

for all $(n,k) \in I$, where

$$I = \{(n,k) \mid n \in \mathbb{N}_0,\ k = -n,\dots,n\}$$

and $\mathbb{N}_0$ denotes the set of non-negative integers. The spherical harmonics

$$Y_n^k(\boldsymbol{\xi}(\varphi,t)) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-k)!}{(n+k)!}}\,P_n^k(t)\mathrm{e}^{\mathrm{i}k\varphi}, \qquad \boldsymbol{\xi}(\varphi,t)\in\mathbb{S}^2,$$

form an orthonormal basis in the Hilbert space $L^2(\mathbb{S}^2)$. Accordingly, any function $f \in L^2(\mathbb{S}^2)$ can be expressed by its Fourier series

$$f = \sum_{n=0}^{\infty}\sum_{k=-n}^{n}\hat{f}(n,k)Y_n^k$$

with the Fourier coefficients

$$\hat{f}(n,k) = \int_{\mathbb{S}^2} f(\boldsymbol{\xi})\overline{Y_n^k(\boldsymbol{\xi})}\,\mathrm{d}\boldsymbol{\xi}.$$

For $s \geq 0$, the Sobolev space $H^s(\mathbb{S}^2)$ is defined as the space of all functions $f \in L^2(\mathbb{S}^2)$ with finite Sobolev norm

$$\|f\|_{H^s(\mathbb{S}^2)}^2 = \sum_{n=0}^{\infty}\sum_{k=-n}^{n}\left(n+\tfrac{1}{2}\right)^{2s}\left|\hat{f}(n,k)\right|^2.$$

Obviously, $H^0(\mathbb{S}^2) = L^2(\mathbb{S}^2)$. Furthermore, we set $L^2_{\mathrm{e}}(\mathbb{S}^2)$ and $H^s_{\mathrm{e}}(\mathbb{S}^2)$ as the respective spaces restricted to even functions.

Before we give the theorem about the range of $\mathcal{U}_z$, we need the following technical lemma.

Lemma 4.6. *Let $z \in [0,1)$. The restriction of $\mathcal{N}_z$, which was defined in (3.2), to an operator*

$$\mathcal{N}_z\colon H^{1/2}_{\mathrm{e}}(\mathbb{S}^2) \to H^{1/2}_{\mathrm{e}}(\mathbb{S}^2)$$

is continuous and bijective.

Proof. The structure of this proof is as follows. At first, we consider the Sobolev space $H^1(\mathbb{S}^2)$, where we compute the norms of f and $\mathcal{N}_z f$, from which we subsequently derive that $\mathcal{N}_z$ is continuous on the Sobolev space $H^1(\mathbb{S}^2)$. Afterwards, we see the continuity of the inverse $\mathcal{N}_z^{-1}$. Hence, $\mathcal{N}_z \colon H^1_{\mathrm{e}}(\mathbb{S}^2) \to H^1_{\mathrm{e}}(\mathbb{S}^2)$ is a continuous bijection. In the last part, we utilize interpolation theory to transfer the obtained continuity to the space $H^{1/2}_{\mathrm{e}}(\mathbb{S}^2)$.

The Sobolev norm in H^1. Let $f \in C^\infty(\mathbb{S}^2)$. In order to show that $\mathcal{N}_z$ is continuous on $H^1(\mathbb{S}^2)$, we use a different characterization of the Sobolev norm (see [12, Theorems 4.12 and 6.12])

$$\|f\|^2_{H^1(\mathbb{S}^2)} = \|\nabla^* f\|^2_{L^2(\mathbb{S}^2)} + \frac{1}{4}\|f\|^2_{L^2(\mathbb{S}^2)},$$

with the surface gradient

$$\nabla^* = \boldsymbol{e}_\varphi \frac{1}{\sqrt{1-t^2}}\frac{\partial}{\partial\varphi} + \boldsymbol{e}_t\sqrt{1-t^2}\frac{\partial}{\partial t},$$

where $\boldsymbol{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)^\top$ and $\boldsymbol{e}_t = (-t\cos\varphi, -t\sin\varphi, \sqrt{1-t^2})^\top$ are the orthonormal tangent vectors of the sphere with respect to the cylinder coordinates (φ, t). Let $f \in C^\infty(\mathbb{S}^2)$. Then we have

$$\|f\|^2_{H^1(\mathbb{S}^2)} = \int_0^{2\pi}\int_{-1}^1 \left[\frac{1}{4}|f(\boldsymbol{\xi}(\varphi,t))|^2 + \frac{1}{1-t^2}\left|\frac{\partial f(\boldsymbol{\xi}(\varphi,t))}{\partial\varphi}\right|^2 + (1-t^2)\left|\frac{\partial f(\boldsymbol{\xi}(\varphi,t))}{\partial t}\right|^2\right] \mathrm{d}t\,\mathrm{d}\varphi. \tag{4.8}$$

As in the proof of Lemma 4.2, we define

$$u = t\sqrt{\frac{1-z^2}{1-z^2t^2}},$$

which implies

$$t = \frac{u}{\sqrt{1-z^2+z^2u^2}}.$$

Hence,

$$\frac{\partial u}{\partial t} = \frac{\sqrt{1-z^2}}{(1-z^2t^2)^{3/2}} = \frac{(1-z^2+z^2u^2)^{3/2}}{1-z^2}.$$

Furthermore, we set

$$v = \frac{1}{\sqrt{1-z^2t^2}} = \sqrt{\frac{1-z^2+z^2u^2}{1-z^2}},$$

and we have

$$\frac{\partial v}{\partial t} = \frac{z^2 t}{(1-z^2t^2)^{3/2}} = \frac{z^2u(1-z^2+z^2u^2)}{(1-z^2)^{3/2}}.$$

Hence, we can write

$$\mathcal{N}_z f(\boldsymbol{\xi}(\varphi,t)) = v f(\boldsymbol{\xi}(\varphi,u)).$$

Thus, we have

$$\|\nabla^*\mathcal{N}_z f\|^2_{L^2(\mathbb{S}^2)} = \int_0^{2\pi}\int_{-1}^1 \left[\frac{v^2}{1-t^2}\left|\frac{\partial f(\boldsymbol{\xi}(\varphi,u))}{\partial\varphi}\right|^2 + (1-t^2)\left|\frac{\partial v}{\partial t}f(\boldsymbol{\xi}(\varphi,u)) + v\frac{\partial f(\boldsymbol{\xi}(\varphi,u))}{\partial u}\frac{\partial u}{\partial t}\right|^2\right]\mathrm{d}t\,\mathrm{d}\varphi.$$

By the above formulas for u and v as well as their derivatives, we obtain

$$\begin{aligned}\|\nabla^*\mathcal{N}_z f\|^2_{L^2(\mathbb{S}^2)} = \int_0^{2\pi}\int_{-1}^{1}\Bigg[&\frac{1-z^2+z^2u^2}{1-z^2}\frac{1-z^2+z^2u^2}{(1-z^2)(1-u^2)}\left|\frac{\partial f(\boldsymbol{\xi}(\varphi,u))}{\partial\varphi}\right|^2\\ &+\frac{(1-z^2)(1-u^2)}{1-z^2+z^2u^2}\Bigg|\frac{z^2u(1-z^2+z^2u^2)}{(1-z^2)^{3/2}}f(\boldsymbol{\xi}(\varphi,u))\\ &\qquad+\sqrt{\frac{1-z^2+z^2u^2}{1-z^2}}\frac{(1-z^2+z^2u^2)^{3/2}}{1-z^2}\frac{\partial f(\boldsymbol{\xi}(\varphi,u))}{\partial u}\Bigg|^2\Bigg]\\ &\cdot\frac{1-z^2}{(1-z^2+z^2u^2)^{3/2}}\,\mathrm{d}u\,\mathrm{d}\varphi\end{aligned}$$

and hence

$$\begin{aligned}\|\nabla^*\mathcal{N}_z f\|^2_{L^2(\mathbb{S}^2)} = \int_0^{2\pi}\int_{-1}^{1}\Bigg[&\frac{\sqrt{1-z^2+z^2u^2}}{(1-z^2)(1-u^2)}\left|\frac{\partial f(\boldsymbol{\xi}(\varphi,u))}{\partial\varphi}\right|^2\\ &+\frac{1-u^2}{1-z^2}\left|\frac{z^2uf(\boldsymbol{\xi}(\varphi,u))}{(1-z^2+z^2u^2)^{1/4}}+(1-z^2+z^2u^2)^{3/4}\frac{\partial f(\boldsymbol{\xi}(\varphi,u))}{\partial u}\right|^2\Bigg]\,\mathrm{d}u\,\mathrm{d}\varphi.\end{aligned}\tag{4.9}$$

Boundedness on H^1. By Lemma 4.2, we know that $\mathcal{N}_z$ is bounded in $L^2(\mathbb{S}^2)$. In order to prove the boundedness $\mathcal{N}_z$ of in $H^1(\mathbb{S}^2)$, we still have to show that $\|\nabla^*\mathcal{N}_z f\|_{L^2(\mathbb{S}^2)}$ is bounded by a multiple of $\|f\|_{H^1(\mathbb{S}^2)}$. By (4.9) and the inequality $|a+b|^2\le 2|a|^2+2|b|^2$ for all $a,b\in\mathbb{C}$, we have the upper bound

$$\begin{aligned}\|\nabla^*\mathcal{N}_z f\|^2_{H^1(\mathbb{S}^2)} \le \int_0^{2\pi}\int_{-1}^{1}\Bigg[&\frac{2z^4u^2(1-u^2)\,|f(\boldsymbol{\xi}(\varphi,u))|^2}{\sqrt{1-z^2+z^2u^2}(1-z^2)}+\frac{\sqrt{1-z^2+z^2u^2}}{(1-z^2)(1-u^2)}\left|\frac{\partial f(\boldsymbol{\xi}(\varphi,u))}{\partial\varphi}\right|^2\\ &+\frac{2(1-z^2+z^2u^2)^{3/2}(1-u^2)}{1-z^2}\left|\frac{\partial f(\boldsymbol{\xi}(\varphi,u))}{\partial u}\right|^2\Bigg]\,\mathrm{d}u\,\mathrm{d}\varphi.\end{aligned}$$

We denote the coefficients of f and its derivatives in the integrand of the last equation with $\alpha_z(u),\beta_z(u)$ and $\gamma_z(u)$, such that

$$\begin{aligned}\|\nabla^*\mathcal{N}_z f\|^2_{L^2(\mathbb{S}^2)} \le \int_0^{2\pi}\int_{-1}^{1}\Bigg[&\alpha_z(u)\,|f(\boldsymbol{\xi}(\varphi,u))|^2+\beta_z(u)\left|\frac{\partial f(\boldsymbol{\xi}(\varphi,u))}{\partial\varphi}\right|^2\\ &+\gamma_z(u)\left|\frac{\partial f(\boldsymbol{\xi}(\varphi,u))}{\partial u}\right|^2\Bigg]\,\mathrm{d}u\,\mathrm{d}\varphi.\end{aligned}\tag{4.10}$$

Comparing (4.10) with (4.8), we obtain

$$\|\nabla^*\mathcal{N}_z f\|^2_{L^2(\mathbb{S}^2)} \le \sup_{u\in(-1,1)}\left(\max\left\{4\alpha_z(u),(1-u^2)\beta_z(u),\frac{\gamma_z(u)}{1-u^2}\right\}\right)\|f\|^2_{H^1(\mathbb{S}^2)}.$$

Thus, if the arguments of the maximum in the previous equation are bounded uniformly with respect to $u\in(-1,1)$, it follows that the operator $\mathcal{N}_z$ is bounded on $H^1(\mathbb{S}^2)$, since the space $C^\infty(\mathbb{S}^2)$ is dense in $H^1(\mathbb{S}^2)$. Firstly, the term

$$4\alpha_z(u)=\frac{8z^4u^2(1-u^2)}{\sqrt{1-z^2+z^2u^2}(1-z^2)}$$

is bounded since its numerator is a polynomial in u and its denominator

$$\sqrt{(u^2-1)\,z^2+1} \geq \sqrt{1-z^2} > 0$$

is bounded away from zero. Furthermore, we see that both the terms

$$(1-u^2)\beta_z(u) = \frac{\sqrt{1-z^2+z^2u^2}}{1-z^2}$$

and

$$\frac{\gamma_z(u)}{1-u^2} = \frac{2(1-z^2+z^2u^2)^{3/2}}{1-z^2}$$

are square roots of polynomials in u and hence uniformly bounded.
Surjectivity on H^1. Lemma 4.2 implies that $\mathcal{N}_z$ is injective. For proving that $\mathcal{N}_z\colon H^1(\mathbb{S}^2) \to H^1(\mathbb{S}^2)$ is surjective, it is sufficient to show that the inverse operator $\mathcal{N}_z^{-1}$ restricted to $H^1(\mathbb{S}^2)$ is continuous on $H^1(\mathbb{S}^2)$. Let $g \in C^\infty(\mathbb{S}^2)$, which is dense in $H^1(\mathbb{S}^2)$. With a computation that is similar to the first part of the proof and therefore skipped, we can obtain

$$\|\mathcal{N}_z^{-1}g\|^2_{H^1(\mathbb{S}^2)} = \int_0^{2\pi}\int_{-1}^{1}\left[\frac{\sqrt{1-z^2}}{4\sqrt{1-z^2t^2}}\,|g(\boldsymbol{\xi}(\varphi,t))|^2 + \frac{\sqrt{1-z^2}\sqrt{1-z^2t^2}}{1-t^2}\left|\frac{\partial g(\boldsymbol{\xi}(\varphi,t))}{\partial\varphi}\right|^2 \right.$$
$$\left. + \frac{1-t^2}{\sqrt{1-z^2}\sqrt{1-z^2t^2}}\left|(1-z^2t^2)\frac{\partial g(\boldsymbol{\xi}(\varphi,t))}{\partial t} - tz^2 g(\boldsymbol{\xi}(\varphi,t))\right|^2\right]\mathrm{d}t\,\mathrm{d}\varphi.$$

Analogously to the first part, this implies that the restriction of the operator $\mathcal{N}_z^{-1}$ to the space $H^1(\mathbb{S}^2) \to H^1(\mathbb{S}^2)$ is continuous.
Interpolation to $H^{1/2}$. Every function f in the Sobolev spaces $H^s(\mathbb{S}^2)$ can be identified with the sequence of its Fourier coefficients $\hat{f}(n,k)$, $(n,k) \in I$. Hence, the Sobolev space $H^s(\mathbb{S}^2)$ is isometrically isomorphic to a weighted L^2-space on the set I with the counting measure μ and the weight $w_s(n,k) = (n+\frac{1}{2})^s$, i.e.

$$H^s(\mathbb{S}^2) \cong L^2_{w_s}(I;\mu) = \left\{ \hat{f} \in L^2(I;\mu) \,\middle|\, \|\hat{f}\|^2_{L^2_{w_s}(I;\mu)} = \sum_{(n,k)\in I}\left|\hat{f}(n,k)\right|^2 w_s(n,k)^2 < \infty \right\}.$$

For $0 \leq s \leq t$ and $\theta \in [0,1]$, we can compute the interpolation space

$$[L^2_{w_s}(I;\mu), L^2_{w_t}(I;\mu)]_\theta = L^2_w(I;\mu),$$

where

$$w(n,k) = w_s(n,k)^{1-\theta}w_t(n,k)^\theta = \left(n+\tfrac{1}{2}\right)^{(1-\theta)s+\theta t} = w_{(1-\theta)s+\theta t}(n,k),$$

see [24, Theorem 1.18.5]. So, for $s=0$, $t=1$ and $\theta = 1/2$, the space $H^{1/2}(\mathbb{S}^2)$ is an interpolation space between $H^0(\mathbb{S}^2)$ and $H^1(\mathbb{S}^2)$. By Lemma 4.2 and the first part of this proof, the operator $\mathcal{N}_z$ is continuous on both $H^0(\mathbb{S}^2) \to H^0(\mathbb{S}^2)$ and $H^1(\mathbb{S}^2) \to H^1(\mathbb{S}^2)$. Hence, it is also continuous $H^{1/2}(\mathbb{S}^2) \to H^{1/2}(\mathbb{S}^2)$. The same holds for the inverse operator $\mathcal{N}_z^{-1}$.

In order to obtain the claimed result on $H^{1/2}_{\mathrm{e}}(\mathbb{S}^2)$, it is left to show that $\mathcal{N}_z$ is invariant for even functions. This follows from the fact that an even function plugged into (3.2) and (4.4) for $\mathcal{N}_z$ and $\mathcal{N}_z^{-1}$, respectively, yields again an even function. ■

Theorem 4.7. *Let $z \in [0,1)$. Define $\tilde{L}^2_{\mathrm{e},z}(\mathbb{S}^2)$ as the subspace of $L^2(\mathbb{S}^2)$ of functions satisfying*

$$f(\boldsymbol{\xi}(\varphi,t)) = f\left(\boldsymbol{\xi}\left(\varphi+\pi, \frac{t-2z+tz^2}{2tz-1-z^2}\right)\right)\frac{1-z^2}{1+z^2-2tz}$$

almost everywhere on $\mathbb{S}^2$. *The spherical transform*

$$\mathcal{U}_z\colon \tilde{L}^2_{\mathrm{e},z}(\mathbb{S}^2) \to H^{1/2}_{\mathrm{e}}(\mathbb{S}^2)$$

is linear, continuous and bijective.

Proof. This proof is based on the decomposition $\mathcal{U}_z = \mathcal{N}_z\mathcal{F}\mathcal{M}_z$ derived in Theorem 3.1. Analogously to the proof of Theorem 4.4, we see that $\mathcal{M}_z^{-1}L^2_{\mathrm{e}}(\mathbb{S}^2) = \tilde{L}^2_{\mathrm{e},z}(\mathbb{S}^2)$. Furthermore, the operator $\mathcal{M}_z\colon \tilde{L}^2_{\mathrm{e},z}(\mathbb{S}^2) \to L^2_{\mathrm{e}}(\mathbb{S}^2)$ is continuous and bijective by Lemma 4.1. It is well-known that the Funk–Radon transform

$$\mathcal{F}\colon L^2_{\mathrm{e}}(\mathbb{S}^2) \to H^{1/2}_{\mathrm{e}}(\mathbb{S}^2)$$

is bijective and continuous, cf. [23, Lemma 4.3]. In Lemma 4.6, we have seen that $\mathcal{N}_z\colon H^{1/2}_{\mathrm{e}}(\mathbb{S}^2) \to H^{1/2}_{\mathrm{e}}(\mathbb{S}^2)$ is continuous and bijective. ■

5 An inversion formula

In the following theorem, we give an inversion formula for the spherical transform $\mathcal{U}_z$. This formula is based on the work of Helgason [8, Section III.1.C], who proved that every even function f can be reconstructed from its Funk–Radon transform $\mathcal{F}f$ via

$$f(\boldsymbol{\eta}) = \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}u}\int_0^u \int_{\langle\boldsymbol{\xi},\boldsymbol{\eta}\rangle^2=1-w^2} \mathcal{F}f(\boldsymbol{\xi})\,\mathrm{d}\ell(\boldsymbol{\xi})\frac{1}{\sqrt{u^2-w^2}}\,\mathrm{d}w\bigg|_{u=1}, \qquad \boldsymbol{\eta}\in\mathbb{S}^2. \tag{5.1}$$

Theorem 5.1. *Let* $z \in [0,1)$ *and* $f \in \tilde{L}^2_{\mathrm{e},z}(\mathbb{S}^2)$. *Then for* $\boldsymbol{\eta}(\psi,v) \in \mathbb{S}^2$

$$f(\boldsymbol{\eta}(\psi,v)) = \frac{1-z^2}{2\pi(1-zv)}\frac{\mathrm{d}}{\mathrm{d}u}\int_0^u\int_{\mathscr{S}_z(v,w)} \frac{\mathcal{U}_z f\left(\boldsymbol{\xi}\left(\varphi,\frac{t}{\sqrt{1-z^2+z^2t^2}}\right)\right)}{\sqrt{1-z^2+z^2t^2}}\,\mathrm{d}\ell(\boldsymbol{\xi}(\varphi,t))\frac{\mathrm{d}w}{\sqrt{u^2-w^2}}\bigg|_{u=1},$$

where $\mathrm{d}\ell$ *is the arc-length on the circle*

$$\mathscr{S}_z(v,w) = \left\{\boldsymbol{\xi}\in\mathbb{S}^2 \,\middle|\, \left\langle \boldsymbol{\xi},\boldsymbol{\eta}\left(\psi,\frac{z-v}{zv-1}\right)\right\rangle = \sqrt{1-w^2}\right\}.$$

Proof. We set $g = \mathcal{U}_z f$. By the decomposition from Theorem 3.1 together with Lemma 4.1, we have

$$\begin{aligned} f(\boldsymbol{\eta}(\psi,v)) &= \mathcal{M}_z^{-1}\mathcal{F}^{-1}\mathcal{N}_z^{-1}g(\boldsymbol{\eta}(\psi,v)) \\ &= \frac{\sqrt{1-z^2}}{1-zv}\mathcal{F}^{-1}\mathcal{N}_z^{-1}g\left(\boldsymbol{\eta}\left(\psi,\frac{v-z}{1-zv}\right)\right). \end{aligned} \tag{5.2}$$

By Helgason's formula (5.1), we obtain

$$f(\boldsymbol{\eta}(\psi,v)) = \frac{\sqrt{1-z^2}}{2\pi(1-zv)}\frac{\mathrm{d}}{\mathrm{d}u}\int_0^u\int_{\langle\boldsymbol{\xi},\boldsymbol{\eta}(\psi,\frac{v-z}{1-zv})\rangle^2=1-w^2} \mathcal{N}_z^{-1}g(\boldsymbol{\xi})\,\mathrm{d}\ell(\boldsymbol{\xi})\frac{\mathrm{d}w}{\sqrt{u^2-w^2}}\bigg|_{u=1}.$$

Plugging (4.4) into the above equation, we conclude that

$$f(\boldsymbol{\eta}(\psi,v)) = \frac{1-z^2}{2\pi(1-zv)}\frac{\mathrm{d}}{\mathrm{d}u}\int_0^u\int_{\mathscr{S}_z(v,w)} \frac{g\left(\boldsymbol{\xi}\left(\varphi,\frac{t}{\sqrt{1-z^2+z^2t^2}}\right)\right)}{\sqrt{1-z^2+z^2t^2}}\,\mathrm{d}\ell(\boldsymbol{\xi}(\varphi,t))\frac{\mathrm{d}w}{\sqrt{u^2-w^2}}\bigg|_{u=1}. \quad ■$$

The inversion of the Funk–Radon transform $\mathcal{F}$ is a well-studied problem. Instead of Helgason's formula we used for Theorem 5.1, other inversion schemes of $\mathcal{F}$ could also be applied to (5.2), like the reconstruction formulas in [7, 18, 14, 4]. For the numerical inversion of the Funk–Radon transform, Louis et al. [11] proposed the mollifier method, which was used with locally supported mollifiers in [17]. The mollifier method was combined with the spherical Fourier transform leading to fast algorithms in [9]. Variational splines were suggested by Pesenson [15].

6 Relation with the stereographic projection

In this section, we take a closer look at the inversion method of the spherical transform used by Salman [21] and describe its connection with our approach. His prove relies on the stereographic projection $\boldsymbol{\pi}\colon \mathbb{S}^2 \to \mathbb{R}^2$. In cylinder coordinates (2.1) on the sphere and polar coordinates

$$\boldsymbol{x}(r,\phi) = (r\cos\phi, r\sin\phi)^\top \in \mathbb{R}^2$$

in the plane $\mathbb{R}^2$, the stereographic projection is expressed by

$$\boldsymbol{\pi}(\boldsymbol{\xi}(\varphi,t)) = \boldsymbol{x}\left(\sqrt{\frac{1+t}{1-t}}, \varphi\right)$$

and conversely

$$\boldsymbol{\pi}^{-1}(\boldsymbol{x}(r,\phi)) = \boldsymbol{\xi}\left(\phi, \frac{r^2-1}{r^2+1}\right).$$

Proposition 6.1. For $z \in [0,1)$, define

$$\sigma_z = \sqrt{\frac{1+z}{1-z}}. \tag{6.1}$$

Let $f \in C^\infty(\mathbb{S}^2)$ be a smooth function supported strictly inside the spherical cap $\{\boldsymbol{\xi} \in \mathbb{S}^2 \mid \xi_3 < z\}$. Then f can be reconstructed from $\mathcal{U}_z f$ via

$$\begin{aligned}(f\circ\boldsymbol{\pi}^{-1})&\left(\frac{2\sigma_z}{1+\sqrt{1+4\left\|\boldsymbol{x}\right\|^2}}\boldsymbol{x}\right)\\ &= \frac{\sqrt{1+4\left\|\boldsymbol{x}\right\|^2}\left((1-z)\left(1+\sqrt{1+4\left\|\boldsymbol{x}\right\|^2}\right)+4(1+z)\left\|\boldsymbol{x}\right\|^2\right)}{8\pi\left(1+\sqrt{1+4\left\|\boldsymbol{x}\right\|^2}\right)}\\ &\qquad\boldsymbol{\Delta}_{\boldsymbol{x}}\int_{-\pi}^{\pi}\int_0^{\pi/2}\frac{\mathcal{U}_z f(\boldsymbol{\xi}(\varphi,\sin\theta))\log\left|x_1\cos\varphi + x_2\sin\varphi - \frac{1}{2}\sqrt{1-z^2}\tan\theta\right|d\theta\,\mathrm{d}\varphi}{\cos\theta},\end{aligned} \tag{6.2}$$

where $\boldsymbol{\Delta}_{\boldsymbol{x}} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplacian with respect to $\boldsymbol{x} = (x_1,x_2)^\top \in \mathbb{R}^2$.

The inversion formula (6.2) was derived in [21] by considering the function $f \circ \boldsymbol{\pi}^{-1} \circ \boldsymbol{\sigma}_z$, where $\boldsymbol{\sigma}_z\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the uniform scaling in the plane defined by $\boldsymbol{\sigma}_z\boldsymbol{x} = \sigma_z\boldsymbol{x}$ with the scaling factor σ_z given in (6.1). By the transformation

$$\boldsymbol{\pi}^{-1}\circ\boldsymbol{\sigma}_z\colon \mathbb{R}^2 \to \mathbb{S}^2,$$

every circle in the plane that intersects the unit circle $\{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$ in two antipodal points of the unit circle is mapped to a circle on the sphere passing through $\boldsymbol{\zeta}_z$ and vice versa.

Afterwards, an inversion formula is applied to the function $f \circ \boldsymbol{\pi}^{-1} \circ \boldsymbol{\sigma}_z$ for the Radon-like transform that integrates a function along the circles intersecting the unit circle in antipodal points.

In this light, we can look in a different way at proof of Theorem 3.1. There we have considered $f \circ \mathbf{h}_z$. The transformation $\mathbf{h}_z$ from (3.9) can be written in terms of the stereographic projection as

$$\mathbf{h}_z = \boldsymbol{\pi}^{-1} \circ \boldsymbol{\sigma}_z \circ \boldsymbol{\pi}.$$

Indeed, we have for any $\boldsymbol{\xi}(\varphi, t) \in \mathbb{S}^2$

$$\begin{aligned}\boldsymbol{\pi}^{-1} \circ \boldsymbol{\sigma}_z \circ \boldsymbol{\pi}(\boldsymbol{\xi}(\varphi, t)) &= \boldsymbol{\pi}^{-1}\left(\boldsymbol{x}\left(\sqrt{\frac{1+z}{1-z}}\sqrt{\frac{1+t}{1-t}}, \varphi\right)\right) \\ &= \boldsymbol{\xi}\left(\varphi, \frac{\frac{1+z}{1-z}\frac{1+t}{1-t} - 1}{\frac{1+z}{1-z}\frac{1+t}{1-t} + 1}\right) = \boldsymbol{\xi}\left(\varphi, \frac{t+z}{1+tz}\right) = \mathbf{h}_z(\boldsymbol{\xi}).\end{aligned}$$

So, like Salman, we first perform the stereographic projection $\boldsymbol{\pi}$ followed by the scaling $\boldsymbol{\sigma}_z^{-1}$ in the plane. But then, we use the inverse stereographic projection $\boldsymbol{\pi}^{-1}$ to come back to the sphere.

Both the stereographic projection $\boldsymbol{\pi}$ and the scaling $\boldsymbol{\sigma}_z$ map circles onto circles. Therefore, $\mathbf{h}_z$ also maps circles to circles. Furthermore, the stereographic projection maps great circles on the sphere to circles that intersect the unit circle in two antipodal points. This way, we have found another way to prove that the transformation $\mathbf{h}_z$ maps great circles onto circles through $\boldsymbol{\zeta}_z$.

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