A new eigenvalue bound for independent sets

J. Harant 1, S. Richter 2

1 Ilmenau University of Technology, Department of Mathematics, Germany
2 Chemnitz University of Technology, Department of Mathematics, Germany
email: jochen.harant@tu-ilmenau.de, sebastian.richter@mathematik.tu-chemnitz.de

Abstract. Let $G$ be a simple, undirected, and connected graph on $n$ vertices with eigenvalues $\lambda_1 \leq ... \leq \lambda_n$. Moreover, let $m$, $\delta$, and $\alpha$ denote the size, the minimum degree, and the independence number of $G$, respectively. W.H. Haemers proved $\alpha \leq \frac{-\lambda_1 \lambda_n}{\delta - \lambda_1 \lambda_n} n$ and, if $\eta$ is the largest Laplacian eigenvalue of $G$, then $\alpha \leq \frac{n - \delta}{\eta} n$ is shown by C.D. Godsil and M.W. Newman. We prove $\alpha \leq 2\sigma - 2\sigma \delta m$ for the largest normalized eigenvalue $\sigma$ of $G$, if $\delta \geq 1$. For $\varepsilon > 0$, an infinite family $F_\varepsilon$ of graphs is constructed such that $2\sigma - 2\sigma m = \alpha < (\frac{2}{3} + \varepsilon) \min\{\frac{-\lambda_1 \lambda_n}{\delta - \lambda_1 \lambda_n}, \frac{2\delta - 2\delta}{\eta} n\}$ for all $G \in F_\varepsilon$. Moreover, a sequence of graphs is presented showing that the difference between $2\sigma - 2\sigma m$ and D.M. Cvetković’s upper bound on $\alpha$ can be arbitrarily small.

Keywords. independence number, eigenvalues

1 Introduction and Result

We use standard notation and terminology of graph theory and consider a finite, simple, and undirected graph $G$ with vertex set $V = \{1, ..., n\}$ and edge set $E$, where $m = |E|$. Let $d_i$ and $\delta$ denote the degree of $i \in V$ in $G$ and the minimum degree of $G$, respectively. Furthermore, we assume that $G$ has no isolated vertices, i.e. $\delta \geq 1$. A set of vertices $I \subseteq V$ in $G$ is independent, if no two vertices in $I$ are adjacent. The independence number $\alpha$ of $G$ is the maximum cardinality of an independent set of $G$.

The independence number is one of the most fundamental and well-studied graph parameters [14]. In view of its computational hardness [11] various bounds on the independence number have been proposed, for a survey see [12].

In this paper, we are interested in upper bounds on $\alpha$ involving eigenvalues of matrices assigned to $G$ (lower bounds on $\alpha$ in terms of eigenvalues can be found in [16]). Let $\lambda_1 \leq ... \leq \lambda_n$ denote the eigenvalues of the adjacency matrix $A$ of $G$. Our starting point is the following Delsarte-Hoffman-bound [4, 8, 10, 13]. If $G$ is an $r$-regular graph, then

$$\alpha \leq \frac{-\lambda_1}{r - \lambda_1} n.$$ 

Note that $\lambda_n = r$ if $G$ is $r$-regular [4]. In [9, 10], W.H. Haemers proved the following extension of the Delsarte-Hoffman-bound for arbitrary graphs.
\[ \alpha \leq \frac{-\lambda_1 \lambda_n - n}{\delta^2 - \lambda_1 \lambda_n} \]

If all eigenvalues of \( G \) are taken into consideration, then D.M. Cvetković [4, 6, 7] proved

\[ \alpha \leq \min\{|\{i \in \{1, \ldots, n\}| \lambda_i \leq 0\}|, \{|\{i \in \{1, \ldots, n\}| \lambda_i \geq 0\}| \}. \]

Let \( D \) be the degree matrix of \( G \), that is an \((n \times n)\) diagonal matrix, where \( d_i \) is the \( i \)-th element of the main diagonal. Moreover, let \( 0 = \eta_1 \leq \ldots \leq \eta_n \) be the eigenvalues of the Laplacian matrix \( L = D - A \) of \( G \) [1].

In [8], C.D. Godsil and M.W. Newman established the following inequality, which is also a consequence of a result in [2] concerning the size of a cut in a graph.

\[ \alpha \leq \frac{\eta_n - \delta}{\eta_n} n. \]

For \( G \) without isolated vertices, the normalized Laplacian is the \((n \times n)\) matrix \( \mathcal{L} = (l_{ij}) \) with \( l_{ij} = 1 \) if \( i = j \), \( l_{ij} = -\frac{1}{\sqrt{d_i d_j}} \) if \( ij \in E \), and \( l_{ij} = 0 \) otherwise. The eigenvalues \( \sigma_1 \leq \ldots \leq \sigma_n \) of \( \mathcal{L} \) are the normalized eigenvalues of \( G \) [3, 5]. It is known that \( \sigma_1 = 0 \) and \( 1 < \sigma_n \leq 2 \) [3, 5].

Our result is the following inequality.

\[ \alpha \leq \frac{2\sigma_n - 2}{\sigma_n \delta} m. \]

For its proof, let \( \{u_1, \ldots, u_n\} \) be an orthonormal basis of \( \mathbb{R}^n \) consisting of eigenvectors of the symmetric matrix \( \mathcal{L} \) such that \( u_i \) is an eigenvector of \( \sigma_i \) for \( i = 1, \ldots, n \).

Moreover, let \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \) and \( y = \mu_1 u_1 + \ldots + \mu_n u_n \) for suitable \( \mu_1, \ldots, \mu_n \in \mathbb{R} \). It follows

\[ y^T \mathcal{L} y = \sigma_2 \mu_2^2 + \ldots + \sigma_n \mu_n^2 = -\sigma_n \mu_1^2 + (\sigma_2 - \sigma_n) \mu_2^2 + \ldots + (\sigma_n - \sigma_n) \mu_n^2, \]

\[ \leq -\sigma_n \mu_1^2 + \sigma_n (\mu_2^2 + \ldots + \mu_n^2) = -\eta_n (y^T u_1)^2 + \sigma_n y^T y. \]

Let \( M \) be an \((n \times n)\) diagonal matrix, where \( \frac{1}{\sqrt{d_i}} \) is the \( i \)-th element of the main diagonal, and \( I \) be the \((n \times n)\) identity matrix. With \( M^T = M \) and \( \mathcal{L} = I - M A M \), we obtain \( y^T M A M y \geq \sigma_n (y^T u_1)^2 + (1 - \sigma_n) y^T y \). We may choose \( u_1^T = \frac{1}{\sqrt{2m}} (\sqrt{d_1}, \ldots, \sqrt{d_n}) \) and, substituting \( y_i = x_i \sqrt{d_i} \) for \( i = 1, \ldots, n \), it follows that

\[ \sigma_n \left( \sum_{i=1}^{n} d_i x_i^2 \right)^2 + 2m (1 - \sigma_n) \sum_{i=1}^{n} d_i x_i^2 \leq 4m \sum_{ij \in E} x_i x_j \quad (1) \]

for arbitrary real numbers \( x_1, \ldots, x_n \). Let \( I \) be a maximum independent set of \( G \) and \( \overline{x} = (x_1, \ldots, x_n) \) with \( x_i = 1 \) if \( i \in I \) and \( x_i = 0 \), otherwise. By inequality (1),

\[ \sigma_n \left( \sum_{i \in I} d_i \right)^2 + 2m (1 - \sigma_n) \sum_{i \in I} d_i = \sigma_n \left( \sum_{i=1}^{n} d_i x_i^2 \right)^2 + 2m (1 - \sigma_n) \sum_{i=1}^{n} d_i x_i^2 \leq 4m \sum_{ij \in E} x_i x_j, \]

hence, with \( \sum_{ij \in E} x_i x_j = 0 \) and \( \sum_{i \in I} d_i \geq \delta \alpha \), it follows \( \alpha \leq \frac{2\sigma_n - 2}{\sigma_n \delta} m. \)
Let us remark that \( \sum_{i=1}^{n} d_i x_i = \sum_{ij \in E} (x_i + x_j) \) and \( \sum_{i=1}^{n} d_i x_i^2 = \sum_{ij \in E} (x_i^2 + x_j^2) \). Hence, if \( G \) is bipartite, then \( \sigma_n = 2 \) [5, 15] and (1) is equivalent to
\[
(\sum_{ij \in E} (x_i + x_j))^2 \leq m \sum_{ij \in E} (x_i + x_j)^2. \tag{2}
\]

Note that inequality (2) is a consequence of the Cauchy-Schwarz inequality and, therefore, (2) is valid also for an arbitrary (not necessarily bipartite) graph \( G \).

Using (2), \( \sum_{i=1}^{n} d_i x_i^2 = (\sum_{ij \in E} (x_i + x_j))^2 = \sum_{ij \in E} (x_i^2 + x_j^2) - (\sum_{ij \in E} (x_i + x_j))^2 \geq m(\sum_{ij \in E} (x_i^2 + x_j^2) - \sum_{ij \in E} (x_i + x_j)^2) = m \sum_{ij \in E} (x_i - x_j)^2 \geq 0 \) and it follows that the coefficient \( c(\sigma_n) \) of \( \sigma_n \) in inequality (1) is not positive. Hence, the left side of (1) is a non-increasing function in \( \sigma_n \) and, if \( \sigma_n < 2 \) and \( c(\sigma_n) < 0 \), then (1) is stronger than (2).

Next, for given \( \varepsilon > 0 \), we present the infinite family \( F_\varepsilon \) mentioned in the abstract.

For a positive integer \( k \), consider the graph \( G_k \) on \( n = 3k + 1 \) vertices obtained from \( k \) pairwise disjoint paths each on 3 vertices, where the vertices of these paths are numbered arbitrarily from 1 up to 3k, an additional vertex \( n = 3k + 1 \), and additional 2k edges connecting \( n \) with the 2k endvertices of these paths. Obviously, \( G_k \) is bipartite, \( n \) has degree 2k; each other vertex has degree 2, \( m = 4k \), \( \delta = 2 \), and, since \( G_k \) is bipartite, \( \sigma_n = 2 \) and \( \lambda_1 = -\lambda_n \) [5, 15]. Moreover, \( \frac{2r_n}{\sigma_n^\delta} m = \alpha = 2k = \frac{2}{3}(n-1) \). Let \( z \in \mathbb{R}^n \) be defined by \( x_i = 1 \) if \( i \) is a neighbour of \( n \), \( x_n = \sqrt{2k} \), and \( x_i = 0 \) otherwise. It follows \( \lambda_n \geq \frac{x_i^2 A z}{\sqrt{2k} z} = \sqrt{2k} \).

Rayleigh’s principle [4], hence, \( \frac{\lambda_{\delta^2} - \lambda_{\delta^2} \lambda \lambda}{\lambda_{\delta^2} - \lambda_{\delta^2} \lambda \lambda} n \mid \lambda_n \leq 0 \leq \lambda_n \) if \( \frac{\lambda_{\delta^2}}{\lambda_{\delta^2} \lambda \lambda} n \mid \lambda_n \leq 0 \leq \lambda_n \). If \( x \) is defined by \( x_n \) if \( i = n \) and \( x_i = 0 \) otherwise, then \( \eta \mid \eta \leq 0 \mid \eta \leq 0 \leq \eta \) and \( \frac{\lambda_{\delta^2}}{\lambda_{\delta^2} \lambda \lambda} n \mid \lambda_n \leq 0 \leq \lambda_n \), \( \frac{\lambda_{\delta^2}}{\lambda_{\delta^2} \lambda \lambda} n \mid \lambda_n \leq 0 \leq \lambda_n \), \( \frac{\lambda_{\delta^2}}{\lambda_{\delta^2} \lambda \lambda} n \mid \lambda_n \leq 0 \leq \lambda_n \), \( \frac{\lambda_{\delta^2}}{\lambda_{\delta^2} \lambda \lambda} n \mid \lambda_n \leq 0 \leq \lambda_n \). Let the integer \( l = l_\varepsilon \geq 2 \) be chosen large enough such that \( \frac{\lambda_{\delta^2}}{\lambda_{\delta^2} \lambda \lambda} n \mid \lambda_n \leq 0 \leq \lambda_n \) and \( \frac{\lambda_{\delta^2}}{\lambda_{\delta^2} \lambda \lambda} n \mid \lambda_n \leq 0 \leq \lambda_n \). It follows \( k \geq \frac{k}{k+2} \) and \( \frac{\lambda_{\delta^2}}{\lambda_{\delta^2} \lambda \lambda} n \mid \lambda_n \leq 0 \leq \lambda_n \), \( \frac{\lambda_{\delta^2}}{\lambda_{\delta^2} \lambda \lambda} n \mid \lambda_n \leq 0 \leq \lambda_n \), \( \frac{\lambda_{\delta^2}}{\lambda_{\delta^2} \lambda \lambda} n \mid \lambda_n \leq 0 \leq \lambda_n \), \( \frac{\lambda_{\delta^2}}{\lambda_{\delta^2} \lambda \lambda} n \mid \lambda_n \leq 0 \leq \lambda_n \). Eventually, we show that the difference between \( \frac{\lambda_{\delta^2}}{\lambda_{\delta^2} \lambda \lambda} n \mid \lambda_n \leq 0 \leq \lambda_n \) and D.M. Cvetković’s bound \( \min\{\{i \in \{1, \ldots, n\} \mid \lambda_i \leq 0\}, \{i \in \{1, \ldots, n\} \mid \lambda_i \geq 0\}\} \) can be arbitrarily small. For two graphs \( G \) on \( V(G) \) and \( G' \) on \( V(G') \), the cartesian product \( G \times G' \) is the graph on \( V(G) \times V(G') \) and two vertices \( (v, v') \) and \( (w, w') \) of \( G \times G' \) are adjacent in \( G \times G' \) if and only if \( v = w \) and \( v' = w' \) or \( v' = w \) and \( w \neq w' \). For a positive integer \( k \), consider the bipartite and 4-regular graph \( H_k = C_{4k} \times C_{4k} \) on \( n = 16k^2 \) vertices, where \( C_{4k} \) is the cycle on \( 4k \) vertices, and it follows \( \frac{\lambda_{\delta^2}}{\lambda_{\delta^2} \lambda \lambda} n \mid \lambda_n \leq 0 \leq \lambda_n \), \( \frac{\lambda_{\delta^2}}{\lambda_{\delta^2} \lambda \lambda} n \mid \lambda_n \leq 0 \leq \lambda_n \), \( \frac{\lambda_{\delta^2}}{\lambda_{\delta^2} \lambda \lambda} n \mid \lambda_n \leq 0 \leq \lambda_n \), \( \frac{\lambda_{\delta^2}}{\lambda_{\delta^2} \lambda \lambda} n \mid \lambda_n \leq 0 \leq \lambda_n \). If \( \lambda \) and \( \lambda' \) are eigenvalues of \( G \) and \( G' \), respectively, then \( \lambda + \lambda' \) is an eigenvalue of \( G \times G' \) [4]. If the graph \( G \) is bipartite, then the set of its eigenvalues is symmetric w.r.t. 0 [4]. Since \( C_{4k} \) has the eigenvalue 0 with multiplicity 2 [4], \( H_k \) has the eigenvalue 0 with multiplicity \( 4k + 2 \), and, consequently,

\[
\min\{\{i \in \{1, \ldots, n\} \mid \lambda_i \leq 0\}, \{i \in \{1, \ldots, n\} \mid \lambda_i \geq 0\}\} = \frac{n}{2} + 2k + 1 \text{ for } G_k.
\]
Acknowledgement. The authors would like to express their gratitude to Horst Sachs, Ilmenau University of Technology, and two anonymous referees for their valuable advice.

References


