Uniform existence of the integrated density of states on metric Cayley graphs

Felix Pogorzelski, Fabian Schwarzenberger and Christian Seifert

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Abstract

Given a finitely generated amenable group we consider ergodic random Schrödinger operators on a Cayley graph with random potentials and random boundary conditions. We show that the normalised eigenvalue counting functions of finite volume parts converge uniformly. The integrated density of states as the limit can be expressed by a Pastur-Shubin formula. The spectrum supports the corresponding measure and discontinuities correspond to the existence of compactly supported eigenfunctions.

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1 Introduction

In this paper we study the integrated density of states for random Schrödinger operators on Cayley graphs over a finitely generated amenable group $\mathcal{G}$. As we are concerned with the metric graph, the functions in question are defined on the edges e of the graph. For such an edge e the operator acts as $(Hf)_e = -f_e'' + V_e f_e$, where V_e is a L^∞-function. Selfadjointness of such an operator is obtained by choosing appropriate boundary conditions. We will only deal with local boundary conditions acting separately on each vertex v and taking into account only the functions of the edges connected with v. Randomness occurs in the choice of the potential and the boundary conditions. More precisely, given a finite set of potentials and a finite set of boundary conditions, we may choose the potential and the boundary conditions by random, such that the operator family $(H_\omega)_{\omega\in\Omega}$ becomes ergodic, where $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space and $\alpha\colon \mathcal{G}\times\Omega\to\Omega$ is an ergodic group action on Ω.

The restriction of such an operator H_ω to a finite subgraph has discrete spectrum and therefore possesses a well-defined eigenvalue counting function. By increasing the finite subgraph, we study the question whether (and with respect to which topology) the associated normalised eigenvalue counting functions converge. The aim of this paper is to show uniform convergence. We furthermore prove that the limit function, which is called *integrated density of states* (IDS), can be expressed via a Pastur-Shubin formula.

In [3] the authors verified this for $\mathcal{G} = \mathbb{Z}^d$ with standard edges. The present paper extends this to a large class of Cayley graphs over amenable groups. The discrete case is treated in [9] and [12] where uniform convergence of the eigenvalue counting functions is proven for operators on $\ell^2(\mathbb{Z}^d)$ and on combinatorial Cayley graphs, respectively.

In section 2 we give some basic features on the geometric setting. Section 3 describes the operator families in question. Restrictions to finite subgraphs are discussed in section 4, where we also state the main results. In section 5 we apply an ergodic theorem obtained in [12] to a sequence of spectral shift functions for an exhaustion of subgraphs. These results are used in section 6 to prove our main theorem.

2 Metric Cayley graphs over amenable groups

Let $\mathcal{G}$ be a group, $\mathcal{S} \subseteq \mathcal{G}$ a finite but not necessarily symmetric set of generators and $\mathrm{id} \in \mathcal{G}$ the unit element. We define the distance between two elements $g, h \in \mathcal{G}$ to be the smallest number of elements in $\mathcal{S} \cup \mathcal{S}^{-1}$ one needs to turn h into g by left multiplication, i. e.

$$d(g,h) := \begin{cases} \min\{k \in \mathbb{N} \mid \exists s_1, \dots, s_k \in \mathcal{S} \cup \mathcal{S}^{-1} \text{ with } s_1 \cdots s_k h = g\} & \text{if } g \neq h \\ 0 & \text{else.} \end{cases}$$

We denote the set of all finite subsets of $\mathcal{G}$ by $\mathcal{F}$. The diameter of a set $Q \in \mathcal{F}$ is given by $\operatorname{diam} Q := \max\{d(g,h) \mid g, h \in Q\}$. For a subset $Q \subseteq \mathcal{G}$ and $g \in \mathcal{G}$ we set $d(g,Q) := \min\{d(g,h) \mid h \in Q\}$. Given $R \in \mathbb{N}$ and $Q \in \mathcal{F}$ we define $\partial^R Q = \{g \in \mathcal{G} \mid g \in Q, d(g, G \setminus Q) \leq R \text{ or } g \notin Q, d(g,Q) \leq R\}$. We assume that $\mathcal{G}$ is amenable, i. e., there exists a sequence $(Q_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{F}$ such that

$$\lim_{n\to\infty} \frac{|\mathcal{S}Q_n \setminus Q_n|}{|Q_n|} = 0.$$

Such a sequence (Q_n) is called *Følner sequence.* A Følner sequence is said to be *tempered* if there exists $C > 0$ such that

$$\left| \bigcup_{k=1}^{n-1} Q_k^{-1} Q_n \right| \leq C|Q_n|$$

holds for all $n \in \mathbb{N}$. We say that a set $Q \in \mathcal{F}$ *symmetrically tiles* $\mathcal{G}$ with grid $T \subseteq \mathcal{G}$ if $T = T^{-1}$ and $\mathcal{G}$ is the disjoint union of the sets Qg, $g \in T$.

Throughout the paper we assume that there exists a tempered Følner sequence (Q_n) such that each Q_n symmetrically tiles $\mathcal{G}$.

Remark 2.1. (a) It is easy to see that for a Følner sequence $(Q_n)_{n\in\mathbb{N}}$

$$\lim_{n\to\infty} \frac{|\partial^R Q_n|}{|Q_n|} = 0$$

holds true for all $R \in \mathbb{N}$, c.f. [12].

(b) Note that each Følner sequence has a tempered subsequence, c.f. [13].

(c) Let T be a finite index subgroup of $\mathcal{G}$ and Q an associated fundamental domain, i.e. Q is a selection of representatives of the left cosets of T in $\mathcal{G}$. Then Q symmetrically tiles $\mathcal{G}$ with grid T. Therefore, each group with a sequence of finite index subgroups $(\mathcal{G}_n)_{n\in\mathbb{N}}$ and associated fundamental domains F_n, $n \in \mathbb{N}$ such that $(F_n)_{n\in\mathbb{N}}$ is a Følner sequence fits in our setting. In [7] Krieger proved in a slight extension of a result of Weiss in [15] that each residually finite, amenable group obeys such sequences.

For a given group $\mathcal{G}$ and a finite set of generators $\mathcal{S}$ we denote the induced (directed) metric Cayley graph by $\Gamma = \Gamma(\mathcal{G},\mathcal{S}) = (\mathcal{V},\mathcal{E},\gamma)$, i.e., $\mathcal{V} = \mathcal{G}$ is the vertex set, $\mathcal{E}$ the set of edges and $\gamma = (\gamma_0,\gamma_1)\colon \mathcal{E} \to \mathcal{V}\times\mathcal{V}$ associates to each edge $e \in \mathcal{E}$ the starting vertex $\gamma_0(e)$ and the end vertex $\gamma_1(e)$. There will be an edge e from v to w if there exists $s \in \mathcal{S}$ such that $w = sv$. Every edge $e \in \mathcal{E}$ will be identified with the interval $[0,1]$.

Example 2.2. Let $\mathcal{G} = \mathbb{Z}^2$ and set

$$\mathcal{S}_1 = \{(1,0),(0,1)\} \quad \text{and} \quad \mathcal{S}_2 = \{(0,0),(1,1),(1,0),(-1,0)\}.$$

Then $\mathcal{S}_1$ and $\mathcal{S}_2$ are generating systems for $\mathcal{G}$. We denote the corresponding metric Cayley graphs by $\Gamma_1 = \Gamma_1(\mathcal{G},\mathcal{S}_1)$ and $\Gamma_2 = \Gamma_2(\mathcal{G},\mathcal{S}_2)$. Note that while Γ_1 is the usual graph of $\mathbb{Z}^2$ with standard edges, Γ_2 contains multiple edges as well as loops.

3 Random Schrödinger Operators on graphs

Let $\mathcal{B} \subseteq L^\infty(0,1)$ be a finite subset. For $e \in \mathcal{E}$ let $V_e \in \mathcal{B}$. In the Hilbert space

$$\mathcal{H}_\Gamma := \bigoplus_{e\in\mathcal{E}} L^2(0,1)$$

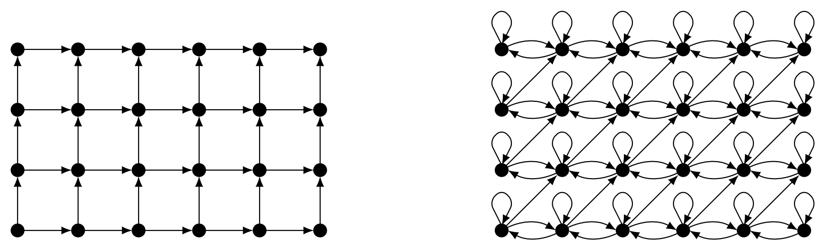

Figure 1: Illustration of $\Gamma_1(\mathcal{G}, \mathcal{S}_1)$ and $\Gamma_2(\mathcal{G}, \mathcal{S}_2)$ from Example 2.2

we define the maximal operator

$$D(\hat{H}) := \bigoplus_{e\in\mathcal{E}} W^{2,2}(0,1),$$
$$(\hat{H}f)_e := -f_e'' + V_e f_e \quad (e \in \mathcal{E}).$$

In order to obtain selfadjoint realisations we need to impose boundary conditions at the vertices. We will not consider the most general boundary conditions, but rather restrict ourselves to so-called local boundary conditions.

For $v \in \mathcal{V}$,

$$\mathcal{E}_{v,j} := \{e \in \mathcal{E} \mid \gamma_j(e) = v\} \qquad (j = 0,1)$$

describe the sets of all edges starting or ending at v, respectively, and

$$\mathcal{E}_v := \big(\mathcal{E}_{v,0} \times \{0\}\big) \cup \big(\mathcal{E}_{v,1} \times \{1\}\big)$$

encodes all edges connected with v (where loops are counted twice).

For $f \in D(\hat{H})$ and $v \in \mathcal{V}$ we define the *trace mapping* (or *boundary value mapping*) $\operatorname{tr}_v f \in \mathbb{K}^{\mathcal{E}_v}$ by

$$(\operatorname{tr}_v f)(e,j) := f_e(j) \quad ((e,j) \in \mathcal{E}_v).$$

Furthermore, define the *signed trace* $\operatorname{str}_v f' \in \mathbb{K}^{\mathcal{E}_v}$ by

$$(\operatorname{str}_v f')(e,j) := (-1)^j f_e'(j) \quad ((e,j) \in \mathcal{E}_v).$$

Remark 3.1. (a) Note that $W^{2,2}(0,1) \subseteq C^1[0,1]$ by standard Sobolev arguments and hence for $f \in D(\hat{H})$ the vectors $\operatorname{tr}_v f$ and $\operatorname{str}_v f'$ are well-defined ($v \in \mathcal{V}$).

(b) The definition of the signed trace gives that the orientation of the edges plays a minor role. In particular only the boundary conditions take into account the direction of the edges.

Definition (local boundary conditions)**.** Let $v \in \mathcal{V}$. Local boundary conditions at v are encoded in a subspace $U_v \subseteq \mathbb{K}^{\mathcal{E}_v} \oplus \mathbb{K}^{\mathcal{E}_v}$ with $\dim U_v = |\mathcal{E}_v|$ such that

$$(f_1' \mid f_2) - (f_1 \mid f_2') = 0 \quad ((f_1, f_1'), (f_2, f_2') \in U_v),$$

where $(\cdot\,|\,\cdot)$ denotes the usual inner product in $\mathbb{K}^{\mathcal{E}_v}$. We say that $f \in D(\hat{H})$ satisfies the *local boundary condition* U_v *at* $v \in V$, if $(\operatorname{tr}_v f, \operatorname{str}_v f') \in U_v$. *Local boundary conditions* are a family $U := (U_v)_{v\in\mathcal{V}}$ of local boundary conditions for each vertex $v \in \mathcal{V}$.

For a local boundary condition U the operator

$$
\begin{aligned}
D(H) &:= \left\{ f \in D(\hat{H}) \mid (\operatorname{tr}_v f, \operatorname{str}_v f')_{v\in\mathcal{V}} \in U \right\}, \\
(Hf)_e &:= (\hat{H}f)_e = -f_e'' + V_e f_e \quad (e \in \mathcal{E})
\end{aligned}
$$

is selfadjoint; cf. [6, 5, 8, 3].

Example 3.2. (a) Dirichlet boundary conditions. Let $U_v^D := \{0\}^{\mathcal{E}_v} \oplus \mathbb{K}^{\mathcal{E}_v}$. Then U_v^D encodes Dirichlet boundary conditions at v, since $\operatorname{tr}_v f = 0$ $(f \in D(H))$.

(b) Neumann boundary conditions. Let $U_v^N := \mathbb{K}^{\mathcal{E}_v} \oplus \{0\}^{\mathcal{E}_v}$. Then U_v^N encodes Neumann boundary conditions at v, since $\operatorname{str}_v f' = 0$ $(f \in D(H))$.

Example 3.3 (Dirichlet-Laplacian)**.** Let $V_e = 0$ for all $e \in \mathcal{E}$. Then the operator H with Dirichlet boundary conditions (U_v^D) is called *Dirichlet Laplacian* and is denoted by $-\Delta_D$. We have

$$
\begin{aligned}
D(-\Delta_D) &= \bigoplus_{e\in\mathcal{E}} W_0^{1,2} \cap W^{2,2}(0,1), \\
(-\Delta_D f)_e &= -f_e'' \quad (e \in \mathcal{E}).
\end{aligned}
$$

Now, we want to introduce randomness in the choice of potentials and boundary conditions.

Note that $\mathcal{G}$ acts on Γ in the following way: For $e \in \mathcal{E}$ and $g \in \mathcal{G}$ there is also a unique edge $e \circ g \in \mathcal{E}$ connecting $\gamma_0(e)g^{-1}$ and $\gamma_1(e)g^{-1}$. Shorthand, we can therefore write

$$
\gamma(e \circ g) = (\gamma_0(e)g^{-1}, \gamma_1(e)g^{-1}).
$$

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $\mathcal{G}$ act ergodically on $(\Omega, \mathcal{A}, \mathbb{P})$, i. e., if $\alpha\colon \mathcal{G} \times \Omega \to \Omega$ is the group action on Ω, then every subset of Ω which is invariant under $(\alpha_g)_{g\in\mathcal{G}}$ has measure either zero or one. Additionally we want α to act measure preserving, i. e. $\mathbb{P}(A) = \mathbb{P}(\alpha_g(A))$ for all $g \in \mathcal{G}$ and all $A \in \mathcal{A}$.

A random potential is a map $V\colon \Omega \to \prod_{e\in\mathcal{E}} \mathcal{B}$ satisfying

$$
V(\alpha_g(\omega))_{e\circ g} = V(\omega)_e \quad (g \in \mathcal{G}, e \in \mathcal{E}). \tag{1}
$$

Since a Cayley graph is very regular (i. e. every vertex has the same degree and for two vertices there exists a bijective mapping between the adjacent edges at these vertices), we can choose local boundary conditions U_{id} at $\mathrm{id} \in \mathcal{V}$ and then shift these boundary conditions to an arbitrary $v \in \mathcal{V}$ to obtain a local boundary condition at v. Hence we can choose random boundary conditions in the following way:

Let $\mathcal{U}$ be a finite set of local boundary conditions at id. A random boundary condition is a map $U\colon \Omega \to \prod_{v\in\mathcal{V}} \mathcal{U}$ satisfying

$$U(\alpha_g(\omega))_v = U(\omega)_{vg} \quad (g \in \mathcal{G}, v \in \mathcal{V}). \tag{2}$$

The family of random Schrödinger operators $(H_\omega)_{\omega\in\Omega}$ on $\mathcal{H}_\Gamma$ is defined by

$$D(H_\omega) := \left\{ f \in D(\hat{H}) \mid (\operatorname{tr}_v f, \operatorname{str}_v f')_{v\in\mathcal{V}} \in U(\omega) \right\}, \tag{3}$$

$$(H_\omega f)_e := -f_e'' + V(\omega)_e f_e \quad (e \in \mathcal{E}), \tag{4}$$

for $\omega \in \Omega$. For each $\omega \in \Omega$, H_ω is selfadjoint and semibounded from below. More precisely, there is $C \geq 0$ such that $H_\omega + C \geq 0$ for all $\omega \in \Omega$.

4 Restrictions to finite subsets

Let $Q \subseteq \mathcal{G}$ be a finite subset. The associated subgraph $\Gamma_Q = (\mathcal{V}_Q, \mathcal{E}_Q, \gamma_Q)$ of Γ is defined as follows:

$$\mathcal{E}_Q := \bigcup_{v\in Q} \mathcal{E}_{v,0}, \quad \mathcal{V}_Q := Q \cup \mathcal{S}Q, \quad \gamma_Q := \gamma|_{\mathcal{E}_Q}.$$

We also define inner vertices $\mathcal{V}_Q^i$ and boundary vertices $\mathcal{V}_Q^\partial$ by

$$\mathcal{V}_Q^i := \{ v \in \mathcal{V}_Q \mid \mathcal{E}_{v,0} \cup \mathcal{E}_{v,1} \subseteq \mathcal{E}_Q \}, \quad \mathcal{V}_Q^\partial := \mathcal{V}_Q \setminus \mathcal{V}_Q^i,$$

and accordingly inner edges $\mathcal{E}_Q^i$ and boundary edges $\mathcal{E}_Q^\partial$ by

$$\mathcal{E}_Q^i := \left\{ e \in \mathcal{E}_Q \mid \gamma_0(e), \gamma_1(e) \in \mathcal{V}_Q^i \right\}, \quad \mathcal{E}_Q^\partial := \mathcal{E}_Q \setminus \mathcal{E}_Q^i.$$

We define the restriction H_ω^Q of H_ω to Γ_Q on

$$\mathcal{H}_{\Gamma_Q} = \bigoplus_{e\in\mathcal{E}_Q} L^2(0,1)$$

by

$$\begin{aligned} D(H_\omega^Q) := \Big\{ f \in \bigoplus_{e\in\mathcal{E}_Q} W^{2,2}(0,1) \mid (\operatorname{tr}_v f, \operatorname{str}_v f') \in U(\omega)_v \quad (v \in \mathcal{V}_Q^i), \\ (\operatorname{tr}_v f, \operatorname{str}_v f') \in U_v^D \quad (v \in \mathcal{V}_Q^\partial) \Big\}, \end{aligned}$$

$$(H_\omega^Q f)_e := -f_e'' + V(\omega)_e f_e \quad (e \in \mathcal{E}_Q).$$

This operator is again selfadjoint and semibounded from below. Furthermore, H_ω^Q has purely discrete spectrum; cf. [8, Theorem 18].

Let us enumerate the eigenvalues $(\lambda_n(H_\omega^Q))_{n\in\mathbb{N}}$ as an increasing sequence, counting their multiplicities. The *eigenvalue counting function* $n_\omega^Q\colon \mathbb{R} \to \mathbb{N}_0$ is defined by

$$n_\omega^Q(\lambda) := \left| \left\{ n \in \mathbb{N} \mid \lambda_n(H_\omega^Q) \leq \lambda \right\} \right| = \operatorname{Tr} \mathbf{1}_{(-\infty,\lambda]}(H_\omega^Q).$$

Then n_ω^Q is monotone increasing and right continuous, i.e. a distribution function. The volume-scaled version of n_ω^Q will be denoted by N_ω^Q, i. e.,

$$N_\omega^Q(\lambda) := \frac{1}{|\mathcal{E}_Q|} n_\omega^Q(\lambda) \quad (\lambda \in \mathbb{R}).$$

It is associated to a pure point measure μ_ω^Q. Note that $|\mathcal{E}_Q| = |\mathcal{S}|\,|Q|$.

We now state the main theorem of this paper.

Theorem 4.1. *Let $(Q_l)_{l\in\mathbb{N}}$ be a Følner sequence in $\mathcal{G}$. Then there is $N\colon \mathbb{R} \to \mathbb{R}$ monotone increasing and right continuous (i. e. a distribution function), such that*

$$\lim_{l\to\infty} \left\| N_\omega^{Q_l} - N \right\|_\infty = 0$$

for $\mathbb{P}$-a. a. $\omega \in \Omega$. In particular, $N_\omega^{Q_l} \to N$ pointwise for $\mathbb{P}$-a. a. $\omega \in \Omega$.

Furthermore, for $\lambda \in \mathbb{R}$ and $Q \subseteq \mathcal{G}$ finite

$$N(\lambda) = \frac{1}{|\mathcal{E}_Q|} \int_\Omega \operatorname{Tr}\left(\mathbf{1}_{\mathcal{E}_Q} \mathbf{1}_{(-\infty,\lambda]}(H_\omega) \right) \, d\mathbb{P}(\omega). \tag{5}$$

Note that $N(\lambda)$ does not depend on the choice of Q.

The distribution function N is called the *integrated density of states (IDS)*. Let μ be the corresponding measure. Theorem 4.1 states that the IDS is the uniform limit of the normalised eigenvalue counting functions on finite subgraphs and can be expressed by a Pastur-Shubin trace formula in (5). The operator Tr denotes the usual trace in L^2.

By ergodicity of $(H_\omega)_{\omega\in\Omega}$ we obtain the following Theorem, which is an analogue of [3, Theorem 5]. For the proof we may apply the general framework of [10, Theorem 5.1].

Theorem 4.2. *There exist subsets $\Sigma, \Sigma_{pp}, \Sigma_{sc}, \Sigma_{ac}, \Sigma_{disc}, \Sigma_{ess} \subseteq \mathbb{R}$ and $\Omega' \subseteq \Omega$ with $\mathbb{P}(\Omega') = 1$ such that $\sigma(H_\omega) = \Sigma$ and $\sigma_\bullet(H_\omega) = \Sigma_\bullet$ for all the spectral types $\bullet \in \{pp, sc, ac, disc, ess\}$ and all $\omega \in \Omega'$.*

As a consequence, we can relate the measure μ with the $\mathbb{P}$-a. s. spectrum Σ of (H_ω), cf. [10, 3].

Corollary 4.3. *Σ is the topological support of μ.*

Denote by

$$D := \left\{ f \in \bigoplus_{e\in\mathcal{E}} L^2(0,1) \mid \exists\, \mathcal{E}' \subseteq \mathcal{E} \text{ finite} : f_e = 0 \quad (e \in \mathcal{E} \setminus \mathcal{E}') \right\}$$

the set of compactly supported L^2-functions on Γ.

Corollary 4.4. *Let*

$$\Sigma_{comp} := \{\lambda \in \mathbb{R} \mid \textit{for } \mathbb{P}\textit{-a. a. } \omega \in \Omega \textit{ exists } f_\omega \in D(H_\omega) \cap D : H_\omega f_\omega = \lambda f_\omega\}\,.$$

Then

$$\Sigma_{comp} = \{\lambda \in \mathbb{R} \mid \mu(\{\lambda\}) > 0\}\,.$$

Remark 4.5. (a) The set $\{\lambda \in \mathbb{R} \mid \mu(\{\lambda\}) > 0\}$ is the set of atoms of μ and equals the set of discontinuities of the IDS.

(b) The proof of Corollary 4.4 follows the lines of [3, Proof of Corollary 7].

5 Convergence of spectral shift functions

The next aim is the application of a Banach-space valued ergodic theorem given in [12]. Therefore it is necessary to prove certain properties of the spectral shift functions. Before this we introduce the notion concerning the colouring of the Cayley graph $\Gamma = (\mathcal{V}, \mathcal{E}, \gamma)$ associated to a given group $\mathcal{G}$ with finite set of generators $\mathcal{S}$.

Let $\mathcal{A}$ be an arbitrary finite set. A map $\mathcal{C} : \mathcal{V} \to \mathcal{A}$ is called a *colouring* of Γ and a map $P : D(P) \to \mathcal{A}$, where $D(P) \in \mathcal{F}$, a *pattern*. Note that, as before, $\mathcal{F}$ denotes the set of all finite subsets of $\mathcal{G}$. We write $\mathcal{P}$ for the set of all patterns and for given $Q \in \mathcal{F}$ we define the set $\mathcal{P}(Q) := \{P \in \mathcal{P} \mid D(P) = Q\}$. Given a pattern P and a set $Q \subseteq D(P)$ the *restriction of P on Q* is the map $P|_Q : Q \to \mathcal{A}$ with $P|_Q(g) = P(g)$ for all $g \in Q$. Equivalently, the *restriction of a colouring* $\mathcal{C}$ to a finite set $Q \in \mathcal{F}$ is given by $\mathcal{C}|_Q : Q \to \mathcal{A}, \mathcal{C}|_Q(g) = \mathcal{C}(g)$ for all $g \in Q$. For $P \in \mathcal{P}$ and $x \in \mathcal{G}$ the *translation of P by x* is defined by $Px : D(P)x \to \mathcal{A}$, $(Px)(g) = P(gx^{-1})$. We say that two patterns $P, P' \in \mathcal{P}$ are *equivalent* (and write $P \sim P'$) if there exists $x \in \mathcal{G}$ with $D(P)x = D(P')$ and $(Px)(g) = P'(g)$ for all $g \in D(P')$. The induced quotient set is denoted by $\tilde{\mathcal{P}}$ and the equivalence class for given $P \in \mathcal{P}$ by $\tilde{P} \in \tilde{\mathcal{P}}$. For given patterns $P_1, P_2 \in \mathcal{P}$ we set $\sharp_{P_1}(P_2)$ to be the number of occurrences of P_1 in P_2, i. e.

$$\sharp_{P_1}(P_2) := |\{P \in \mathcal{P} \mid P \sim P_1, D(P) \subseteq D(P_2)\}|.$$

Definition. A function $b : \mathcal{F} \to [0, \infty)$ is called *boundary term* if $b(Q) = b(Qx)$ for all $Q \in \mathcal{F}$ and $x \in \mathcal{G}$, $\lim_{n\to\infty} |Q_n|^{-1} b(Q_n) = 0$ for any Følner sequence (Q_n) and $|Q|^{-1} b(Q)$ is uniformly bounded for all $Q \in \mathcal{F}$.

Definition. Let $(X, \|\cdot\|)$ be a Banach space and a function $F : \mathcal{F} \to X$ be given. F is called

(i) *almost additive* if there exists a boundary term $b : \mathcal{F} \to [0, \infty)$ such that for any pairwise disjoint subsets Q_k, $k = 1, \dots, m$

$$\left\| F(Q) - \sum_{k=1}^{m} F(Q_k) \right\| \leq \sum_{k=1}^{m} b(Q_k)$$

holds, where $Q = \bigcup_{k=1}^{m} Q_k$;

(ii) $\mathcal{C}$*-invariant* if $F(Q) = F(Qx)$ for all $x \in \mathcal{G}$ and all $Q \in \mathcal{F}$ with $\mathcal{C}|_Q \sim \mathcal{C}|_{Qx}$.

For a given almost additive and $\mathcal{C}$-invariant function $F : \mathcal{F} \to X$ we define a function $\tilde{F} : \tilde{\mathcal{P}} \to X$ by setting

$$\tilde{F}(\tilde{P}) = \begin{cases} F(Q) & \text{if } \exists\, Q \in \mathcal{F} \text{ such that } \widetilde{\mathcal{C}|_Q} = \tilde{P}, \\ 0 & \text{else.} \end{cases}$$

Note that this is well-defined by $\mathcal{C}$-invariance of F.

Theorem 5.1. *Let $\mathcal{G}$ be an amenable group generated by a finite set $\mathcal{S}$, $\Gamma = (\mathcal{V}, \mathcal{E}, \gamma)$ the associated Cayley graph, $\mathcal{A}$ a finite set and $\mathcal{C} : \mathcal{V} \to \mathcal{A}$ be an arbitrary colouring. Assume that there exists a Følner sequence (Q_n) in $\mathcal{G}$ such that each Q_n symmetrically tiles $\mathcal{G}$. Let the frequencies $\nu_P := \lim_{j\to\infty} |Q_j|^{-1} \sharp_P(\mathcal{C}|_{Q_j})$ exist for all patterns $P \in \bigcup_{n\in\mathbb{N}} \mathcal{P}(Q_n)$. Furthermore let $(X, \|\cdot\|)$ be a Banach-space and $F : \mathcal{F} \to X$ an almost additive and $\mathcal{C}$-invariant mapping. Then the limits*

$$\lim_{j\to\infty} \frac{F(Q_j)}{|Q_j|} = \lim_{n\to\infty} \sum_{P \in \mathcal{P}(Q_n)} \nu_P \frac{\tilde{F}(\tilde{P})}{|Q_n|}$$

exist and are equal.

See [12] for a proof of Theorem 5.1. Now we show that in our situation the assumptions of the theorem are fulfilled almost surely. For $\omega \in \Omega$ define the map $\mathcal{C}_\omega : \mathcal{V} \to \mathcal{A}$ by

$$\mathcal{C}_\omega(v) := \big((V(\omega)_e)_{e \in \mathcal{E}_{v,0}}, U(\omega)_v \big) \quad \text{where} \quad \mathcal{A} := (\oplus_{e \in \mathcal{E}_{\mathrm{id},0}} \mathcal{B}) \times \mathcal{U} = (\oplus_{s \in \mathcal{S}} \mathcal{B}) \times \mathcal{U}. \tag{6}$$

To show the existence of the frequencies ν_P we need the following theorem, which is a special case of the Lindenstrauss' pointwise ergodic theorem in [13].

Theorem 5.2. *Let $\mathcal{G}$ act from the left on a measure space $(\Omega, \mathcal{A}, \mathbb{P})$ by an ergodic and measure preserving transformation α and let (Q_j) be a tempered Følner sequence. Then for any $f \in L^1(\mathbb{P})$*

$$\lim_{j\to\infty} \frac{1}{|Q_j|} \sum_{g \in Q_j} f(\alpha_g(\omega)) = \int_\Omega f(\omega) d\mathbb{P}(\omega)$$

holds for $\mathbb{P}$-a. a. $\omega \in \Omega$.

Lemma 5.3. *Let (Q_j) be a Følner sequence and $\mathcal{C}_\omega$ and $\mathcal{A}$ be given as in (6) for all $\omega \in \Omega$. Then there exists a set $\tilde{\Omega} \subseteq \Omega$ with $\mathbb{P}(\tilde{\Omega}) = 1$, such that for each $P \in \bigcup_{n\in\mathbb{N}} \mathcal{P}(Q_n)$ and $\omega \in \tilde{\Omega}$ the limit*

$$\nu_P = \lim_{j\to\infty} \frac{\sharp_P(\mathcal{C}_\omega|_{Q_j})}{|Q_j|}$$

exists and is independent of $\omega \in \tilde{\Omega}$.

Proof. Let $P : Q \to \mathcal{A}$ be given for some $Q \in \mathcal{F}$ with $\operatorname{diam} Q = R$. W.l.o.g. we may assume that $\mathrm{id} \in Q$, which is possible since we are interested in counting translates of P. Obviously the following inequalities hold

$$\sum_{g\in Q_n\setminus\partial^R Q_n} \mathbf{1}_{A(\omega)}(g) \le \sharp_P(\mathcal{C}_\omega|_{Q_n}) \le \sum_{g\in Q_n} \mathbf{1}_{A(\omega)}(g),$$

where $A(\omega) := \{g \in \mathcal{G} \mid P(v) = \mathcal{C}_\omega(vg) \text{ for all } v \in Q\}$. By the properties of U and V given in (1) and (2) we have for given $g, v \in \mathcal{G}$, $\omega \in \Omega$

$$\mathcal{C}_\omega(vg) = \big((V(\omega)_e)_{e\in\mathcal{E}_{vg,0}}, U(\omega)_{vg}\big) = \big((V(\alpha_g(\omega))_e)_{e\in\mathcal{E}_{v,0}}, U(\alpha_g(\omega))_v\big) = \mathcal{C}_{\alpha_g(\omega)}(v).$$

Therefore,

$$\mathbf{1}_{A(\omega)}(g) = \mathbf{1}_{\{g\in\mathcal{G}|P(v)=\mathcal{C}_{\alpha_g(\omega)}(v) \text{ for all } v\in Q\}}(g) = f_P(\alpha_g(\omega)),$$

where

$$f_P(\omega) = \begin{cases} 1 & \text{if } P(v) = \mathcal{C}_\omega(v) \text{ for all } v \in Q, \\ 0 & \text{else,} \end{cases}$$

and hence

$$\sum_{g\in Q_n\setminus\partial^R Q_n} f_P(\alpha_g(\omega)) \le \sharp_P(\mathcal{C}_\omega|_{Q_n}) \le \sum_{g\in Q_n} f_P(\alpha_g(\omega)).$$

This gives

$$\limsup_{n\to\infty} \frac{\sharp_P(\mathcal{C}_\omega|_{Q_n})}{|Q_n|} \le \limsup_{n\to\infty} \frac{1}{|Q_n|} \sum_{g\in Q_n} f_P(\alpha_g(\omega))$$

and

$$\liminf_{n\to\infty} \frac{\sharp_P(\mathcal{C}_\omega|_{Q_n})}{|Q_n|} \ge \liminf_{n\to\infty} \frac{1}{|Q_n|} \sum_{g\in Q_n\setminus\partial^R Q_n} f_P(\alpha_g(\omega)) = \liminf_{n\to\infty} \frac{1}{|Q_n|} \sum_{g\in Q_n} f_P(\alpha_g(\omega)),$$

where we used that (Q_n) is a Følner sequence, cf. Remark 2.1. As α is an ergodic an measure preserving action Theorem 5.2 yields a set $\Omega_P \subseteq \Omega$ of full measure such that the limits

$$\lim_{n\to\infty} \frac{\sharp_P(\mathcal{C}_\omega|_{Q_n})}{|Q_n|} = \lim_{n\to\infty} \frac{1}{|Q_n|} \sum_{g\in Q_n} f_P(\alpha_g(\omega)) = \int_\Omega f_P(\omega) d\mathbb{P}(\omega)$$

exist and are equal for all $\omega \in \Omega_P$. The desired set $\tilde{\Omega}$ is the (countable) intersection of these Ω_P for $P \in \bigcup_{n\in\mathbb{N}} \mathcal{P}(Q_n)$. $\square$

We now focus on the spectral shift function. Since the operators H_ω^Q are unbounded, the eigenvalue counting functions n_ω^Q are unbounded as well. However, the spectral shift function for two realisations H_1^Q and H_2^Q with different boundary conditions is bounded, which will be shown in Lemma 5.4.

Definition. Let $\mathcal{H}$ be a Hilbert space and H_1, H_2 be selfadjoint, lowerbounded operators with discrete spectra. Then the spectral shift function is defined by

$$\xi_{H_1,H_2}(\lambda) := n_{H_2}(\lambda) - n_{H_1}(\lambda) \quad (\lambda \in \mathbb{R}).$$

Thus, to obtain properties of n_{H_2} it suffices to study properties of n_{H_1} and ξ_{H_1,H_2}.

Lemma 5.4. *Let H_0 be a densely defined, closed symmetric and lower bounded operator with deficiency index k. Let H_1 and H_2 be two selfadjoint extensions of H_0 with discrete spectrum. Then*

$$|\xi_{H_1,H_2}| \le k.$$

Proof. By the min-max principle, for any selfadjoint operator H we have

$$n_H(\lambda) = \max\{\dim X \mid X \subseteq D(H) \text{ linear subspace, } H|_X \le \lambda\} \quad (\lambda \in \mathbb{R}),$$

cf. [3]. Now, for $\lambda \in \mathbb{R}$,

$$\begin{aligned} n_{H_2}(\lambda) &= \max\{\dim X \mid X \subseteq D(H_2) \text{ linear subspace, } H_2|_X \le \lambda\} \\ &\le \max\{\dim X \mid X \subseteq D(H_0) \text{ linear subspace, } H_2|_X \le \lambda\} + k \\ &= \max\{\dim X \mid X \subseteq D(H_0) \text{ linear subspace, } H_1|_X \le \lambda\} + k \\ &\le \max\{\dim X \mid X \subseteq D(H_1) \text{ linear subspace, } H_1|_X \le \lambda\} + k \\ &= n_{H_1}(\lambda) + k. \end{aligned}$$

□

Changing the boundary conditions of a selfajoint operator on a graph at one vertex v yields a perturbation of rank at most $2\,|\mathcal{E}_v|$. Hence, the spectral shift function of two selfadjoint operators H_1 and H_2 on a graph which differ only by the boundary conditions at a finite vertex set Q satisfies

$$|\xi_{H_1,H_2}| \le 2 \bigcup_{v \in Q} |\mathcal{E}_v| = 4\,|Q|\,|\mathcal{S}|\,. \tag{7}$$

In section 4 we defined the eigenvalue counting function n_ω^Q for the restriction of the operator H_ω to the subgraph Γ_Q generated by the set $Q \in \mathcal{F}$. Similarly we denote the eigenvalue counting function for the Dirichlet Laplacian $-\Delta_D$ restricted to Γ_Q by n_D^Q. The Dirichlet boundary conditions induce that n_D^Q decomposes into a sum of counting functions, i. e.,

$$n_D^Q(\lambda) = \sum_{e \in \mathcal{E}_Q} n_D(\lambda) = |\mathcal{E}_Q| n_D(\lambda) = |Q||\mathcal{S}| n_D(\lambda), \tag{8}$$

where n_D is the eigenvalue counting function of the Dirichlet Laplacian on the space $L^2(0,1)$. We are interested in the spectral shift function

$$\xi^Q_\omega(\lambda) := n^Q_\omega(\lambda) - n^Q_D(\lambda) = |Q||\mathcal{S}|(N^Q_\omega(\lambda) - n_D(\lambda)). \tag{9}$$

Denote the Banach space of the right-continuous, bounded functions $f : \mathbb{R} \to \mathbb{R}$ equipped with supremum norm by $\mathcal{B}(\mathbb{R})$. We study the behaviour of the functions $\xi^{Q_n}_\omega$ as $n \to \infty$ as elements of $\mathcal{B}(\mathbb{R})$, where (Q_n) is a Følner sequence. To this end we prove that $\xi_\omega : \mathcal{F} \to \mathcal{B}(\mathbb{R})$, $Q \mapsto \xi^Q_\omega$ is almost additive and $\mathcal{C}$-invariant, which makes it possible to apply Theorem 5.1.

Lemma 5.5. *Let $\omega \in \Omega$ and $\xi_\omega : \mathcal{F} \to \mathcal{B}(\mathbb{R})$, $Q \mapsto \xi^Q_\omega$, where ξ^Q_ω is given as in* (9)*. Then ξ_ω is almost additive and $\mathcal{C}_\omega$-invariant.*

Proof. Let $\omega \in \Omega$ and $Q_i \in \mathcal{F}$, $i = 1, \dots, k$ pairwise disjoint be given and set $Q := \bigcup_{i=1}^k Q_i$. Then

$$\begin{aligned}\left\| \xi^Q_\omega - \sum_{i=1}^k \xi^{Q_i}_\omega \right\| &= \left\| n^Q_\omega - n^Q_D + \sum_{i=1}^k (n^{Q_i}_\omega - n^{Q_i}_D) \right\| \\ &\le \left\| n^Q_\omega - \sum_{i=1}^k n^{Q_i}_\omega \right\| + \left\| n^Q_D - \sum_{i=1}^k n^{Q_i}_D \right\|\end{aligned}$$

holds, where we denote by $\|\cdot\|$ the supremum norm. Equation (8) yields $n^Q_D = \sum_{i=1}^k n^{Q_i}_D$, therefore it remains to prove almost additivity for $n_\omega \colon \mathcal{F} \to \mathcal{B}(\mathbb{R})$, $Q \mapsto n^Q_\omega$. Note that $\sum_{i=1}^k n^{Q_i}_\omega$ is the eigenvalue counting function of the operator $\oplus_{i=1}^k H^{Q_i}_\omega$, which equals H^Q_ω up to the boundary conditions on the vertices $\bigcup_{i=1}^k \mathcal{V}^\partial_{Q_i}$. Now, (7) gives

$$\left\| n^Q_\omega - \sum_{i=1}^k n^{Q_i}_\omega \right\| \le 4\,|\mathcal{S}| \left| \bigcup_{i=1}^k \mathcal{V}^\partial_{Q_i} \right| \le 4\,|\mathcal{S}| \cdot \sum_{i=1}^k |\partial^1 Q_i|$$

which proves almost additivity of ξ^Q_ω with boundary term $b(Q_i) := 4\,|\mathcal{S}|\,|\partial^1 Q_i|$. The $\mathcal{C}$-invariance of ξ_ω follows directly from its definition. □

Note that almost additivity and $\mathcal{C}$-invariance easily imply boundedness, see [12] for instance.

Corollary 5.6. *Let $\mathcal{G}$ be an amenable group generated by a finite set $\mathcal{S}$, $\Gamma = (\mathcal{V}, \mathcal{E}, \gamma)$ the associated Cayley graph, (Q_n) a tempered Følner sequences such that each Q_n symmetrically tiles $\mathcal{G}$. Then the limit*

$$\lim_{j\to\infty} \frac{\xi^{Q_j}_\omega}{|Q_j||\mathcal{S}|}$$

exists in $\mathcal{B}(\mathbb{R})$ for almost all $\omega \in \Omega$ and is independent of ω.

6 Proof of main theorem

We now prove our main Theorem.

Proof of Theorem 4.1. (i) First, we show convergence of $(|\mathcal{E}_{Q_l}|^{-1} n_\omega^{Q_l})_{l\in\mathbb{N}}$. By Corollary 5.6, the sequence $(|\mathcal{E}_{Q_l}|^{-1} \xi_\omega^{Q_l})_{l\in\mathbb{N}}$ converges uniformly. Hence, there is $N\colon \mathbb{R}\to\mathbb{R}$ such that

$$\frac{1}{|\mathcal{E}_{Q_l}|} n_\omega^{Q_l} = \frac{1}{|\mathcal{E}_{Q_l}|}\xi_\omega^{Q_l} + n_D \to N \quad \text{as} \quad l\to\infty$$

uniformly $\mathbb{P}$-a. s.

(ii) Let $Q\subseteq\mathcal{G}$ finite. Define $\tilde{N}\colon\mathbb{R}\to[0,\infty]$ by

$$\tilde{N}(\lambda) := \frac{1}{|\mathcal{E}_Q|}\int_\Omega \operatorname{Tr}\left(\mathbf{1}_{\mathcal{E}_Q}\mathbf{1}_{(-\infty,\lambda]}(H_\omega)\right)\, d\mathbb{P}(\omega).$$

We show independence of $\tilde{N}$ of the choice of Q: by the invariance assumptions, we obtain that

$$\frac{1}{\left|\mathcal{E}_{\{x\}}\right|}\int_\Omega \operatorname{Tr}\left(\mathbf{1}_{\mathcal{E}_{\{x\}}}\mathbf{1}_{(-\infty,\lambda]}(H_\omega)\right)\, d\mathbb{P}(\omega) = \frac{1}{|\mathcal{S}|}\int_\Omega \operatorname{Tr}\left(\mathbf{1}_{\mathcal{E}_{\{x\}}}\mathbf{1}_{(-\infty,\lambda]}(H_\omega)\right)\, d\mathbb{P}(\omega)$$

does not depend on x. Hence, independence of Q follows.

(iii) We show the equality (5), i. e., $\tilde{N} = N$. Let $\lambda\in\mathbb{R}$ and $Q\subseteq\mathcal{G}$ finite. Then

$$\begin{aligned}\tilde{N}(\lambda) &= \frac{1}{|\mathcal{E}_Q|}\int_\Omega \operatorname{Tr}\left(\mathbf{1}_{\mathcal{E}_Q}\mathbf{1}_{(-\infty,\lambda]}(H_\omega)\right)\, d\mathbb{P}(\omega)\\ &= \lim_{l\to\infty}\frac{1}{|\mathcal{E}_{Q_l}|}\int_\Omega \operatorname{Tr}\left(\mathbf{1}_{\mathcal{E}_{Q_l}}\mathbf{1}_{(-\infty,\lambda]}(H_\omega)\right)\, d\mathbb{P}(\omega),\end{aligned}$$

and

$$\begin{aligned}N(\lambda) &= \lim_{l\to\infty}\frac{1}{|\mathcal{E}_{Q_l}|}\operatorname{Tr}\left(\mathbf{1}_{(-\infty,\lambda]}(H_\omega^{Q_l})\right)\\ &= \int_\Omega \lim_{l\to\infty}\frac{1}{|\mathcal{E}_{Q_l}|}\operatorname{Tr}\left(\mathbf{1}_{(-\infty,\lambda]}(H_\omega^{Q_l})\right)\, d\mathbb{P}(\omega)\\ &= \lim_{l\to\infty}\frac{1}{|\mathcal{E}_{Q_l}|}\int_\Omega \operatorname{Tr}\left(\mathbf{1}_{(-\infty,\lambda]}(H_\omega^{Q_l})\right)\, d\mathbb{P}(\omega),\end{aligned}$$

since $\tilde{N}$ does not depend on the choice of Q, $\mathbb{P}$ is a probability measure and N is the uniform limit $\mathbb{P}$-a. s.

It suffices to show that the measures associated with N and $\tilde{N}$, respectively, are equal, which in turn follows by vague convergence of the approximating measures $(\mu_l^\lambda)_{l\in\mathbb{N}}$ and $(\tilde{\mu}_l^\lambda)_{l\in\mathbb{N}}$, respectively, defined by

$$\langle f, \mu_l^\lambda \rangle := \frac{1}{|\mathcal{E}_{Q_l}|} \int_\Omega \operatorname{Tr}\left(f(H_\omega^{Q_l})\right) \, d\mathbb{P}(\omega),$$
$$\langle f, \tilde{\mu}_l^\lambda \rangle := \frac{1}{|\mathcal{E}_{Q_l}|} \int_\Omega \operatorname{Tr}\Big(\mathbf{1}_{\mathcal{E}_{Q_l}} f(H_\omega)\Big) \, d\mathbb{P}(\omega),$$

for $f \in C_0(\mathbb{R})$.

By the Stone-Weierstraß Theorem ([2, Theorem A.10.1]) it suffices to show that

$$\int_\Omega \frac{1}{|\mathcal{E}_{Q_l}|} \operatorname{Tr}\Big(\mathbf{1}_{\mathcal{E}_{Q_l}} f(H_\omega) - f(H_\omega^{Q_l})\Big) \, d\mathbb{P}(\omega) \to 0 \quad \text{as} \quad l \to \infty$$

for all f of the form $f(t) := (t-z)^{-1}$ with $z \in \mathbb{C} \setminus \mathbb{R}$, since

$$\operatorname{lin}\left\{t \mapsto (t-z)^{-1};\ z \in \mathbb{C} \setminus \mathbb{R}\right\}$$

is dense in $C_0(\mathbb{R})$.

For $l \in \mathbb{N}$ we can split Γ into Γ_{Q_l} and $\Gamma_{\mathcal{G}\setminus Q_l}$. Then H_ω and $H_\omega^{Q_l} \oplus H_\omega^{\mathcal{G}\setminus Q_l}$ differ only by the boundary conditions on the set $\mathcal{V}_{Q_l}^\partial$. Thus, by the second resolvent identity,

$$D := f(H_\omega) - f(H_\omega^{Q_l} \oplus H_\omega^{\mathcal{G}\setminus Q_l})$$

is an operator of rank at most $4\,|\mathcal{S}|\,\left|\mathcal{V}_{Q_l}^\partial\right|$. Moreover, D is bounded by $2\,|\operatorname{Im} z|^{-1}$, since f is bounded by $|\operatorname{Im} z|^{-1}$. Therefore,

$$\begin{aligned}\left|\operatorname{Tr}\Big(\mathbf{1}_{\mathcal{E}_{Q_l}} f(H_\omega) - f(H_\omega^{Q_l})\Big)\right| &= \left|\operatorname{Tr}\Big(\mathbf{1}_{\mathcal{E}_{Q_l}}\left(f(H_\omega) - f(H_\omega^{Q_l} \oplus H_\omega^{\mathcal{G}\setminus Q_l})\right)\Big)\right| \\ &\le \frac{8\,|\mathcal{S}|}{|\operatorname{Im} z|}\left|\mathcal{V}_{Q_l}^\partial\right|.\end{aligned}$$

As (Q_l) is a Følner sequence, $\mathcal{V}_{Q_l}^\partial \subseteq \partial^1 Q_l$ and $|\mathcal{E}_{Q_l}| = |\mathcal{S}||Q_l|$ we obtain

$$\frac{1}{|\mathcal{E}_{Q_l}|}\left|\operatorname{Tr}\Big(\mathbf{1}_{\mathcal{E}_{Q_l}} f(H_\omega) - f(H_\omega^{Q_l})\Big)\right| \le \frac{8\,|\mathcal{S}|}{|\operatorname{Im} z|}\frac{\left|\mathcal{V}_{Q_l}^\partial\right|}{|\mathcal{E}_{Q_l}|} = \frac{8\left|\mathcal{V}_{Q_l}^\partial\right|}{|\operatorname{Im} z|\,|Q_l|} \to 0 \quad \text{as} \quad l \to \infty.$$

As $\mathbb{P}$ is a probability measure, Lebesgue's dominated convergence theorem yields the assertion. □

7 Application to Heisenberg group

In the following we discuss the above results in the case where $\mathcal{G}$ equals the discrete Heisenberg group H_3, which consists of the elements

$$(a,b,c) := \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix}, \quad (a,b,c \in \mathbb{Z})$$

The group action is induced by the usual matrix multiplication. H_3 is an example of a non-abelian group, which is of polynomial growth. Therefore it is amenable as well as residually finite. One can show, see [12], that H_3 is generated by $\mathcal{S} = \{(1,0,0),(0,1,0)\}$ and that for each $n \in \mathbb{N}$ the set

$$Q_n := \{(a,b,c) \mid 0 \leq a,b < n, 0 \leq c < n^2\}$$

symmetrically tiles H_3 with grid $T_n = \{(a,b,c) \mid a,b \in n\mathbb{Z}, c \in n^2\mathbb{Z}\}$. Furthermore (Q_n) is a Følner sequence. We denote the associated metric Cayley graph by $\Gamma = \Gamma(\mathcal{G},\mathcal{S}) = (\mathcal{V},\mathcal{E},\gamma)$.

Let $(\Omega,\mathcal{A},\mathbb{P})$ be a probability space and $(H_\omega)_{\omega\in\Omega}$ a random Schrödinger operator on $\mathcal{H}_\Gamma = \bigoplus_{e\in\mathcal{E}} L^2(0,1)$ defined as in (3) and (4).

Then Theorem 4.1 proves that for increasing n the eigenvalue counting functions $N_\omega^{Q_n}$ given by

$$N_\omega^Q(\lambda) := \frac{1}{|\mathcal{E}_Q|} n_\omega^Q(\lambda) = \frac{1}{|\mathcal{E}_Q|} \left| \left\{ i \in \mathbb{N} \mid \lambda_i(H_\omega^Q) \leq \lambda \right\} \right| \quad (\lambda \in \mathbb{R}, Q \in \mathcal{F}).$$

converge for $\mathbb{P}$-a. a. $\omega \in \Omega$ uniformly in the energy variable to the *integrated density of states* $N : \mathbb{R} \to \mathbb{R}$ defined by

$$N(\lambda) := \frac{1}{|\mathcal{E}_Q|} \int_\Omega \operatorname{Tr}\left(\mathbf{1}_{\mathcal{E}_Q} \mathbf{1}_{(-\infty,\lambda]}(H_\omega)\right) \, d\mathbb{P}(\omega) \quad (\lambda \in \mathbb{R}),$$

where $Q \subseteq \mathcal{G}$ is an arbitrary finite set. Note that for $Q \in \mathcal{F}$ as usual $(\lambda_i(H_\omega^Q))_{i\in\mathbb{N}}$ is the increasing sequence of eigenvalues of H_ω^Q counted by multiplicity.

A Trace class operators on $L(\mathcal{H}_\Gamma)$

We show that the integral in the Pastur-Shubin formula is finite, i. e., that the operator $\mathbf{1}_{\mathcal{E}_Q}\mathbf{1}_{(-\infty,\lambda]}(H_\omega)$ is trace class for all $\omega \in \Omega$.

Let H be a selfadjoint and semibounded Schrödinger operator on $\mathcal{H}_\Gamma$ as in section 3. Let $Q \subseteq \mathcal{G}$ be finite. Since $H^Q \oplus H^{\mathcal{G}\setminus Q} - H$ is of finite rank (they differ only on the boundary conditions at $\mathcal{V}_Q^\partial$), also

$$(H+c)^{-1} - (H^Q \oplus H^{\mathcal{G}\setminus Q} + c)^{-1} = (H+c)^{-1}(H - H^Q \oplus H^{\mathcal{G}\setminus Q})(H^Q \oplus H^{\mathcal{G}\setminus Q} + c)^{-1}$$

has finite rank for sufficiently large $c > 0$. Hence,

$$\mathbf{1}_{\mathcal{E}_Q}((H+c)^{-1} - (H^Q \oplus H^{\mathcal{G}\setminus Q} + c)^{-1})$$

has finite rank and is therefore trace class.

By [11, Proposition 5.3 (ii)], $(H^Q + c)^{-1/2}$ is a continuous linear mapping from $\mathcal{H}_{\Gamma_Q}$ to $\bigoplus_{e\in\mathcal{E}_Q} L^\infty(0,1)$ for sufficiently large $c > 0$. Hence, by [14, Satz 6.14], $\mathbf{1}_{\mathcal{E}_Q}(H^Q + c)^{-1/2}$ is Hilbert Schmidt. But

$$(\mathbf{1}_{\mathcal{E}_Q}(H^Q + c)^{-1/2})^* = (H^Q + c)^{-1/2}\mathbf{1}_{\mathcal{E}_Q}$$

is again Hilbert-Schmidt, so

$$\mathbf{1}_{\mathcal{E}_Q}(H^Q+c)^{-1}\mathbf{1}_{\mathcal{E}_Q} = \mathbf{1}_{\mathcal{E}_Q}(H^Q+c)^{-1/2}(H^Q+c)^{-1/2}\mathbf{1}_{\mathcal{E}_Q}$$

is trace class and therefore also trace class on $\mathcal{H}_\Gamma$.

Since

$$\mathbf{1}_{\mathcal{E}_Q}(H^Q \oplus H^{\mathcal{G}\setminus Q}+c)^{-1} = \mathbf{1}_{\mathcal{E}_Q}(H^Q+c)^{-1/2}(H^Q+c)^{-1/2}\mathbf{1}_{\mathcal{E}_Q},$$

we conclude that

$$\mathbf{1}_{\mathcal{E}_Q}(H+c)^{-1} = \mathbf{1}_{\mathcal{E}_Q}((H+c)^{-1}-(H^Q \oplus H^{\mathcal{G}\setminus Q}+c)^{-1}) + \mathbf{1}_{\mathcal{E}_Q}(H^Q \oplus H^{\mathcal{G}\setminus Q}+c)^{-1}$$

is trace class.

Now,

$$\begin{aligned}\mathbf{1}_{\mathcal{E}_Q}\mathbf{1}_{(-\infty,\lambda]}(H) &= \mathbf{1}_{\mathcal{E}_Q}(H+c)^{-1}(H+c)\mathbf{1}_{(-\infty,\lambda]\cap\sigma(H)}(H) \\ &= \mathbf{1}_{\mathcal{E}_Q}(H+c)^{-1}\left(z \mapsto (z+c)\mathbf{1}_{(-\infty,\lambda]\cap\sigma(H)}(z)\right)(H).\end{aligned}$$

Since $\left(z \mapsto (z+c)\mathbf{1}_{(-\infty,\lambda]\cap\sigma(H)}(z)\right)$ is bounded, this operator is trace class as well.

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References

[1] T. Adachi: *A note on the Følner condition for amenability.* Nagoya Math. J. **131**, 67–74 (1993).

[2] A. Deitmar and S. Echterhoff: *Principles of Harmonic Analysis.* Springer (2009).

[3] M.J. Gruber and D.H. Lenz and I. Veselić: *Uniform Existence of the Integrated Density of the States for Random Schrödinger Operators on metric Graphs over* $\mathbb{Z}^d$. J. Funct. Anal. **253**(2), 515–533 (2007).

[4] M.J. Gruber and D.H. Lenz and I. Veselić: *Uniform Existence of the Integrated Density of the States for combinatorial and metric Graphs over* $\mathbb{Z}^d$. Proc. Symp. Pure Math. **77**, 87–108 (2008).

[5] V. Kostrykin and J. Potthoff and R. Schrader: *Contraction semigroups on metric graphs.* Proc. Symp. Pure Math. **77**, 423–458 (2008).

[6] V. Kostrykin and R. Schrader: *Kirchhoff's Rule for Quantum Wires*. J. Phys. A: Math. Gen. **32**, 595–630 (1999).

[7] F. Krieger: *Sous-décalages de Toeplitz sur les groupes moyennables résiduallement finis.* J. London Math. Soc. **75**(2), 447–462 (2007).

[8] P. Kuchment: *Quantum graphs: I. Some basic structures*. Waves Random Media **14**, 107–128 (2004).

[9] D. Lenz, P. Müller and I. Veselić: *Uniform existence of the integrated density of states for models on $\mathbb{Z}^d$.* Positivity **12**(4), 571–589 (2008).

[10] D.H. Lenz and N. Peyerimhoff and I. Veselić: *Groupoids, von Neumann Algebras and the Integrated Density of States*. Math Phys Anal Geom **10**, 1–41 (2007).

[11] D. Lenz and C. Schubert and P. Stollmann: *Eigenfunction Expansions for Schrödinger Operators on Metric Graphs*. Integr. equ. oper. theory **62**(4), 541–533 (2008).

[12] D.H. Lenz and F. Schwarzenberger and I. Veselić: *A Banach space-valued ergodic theorem and the uniform approximation of the integrated density of states*. Geom. Dedicata **150**, 1–34 (2011).

[13] E. Lindenstrauss: *Pointwise theorems for amenable groups.* Invent. Math. **146**(2), 259–295 (2001).

[14] J. Weidmann: *Lineare Operatoren in Hilberträumen. Teil I: Grundlagen.* Teubner (2000).

[15] B. Weiss: *Monotileable amenable groups.* Amer. Math. Soc. Transl. **202**(2), 257–262 (2001).