Wegner estimate for alloy-type models with sign-changing exponentially decaying single-site potentials

Norbert Peyerimhoff* Martin Tautenhahn† Ivan Veselić†

Abstract

We study Schrödinger operators on $L^2(\mathbb{R}^d)$ and $\ell^2(\mathbb{Z}^d)$ with a random potential of alloy-type. The single-site potential is assumed to be exponentially decaying. Wegner estimates are bounds on the average number of eigenvalues in an energy interval of finite box restrictions of these types of operators. In the described situation a Wegner estimate which is polynomial in the volume of the box and linear in the size of the energy interval holds. It is applicable as an ingredient for a localisation proof via multiscale analysis.

Keywords: random Schrödinger operators, alloy-type model, discrete alloy-type model, integrated density of states, Wegner estimate, single-site potential

1 Model and results

In the present paper we are interested in spectral properties of random Schrödinger operators of alloy-type in the continuous and the discrete setting, respectively. We first introduce the continuous model.

The *alloy-type model* is given by the family of Schrödinger operators

$$H_\omega := H_0 + V_\omega, \quad H_0 := -\Delta + V_0, \quad \omega \in \Omega$$

on $L^2(\mathbb{R}^d)$, where $-\Delta$ is the negative Laplacian, V_0 a $\mathbb{Z}^d$-periodic potential, and V_ω denotes the multiplication by the $\mathbb{Z}^d$-metrically transitive random field

$$V_\omega(x) := \sum_{k\in\mathbb{Z}^d} \omega_k U(x-k).$$

The so-called *coupling constants* ω_k, $k \in \mathbb{Z}^d$, are assumed to be independent identically distributed (i. i. d.) and the distribution μ of ω_0 has a density ρ of bounded support and of finite variation.

Recall that the space of functions of finite total variation $\mathrm{BV}(\mathbb{R})$ is the set of integrable functions $f : \mathbb{R} \to \mathbb{R}$ whose distributional derivatives are signed measures with finite variation, i. e.

$$\mathrm{BV}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \mid f \in L^1(\mathbb{R}),\ Df \text{ is a signed measure},\ |Df|(\mathbb{R}) < \infty\}.$$

*Department of Mathematical Sciences, Durham University, DH1 3LE, Great Britain
†Technische Universität Chemnitz, Fakultät für Mathematik, D-09107 Chemnitz, Germany

To say that a distributional derivative Df of a function $f \in L^1_{\text{loc}}(\mathbb{R})$ is a signed measure means that there exists a regular signed Borel measure μ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \phi \, \mathrm{d}\mu = - \int_{\mathbb{R}} f \phi' \mathrm{d}x$$

for all $\phi \in C_c^\infty(\mathbb{R})$. A norm on $\mathrm{BV}(\mathbb{R})$ is defined by $\|f\|_{\mathrm{BV}(\mathbb{R})} = \|f\|_{L^1(\mathbb{R})} + \|f\|_{\mathrm{Var}}$, where

$$\|f\|_{\mathrm{Var}} = |Df|(\mathbb{R}) = \sup\Big\{ \int_{\mathbb{R}} f v' \mathrm{d}x \colon v \in C_c^\infty(\mathbb{R}), \ |v| \le 1 \Big\}.$$

Note that if $f \in W^{1,1}(\mathbb{R})$ then $f \in \mathrm{BV}(\mathbb{R})$. In particular, we have $\|f\|_{W^{1,1}(\mathbb{R})} = \|f\|_{\mathrm{BV}(\mathbb{R})}$ and $\|f'\|_{L^1(\mathbb{R})} = \|f\|_{\mathrm{Var}}$.

The probability space $\Omega := \times_{k \in \mathbb{Z}^d} \operatorname{supp} \rho$ is equipped with the product measure $\mathbb{P} := \otimes_{k \in \mathbb{Z}^d} \mu$. The corresponding expectation is denoted by $\mathbb{E}$. The function $U : \mathbb{R}^d \to \mathbb{R}$ is called *single-site potential*. Throughout this paper we assume that V_0 and V_ω are infinitesimally bounded with respect to Δ and that the corresponding constants can be chosen uniformly in $\omega \in \Omega$. This is satisfied if U is a so-called *generalised step-function*.

Definition 1.1 (Generalised step-function)**.** Let $L_c^p(\mathbb{R}^d) \ni w \geq \kappa \chi_{(-1/2,1/2)^d}$ with $\kappa > 0$ and $p = 2$ for $d \leq 3$ and $p > d/2$ for $d \geq 4$, where $L_c^p(\mathbb{R})$ denotes the vector space of $L^p(\mathbb{R})$ functions with compact support. Let $u \in \ell^1(\mathbb{Z}^d; \mathbb{R})$. A function of the form $U : \mathbb{R}^d \to \mathbb{R}$,

$$U(x) = \sum_{k \in \mathbb{Z}^d} u(k) w(x-k),$$

is called *generalised step-function* and the function $u : \mathbb{Z}^d \to \mathbb{R}$ a *convolution vector*.

Recall that any real-valued function on $\mathbb{R}^d$ that is uniformly locally L^p, with $p = 2$ for $d \leq 3$ and $p > d/2$ for $d \geq 4$, is infinitesimally bounded with respect to the self-adjoint Laplacian Δ on $W^{2,2}(\mathbb{R}^d)$, see e. g. [RS78, Theorem XIII.96]. This is indeed satisfied for V_ω if U is a generalised step-function, since for any unit cube $C \subset \mathbb{R}^d$

$$\begin{aligned}\int_C |V_\omega(x)|^p \mathrm{d}x &= \int_C \Big| \sum_{k \in \mathbb{Z}^d} \omega_k \sum_{l \in \mathbb{Z}^d} u(l-k) w(x-l) \Big|^p \mathrm{d}x \\ &\leq \omega_+ c_p \|u\|_{\ell^1(\mathbb{Z}^d)} \sum_{l \in \mathbb{Z}^d} \int_C |w(x-l)|^p \mathrm{d}x = \omega_+ c_p \|u\|_{\ell^1(\mathbb{Z}^d)} \|w\|^p_{L^p(\mathbb{R})},\end{aligned}$$

where $\omega_+ = \sup\{|t| : t \in \operatorname{supp} \rho\}$ and c_p is some constant depending on p and the support of w. Notice that the upper bound is uniformly in $\omega \in \Omega$. Hence, V_0 and V_ω are infinitesimally bounded with respect to Δ and the corresponding constants can be chosen uniformly in $\omega \in \Omega$. Therefore, H_ω is self-adjoint (on the domain of Δ) and bounded from below (uniformly in $\omega \in \Omega$).

Let us now introduce the discrete analogue of the alloy-type model. A *discrete alloy-type model* is a family of operators $h_\omega = h_0 + v_\omega$ on $\ell^2(\mathbb{Z}^d)$. Here h_0 is the negative discrete Laplacian on $\mathbb{Z}^d$. The random part v_ω is a multiplication operator by the function

$$v_\omega(x) = \sum_{k \in \mathbb{Z}^d} \omega_k u(x-k),$$

where ω_k, $k \in \mathbb{Z}^d$, is, again, an i. i. d. sequence of bounded random variables each distributed with the density ρ and $u \in \ell^1(\mathbb{Z}^d;\mathbb{R})$ a single-site potential. Notice that in the continuous setting, the (discrete) single-site potential u plays the role of a convolution vector to generate the (continuous) single-site potential U in form of a generalised step-function. With other words, the convolution vector of a generalised step-function serves as a single-site potential for our discrete model.

The estimates we want to prove concern finite box restrictions of the operator H_ω or h_ω, $\omega \in \Omega$. For $l > 0$ and $j \in \mathbb{Z}^d$ we denote by

$$\Lambda_l(j) := (-l,l)^d + j \subset \mathbb{R}^d$$

the open cube of side length $2l$ centered at j. We will use the notation $\Lambda_l = \Lambda_l(0)$. By H_ω^Λ we denote the restriction of the operator H_ω to a bounded open set $\Lambda \subset \mathbb{R}^d$ with Dirichlet boundary conditions on $\partial\Lambda$. In the special case when Λ is a cube, H_ω^Λ will denote the restriction of H_ω to Λ either with Dirichlet or with periodic boundary conditions. Let $P_B(H_\omega^\Lambda)$ denote the spectral projection for the operator H_ω^Λ associated with a Borel set $B \subset \mathbb{R}$. If $\Lambda = \Lambda_l$ we will write H_ω^l and $P_B(H_\omega^l)$ instead of $H_\omega^{\Lambda_l}$ and $P_B(H_\omega^{\Lambda_l})$. Analogously for the discrete model, for $l > 0$ and $j \in \mathbb{Z}^d$ let

$$L_l(j) = \big([-l,l]^d + j\big) \cap \mathbb{Z}^d$$

and $L_l = L_l(0)$. For $L \subset \mathbb{Z}^d$ finite we denote the canonical inclusion $\ell^2(L) \to \ell^2(\mathbb{Z}^d)$ by ι_L and the adjoint restriction $\ell^2(\mathbb{Z}^d) \to \ell^2(L)$ by π_L. The restriction of h_ω to L is defined by $h_\omega^L := \pi_L h_0 \iota_L + \pi_L v_\omega \iota_L \colon \ell^2(L) \to \ell^2(L)$. Let $P_B(h_\omega^L)$ denote the spectral projection for the operator h_ω^L associated with a Borel set $B \subset \mathbb{R}$ and if $L = L_l$ we will write h_ω^l and $P_B(h_\omega^l)$ instead of $h_\omega^{L_l}$ and $P_B(h_\omega^{L_l})$.

Now we are in the position to state our bounds on the expected number of eigenvalues of finite box Hamiltonians H_ω^l and h_ω^l in a bounded energy interval $[E-\varepsilon, E+\varepsilon] \subset \mathbb{R}$. They are called Wegner estimates [Weg81] and are inequalities of the type

$$\forall\, l \in \mathbb{N}, E \in \mathbb{R}, \varepsilon > 0\colon \quad \mathbb{E}\left\{\operatorname{Tr}\big(P_{[E-\varepsilon,E+\varepsilon]}(h_\omega^l)\big)\right\} \leq C_{\mathrm{W}}\, |I|^a\, (2l+1)^{bd} \tag{1}$$

with some (Wegner-)constant C_{W}, some $a \leq 1$ and some $b \geq 1$. The exponent a determines the quality of the estimate with respect to the *length of the energy interval* and b the quality with respect to the *volume of the cube L_l*. The best possible estimate is obtained in the case $a = 1$ and $b = 1$.

For $k \in \mathbb{Z}^d$ we denote by $\|k\|_1 = \sum_{r=1}^d |k_r|$ the ℓ^1-norm of k.

Theorem 1.2 (Wegner estimate, continuous model)**.** *Assume that U is a generalised step-function and that there are constants $C, \alpha \in (0,\infty)$ such that $|u(k)| \leq Ce^{-\alpha\|k\|_1}$ for all $k \in \mathbb{Z}^d$. Then there exists $C(U) > 0$ and $I_0 \in \mathbb{N}_0^d$ both depending only on U such that for any $l \in \mathbb{N}$ and any bounded interval $I := [E_1, E_2] \subset \mathbb{R}$*

$$\mathbb{E}\left\{\operatorname{Tr}\big(P_I(H_\omega^l)\big)\right\} \leq \mathrm{e}^{E_2} C(U) \|\rho\|_{\mathrm{Var}} |I| (2l+1)^{2d+|I_0|}$$

A precise definition of $I_0 \in \mathbb{N}_0^d$ is given in Eq. (10)*.*

Theorem 1.3 (Wegner estimate, discrete model)**.** *Assume there are constants $C, \alpha \in (0,\infty)$ such that $|u(k)| \leq Ce^{-\alpha\|k\|_1}$ for all $k \in \mathbb{Z}^d$. Then there exists $C(u) > 0$ and $I_0 \in \mathbb{N}_0^d$ both depending only on u such that for any $l \in \mathbb{N}$ and any bounded interval $I \subset \mathbb{R}$*

$$\mathbb{E}\left\{\operatorname{Tr}\big(P_I(h_\omega^l)\big)\right\} \leq C(u) \|\rho\|_{\mathrm{Var}} |I| (2l+1)^{2d+|I_0|}.$$

A precise definition of $I_0 \in \mathbb{N}_0^d$ is given in Eq. (10)*.*

The main point of Theorem 1.2 and 1.3 is that no assumption on u (apart from exponential decay) is required. In particular, the sign of u can change arbitrarily. Also, note that the result holds on the whole energy axis.

Remark 1.4 (Continuous model). Let us compare the result of Theorem 1.2 to earlier ones of Wegner estimates for alloy-type models with sign-changing single-site potential.

The papers [Klo95, HK02] concern alloy-type Schrödinger operators on $L^2(\mathbb{R}^d)$. The main result is a Wegner estimate for energies in a neighbourhood of the infimum of the spectrum. It applies to arbitrary non-vanishing single site potentials $u \in C_c(\mathbb{R}^d)$ and coupling constants with a piecewise absolutely continuous density. The upper bound is linear in the volume of the box and Hölder-continuous in the energy variable.

The papers [Ves02, KV06, Ves10b] establish Wegner estimates for both alloy-type Schrödinger operators on $L^2(\mathbb{R}^d)$ and discrete alloy-type Schrödinger operators on $\ell^2(\mathbb{Z}^d)$. We will discuss now the results of [Ves02, KV06, Ves10b] refering to operators on $L^2(\mathbb{R}^d)$. These papers give Wegner estimates that are linear in the volume of the box and Lipschitz continuous in the energy variable. The bounds are valid for all compact intervals along the energy axis. They apply to single-site potentials $U \in L^\infty_c(\mathbb{R}^d)$ of a generalised step function form with a convolution vector satisfying

$$s\colon \theta \mapsto s(\theta) := \sum_{k\in\mathbb{Z}^d} u(k)\mathrm{e}^{-\mathrm{i}k\cdot\theta} \text{ does not vanish on } [0,2\pi)^d.$$

Remark 1.5 (Discrete model). Theorem 1.3 generalises the results established in [Ves10a]. There the same result as in Theorem 1.3 has been established, under the additional assumption that at least one of the following conditions holds:

(i) $\bar{u} := \sum_{k\in\mathbb{Z}^d} u(k) \neq 0$, or

(ii) u is finitely supported, or

(iii) the space dimension satisfies $d = 1$.

If condition (i) is satisfied, the Wegner bound of [Ves10a] holds for all $u \in \ell^1(\mathbb{Z}^d)$ not necessarily of exponential decay. A particularly important case in [Ves10a] is the one when both conditions (i) and (ii) hold. In this situation the exponent of the length scale can be chosen to be equal to the space dimension d. This corresponds to the volume exponent $b = 1$ in Ineq. (1), and yields the Lipschitz continuity of the integrated density of states. This is the distribution function $N\colon \mathbb{R} \to \mathbb{R}$ obtained as the limit

$$\lim_{L\to\infty} \frac{1}{(2L+1)^d} \mathbb{E}\left\{\mathrm{Tr}\left[\chi_{(-\infty,E]}(H_{\omega,L})\right]\right\} = N(E) \quad \text{at all continuity points of } N.$$

Consequently its derivative, the density of states, exists for almost all $E \in \mathbb{R}$. While in the situation where (i) or (ii) holds, a Wegner estimate was already established in [Ves10a], the improved proof presented here allows more explicit control of the exponent b.

Remark 1.6 (Localisation). Our bound is linear in the energy-interval length and polynomial in the volume of the cube. This implies that the Wegner bound can be used for a localisation proof via multiscale analysis, see e. g. [FS83, vDK89, Kir08]. More precisely, if an appropriate initial scale estimate is available, the multiscale analysis —

using as an ingredient the Wegner estimate as given in Theorem 1.3 — yields Anderson localisation. As the Wegner bound is valid on the whole energy axis one can prove Anderson localisation in any energy region where the initial scale estimate holds. For indefinite alloy-type models, such energy regions have been identified in e. g. [Klo95], [Ves02, §7], and [KN09].

If the single site potential does not have compact support, one has to use an enhanced version of the multiscale analysis and so-called uniform Wegner estimates to prove localisation, see [KSS98]. However, there exist criteria which allow one to turn a standard Wegner estimate into a uniform one, see e. g. Lemma 4.10.2 in [Ves08].

Recently Krüger [Krü] has obtained results on localisation for a class of discrete alloy type models which includes the ones considered here. The results rely on the multiscale analysis and the use of Cartan's lemma in the spirit as is has been used earlier, e. g. in [Bou09].

2 Abstract Wegner estimates; proof of Theorem 1.2 and 1.3

In [KV06] an abstract Wegner estimate for the continuous model was established, which we will be able to use in our situation. Let us first fix some notation. For an open set $\Lambda \subset \mathbb{R}^d$, $\tilde{\Lambda}$ is the set of lattice sites $j \in \mathbb{Z}^d$ such that the characteristic function of the cube $\Lambda_{1/2}(j)$ does not vanish identically on Λ. For $j \in \mathbb{Z}^d$ we denote by χ_j the characteristic function of the cube $\Lambda_{1/2}(j)$.

Theorem 2.1 ([KV06]). *Assume there is a number $l_0 \in \mathbb{N}$ such that for arbitrary $l \geq l_0$ and every $j \in \tilde{\Lambda}_l$ there is a compactly supported sequence $t_{j,l} \in \ell^1(\mathbb{Z}^d;\mathbb{R})$ such that*

$$\sum_{k\in\mathbb{Z}^d} t_{j,n}(k)U(x-k) \geq \chi_j(x) \quad \text{for all} \quad x \in \Lambda_l.$$

Let further $I := [E_1, E_2]$ be an arbitrary interval. Then for any $l \geq l_0$

$$\mathbb{E}\{\operatorname{Tr} P_I(H_\omega^l)\} \leq C\mathrm{e}^{E_2}\|\rho\|_{\mathrm{Var}}|I| \sum_{j\in\tilde{\Lambda}_l} \|t_{j,l}\|_{\ell^1(\mathbb{Z}^d)},$$

where C is a constant independent of l and I.

In [KV06] this theorem was stated for compactly supported $U : \mathbb{R}^d \to \mathbb{R}$. However, the proof directly applies for non-compactly supported single-site potentials. Theorem 2.1 also adopts for the discrete alloy-type model. For completeness we will give a short proof of the discrete version.

Theorem 2.2. *Assume there is a number $l_0 \in \mathbb{N}$ such that for arbitrary $l \geq l_0$ and every $j \in L_l$ there is a compactly supported sequence $t_{j,l} \in \ell^1(\mathbb{Z}^d)$ such that*

$$\sum_{k\in\mathbb{Z}^d} t_{j,l}(k)u(x-k) \geq \delta_j(x) \quad \text{for all} \quad x \in L_l.$$

Let further $I := [E_1, E_2]$ be an arbitrary interval. Then for any $l \geq l_0$

$$\mathbb{E}\{\operatorname{Tr} P_I(h_\omega^l)\} \leq \frac{1}{2}\|\rho\|_{\mathrm{Var}}|I| \sum_{j\in L_l} \|t_{j,l}\|_{\ell^1(\mathbb{Z}^d)}.$$

For the proof of Theorem 2.2 we will use an estimate on averages of spectral projections of certain selfadjoint operators. More presicely, let $\mathscr{H}$ be a Hilbert space and consider the following operators on $\mathscr{H}$. Let H be self-adjoint, W symmetric and H-bounded, J bounded and non-negative with $J^2 \leq W$, $H(\zeta) = H + \zeta W$ for $\zeta \in \mathbb{R}$, and $P_I(H(\zeta))$ the corresponding spectral projection onto an Interval $I \subset \mathbb{R}$. Then, for any $g \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, $\psi \in \mathscr{H}$ with $\|\psi\| = 1$ and bounded interval $I \subset \mathbb{R}$,

$$\int_{\mathbb{R}} \langle \psi, JP_I(H(\zeta))J\psi \rangle g(\zeta) \mathrm{d}\zeta \leq \|g\|_\infty |I|. \tag{2}$$

For a proof of Ineq. (2) we refer to [CH94] where compactly supported g is considered. The non-compactly supported case was first treated in [FHLM97], see also [Ves08, Lemma 5.3.2] for a detailed proof.

Proof of Theorem 2.2. In order to estimate the terms of the sum in the expectation $\mathbb{E}\{\mathrm{Tr} P_I(h^l_\omega)\} = \sum_{j \in L_l} \mathbb{E}\{\|P_I(h^l_\omega)\delta_j\|\}$ we fix $l \geq l_0$ and $j \in L_l$, and set $\Sigma = \mathrm{supp}\, t_{j,l} \subset \mathbb{Z}^d$ and $t = t_{j,l}$. Recall that

$$h^l_\omega = -\pi_{L_l} \Delta \iota_{L_l} + \sum_{k \in \mathbb{Z}^d \setminus \Sigma} \omega_k u(\cdot - k) + \sum_{k \in \Sigma} \omega_k u(\cdot - k).$$

We pick some $o \in \Sigma$ with $t(o) \neq 0$ and denote by M the finite dimensional linear transformation $(\eta_k)_{k \in \Sigma} \mapsto (\omega_k)_{k \in \Sigma} = M(\eta_k)_{k \in \Sigma}$ defined as follows: $\omega_o = t(o)\eta_o$ and $\omega_k = t(k)\eta_o + t(o)\eta_k$ for $k \in \Sigma \setminus \{o\}$. Note that M is invertible and $|\det M| = |t(o)|^{|\Sigma|}$. With this transformation there holds for arbitrary fixed $(\omega_k)_{k \in \mathbb{Z}^d \setminus \Sigma}$

$$\int_{\mathbb{R}^{|\Sigma|}} \|P_I(h^l_\omega)\delta_j\| \prod_{k \in \Sigma} \rho(\omega_k) \mathrm{d}\omega_k = \int_{\mathbb{R}^{|\Sigma|}} \|P_I(h^l_\eta)\delta_j\| k(\eta) \mathrm{d}\eta,$$

where $\mathrm{d}\eta = \prod_{k \in \Sigma} \mathrm{d}\eta_k$, $k(\eta) = |t(o)|^{|\Sigma|} \rho(t(o)\eta_o) \prod_{k \in \Sigma \setminus \{o\}} \rho(t(k)\eta_o + t(o)\eta_k)$, and

$$h^l_\eta = -\pi_{L_l} \Delta^L \iota_{L_l} + \sum_{k \in \mathbb{Z}^d \setminus \Sigma} \omega_k u(\cdot - k) + t(o) \sum_{k \in \Sigma \setminus \{o\}} \eta_k u(\cdot - k) + \eta_o \sum_{k \in \Sigma} t(k) u(\cdot - k).$$

We denote by $P_j : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ the orthogonal projection given by $P_j \phi = \phi(j)\delta_j$ and apply Ineq. (2) with the choice $H = h^l_\eta - \eta_o \sum_{k \in \Sigma} t(k) u(\cdot - k)$, $W = \sum_{k \in \Sigma} t(k) u(\cdot - k)$, $\zeta = \eta_o$ and $J = P_j$. This gives by Lebesgue's theorem

$$\int_{\mathbb{R}^{|\Sigma|}} \|P_I(h^l_\omega)\delta_j\| \prod_{k \in \Sigma} \rho(\omega_k) \mathrm{d}\omega_k \leq |I| \int_{\mathbb{R}^{|\Sigma|-1}} \sup_{\eta_o \in \mathbb{R}} |k(\eta)| \prod_{k \in \Sigma \setminus \{o\}} \mathrm{d}\eta_k. \tag{3}$$

If $\rho \in W^{1,1}(\mathbb{R})$, we use $\sup_{\eta_o \in \mathbb{R}} |k(\eta)| \leq \frac{1}{2} \int_{\mathbb{R}} |\partial_o k| \mathrm{d}\eta_o$. By the product rule we obtain for the partial derivative (while substituting back into original coordinates)

$$\frac{\partial}{\partial \eta_o} k(\eta) = |t(o)|^{|\Sigma|} \sum_{k \in \Sigma} t(k) \rho'(\omega_k) \prod_{j \in \Sigma \setminus \{k\}} \rho(\omega_j).$$

Hence, the right hand side of Ineq. (3) is bounded by $\frac{1}{2}|I| \|\rho'\|_{L^1(\mathbb{R})} \sum_{k \in \Sigma} |t(k)|$. Since all the steps were independent of $j \in L_l$, we in turn obtain the statement of the theorem in the case $\rho \in W^{1,1}(\mathbb{R})$. For ρ of bounded total variation (and compact support) we use the fact that there is sequence $\rho_k \in C_c^\infty(\mathbb{R})$, $k \in \mathbb{N}$, such that $\|\rho_k\|_{L^1(\mathbb{R})} = 1$ for all $k \in \mathbb{N}$, $\lim_{k\to\infty}\|\rho_k\|_{\mathrm{Var}} = \|\rho\|_{\mathrm{Var}}$ and $\lim_{k\to\infty}\|\rho_k - \rho\|_{L^1(\mathbb{R})} = 0$, see e. g. [Zie89] or Lemma 2.3 below. Since $\|\rho_k\|_{\mathrm{Var}} = \|\rho_k'\|_{L^1(\mathbb{R})}$ for $\rho_k \in C_c^\infty(\mathbb{R})$, the same consideration as above gives

$$\int_{\mathbb{R}^{|\Sigma|}} \|P_I(h_\omega^l)\delta_j\| \prod_{i\in\Sigma} \rho_k(\omega_i)\mathrm{d}\omega_i \leq \frac{1}{2}|I|\|\rho_k\|_{\mathrm{Var}} \sum_{k\in\Sigma}|t(k)| \tag{4}$$

for all $k \in \mathbb{N}$. By a limiting argument, see [KV06] for details, one obtains Ineq. (4) with ρ_k replaced by ρ. This proves the theorem. $\square$

Lemma 2.3. *Let $u : \mathbb{R} \to \mathbb{R}_0^+$ be a function of finite variation and bounded support. Assume additionally $\|u\|_{L^1(\mathbb{R})} = 1$. Then there exists a sequence $u_k \in C_c^\infty$, $k \in \mathbb{N}$, such that $\|u_k\|_{L^1(\mathbb{R})} = 1$ for all $k \in \mathbb{N}$,*

$$\lim_{k\to\infty}\|u_k\|_{\mathrm{Var}} = \|u\|_{\mathrm{Var}} \tag{5}$$

and

$$\lim_{k\to\infty}\|u_k - u\|_{L^1(\mathbb{R})} = 0. \tag{6}$$

Proof. Let $\phi \in C_c^\infty(\mathbb{R})$ be non-negative with $\operatorname{supp}\phi \subset [-1,1]$ and $\|\phi\|_{L^1(\mathbb{R})} = 1$. For $\varepsilon > 0$ set $\phi_\varepsilon : \mathbb{R} \to \mathbb{R}_0^+$, $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(x/\varepsilon)$. The function ϕ_ε belongs to $C_c^\infty(\mathbb{R})$ and fulfills $\|\phi_\varepsilon\|_{L^1(\mathbb{R})} = 1$. Now consider $u_\varepsilon : \mathbb{R} \to \mathbb{R}_0^+$,

$$u_\varepsilon(x) = \int_{\mathbb{R}} \phi_\varepsilon(x-y)u(y)\mathrm{d}y.$$

Obviously, $u_\varepsilon \in C_c^\infty(\mathbb{R})$ and by Fubini's theorem $\|u_\varepsilon\|_{L^1(\mathbb{R})} = 1$. The proof of the relation (6) is due to Theorem 1.6.1 in [Zie89]. For the proof of the relation (5), first note

$$\begin{aligned}
\|u\|_{\mathrm{Var}} = |Du|(\mathbb{R}) &= \sup\Big\{\int_{\mathbb{R}} uv'\mathrm{d}x \colon v \in C_c^\infty(\mathbb{R}),\ |v| \leq 1\Big\} \\
&= \sup\Big\{\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}} u_\varepsilon v'\mathrm{d}x \colon v \in C_c^\infty(\mathbb{R}),\ |v| \leq 1\Big\} \\
&\leq \liminf_{\varepsilon\searrow 0}|Du_\varepsilon|(\mathbb{R}) = \liminf_{\varepsilon\searrow 0}\|u_\varepsilon\|_{\mathrm{Var}},
\end{aligned}$$

since u_ε converges to u in $L^1(\mathbb{R})$ and v' is bounded. Let now $\psi \in C_c^\infty(\mathbb{R})$ with $|\psi| \leq 1$ and set $\psi_\varepsilon = \phi_\varepsilon * \psi$. Then we have by Fubini's theorem

$$\|u\|_{\mathrm{Var}} \geq \Big|\int_{\mathbb{R}} u\psi_\varepsilon'\mathrm{d}x\Big| = \Big|\int_{\mathbb{R}} u(\psi*\phi_\varepsilon)'\mathrm{d}x\Big| = \Big|\int_{\mathbb{R}} \psi'(u*\phi_\varepsilon)\mathrm{d}x\Big| = \Big|\int_{\mathbb{R}} u_\varepsilon\psi'\mathrm{d}x\Big|.$$

Taking supremum over all such ψ gives $\|u\|_{\mathrm{Var}} \geq \|u_\varepsilon\|_{\mathrm{Var}}$. This proves the lemma. $\square$

Assume that there are $C, \alpha \in (0,\infty)$ such that $|u(x)| \leq C\mathrm{e}^{-\alpha\|x\|_1}$ for all $x \in \mathbb{Z}^d$. Let I_0 and $c_u \neq 0$ be as in Eq. (10). In Section 3 we will construct for each $l \in \mathbb{N}$ a number $R_l \in \mathbb{N}$ given in Eq. (13) such that

$$\frac{2}{c_u}\sum_{k\in L_{R_l}} k^{I_0}u(x-k) \geq 1 \quad \text{for all } x \in L_l. \tag{7}$$

This fact is proven in Proposition 3.4 and we will apply it for the continuous model if U is a generalised step-function with a exponential decaying convolution vector and for the discrete model with exponential decaying single-site potential to verify the hypothesis of Theorem 2.1 and 2.2.

Proof of Theorem 1.3. By Ineq. (7) (respectively Proposition 3.4), the hypothesis of Theorem 2.2 is satisfied with the choice $l_0 = 1$ and $t_{j,l} \in \ell^1(\mathbb{Z}^d)$ given by

$$t_{j,l}(k) = \begin{cases} 2k^{I_0}/c_u & \text{if } k \in L_{R_l}, \\ 0 & \text{else,} \end{cases}$$

for $l \in \mathbb{N}$ and $j \in L_l$. It follows for all $l \in \mathbb{N}$ and $j \in L_l$ that

$$\sum_{j \in L_l} \|t_{j,l}\|_{\ell^1(\mathbb{Z}^d)} = \frac{2}{|c_u|}(2l+1)^d \sum_{k \in L_{R_l}} |k^{I_0}| \leq \frac{2}{|c_u|}(2l+1)^d (2R_l+1)^d R_l^{|I_0|}.$$

Recall that by Proposition 3.4, $R_l = \max\{2l + D, D'\} < 2l + D + D'$ with D and D' depending only on the single-site potential u. Hence there is a constant $C(u) > 0$ depending only on the single site potential u such that

$$\sum_{j \in L_l} \|t_{j,l}\|_{\ell^1(\mathbb{Z}^d)} \leq C(u)(2l+1)^{2d+|I_0|}.$$

By Theorem 2.2, this completes the proof. □

Proof of Theorem 1.2. Recall that U is a generalized step function and that w has compact support. Also recall, that for an open set $\Lambda \subset \mathbb{R}^d$, $\tilde{\Lambda}$ is the set of lattice sites $j \in \mathbb{Z}^d$ such that the characteristic function of the cube $\Lambda_{1/2}(j)$ does not vanish identically on Λ. We choose $r = \sup\{|r| + 1 : w(r) \neq 0\}$. Let $l_0 = 1$ and $t_{j,l} \in \ell^1(\mathbb{Z}^d)$ given by

$$t_{j,l}(k) = \begin{cases} 2k^{I_0}/(c_u \kappa) & \text{if } k \in L_{R_{l+r}}, \\ 0 & \text{else,} \end{cases}$$

for $l \in \mathbb{N}$ and $j \in \tilde{\Lambda}_l$. By Ineq. (7) (respectively Proposition 3.4) we have for all $l \in \mathbb{N}$, $j \in \tilde{\Lambda}_l$ and $x \in \Lambda_l$

$$\begin{aligned} \sum_{k \in \mathbb{Z}^d} t_{j,n}(k) U(x-k) &= \sum_{i \in \mathbb{Z}^d} w(x-i) \sum_{k \in \mathbb{Z}^d} t_{j,l}(k) u(i-k) \\ &\geq \frac{1}{\kappa} \sum_{i \in L_{l+r}} w(x-i) + \sum_{i \in \mathbb{Z}^d \setminus L_{l+r}} w(x-i) \sum_{k \in \mathbb{Z}^d} t_{j,l}(k) u(i-k) \\ &= \frac{1}{\kappa} \sum_{i \in L_{l+r}} w(x-i) \geq \chi_j(x). \end{aligned}$$

Here we have used that $w(x-i) = 0$ for $x \in \Lambda_l$ and $i \notin L_{l+r}$. Hence the assumption of Theorem 2.1 is satisfied. Analogous to the proof of Theorem 1.3 there is a constant $C(U)$ depending only on the single-site potential U such that

$$\sum_{j \in \tilde{\Lambda}_l} \|t_{j,l}\|_{\ell^1(\mathbb{Z}^d)} \leq C(U)(2l+1)^{2d+|I_0|}.$$

This completes the proof by using Theorem 2.1. □

3 Positive combinations of translated single-site potentials

In this section we consider (possibly infinite) linear combinations of translates of the (discrete) single site potential u. In this section we assume that u decays exponentially and is distinct from the zero function. Under these hypotheses we identify a sequence of coefficients such that the resulting linear combination is uniformly positive on the whole space $\mathbb{Z}^d$ (cf. Proposition 3.3) or some finite set $\Lambda \subset \mathbb{Z}^d$ (cf. Proposition 3.4).

Remark 3.1 (Preliminaries). We first recall the following two standard norms for vectors $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$:

$$\|z\|_\infty = \max\{|z_1|, |z_2|, \ldots, |z_d|\}, \quad \|z\|_1 = \sum_{r=1}^{d} |z_r|.$$

Moreover, we use the following multi-index notation: If $I = (i_1, \ldots, i_d) \in \mathbb{Z}^d$ and $z \in \mathbb{C}^d$, we define

$$z^I = z_1^{i_1} \cdot z_2^{i_2} \cdot \ldots \cdot z_d^{i_d},$$

and if $I \in \mathbb{N}_0^d$, we define

$$|I| = \sum_{r=1}^{d} i_r, \quad D_z^I = \frac{\partial^{i_1}}{\partial z_1^{i_1}} \cdot \frac{\partial^{i_2}}{\partial z_2^{i_2}} \cdot \ldots \cdot \frac{\partial^{i_d}}{\partial {z_d}^{i_d}}, \quad I! = i_1! \cdot i_2! \cdot \ldots \cdot i_d!.$$

We also introduce comparison symbols for multi-indices: If $I, J \in \mathbb{N}_0^d$, we write $J \le I$ if we have $j_r \le i_r$ for all $r = 1, 2, \ldots, d$, and we write $J < I$ if $J \le I$ and $|J| < |I|$. For $J \le I$, we use the short hand notation

$$\binom{I}{J} = \binom{i_1}{j_1} \cdot \binom{i_2}{j_2} \cdot \ldots \cdot \binom{i_d}{j_d}.$$

Finally, $\mathbf{0}, \mathbf{1}$ denote the vectors $(0, \ldots, 0)$ and $(1, \ldots, 1) \in \mathbb{C}^d$, respectively.

We also recall the following facts from multidimensional complex analysis. Let $D \subset \mathbb{C}^d$ be open. We call a complex valued function $f : D \to \mathbb{C}$ holomorphic, if every point $w \in D$ has an open neighbourhood U, $w \in U \subset D$, such that f has a power series expansion around w, which converges to $f(z)$ for all $z \in U$. Osgood's lemma tells us that, if $f : D \to \mathbb{C}$ is continuous and holomorphic in each variable separately (in the sense of one-dimensional complex analysis), then f is holomorphic, see [GR09].

Let $f_n : D \to \mathbb{C}$ be a sequence of holomorphic functions. We say that $\sum_n f_n$ converges normally in D, if for every $w \in D$ there is an open neighbourhood U, $w \in U \subset D$, such that $\sum_n \|f_n\|_{U,\infty} < \infty$. Normally convergent sequences of holomorphic functions can be rearranged arbitrarily, the limit is again holomorphic, and differentiation can be carried out termwise, which follows from Weierstrass' theorem, see [Rem84, p. 226] for the one-dimensional case and [Nar95, p. 7] for the higher dimensional case.

Remark 3.2 (Notation). Let $u : \mathbb{Z}^d \to \mathbb{R}$ be a non-vanishing exponentially decaying function, i. e., there are constants $C, \alpha > 0$ such that

$$|u(k)| \le C\mathrm{e}^{-\alpha\|k\|_1} \tag{8}$$

for all $k \in \mathbb{Z}^d$. For $\delta \in (0, 1-\mathrm{e}^{-\alpha})$ we consider the associated generating function $F : D_\delta \subset \mathbb{C}^d \to \mathbb{C}$,

$$D_\delta = \{z \in \mathbb{C}^d : |z_1 - 1| < \delta, \dots, |z_d - 1| < \delta\}, \quad F(z) = \sum_{k \in \mathbb{Z}^d} u(-k) z^k. \tag{9}$$

Notice that the sum $\sum_{k \in \mathbb{Z}^d} u(-k) z^k$ is normally convergent in D_δ by our choice of δ and the exponential decay condition (8). By Weierstrass' theorem, F is a holomorphic function. Since F is holomorphic and not identically zero, we have $(D_z^I F)(\mathbf{1}) \neq 0$ for at least one $I \in \mathbb{N}_0^d$. Therefore, there exists a multi-index $I_0 \in \mathbb{N}_0^d$ (not necessarily unique), such that we have

$$(D_z^I F)(\mathbf{1}) = \begin{cases} c_u \neq 0, & \text{if } I = I_0, \\ 0, & \text{if } I < I_0. \end{cases} \tag{10}$$

(Such a I_0 can be found by diagonal inspection: Let $n \geq 0$ be the largest integer such that $D_z^I F(\mathbf{1}) = 0$ for all $|I| < n$. Then choose a multi-index $I_0 \in \mathbb{N}_0^d$, $|I_0| = n$ with $(D_z^{I_0} F)(\mathbf{1}) \neq 0$.)

Proposition 3.3. *Let u, F, c_u and I_0 be as in* (8), (9), *and* (10). *Let further $I \in \mathbb{N}_0^d$ with $I \leq I_0$, and define $a : \mathbb{Z}^d \to \mathbb{Z}$ by*

$$a(k) = k^I.$$

Then we have for all $x \in \mathbb{Z}^d$

$$\sum_{k \in \mathbb{Z}^d} a(k) u(x-k) = \begin{cases} 0, & \text{if } I < I_0, \\ c_u, & \text{if } I = I_0. \end{cases} \tag{11}$$

Proof. We introduce, again, a bit of notation. For $s \in \mathbb{C}^d$ and $k \in \mathbb{Z}^d$ let

$$e^s = (e^{s_1}, \dots, e^{s_d}) \quad \text{and} \quad \langle k, s \rangle = \sum_{r=1}^d k_r s_r.$$

Let $I \leq I_0$. Then the chain rule yields (for all $s \in C_\delta := \{s \in \mathbb{C}^d : \mathrm{e}^s \in D_\delta\}$)

$$\begin{aligned} D_s^I (F(e^s)) &= \sum_{J \leq I} c_J \, (D_z^J F)(e^s) \, e^{\langle J, s \rangle} \\ &= (D_z^I F)(e^s) \, e^{\langle I, s \rangle} + \sum_{J < I} c_J \, (D_z^J F)(e^s) \, e^{\langle J, s \rangle}, \end{aligned}$$

with suitable integers $c_J \geq 1$ and, in particular, $c_I = 1$. This and Eq. (10) imply that

$$D_s^I (F(e^s)) \big|_{s=\mathbf{0}} = \begin{cases} 0, & \text{if } I < I_0, \\ c_{I_0} \, (D_z^{I_0} F)(\mathbf{1}) = c_u, & \text{if } I = I_0. \end{cases} \tag{12}$$

Next, we use the identity $a(k) = k^I = D_s^I e^{\langle k, s \rangle} \big|_{s=\mathbf{0}}$. Note that the series $\sum_{k \in \mathbb{Z}^d} u(x - k) e^{\langle k, s \rangle}$ converges normally on the domain

$$E_\alpha = \{s \in \mathbb{C}^d \mid -\alpha < \mathrm{Re}(s_j) < \alpha \text{ for all } j = 1, 2, \dots, d\},$$

Therefore, we can rearrange arbitrarily, differentiate componentwise, and obtain for all $s \in C_\delta \cap E_\alpha$ by substitution $\nu = k - x$ and the product rule

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^d} u(x-k) D_s^I e^{\langle k,s\rangle} &= D_s^I \sum_{k\in\mathbb{Z}^d} u(x-k) e^{\langle k,s\rangle} \\
&= D_s^I \Big(e^{\langle x,s\rangle} \sum_{\nu\in\mathbb{Z}^d} u(-\nu) e^{\langle \nu,s\rangle} \Big) = D_s^I \Big(F(e^s) e^{\langle x,s\rangle} \Big) \\
&= \sum_{J\le I} \binom{I}{J} \big(D_s^J F(e^s) \big) D_s^{I-J} e^{\langle x,s\rangle}.
\end{aligned}$$

Finally, evaluating at $s = \mathbf{0}$ and using (12) yields

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^d} a(k) u(x-k) &= \sum_{J\le I} \binom{I}{J} \big(D_s^J F(e^s) \big)\big|_{s=\mathbf{0}} \big(D_s^{I-J} (e^{\langle x,s\rangle}) \big)\big|_{s=\mathbf{0}} \\
&= \begin{cases} 0, & \text{if } I < I_0, \\ c_u, & \text{if } I = I_0. \end{cases}
\end{aligned}$$
□

In Proposition 3.3 we identified a sequence of coefficients such that the associated linear combination of translated single site potentials is positive on the whole of $\mathbb{Z}^d$. However, the sequence cannot be used for Theorems 2.1 and 2.1 directly. This problem can be resolved if we take into consideration that the positivity in Theorem 2.1 and 2.2 concerns lattice sites in L_n only.

Recall that the constants d, α, C and c_u are all determined by the choice of the exponentially decreasing function $u : \mathbb{Z}^d \to \mathbb{R}$. Now we choose $I = I_0$ in Proposition 3.3. The next proposition tells us, for all integer vectors x in the box Λ_l, how far we have to exhaust $\mathbb{Z}^d$ in the sum (11), in order to guarantee that the result is $\ge \frac{c_u}{2}$ (assuming for a moment that $c_u > 0$). The exhaustion is described by the integer indices in another box Λ_R, and the proposition describes the relation between the sizes l and R. For large enough l, this relation is linear.

Proposition 3.4. *Let u, c_u and I_0 be as in* (8) *and* (10)*. Let further $l \in \mathbb{N}$ and define*

$$R_l = \max\left\{ 2l + \frac{2}{\alpha} \ln \frac{2\,3^d C}{|c_u|(1-e^{-\alpha/2})}, \frac{8(d+|I_0|)^2}{\alpha^2} \right\}. \tag{13}$$

Then we have, for all $x \in L_l$:

$$\frac{2}{c_u} \sum_{k\in L_{R_l}} k^{I_0} u(x-k) \ge 1.$$

Proof. We know from Proposition 3.3 that

$$\frac{1}{c_u} \sum_{k\in\mathbb{Z}^d} k^{I_0} u(x-k) = 1,$$

for all $x \in \mathbb{Z}^d$. Thus we need to prove, for $x \in L_l = \mathbb{Z}^d \cap [-l,l]^d$, that

$$\Big| \sum_{k\in\mathbb{Z}^d \setminus L_{R_l}} k^{I_0} u(x-k) \Big| \le \frac{|c_u|}{2}. \tag{14}$$

Using the triangle inequality $\|k-x\|_\infty + \|x\|_\infty \geq \|k\|_\infty$, $\|k\|_\infty \leq \|k\|_1$, and that u is exponentially decreasing, we obtain

$$\begin{aligned}\Big|\sum_{k\in\mathbb{Z}^d\setminus L_{R_l}} k^{I_0}\, u(x-k)\Big| &\leq C \sum_{\|k\|_\infty \geq R_l} \|k\|_\infty^{|I_0|}\, \mathrm{e}^{-\alpha\|x-k\|_\infty} \\ &\leq Ce^{\alpha\|x\|_\infty} \sum_{\|k\|_\infty\geq R_l} \|k\|_\infty^{|I_0|}\, e^{-\alpha\|k\|_\infty} \\ &\leq Ce^{\alpha l} \sum_{r=R_l}^{\infty} (2r+1)^d\, r^{|I_0|}\, e^{-\alpha r} \\ &\leq C3^d e^{\alpha l} \sum_{r=R_l}^{\infty} r^{d+|I_0|} e^{-\alpha r}.\end{aligned}$$

Using Lemma 3.5 below and $r \geq R_l \geq \frac{8(d+|I_0|)^2}{\alpha^2}$, we conclude that $r^{d+|I_0|} \leq e^{\alpha r/2}$, which implies that

$$\Big|\sum_{k\in\mathbb{Z}^d\setminus L_{R_l}} k^{I_0}\, u(x-k)\Big| \leq C3^d e^{\alpha l} \sum_{r=R_l}^{\infty} e^{-\alpha r/2} = C3^d e^{\alpha l} \frac{e^{-\alpha R_l/2}}{1-e^{-\alpha/2}}.$$

Finally, using $R_l \geq 2l + \frac{2}{\alpha}\ln(2\cdot 3^d C/(|c_u|(1-e^{-\alpha/2}))$, we conclude Ineq. (14) which ends the proof. □

Lemma 3.5. *Let $M, \alpha > 0$. Then*

$$n \geq \frac{8M^2}{\alpha^2} \quad \Rightarrow \quad n^M < e^{\alpha n/2}.$$

Proof. If $n \geq \frac{8M^2}{\alpha^2}$ then

$$n \leq \frac{\alpha^2 n^2}{8M^2}.$$

Since

$$e^{\frac{\alpha n}{2M}} = 1 + \frac{\alpha n}{2M} + \frac{\alpha^2 n^2}{8M^2} + \cdots > \frac{\alpha^2 n^2}{8M^2},$$

we conclude that $n \leq e^{\frac{\alpha n}{2M}}$, or, equivalently, $n^M \leq e^{\frac{\alpha n}{2}}$. □

References

[Bou09] J. Bourgain, *An approach to Wegner's estimate using subharmonicity*, J. Stat. Phys. **134** (2009), no. 5-6, 969–978.

[CH94] J.-M. Combes and P. D. Hislop, *Localization for some continuous, random hamiltonians in d-dimensions*, J. Funct. Anal. **124** (1994), no. 1, 149–180.

[FHLM97] W. Fischer, T. Hupfer, H. Leschke, and P. Müller, *Existence of the density of states for multi-dimensional continuum Schrödinger operators with Gaussian random potentials*, Commun. Math. Phys. **190** (1997), no. 1, 133–141.

[FS83] J. Fröhlich and T. Spencer, *Absence of diffusion in the Anderson tight binding model for large disorder or low energy*, Commun. Math. Phys. **88** (1983), no. 2, 151–184.

[GR09] R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, AMS Chelsea Publishing, Providence, RI, 2009, Reprint of the 1965 original.

[HK02] P. D. Hislop and F. Klopp, *The integrated density of states for some random operators with nonsign definite potentials*, J. Funct. Anal. **195** (2002), no. 1, 12–47.

[Kir08] W. Kirsch, *An invitation to random Schrödinger operators*, Random Schrödinger operators, Panor. Synthèses, vol. 25, Soc. Math. France, 2008, With an appendix by Frédéric Klopp, pp. 1–119.

[Klo95] F. Klopp, *Localization for some continuous random Schrödinger operators*, Commun. Math. Phys. **167** (1995), no. 3, 553–569.

[KN09] F. Klopp and S. Nakamura, *Spectral extrema and Lifshitz tails for non-monotonous alloy type models*, Commun. Math. Phys. **287** (2009), no. 3, 1133–1143.

[Krü] H. Krüger, *Localization for random operators with non-monotone potentials with exponentially decaying correlations*, arXiv:1006.5233v1 [math.SP].

[KSS98] W. Kirsch, P. Stollmann, , and G. Stolz, *Anderson localization for random Schrödinger operators with long range interactions*, Commun. Math. Phys. **195** (1998), no. 3, 495–507.

[KV06] V. Kostrykin and I. Veselić, *On the Lipschitz continuity of the integrated density of states for sign-indefinite potentials*, Math. Z. **252** (2006), no. 2, 367–392.

[Nar95] R. Narasimhan, *Several complex variables*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1995, Reprint of the 1971 original.

[Rem84] R. Remmert, *Funktionentheorie 1*, Springer Verlag, Berlin, 1984.

[RS78] M. Reed and B. Simon, *Methods of modern mathematical physics IV: Analysis of operators*, Academic press, London, 1978.

[vDK89] H. v. Dreifus and A. Klein, *A new proof of localization in the Anderson tight binding model*, Comm. Math. Phys. **124** (1989), no. 2, 285–299.

[Ves02] I. Veselić, *Wegner estimate and the density of states of some indefinite alloy type Schrödinger operators*, Lett. Math. Phys. **59** (2002), no. 3, 199–214.

[Ves08] I. Veselić, *Existence and regularity properties of the integrated density of states of random schrdinger operators*, Lecture Notes in Mathematics, vol. 1917, 2008.

[Ves10a] I. Veselić, *Wegner estimate for discrete alloy-type models*, Ann. Henri Poincaré **11** (2010), no. 5, 991–1005.

[Ves10b] I. Veselić, *Wegner estimates for sign-changing single site potentials*, Math. Phys. Anal. Geom. **13** (2010), no. 4, 299–313.

[Weg81] F. Wegner, *Bounds on the DOS in disordered systems*, Z. Phys. B **44** (1981), 9–15.

[Zie89] W. P. Ziemer, *Weakly differentiable functions*, Springer, New York, 1989.