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Preprint 2007-25
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Abstract

Studying first the Euclidean subcase, we show that the Minkowskian width function of a convex body in an n-dimensional (normed linear or) Minkowski space satisfies a specified Lipschitz condition.

AMS Subject Classification (AMS 2000): 46B20, 52A20, 52A21

Keywords: convex body, diameter, Lipschitz condition, Minkowski space, normed linear space, (Minkowskian) width function

1 Introduction

The study of width functions of convex bodies was already stimulated in the classical monograph [3] (see § 33 there). These functions play an important role in the fields of geometric convexity, geometric tomography, geometric inequalities, and Minkowski geometry; cf. [12], [6], [4], and [13], respectively. More precisely, width functions of convex bodies are basic for the following topics and notions from these fields: support functions of convex bodies (see [12], § 1.7), the difference body and the central symmetral of a convex body (and therefore also the related maximum chord-length function, cf. [6], § 3.2 and [1]), bodies of constant width (see the surveys [5], [8], and [10]) and the related class of reduced bodies ([7], [9], and [2]), diameter and thickness as extremal values of width functions (leading to famous topics like the isodiametric problem, or the theorems of Jung and Steinhagen; cf. [3], § 44, [4], § 11, and [11]), and problems involving the mean width of convex bodies (see again [4], § 11).

In what follows, let $K$ denote a convex body in $\mathbb{R}^n$ for some $n \geq 2$, i.e., a compact, convex set whose affine hull $\text{aff}(K)$ equals $\mathbb{R}^n$. The $n$-dimensional Euclidean unit ball is denoted by $E = E_n$. Hence, if $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^n$, one has

$$E_n = \{ v \in \mathbb{R}^n : \langle v, v \rangle \leq 1 \}.$$ 

Moreover, we put, as usual, $S^{n-1} := \partial E_n$.

Let $B$ denote the unit ball of an arbitrary (normed linear or) Minkowski space on $\mathbb{R}^n$, i.e., $B$ is a convex body in $\mathbb{R}^n$ centered at the origin. Thus the induced Minkowskian norm $\| \cdot \|_B$ satisfies

$$B = \{ v \in \mathbb{R}^n : \| v \|_B \leq 1 \}.$$
For \( u \in S^{n-1} \), let \( H(K, u) \) denote the supporting hyperplane of \( K \) with outward normal vector \( u \) in the Euclidean sense.

The Minkowskian width function \( w_B(K, \cdot) : S^{n-1} \to \mathbb{R}^+ \) is defined by

\[
w_B(K, u) := \min \{ \|x - y\|_B : x \in H(K, u), y \in H(K, -u) \}.
\]

(1)

This means: \( w_B(K, u) \) is the Minkowskian distance between \( H(K, u) \) and \( H(K, -u) \). To prove that \( w_B(K, \cdot) \) satisfies a specified Lipschitz Condition, we study first the Euclidean case \( B = E = E_n \). The Euclidean norm is denoted by \( \| \cdot \|_E \). For brevity, we write

\[
w(u) := w_E(K, u) \text{ for } u \in S^{n-1}.
\]

(2)

Furthermore, the diameter \( \text{diam}K \) and the thickness \( \Delta(K) \) in the Euclidean sense are defined by

\[
diam K := \max_{x, y \in K} \|x - y\|_E = \max_{u \in S^{n-1}} w(u) \quad \text{and} \quad \Delta(K) := \min_{u \in S^{n-1}} w(u),
\]

(3)

(4)

respectively.

\section{Results and proofs}

As announced, we start with the Euclidean subcase.

\textbf{Proposition.} For all \( u, v \in S^{n-1} \), the inequality

\[
|w(v) - w(u)| \leq \text{diam}K \cdot \|v - u\|_E
\]

(5)

holds.

\textbf{Proof:} We may assume that \( u \neq v \). In case \( 0 < \theta(u, v) \leq \pi \) one has \( \|v - u\|_E \leq \|v + u\|_E \).

Since \( w(u) = w(-u) \), we can therefore also suppose that \( \alpha := \theta(u, v) \leq \frac{\pi}{2} \), and hence \( \langle u, v \rangle = \cos \alpha \geq 0 \).

Put

\[
H_1 := H(K, u), \quad H'_1 := H(K, -u),
\]

\[
H_2 := H(K, v), \quad H'_2 := H(K, -v);
\]

\[
z := \frac{1}{\|v - \langle u, v \rangle u\|_E} \cdot (v - \langle u, v \rangle u) \in S^{n-1},
\]

\[
H_0 := H(K, z), \quad H'_0 := H(K, -z).
\]
Moreover, let $P_0 \subset \mathbb{R}^n$ denote the homogeneous plane spanned by the unit vectors $u$ and $v$. Without loss of generality, we may suppose that

$$F := K \cap P_0 \neq \emptyset.$$  

Furthermore, put

$$L_i := H_i \cap P_0, \quad L'_i := H'_i \cap P_0 \quad \text{for} \quad 0 \leq i \leq 2.$$  

Then all $L_i, L'_i$ are affine lines in $P_0$, and $F$ is contained in the 2-dimensional strips $\text{conv}(L_i \cup L'_i)$ for $0 \leq i \leq 2$, where $\text{conv}$ denotes convex hull.

Note that $F$ does not necessarily touch the lines $L_i, L'_i$. We merely know that $K$ touches all 6 hyperplanes $H_i, H'_i$ for $0 \leq i \leq 2$. Since $\langle u, z \rangle = 0$, the following holds: The lines $L_0, L'_0$ are parallel to the homogeneous line $\mathbb{R} \cdot u$, while the lines $L_1, L'_1$ are parallel to the homogeneous line $\mathbb{R} \cdot z$. Hence, the four points $a_1, a_2, a_3, a_4 \in P_0$ given by

$$\{a_1\} = L'_0 \cap L'_1, \quad \{a_2\} = L'_0 \cap L_1, \quad \{a_3\} = L_0 \cap L_1, \quad \{a_4\} = L_0 \cap L'_1$$

are the vertices of a rectangle. Without loss of generality, we may assume that

$$a_1 = 0, \quad a_2 = d \cdot u, \quad a_3 = d \cdot u + h \cdot z, \quad a_4 = h \cdot z,$$

where $d := w(u)$ and $h := w(z)$.

Note that, for $0 \leq i \leq 2$, $L_i$ and $L'_i$ have the same Euclidean distance as $H_i$ and $H'_i$, because $\{u, v, z\} \subseteq P_0$.

Let $H_3$ or $H'_3$ denote the hyperplanes in $\mathbb{R}^n$ that are parallel to $H_2 = H(K, v)$ and pass through $a_1$ or $a_3$, respectively. Then one has

$$K \subseteq \text{conv}(H_0 \cup H'_0) \cap \text{conv}(H_1 \cup H'_1) \subseteq \text{conv}(H_3 \cup H'_3)$$

Figure 1

Let $H_3$ or $H'_3$ denote the hyperplanes in $\mathbb{R}^n$ that are parallel to $H_2 = H(K, v)$ and pass through $a_1$ or $a_3$, respectively. Then one has

$$K \subseteq \text{conv}(H_0 \cup H'_0) \cap \text{conv}(H_1 \cup H'_1) \subseteq \text{conv}(H_3 \cup H'_3)$$

3
and, hence,

\[ w(v) \leq \langle a_3, v \rangle. \]

Since \( 0 < \alpha \leq \frac{\pi}{2} \), we have

\[ v = \cos \alpha \cdot u + \sin \alpha \cdot z. \]

Therefore we get

\[
\|v - u\|_E = \sqrt{(1 - \cos \alpha)^2 + \sin^2 \alpha} = \sqrt{2 - 2 \cdot \cos \alpha},
\]

\[ w(v) - w(u) \leq \cos \alpha \cdot d + \sin \alpha \cdot h - d = \sin \alpha \cdot h - (1 - \cos \alpha) \cdot d. \]

This implies

\[
\frac{w(v) - w(u)}{\|v - u\|_E} < h \cdot \frac{\sin \alpha}{\sqrt{2 - 2 \cdot \cos \alpha}}
\]

\[ = h \cdot \sqrt{\frac{1 - \cos^2 \alpha}{2 \cdot (1 - \cos \alpha)}}
\]

\[ = h \cdot \sqrt{\frac{1}{2} \cdot (1 + \cos \alpha)}
\]

\[ \leq h \leq \text{diam} K. \]

By exchanging the roles of \( u \) and \( v \), (5) follows.

**Remarks.**

i) As pointed out to us by Rolf Schneider, Lemma 1.8.10 in [12] implies the following, slightly weaker Lipschitz Condition:

\[ |w(v) - w(u)| \leq 2 \cdot R \cdot \|v - u\|_E. \]  

(6)

Here \( R \) denotes the circumradius of \( K \); that is the radius of the uniquely determined smallest Euclidean ball containing \( K \).

ii) The estimate (5) is sharp in the following sense: For every \( \eta > 0 \), there exist a compact and convex body \( K \) as well as \( u, v \in S^{n-1} \) satisfying

\[ |w(v) - w(u)| > (1 - \eta) \cdot \text{diam} K \cdot \|v - u\|_E. \]  

(7)

Namely, let \( K \subseteq \mathbb{R}^2 \) denote the rectangle with vertices

\[ (0, 0), (d, 0), (d, h), (0, h), \]
where $0 < d < h$.

If $u = (1, 0)$, then we get, similarly as in the above proof:

$$
\lim_{v \to u, v \in S^{n-1} \setminus \{u\}} \frac{|w(v) - w(u)|}{\|v - u\|_E} = \lim_{\alpha \to 0, \alpha > 0} \frac{|\sin \alpha \cdot h - (1 - \cos \alpha) \cdot d|}{\sqrt{2 - 2 \cdot \cos \alpha}} = h \cdot \lim_{\alpha \to 0} \sqrt{\frac{1}{2} \cdot (1 + \cos \alpha)} = h.
$$

Hence, if $\frac{h}{d}$ is so large that

$$h > (1 - \eta) \cdot \sqrt{k^2 + d^2} = (1 - \eta) \cdot \text{diam} K,$$

then (7) holds for $u = (1, 0)$ and $v = (\cos \alpha, \sin \alpha)$, if $\alpha \in \mathbb{R}^+$ is small enough. \hfill \square

Now we return to arbitrary Minkowskian norms $\| \cdot \|_B$. Recall that all $u \in S^{n-1}$ satisfy

$$w_B(K, u) = 2 \cdot \frac{w_E(K, u)}{w_E(B, u)}.
$$

(8)

See, for instance, [1] and [2]. Based on our Proposition and (8), we can now also prove the following

**Theorem.** For every convex body $K$ in $\mathbb{R}^n$, $n \geq 2$, and every Minkowskian norm $\| \cdot \|_B$ on $\mathbb{R}^n$ one has

$$|w_B(K, v) - w_B(K, u)| \leq 2 \cdot (\Delta(B)^{-2} \cdot \text{diam} K \cdot (\Delta(B) + \text{diam} B) \cdot \|v - u\|_E
$$

$$\leq 4 \cdot (\Delta(B)^{-2} \cdot \text{diam} B \cdot \text{diam} K \cdot \|v - u\|_E
$$

(9)

for all $u, v \in S^{n-1}$.

**Proof:** The second estimate in (9) is trivial, because $\Delta(B) \leq \text{diam} B$. Now assume that $u, v \in S^{n-1}$ are fixed. Our Proposition, applied to the convex bodies $K$ and $B$, yields:

$$|w_E(K, v) - w_E(K, u)| \leq \text{diam} K \cdot \|v - u\|_E,$$

$$|w_E(B, v) - w_E(B, u)| \leq \text{diam} B \cdot \|v - u\|_E.$$

Together with (8), (3), and (4) we obtain:

$$|w_B(K, v) - w_B(K, u)| = 2 \cdot \left| \frac{w_E(K, v) - w_E(K, u)}{w_E(B, v) - w_E(B, u)} \right|$$

$$= 2 \cdot \left( \frac{|w_E(K, v) - w_E(K, u)|}{w_E(B, v)} + w_E(K, u) \cdot \frac{|w_E(B, u) - w_E(B, v)|}{w_E(B, v) \cdot w_E(B, u)} \right)$$

$$\leq 2 \cdot \left( \frac{|w_E(K, v) - w_E(K, u)|}{w_E(B, v)} + w_E(K, u) \cdot \frac{|w_E(B, u) - w_E(B, v)|}{w_E(B, v) \cdot w_E(B, u)} \right)$$

$$\leq 2 \cdot (\Delta(B)^{-1} \cdot \text{diam} K + \Delta(B)^{-2} \cdot \text{diam} K \cdot \text{diam} B) \cdot \|v - u\|_E$$

$$= 2 \cdot (\Delta(B)^{-2} \cdot \text{diam} K \cdot (\Delta(B) + \text{diam} B) \cdot \|v - u\|_E.$$

\hfill \square
References


