Multi-parameter regularization - convergence and convergence rates results

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Abstract

In this paper we present a multi-parameter regularization approach for solving nonlinear ill-posed problems when a 'vector' of data is given. Based on the the convergence analysis for nonlinear Tikhonov regularization we show stability and convergence of the method. Additionally we prove convergence rates results by using Bregman distances and suggest a numerical algorithm for solving the underlying minimization problem in an efficient way. The advantage of considering multi-parameter regularization approaches is illustrated by an example arising in mathematical finance.

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1 Introduction

Classical regularization theory are based on ill-posed operator equations

$$F(x) = y^\delta,$$

with given (single) noisy data y^δ and (single) noise level δ. Thereby F denotes a (nonlinear) mapping between Banach spaces. Usually a regularization strategy is applied in order to find an approximate solution of this equation in a stable way, see e.g. [6] for an overview over different regularization methods for linear and nonlinear ill-posed problems. The balance between stability and approximation can be expressed by the regularization parameter $\alpha > 0$. However, there are often identification problems, where the measurement y^δ can be considered as a 'vector' $y^\delta = (y_1^\delta, \ldots, y_l^\delta)$ of different types of data. Thereby the exactness of the different measurements may be vary, so it is natural to consider the noise level as vector $\underline{\delta} := (\delta_1, \ldots, \delta_l)^T \in \mathbb{R}^l$ as well. This gives us the motivation for using a vector $\underline{\alpha} = (\alpha_1, \ldots, \alpha_l)^T \in \mathbb{R}^l$ as regularization parameter.

Let $\mathcal{X}$, $\mathcal{Y}_j$, $1 \le j \le l$, denote reflexive Banach spaces. Moreover, let $F_j : \mathcal{D}(F_j) \subseteq \mathcal{X} \longrightarrow \mathcal{Y}_j$, $1 \le j \le l$, define (nonlinear) operators which describe the relation between the parameter $x \in \mathcal{X}$ to be identified and the observations $y_j \in \mathcal{Y}_j$, $1 \le j \le l$. We introduce $\mathcal{D} := \bigcap \mathcal{D}(F_j) \subset \mathcal{X}$ as the set of all feasible parameters and suppose $\mathcal{D} \neq \emptyset$. For given exact data we have to solve the system of equation

$$F_j(x) = y_j, \qquad 1 \le j \le l,\ x \in \mathcal{D}, \tag{1}$$

whereas in the case of noisy data y_j^δ with $\|y_j - y_j^\delta\| \le \delta_j$, $1 \le j \le l$, we have the perturbated system

$$F_j(x) = y_j^\delta, \qquad 1 \le j \le l,\ x \in \mathcal{D}. \tag{2}$$

In many applications such problems are ill-posed. Even if we suppose unique solvability of the system (1) respectively (2), a solution $x \in \mathcal{D}$ does not depend continuously on the given data. That means, if $x^\dagger$ denotes the unique solution of (1) and x^δ is the unique solution of (2) then the distance $\|x^\dagger - x^\delta\|$ might become arbitrarily large even the noise levels δ_j, $1 \le j \le l$, are arbitrarily small.

Therefore regularization strategies has to be applied. One standard approach is to introduce a stabilizing functional $J : \mathcal{D} \subseteq \mathcal{X} \longrightarrow \mathbb{R}$ and replace (2) by the constrained minimizing problem

$$\begin{cases} J(x) \to \min \quad \text{subject to} \\ x \in \mathcal{D}, \\ \left\|F_j(x) - y_j^\delta\right\|_{\mathcal{Y}_j} \le \delta_j, \quad 1 \le j \le l. \end{cases} \tag{3}$$

Such problems were originally considered in [10, Chapter 4.2] for discrete data, see also [4] for an application in Hilbert spaces.

As in classical regularization theory, alternatively a Tikhonov-like approach

$$\sum_{j=1}^{l} \lambda_j \left\|F_j(x) - y_j^\delta\right\|_{\mathcal{Y}_j}^2 + J(x) \to \min \quad \text{subject to } \ x \in \mathcal{D}, \tag{4}$$

with parameters $\lambda_j > 0$ $1 \le j \le l$, might be used. These parameters can be considered as Lagrange multipliers while using a Lagrange method for solving (3). On the other hand, the parameter $\alpha_j := \lambda_j^{-1}$, $1 \le j \le l$, play the role of l regularization parameters. Therefore we can speak from a *multi-parameter regularization* approach.

The paper is organized as follows: based on standard assumptions we show the well-posedness of (3). In fact it will be shown, that the analysis is quite similar as in the case of a single data. In section 3 we consider convergence rates in terms of Bregman distances which provide a concept for generalizing convergence rates for arbitrary penalty functionals . Section 4 presents a algorithm for solving (3) in Hilbert spaces. The algorithm is closely related to a special variant of a multiplier method for inequality constrained minimization problems. Afterwards we show convergences rates for (4) again in terms of Bregman distances. Finally we shortly present an application which arises in financial mathematics.

2 Stability results

First of all we want to show the well-posedness of the minimizing problem (3). In particular, we give conditions for the existence of a solution x^δ of (3), which depends stable on the given data y_j^δ (and the noise-level δ_j) for $1 \leq j \leq l$. As we will see, the analysis does not much differ from the one for proving existence, stability and convergence of minimizers x_α^δ of the Tikhonov functional

$$\left\|F(x) - y^\delta\right\|_{\mathcal{Y}}^2 + \alpha J(x) \tag{5}$$

in classical regularization theory, see [5] for $J(x) := \|x - x^*\|^2$, [12] for $J(x) := \|D(x-x^*)\|^2$ with closed linear operator D or [16] for more general penalty terms $J(x)$). Although the basic assumptions slightly differ the main ideas of the proofs are in general the same,see also [1]. Recently, the well-posedness of (5) in more general topologies was shown in [11].

The assumptions under considerations are in detail:

(A1) For exact data there exists a solution $x^\dagger$ of (1), i.e. $F_j(x^\dagger) = y_j$ for all $1 \leq j \leq l$.

(A2) The operators F_j, $1 \leq j \leq l$, are continuous and weakly sequentially closed, i.e. for $x_n \in \mathcal{D}(F_j)$, $x_n \rightharpoonup x$, $F_j(x_n) \rightharpoonup y_j$ we have $x \in \mathcal{D}(F_j)$ and $F_j(x) = y_j$.

(A3) The penalty functional J is nonnegative and weakly sequentially lower semi-continuous, i.e. $x_n \rightharpoonup x$ implies $J(x) \leq \liminf\limits_{n\to\infty} J(x_n)$.

(A4) The sets

$$\mathcal{A}(C) := \left\{ x \in \mathcal{X} \,:\, J(x) + \sum_{j=1}^{l} \|F_j(x)\|_{\mathcal{Y}_j} \leq C \right\}$$

are bounded in $\mathcal{X}$ for all $C \geq 0$.

(A5) The properties $x_n \rightharpoonup x$, $J(x_n) \to J(x)$ imply $x_n \to x$.

Note, that assumption (A1) seems to be natural for identification problems. If we have (exact) observations y_j, $1 \leq j \leq l$, we can assume a 'cause' $x^\dagger \in \mathcal{D}$ of the given data. This might not remain true for given noisy data y_j^δ, $1 \leq j \leq l$. Hence, a solution of (2) does not need exist.

With

$$\mathcal{M}_\delta := \left\{ x \in \mathcal{X} \,:\, \|F_j(x) - y_j^\delta\| \leq \delta_j,\ 1 \leq j \leq l \right\} \tag{6}$$

we denote the set of feasible elements. Since $x^\dagger \in \mathcal{M}_\delta$ for all $\delta \geq 0$ the sets are nonempty. By the continuity of the operators F_j, $1 \leq j \leq l$, the sets $\mathcal{M}_\delta$ are closed.

We show the existence of a solution x^δ of the problem (3).

Lemma 2.1 *Under the conditions (A1)-(A4) there exists a solution x^δ of (3).*

Proof. Let $\{x_n\} \subset \mathcal{M}_\delta$ be a minimizing sequence, i.e. $J(x_{n+1}) \leq J(x_n)$. Hence, because of (A4), the sequence $\{(x_n, F_1(x_n), \ldots, F_l(x_n))\}$ is bounded in $\mathcal{X} \times \mathcal{Y}_1 \times \ldots \times \mathcal{Y}_l$.

Consequently, there exists a weak subsequence $\{x_{n_k}\}$ with $x_{n_k} \rightharpoonup x_{n_k}$ and $F_j(x_{n_k}) \rightharpoonup \hat{y}_j$, $1 \le j \le l$. By (A2) and the lower semi-continuity of the norm we have

$$\|F_j(x) - y_j\| \le \liminf_{k\to\infty} \|F_j(x_{n_k}) - y_j\| \le \delta_j$$

which gives $x \in \mathcal{M}_\delta$. Hence, by (A3), $J(x) \le \liminf_{k\to\infty} J(x_{n_k})$, which shows, that x is a minimizer of (3). ■

Under certain conditions to perturbations in the side constraints we can prove stable dependence of a solution of (3) from the given data. A similar proof for discrete data can be found in [10, Theorem 4.24].

Lemma 2.2 *Let (A1)-(A4) hold. Moreover, let be $y_j^{(n)}$ with $\|y_j^{(n)} - y_j^\delta\| \le c_j^{(n)}$ with $c_j^{(n)} \to 0$ for $n \to \infty$. Let $\{x_n^\delta\}$ denotes a sequence of solutions of (3), where y_j^δ and δ_j are replaced by $y_j^{(n)}$ and $c_j^{(n)} + \delta_j$. Then there exists a subsequence $\{x_{n_k}^\delta\}$ converging weakly to a minimizer x^δ of (3). If the minimizer is unique then $x_n^\delta \rightharpoonup x^\delta$.*

Proof. For given n let $\mathcal{M}_\delta^{(n)}$ denote the set of all feasible elements. By construction we have $x^\delta \in \mathcal{M}_\delta^{(n)}$ and $\mathcal{M}_\delta^{(n)} \to \mathcal{M}_\delta$. Hence, $J(x_n^\delta) \le J(x^\delta)$ for all n. Same arguments as above we have a subsequence $\{x_{n_k}^\delta\}$ with $x_{n_k}^\delta \rightharpoonup x \in \mathcal{M}_\delta$. This implies

$$J(x^\delta) \le J(x) \le \liminf_{k\to\infty} J(x_{n_k}^\delta) \le J(x^\delta).$$

Now, $J(x) = J(x^\delta)$ which shows, that x is a minimizer of (3). The uniqueness of the minimizer says $x = x^\delta$ which trivially shows the weak convergence of the whole sequence $\{x_n\}$. ■

The introduction of the variables $c_j^{(n)}$ has technical reasons. They only ensure that for fixed n^* the element $x_{n^*}^\delta$ belongs to $\mathcal{M}_\delta^{(n)}$ for $n \le n^*$, which is used in the proof.

The additional assumption (A5) clearly gives strong convergence.

Lemma 2.3 *Let the assumptions of Lemma 2.2 hold. If additionally (A5) holds, then the convergence is strong.*

Finally we want to prove the convergence of a solution of (3) to a solution $x^\dagger$ of (1) for $\delta_j \to 0$, $1 \le j \le l$. Moreover, such a solution hat the property that it minimizes the penalty functional J under all solutions of (1). Therefore we introduce the following notation.

Definition 2.1 *We call $x^\dagger$ a J-minimizing solution of (1) if*

$$J(x^\dagger) = \min\{J(x) \,:\, F_j(x) = y_j,\ 1 \le j \le l\}.$$

The previous Lemmas 2.2 and 2.3 now immediately show the convergence x^δ to a J-minimizing solution of (1) for $\delta \to 0$. In fact, we have to replace $\delta_j + c_j^{(n)}$ by δ_j and y_j^δ by y for $\le j \le l$ and repeat the proof of Lemma 2.2. The corresponding statement is given below.

Lemma 2.4 *Let (A1)-(A4) hold and $\delta_j \to 0$, i.e. $y_j^\delta \to y_j$, $1 \le j \le l$. If x^δ denote the corresponding solutions of (3), then there exists a subsequence $\{\tilde{x}^\delta\}$ converging weakly to a J-minimizing solution $x^\dagger$ of (1). Moreover, if the J-minimizing element $x^\dagger$ is unique, then $x^\delta \rightharpoonup x^\dagger$. If additional (A5) holds, the convergence is strong.*

3 Convergence rates

Now we consider the rate of convergence of x^δ to $x^\dagger$ for $\delta_j \to 0$, $1 \le j \le l$. Thereby we formulate convergence rates in terms of Bregman distances. This concept was introduced in [3], see also [11] for some newer results. For simplicity we assume Fréchet- differentiability of the functional J. Then we can define the Bregman distance.

Definition 3.1 *Let be J convex and Fréchet-differentiable with Fréchet-derivative $J'(x) \in \mathcal{X}^*$ for $x \in \mathcal{D}$. Then we define the Bregman distance $D(\tilde{x}, x)$ for two elements $\tilde{x}, x \in \mathcal{D}$ as*

$$D(\tilde{x}, x) := J(\tilde{x}) - J(x) - \langle J'(x), \tilde{x} - x \rangle_{\mathcal{X}^*,\mathcal{X}}\,,$$

where $\langle \cdot, \cdot \rangle_{\mathcal{X}^*,\mathcal{X}}$ denotes the duality product in $\mathcal{X}$.

The convexity of J implies $D(\tilde{x}, x) \ge 0$ for all $\tilde{x}, x \in \mathcal{D}$. If we additional assume strict convexity, then $D(\tilde{x}, x) > 0$ for $\tilde{x} \neq x$. The idea of Bregman distances originally was applied to non-smooth functionals J, where $D(\tilde{x}, x)$ denote sets of subgradients of J at the point x, see [3]. Note, that all assertions stated below remain true in that case.

For the proof of convergence rates we suppose the following condition, which was firstly introduces in [11].

(A6) There exist $0 \le \gamma < 1$ and $\underline{\beta} = (\beta_1, \ldots, \beta_l)^T \in \mathbb{R}^l$ with $\beta_j \ge 0$, $1 \le j \le l$, such that

$$|\langle J'(x^\dagger), x - x^\dagger \rangle| \le \gamma D(x, x^\dagger) + \sum_{j=1}^{l} \beta_j \|F_j(x) - F_j(x^\dagger)\|_{\mathcal{Y}_j} \tag{7}$$

for all $x \in \mathcal{B}_r(x^\dagger) \cap \mathcal{D}$ with $\mathcal{B}_r(x^\dagger) := \{x \in \mathcal{X} \,:\, \|x - x^\dagger\| < r\}$ and $r > 0$ sufficiently large.

Under this additional assumption we can easily prove convergence rates depending on $\|\underline{\delta}\|_2$ where $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^l$.

Theorem 3.1 *Assume (A1)-(A6) to be hold. Then with $\underline{\delta} := (\delta_1, \ldots, \delta_l)^T \in \mathbb{R}^l$ we have*

$$D(x^\delta, x^\dagger) \le \frac{2}{1-\gamma} \|\underline{\beta}\|_2 \|\underline{\delta}\|_2 \tag{8}$$

Proof. The minimizing property $J(x^\delta) \le J(x^\dagger)$ implies

$$D(x^\delta, x^\dagger) \;\le\; \left\langle J'(x^\dagger), x^\delta - x^\dagger \right\rangle_{\mathcal{X}^*,\mathcal{X}}$$

$$
\begin{aligned}
&\leq \gamma D(x^\delta, x^\dagger) + \sum_{j=1}^{l} \beta_j \|F_j(x^\delta) - F_j(x^\dagger)\|_{\mathcal{Y}_j} \\
&\leq \gamma D(x^\delta, x^\dagger) + \sum_{j=1}^{l} \beta_j \left(\|F_j(x^\delta) - y_j^\delta\|_{\mathcal{Y}_j} + \|y_j^\delta - y_j\|_{\mathcal{Y}_j}\right) \\
&\leq \gamma D(x^\delta, x^\dagger) + 2\|\underline{\beta}\|_2 \|\underline{\delta}\|_2
\end{aligned}
$$

which shows the assertion. ■

In order to prove convergence rates in the norm of $\mathcal{X}$ we need additional conditions to the operators F_j, $1 \leq j \leq l$, and/or the penalty functional J. If, for example, J strongly convex in $x^\dagger$, then (8) implies

$$
\left\|x^\delta - x^\dagger\right\|_{\mathcal{X}}^2 \leq c\, D(x^\delta, x^\dagger) \leq \frac{2c}{1-\gamma} \|\underline{\beta}\|_2 \|\underline{\delta}\|_2
$$

for a constant $c > 0$. On the other hand, if $J(x) = \|x - x^*\|^2$ then obviously $D(\tilde{x}, x) = \|\tilde{x} - x\|^2$ so that (8) already presents convergence rates in the norm of $\mathcal{X}$ in that case. In the case $J(x) = \|D(x - x^*)\|^2$ conditions for convergence rates can be found in [14] for regularization of linear operator equations and in [12] in the nonlinear case.

We shortly discuss the assumption (A6). It can be fulfilled by assuming a source condition

$$
J'(x^\dagger) = \sum_{j=1}^{l} G_j^* \omega_j, \qquad \omega_j \in \mathcal{Y}_j^*,\ 1 \leq j \leq l \tag{9}
$$

where $G_j \in L(\mathcal{X}, \mathcal{Y}_j)$ is a linear approximation of F_j with the residual

$$
r_j(x, x^\dagger) := F_j(x) - F_j(x^\dagger) - G(x - x^\dagger) \tag{10}
$$

satisfying an additional restriction. We give three examples, which show that (A6) is closely related to assumptions which are uses in classical regularization theory.

Lemma 3.1 *Assume (9) holds with*

$$
\|r_j(x, x^\dagger)\|_{\mathcal{Y}_j} \leq C_j D(x, x^\dagger) \quad \textit{and} \quad C := \sum_{j=1}^{l} \|\omega_j\|_{\mathcal{Y}_j^*} C_j < 1
$$

Then (A6) hold with $\gamma = C$ and $\beta_j := \|\omega_j\|_{\mathcal{Y}_j^*}$, $1 \leq j \leq l$.

Proof. We have

$$
\begin{aligned}
\left\langle J'(x^\dagger), x - x^\dagger \right\rangle_{\mathcal{X}^*, \mathcal{X}} &= \sum_{j=1}^{l} \left\langle \omega_j, G_j(x - x^\dagger) \right\rangle_{\mathcal{Y}_j^*, \mathcal{Y}_j} \\
&= \sum_{j=1}^{l} \left\langle \omega_j, F_j(x) - F_j(x^\dagger) - r_j(x, x^\dagger) \right\rangle_{\mathcal{Y}_j^*, \mathcal{Y}_j} \\
&\leq \sum_{j=1}^{l} \|\omega_j\|_{\mathcal{Y}_j^*} \left(\|F_j(x) - F_j(x^\dagger)\|_{\mathcal{Y}_j} + \|r_j(x, x^\dagger)\|_{\mathcal{Y}_j}\right) \\
&\leq \sum_{j=1}^{l} \|\omega_j\|_{\mathcal{Y}_j^*} \|F_j(x) - F_j(x^\dagger)\|_{\mathcal{Y}_j} + \left(\sum_{j=1}^{l} C_j \|\omega_j\|_{\mathcal{Y}_j^*}\right) D(x, x^\dagger). \quad ■
\end{aligned}
$$

Assume that $J(x) = \|x - x^*\|^2$ and that the operators F_j are Fréchet-differentiable. Then the conditions of Lemma 3.1 are similar to the one for classical Tikhonov regularization (see e.g. [5]), where Lipschitz continuity of the Fréchet-derivative was supposed. We only have to set $G_j = F_j'(x^\dagger)$, $1 \le j \le l$. This shows, that condition (7) is a generalization of classical assumptions for proving convergence rates. On the other hand, the Lipschitz continuity of the Fréchet-derivative is not sufficient to prove convergence (rates) for iterative regularization methods. Hence different assumptions to restrict the nonlinearity of the operators were used (see [7] for a brief overview over different iterative regularization methods and their convergence analysis). So we give two more conditions, under which (7) is satisfied.

Lemma 3.2 *Assume (9) holds with*

$$\|r_j(x, x^\dagger)\|_{\mathcal{Y}_j} \le C_j \|F_j(x) - F_j(x^\dagger)\|_{\mathcal{Y}_j}, \qquad 1 \le j \le l.$$

Then (A6) hold with $\gamma = 0$ and $\beta_j := \|\omega_j\|_{\mathcal{Y}_j^*}(C_j + 1)$, $1 \le j \le l$.

Proof. Following the lines of the proof above we conclude

$$\begin{aligned} \left\langle J'(x^\dagger), x - x^\dagger \right\rangle_{\mathcal{X}^*,\mathcal{X}} &\le \sum_{j=1}^{l} \|\omega_j\|_{\mathcal{Y}_j^*} \left(\|F_j(x) - F_j(x^\dagger)\|_{\mathcal{Y}_j} + \|r_j(x, x^\dagger)\|_{\mathcal{Y}_j} \right) \\ &\le \sum_{j=1}^{l} \|\omega_j\|_{\mathcal{Y}_j^*} (C_j + 1) \|F_j(x) - F_j(x^\dagger)\|_{\mathcal{Y}_j}. \qquad \blacksquare \end{aligned}$$

For $C_j < \frac{1}{2}$ the condition above is known as so-called η-condition, which was firstly used to prove convergence of the nonlinear Landweber iteration method [8]. Alternatively we can use the operators G_j to restrict the residuals $r_j(x, x^\dagger)$.

Lemma 3.3 *Assume (9) holds with*

$$\|r_j(x, x^\dagger)\|_{\mathcal{Y}_j} \le C_j \|G(x - x^\dagger)\|_{\mathcal{Y}_j}, \qquad 1 \le j \le l.$$

and $C_j < 1$. Then (A6) hold with $\gamma = 0$ and $\beta_j := \|\omega_j\|_{\mathcal{Y}_j^*}(1 - C_j)^{-1}$, $1 \le j \le l$.

Proof. By applying triangle and inverse triangle inequality we have

$$(1 - C_j)\|G_j(x - x^\dagger)\|_{\mathcal{Y}_j} \le \|F_j(x) - F_j(x^\dagger)\|_{\mathcal{Y}_j} \le (1 + C_j)\|G_j(x - x^\dagger)\|_{\mathcal{Y}_j}.$$

Taking the first part we have

$$\begin{aligned} \left\langle J'(x^\dagger), x - x^\dagger \right\rangle_{\mathcal{X}^*,\mathcal{X}} &\le \sum_{j=1}^{l} \|\omega_j\|_{\mathcal{Y}_j^*} \|G_j(x - x^\dagger)\|_{\mathcal{Y}_j} \\ &\le \sum_{j=1}^{l} \frac{\|\omega_j\|_{\mathcal{Y}_j^*}}{1 - C_j} \|F_j(x) - F_j(x^\dagger)\|_{\mathcal{Y}_j}. \qquad \blacksquare \end{aligned}$$

These three examples shows, that condition (7) provides an interesting and reasonable way for presenting convergence rates results.

4 Multi-parameter regularization in Hilbert spaces

In this section we present an efficient approach for solving the minimizing problem (3). Therefore, let us now assume for simplicity, that $\mathcal{X}$ and $\mathcal{Y}_j$, $1 \leq j \leq l$, are Hilbert spaces. A generalization of the idea to arbitrary Banach spaces is in principle possible. Moreover, we suppose, that the functional J has s special structure, i.e.

$$J(x) := \|P(x)\|_{\mathcal{Z}}^2,$$

where $P : \mathcal{X} \longrightarrow \mathcal{Z}$ denotes a (nonlinear) mapping from $\mathcal{X}$ into another Hilbert space $\mathcal{Z}$.

We apply a Lagrange technique for solving (3). By reformulation of the inequality constraints we introduce the Lagrangian functional

$$L(x, \underline{\lambda}) := J(x) + \sum_{j=1}^{l} \lambda_j \left(\left\| F_j(x) - y_j^\delta \right\|_{\mathcal{Y}_j}^2 - \delta_j^2 \right) \tag{11}$$

as well as the Tikhonov-like functional

$$\tilde{T}_\lambda(x) := J(x) + \sum_{j=1}^{l} \lambda_j \left\| F_j(x) - y_j^\delta \right\|_{\mathcal{Y}_j}^2 \tag{12}$$

The latter definition gives the motivation of calling this approach *multi-parameter regularization.* For $\lambda_j > 0$, the Lagrangian multiplier λ_j can be considered as inverse of regularization parameter $\alpha_j := \lambda_j^{-1}$, so that we have up to l regularization parameter instead of single parameter as common in classical regularization theory.

For finding a solution x^δ of (3), the following algorithm was suggested in [10, Algorithm 4.26]:

Algorithm 1

1. give initial guess $x_0^\delta \in \mathcal{X}$, $\underline{\lambda}^{(0)} := (\lambda_1^{(0)}, \ldots, \lambda_l^{(0)})^T \geq 0$, set $k := 0$
2. find a solution x_{k+1}^δ of the (unconstrained) minimizing problem $\tilde{T}_{\lambda^{(k)}}(x) \rightarrow \min$
3. set
$$\lambda_j^{(k+1)} := \lambda_j^{(k)} \max \left(\frac{\|F_j(x_{k+1}^\delta) - y_j^\delta\|^\nu}{\delta_j^\nu}, \varepsilon \right) \tag{13}$$
   for $0 < \varepsilon \ll 1$ and arbitrary $\nu > 0$.
4. if $\|\underline{\lambda}^{(k+1)} - \underline{\lambda}^{(k)}\| < TOL$, then STOP, else $k := k + 1$ and return to 2.

It can be shown, that if we have convergence $x_k^\delta \rightarrow x^\delta$ and $\underline{\lambda}^{(k)} \rightarrow \underline{\lambda}$ for $k \rightarrow \infty$, then the x^δ is in fact a solution of (3) ([10, Theorem 4.24], see also [4, Theorem 8]).

Theorem 4.1 *As $k \rightarrow \infty$ let $\underline{\lambda}^{(k)} \rightarrow \underline{\lambda} \geq 0$ and $x_k^\delta \rightarrow x^\delta \in \mathcal{D} \subset \mathcal{X}$. Then the pair $(x^\delta, \underline{\lambda}) \in \mathcal{X} \times \mathbb{R}^l$ forms a saddle point of the Lagrangian functional (11) and hence x^δ is an optimal solution of the problem (3).*

The approach considered above has two disadvantages in numerical realization. First, we have to solve a nonlinear optimization problem in each iteration, which can become expensive. The second point is the slow convergence of the iteration (13) for updating the Lagrangian multipliers. In fact, formula (13) is based on a fixed point equation approach so we can expect only linear speed of convergence. This matter leads to an increasing numbers of iterations which raise the numerical costs again.

An alternative approach can be found, that the algorithm above is closely related to classical multiplier methods for constrained minimization problems for some ν. Therefore we can apply, what is known as *diagonalized method of multipliers* (see [2, pp.240]) in combination with a Gauß-Newton approximation of second derivatives. Let be F_j and P Fréchet-differentiable with derivatives F_j' and P'. Then for given x_k^δ we linearize

$$F_j(x_k^\delta + \Delta x) \approx F_j(x_k^\delta) + F_j'(x_k^\delta)\Delta x,\ 1 \le j \le l, \quad \text{and} \quad P(x_k^\delta + \Delta x) \approx P(x_k^\delta) + P'(x_k^\delta)\Delta x,$$

and consider the quadratic problem

$$\tilde{T}_{\lambda^{(k)}}(x_k^\delta + \Delta x) \approx \left\| P'(x_k^\delta)\Delta x + P(x_k^\delta) \right\|_{\mathcal{Z}}^2 + \sum_{j=1}^{l} \lambda_j^{(k)} \left\| F_j'(x_k^\delta)\Delta x - \left(y_j^\delta - F_j'(x_k^\delta)\right) \right\|_{\mathcal{Y}_j}^2 \to \min .$$

A solution Δx can be found by solving the normal equation

$$\left[P'(x_k^\delta)^* P'(x_k^\delta) + \sum_{j=1}^{l} \lambda_j F_j'(x_k^\delta)^* F_j'(x_k^\delta) \right] \Delta x = \sum_{j=1}^{l} \lambda_j F_j'(x_k^\delta)^* \left(y_j^\delta - F_j(x_k^\delta)\right) - P'(x_k^\delta)^* P(x_k^\delta), \tag{14}$$

where $F_j'(x_k^\delta)^*$ and $P'(x_k^\delta)^*$ denote the (Hilbert space-)adjoint operators of $F_j'(x_k^\delta)$ and $P'(x_k^\delta)$. Now we can reformulate the algorithm, which gives in fact an improvement.

Algorithm 2

1. give initial guess $x_0^\delta \in \mathcal{X}$, $\underline{\lambda}^{(0)} := (\lambda_1^{(0)}, \ldots, \lambda_l^{(0)})^T \ge 0$, set $k := 0$
2. solve (14) and set $x_{k+1}^\delta := x_k^\delta + \Delta x$
3. update the Lagrangian multiplier by (13)
4. if $\|\underline{\lambda}^{(k+1)} - \underline{\lambda}^{(k)}\|_2 < TOL1$ and $\|\Delta x\|_{\mathcal{X}} < TOL2$, then STOP, else $k := k+1$ and return to 2.

The algorithm can be basically applied in general reflexive Banach spaces. The Hilbert space properties are only used for establishing the normal equation (14). So we only have replace (14) by an appropriate condition for calculating Δx in Banach spaces. This can be done for example by applying dual mappings, see [17, Chapter 47.12].

In principle, the problem (3) can be solved by arbitrary optimization method for inequality constrained minimization problems. We refer to [2] for further variants and to [15] for aspects of their numerical realization.

5 Tikhonov-Multi-parameter regularization

We consider $\underline{\lambda} := (\lambda_1, \ldots, \lambda_l)^T \in \mathbb{R}^l$ with $\|\underline{\lambda}\|_\infty = 1$ and assume $\lambda_j > 0$ for all $1 \le j \le l$. An alternative regularization approach to (3) can be achieved by applying a Tikhonov-like regularization approach, i.e. we search a solution x_α^δ of the minimizing problem

$$T_\alpha(x) := \sum_{j=1}^{l} \lambda_j \left\| F_j(x) - y_j^\delta \right\|_{\mathcal{Y}_j}^2 + \alpha J(x) \to \min . \tag{15}$$

We have two interpretations of (15) concerning convergence studies. First we can assume $\lambda_1, \ldots, \lambda_l$ to be fixed and suppose $\alpha > 0$ to be the (single) regularization parameter. Then, by introducing the Banach space $\mathcal{Y} := \mathcal{Y}_1 \times \ldots \times \mathcal{Y}_l$ with weighted norm

$$\|\underline{y}\|_{\mathcal{Y}} := \left(\sum_{j=1}^{l} \lambda_j \|y_j\|_{\mathcal{Y}_j}^2 \right)^{\frac{1}{2}}, \quad \underline{y} = (y_1, \ldots, y_l),\ y_j \in \mathcal{Y}_j,\ 1 \le j \le l,$$

and the operator $F : \mathcal{D} \subset \mathcal{X} \longrightarrow \mathcal{Y}$ as $F(x) := (F_1(x), \ldots, F_l(x))$ we can apply stability and convergence results of classical Tikhonov regularization [16]. Here we have also one noise level δ. On the other hand, the additional parameter α in (15) can be considered as scaling parameter of $\tilde{T}_\lambda(x)$, so we in fact consider $\tilde{T}_\lambda(x) \to \min$ and $\alpha := \|\underline{\lambda}\|_\infty^{-1}$ is a parameter which is only introduced to present convergence analysis in a proper way. The vector $\underline{\alpha} := (\alpha_1, \ldots, \alpha_l)^T \in \mathbb{R}^l$ with $\alpha_j := \frac{\alpha}{\lambda_j}$, $1 \le j \le l$, play the role of the regularization parameter. The convergence rates results below will show, that it makes sense to have the second idea in mind.

By standard arguments it can be easily shown, that under the conditions (A1)-(A5) the problem (15) has at least one solution x_α^δ, which depends stable on the given data. For the sake of completeness we show the prove of convergence of x_α^δ to a solution $x^\dagger$ of (1) for $\delta \to 0$. Moreover we consider the case of incomplete minimization, i.e. we have $x_\alpha^{\delta,\eta}$ which satisfies

$$T_\alpha(x_\alpha^{\delta,\eta}) \le T_\alpha(x_\alpha^\delta) + \eta \tag{16}$$

for given $\eta \ge 0$.

Proposition 5.1 *Assume (A1)-(A5) to be hold. Let be $y_j^\delta \in \mathcal{Y}_j$ with $\|y_j - y_j^\delta\|_{\mathcal{Y}_j} \le \delta_j$, $\underline{\delta} := (\delta_1, \ldots, \delta_l)^T$ and let $\alpha := \alpha(\underline{\delta}, \eta)$ be such that*

$$\alpha(\underline{\delta}, \eta) \to 0, \ \frac{\|\underline{\delta}\|_2^2}{\alpha(\underline{\delta}, \eta)} \to 0 \ \textit{and} \ \frac{\eta}{\alpha(\underline{\delta}, \eta)} \to 0$$

as $\underline{\delta} \to 0$ and $\eta \to 0$. Then every sequences $\{x_{\alpha_k}^{\delta_k,\eta_k}\}$ with $\underline{\delta}_k \to 0$ and $\eta_k \to 0$, $\alpha_k = \alpha(\delta_k, \eta_k)$ with $x_{\alpha_k}^{\delta_k,\eta_k}$ satisfies (16) has a convergent subsequence. Each limit $\hat{x}$ of such subsequence is a J-minimizing solution of (1). If – in addition – the solution $x^\dagger$ of (1) is unique, then

$$\lim_{k \to \infty} x_{\alpha_k}^{\delta_k,\eta_k} = x^\dagger .$$

PROOF. We set $x_k := x_{\alpha_k}^{\delta_k,\eta_k}$. By definition of x_k we have

$$\begin{aligned}\sum_{j=1}^{l} \lambda_j \left\| F_j(x_k) - y_j^{\delta_{j,k}} \right\|_{\mathcal{Y}_j}^2 + \alpha_k J(x_k) \;&\leq\; \sum_{j=1}^{l} \lambda_j \left\| F_j(x^\dagger) - y_j^{\delta_{j,k}} \right\|_{\mathcal{Y}_j}^2 + \alpha_k J(x^\dagger) + \eta_k \\ &\leq\; \|\underline{\delta}_k\|_2^2 + \alpha_k J(x^\dagger) + \eta_k.\end{aligned}$$

Taking the limit, we have $F_j(x_k) \to y_j$ as $k \to \infty$ and

$$J(x_k) \leq \frac{\|\underline{\delta}_k\|_2^2}{\alpha_k} + J(x^\dagger) + \frac{\eta_k}{\alpha_k}. \tag{17}$$

In particular, the sequence $\{J(x_k), F_1(x_k), \ldots, F_l(x_k)\}$ remain bounded. Consequently there exists a weak convergent subsequence $x_{k_i} \rightharpoonup \hat{x} \in \mathcal{D}(F)$. Moreover, by (17) with $\hat{x}$ instead of $x^\dagger$ we obtain

$$J(\hat{x}) \leq \liminf_{i\to\infty} J(x_{k_i}) \leq \limsup_{i\to\infty} J(x_{k_i}) \leq \lim_{i\to\infty} \frac{\|\underline{\delta}_k\|_2^2}{\alpha_k} + J(\hat{x}) + \frac{\eta_k}{\alpha_k} = J(\hat{x})$$

and hence $J(x_{k_i}) \to J(\hat{x})$. Condition (A5) now implies $x_{k_i} \to \hat{x}$. Moreover, by (17) we conclude

$$J(\hat{x}) \leq \limsup_{i\to\infty} J(x_{k_i}) \leq J(x^\dagger) \leq J(\hat{x}).$$

In particular, $\hat{x}$ is a J-minimizing solution (1). If the solution $x^\dagger$ is unique, i.e. $\hat{x} = x^\dagger$, the convergence $x_k \to x^\dagger$ follows from the fact, that each weak convergent subsequence of $\{x_k\}$ converges to $x^\dagger$. ■

We also want to prove convergence rates.

Theorem 5.1 *Assume (A1)-(A6) to be hold. Then – for a choice $\alpha \sim \|\underline{\delta}\|_2$ – we obtain a convergence rate*

$$D(x_\alpha^\delta, x^\dagger) \sim \mathcal{O}\left(\|\underline{\delta}\|_2\right).$$

PROOF. Starting with $T_\alpha(x_\alpha^\delta) \leq T_\alpha(x^\dagger)$ we obtain

$$\begin{aligned}&\sum_{j=1}^{l} \lambda_j \left\| F_j(x_\alpha^\delta) - y_j^\delta \right\|_{\mathcal{Y}_j}^2 + \alpha D(x_\alpha^\delta, x^\dagger) \\ &\quad\leq \sum_{j=1}^{l} \lambda_j \left\| F_j(x^\dagger) - y_j^\delta \right\|_{\mathcal{Y}_j}^2 + \alpha \left\langle J'(x^\dagger), x_\alpha^\delta - x^\dagger \right\rangle_{\mathcal{X}^*,\mathcal{X}} \\ &\quad\leq \sum_{j=1}^{l} \lambda_j \delta_j^2 + \alpha\gamma D(x_\alpha^\delta, x^\dagger) + \alpha \sum_{j=1}^{l} \beta_j \left(\left\| F_j(x_\alpha^\delta) - y_j^\delta \right\|_{\mathcal{Y}_j} + \delta_j \right) \\ &\quad\leq \alpha\gamma D(x_\alpha^\delta, x^\dagger) + \sum_{j=1}^{l} \left(\lambda_j \delta_j^2 + \alpha\beta_j\delta_j + \frac{\alpha^2\beta_j^2}{4\lambda_j} + \lambda_j \left\| F_j(x_\alpha^\delta) - y_j^\delta \right\|_{\mathcal{Y}_j}^2 \right).\end{aligned}$$

With the choice $\alpha := c\|\underline{\delta}\|_2$ we have

$$D(x_\alpha^\delta, x^\dagger) \leq \frac{\|\underline{\delta}\|_2}{1-\gamma} \left(\frac{1}{c} + \|\underline{\beta}\|_2 + c \sum_{j=1}^{l} \frac{\beta_j^2}{\lambda_j} \right),$$

which proves the desired convergence rate. ■

Following the lines of the proof we have

$$D(x_\alpha^\delta, x^\dagger) \le \frac{1}{1-\gamma} \sum_{j=1}^{l} \left(\frac{\sqrt{\lambda_j}\delta_j}{\sqrt{\alpha}} + \sqrt{\alpha}\frac{\beta_j}{2\sqrt{\lambda_j}} \right)^2 . \qquad (18)$$

The error bound becomes minimal if we set

$$\alpha_j^{-1} = \frac{\lambda_j}{\alpha} := \frac{\beta_j}{2\delta_j}, \qquad 1 \le j \le l.$$

Then from (18) we derive the error bound

$$D(x_\alpha^\delta, x^\dagger) \le \frac{2}{1-\gamma} \|\underline{\delta}\|_2 \|\underline{\beta}\|_2,$$

which coincides with the error bound (8) in Theorem 3.1. Consequently, by solving (3) with Lagrangian methods we obtain Lagrangian multipliers $\lambda_1, \dots, \lambda_l$ which provide an 'optimal' error bound for $D(x_\alpha^\delta, x^\dagger)$ under the assumptions under consideration.

Another point seems to be quite remarkable: the 'optimal' choice of the regularization parameter $\alpha_j = \frac{\alpha}{\lambda_j}$ does not depend only on the noise level δ_j but on the ratio $\frac{\delta_j}{\beta_j}$. Hence it seems to be difficult to find an optimal choice of $\underline{\lambda}$ a priori when considering (15) with fixed $\underline{\lambda}$ and α as the only regularization parameter. Hence Algorithm 1 can be interpreted as multi-parameter Tikhonov regularization approach in combination with an a-posteriori parameter choice strategy. The parameter choice strategy can be considered as a generalization of the discrepancy principle of Morozov, see e.g. [6, Chapter 4.3].

6 An illustrating example

As example we consider the inverse problem of determining the time-dependent volatility from given prices of European call options. At present time $t^* = 0$ let the price $X^* > 0$ of the asset and the risk-less interest rate $r \ge 0$ be given and fixed. Following the Black-Scholes analysis (see e.g. [13, pp.71]) the price $y(K,t)$ for call option with strike $K > 0$ and maturity $t \in [0,T]$ is given by

$$y(K,t) := U_{BS}\left(X^*, K, r, t, \int_0^t \sigma^2(\tau)\, d\tau \right),$$

where the so-called Black-Scholes function is given by

$$U_{BS}(X,K,r,t,s) := \begin{cases} X\,\Phi(d_1) - Ke^{-r\,t}\Phi(d_2), & s > 0, \\ \max\left(X - Ke^{-r\,t}, 0\right) & s = 0, \end{cases}$$

with

$$d_1 := \frac{\ln\frac{X}{K} + r\,t + \frac{s}{2}}{\sqrt{s}}, \qquad d_2 := d_1 - \sqrt{s}$$

and the cumulative density function of the standard normal distribution

$$\Phi(\xi) := \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\xi} e^{-\frac{\tau^2}{2}}\, d\tau.$$

Thereby the volatility function $\sigma(t)$, $t \in [0, T]$, is an additional market parameter, which has to be determined from given option price data.

In [9] the following inverse problem was analyzed: for fixed strike price $K = K^* > 0$ and (noisy) data $y^*(t) = y(K^*, t)$, $t \in [0, T]$, a positive function $x \in L^2(0, T)$ (with $x = \sigma^2$) as approximate solution of the nonlinear equation

$$U_{BS}\left(X^*, K^*, r, t, \int_0^t x(\tau)\, d\tau\right) = y^*(t), \qquad t \in [0, T],$$

has to be estimated. In particular, only one strike price K^* was considered for the identification of x. On the other hand, data usually is given in form of several price functions $y_j(t) = y_j(K_j, t)$, $t \in [0, T]$, for different strikes $K_j > 0$, $1 \le j \le l$. So there no reason to pick out a single data for the identification problem. Moreover, by using all given data we can expect stabilizing effects which promise better reconstruction chances. Hence, the multi-parameter approach can be naturally formulated as follows: we set $X = \mathcal{Y}_1 = \ldots = \mathcal{Y}_l = L^2(0, T)$, introduce the operators $F_j : \mathcal{D} \subset L^2(0, T) \longrightarrow L^2(0, T)$ as

$$[F_j(x)](t) := U_{BS}\left(X^*, K_j, r, t, \int_0^t x(\tau)\, d\tau\right), \qquad t \in [0, T],$$

whereas the domain $\mathcal{D}$ contains all function in $L^2(0, T)$ which are positive a.e. on $[0, T]$. In fact, here it makes sens to deal with different noise levels. It is well-known, that option prices with $K > X^*$ or $K < X^*$ are less sensible with respect to perturbation of the volatility function that options with $K \approx X^*$. So a higher noise level δ_j can be assumed as larger or smaller the ratio $\frac{X_j}{K^*}$ for $1 \le j \le l$ becomes.

7 Conclusions

We have shown that the stability and convergence theory for nonlinear Tikhonov regularization can be easily generalized to a multi-parameter approach when we consider the given data as a 'vector' of l different measurements. The convergence rates results indicate, that is advantageously not to fix the parameter $\underline{\lambda}$ in the Tikhonov functional $T_\alpha(x)$ a-priori. There is a good chance to obtain better approximate solutions by using a vector of regularization parameter in combination with an a-posteriori parameter choice strategy for choosing the optimal regularization parameter. In particular, solving the minimization problem (3) by Lagrangian methods as suggested in Section 4 can be interpreted as one possibility for a multi-parameter Tikhonov regularization strategy with an a-posteriori choice of the regularization parameter based on the discrepancy principle of Morozov.

A Multi-parameter regularization and Multiplier methods

We show that Algorithm 1 is closely related to multiplier methods for inequality constrained optimization problems. Let again $\mathcal{X}$ be a Banach space. We consider the problem

$$\begin{cases} J(x) \to \min \\ g_j(x) \le 0, \quad 1 \le j \le l. \end{cases} \tag{19}$$

with objective functional $J : \mathcal{X} \longrightarrow \mathbb{R}$ and functionals $g_j : \mathcal{X} \longrightarrow \mathbb{R}$, $1 \le j \le l$, which describe the side constraints. For given Lagrange multiplier $\underline{\lambda} = (\lambda_1, \ldots, \lambda_l)^T \in \mathbb{R}^l$ with $\lambda_j \ge 0$ and constant $c > 0$, the augmented Lagrangian is given as

$$M(x, \underline{\lambda}, c) := J(x) + \frac{1}{2c} \sum_{j=1}^{l} \left(\max\{0, \lambda_j + c\, g_j(x)\}^2 - \lambda_j^2 \right). \tag{20}$$

The Multiplier method is now given as follows: for given $\underline{\lambda}_k \ge 0$, $c_k > 0$ we estimate x_k as solution of

$$M(x, \underline{\lambda}_k, c_k) \to \min \quad \text{subject to } x \in \mathcal{X}, \tag{21}$$

and update the Lagrange multiplier via

$$\lambda_j^{(k+1)} := \max\left\{0, \lambda_j^{(k)} + c_k g_j(x_k)\right\}, \quad 1 \le j \le l. \tag{22}$$

The parameter c_{k+1} is chosen by some rule (usually $c_{k+1} \ge c_k$ recommended). We introduce a slight modification, which seems to be quite natural. Instead of a single parameter $c \in \mathbb{R}$ we choose a vector $\underline{c} = (c_1, \ldots, c_l)^T \in \mathbb{R}^l$ with $c_j > 0$ and replace (20) by

$$M(x, \underline{\lambda}, \underline{c}) := J(x) + \sum_{j=1}^{l} \frac{1}{2c_j} \left(\max\{0, \lambda_j + c_j g_j(x)\}^2 - \lambda_j^2 \right) \tag{23}$$

Now, for $\nu > 0$, we set

$$g_j(x) := 2\delta_j^\nu \left(\left\| F_j(x) - y_j^\delta \right\|_{\mathcal{Y}_j}^\nu - \delta_j^\nu \right).$$

and

$$c_j := \frac{\lambda_j}{2\delta_j^{2\nu}}.$$

Hence

$$\begin{aligned} M(x, \underline{\lambda}, \underline{c}) &= J(x) + \sum_{j=1}^{l} \frac{1}{2c_j} \left(\max\left\{0, \lambda_j + 2c_j\delta_j^\nu \left(\left\| F_j(x) - y_j^\delta \right\|_{\mathcal{Y}_j}^\nu - \delta_j^\nu \right) \right\}^2 - \lambda_j^2 \right) \\ &= J(x) + \sum_{j=1}^{l} \frac{\delta_j^{2\nu}}{\lambda_j} \left(\max\left\{0, \lambda_j + \frac{\lambda_j}{\delta_j^\nu} \left(\left\| F_j(x) - y_j^\delta \right\|_{\mathcal{Y}_j}^\nu - \delta_j^\nu \right) \right\}^2 - \lambda_j^2 \right) \\ &= J(x) + \sum_{j=1}^{l} \delta_j^{2\nu} \lambda_j \left(\max\left\{0, \frac{\delta_j^\nu + \left\| F_j(x) - y_j^\delta \right\|_{\mathcal{Y}_j}^\nu - \delta_j^\nu}{\delta_j^\nu} \right\}^2 - 1 \right) \\ &= J(x) + \sum_{j=1}^{l} \lambda_j \left(\left\| F_j(x) - y_j^\delta \right\|_{\mathcal{Y}_j}^{2\nu} - \delta_j^{2\nu} \right), \end{aligned}$$

which coincides with the Lagrangian (11) for $\nu = 1$. Moreover, for the parameter update (22) we conclude

$$\begin{aligned}
\lambda_j^{(k+1)} &:= \max\left\{0, \lambda_j^{(k)} + 2c\delta_j^\nu \left(\left\|F_j(x) - y_j^\delta\right\|_{\mathcal{Y}_j}^\nu - \delta_j^\nu\right)\right\} \\
&= \max\left\{0, \lambda_j^{(k)} \frac{\left\|F_j(x) - y_j^\delta\right\|_{\mathcal{Y}_j}^\nu}{\delta^\nu}\right\} \\
&= \lambda_j^{(k)} \max\left\{0, \frac{\left\|F_j(x) - y_j^\delta\right\|_{\mathcal{Y}_j}^\nu}{\delta^\nu}\right\},
\end{aligned}$$

which is equal to (13) with $\varepsilon = 0$ for the case $\nu = 1$.

Remark A.1 *The functional (20) can be derived after some calculations from the (classical) augmented Lagrangian with quadratic penalty term. By a similar calculation it can be shown, that the choice $\nu = 2$ in (13) can be interpreted as a variant of the multiplier method with exponential penalty functional (see e.g. [2, Chapter 5] for theoretical aspects of multiplier methods with non-quadratic penalty terms).*

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