A note on stability results for scattered data interpolation on Euclidean spheres

Stefan Kunis*

April 16, 2007

The present work considers the interpolation of the scattered data on the d-sphere by spherical polynomials. We prove bounds on the conditioning of the problem which rely only on the used polynomial degree and the separation distance of the sampling nodes. To this end, we establish a packing argument for well separated sampling nodes and construct strongly localised polynomials on spheres. Numerical results illustrate our theoretical findings.

Key words and phrases: interpolation, localised polynomials, separation distance, zonal functions.

1 Introduction

Recently, the least squares approximation and interpolation of scattered data on the sphere has attracted much attention, see e.g. [5, 7]. Central themes in the study of scattered data interpolation/approximation methods are their convergence rates, see e.g. [22] and references therein, and the conditioning of proposed solution schemes. We consider condition numbers of interpolation matrices as in [16, 23], but focus on *polynomial* interpolation, i.e., interpolation by spherical harmonics [15, 19]. Thus, the interpolant $f : \mathbb{S}^{d-1} \to \mathbb{C}$ takes the form

$$f(\boldsymbol{\xi}) = \sum_{l=1}^{N_d} \hat{f}_l Y_l(\boldsymbol{\xi}),$$

where Y_l denotes a basis for the N_d-dimensional space of all spherical polynomials up to degree N. For given data points $(\boldsymbol{\xi}_j, y_j) \in \mathbb{S}^{d-1} \times \mathbb{C}$, $j = 0, \ldots, M-1$, we aim to solve the under-determined interpolation problem $f(\boldsymbol{\xi}_j) = y_j$. In matrix-vector notation, we wish to solve

$$\boldsymbol{Y}\hat{\boldsymbol{f}} = \boldsymbol{y}$$

*kunis@mathematik.tu-chemnitz.de, Chemnitz Universitiy of Technology, Department of Mathematics

for the vector $\hat{\boldsymbol{f}} \in \mathbb{C}^{N_d}$, where $Y_{j,l} = Y_l\left(\boldsymbol{\xi}_j\right)$ are the entries of the rectangular spherical Fourier matrix $\boldsymbol{Y} \in \mathbb{C}^{M \times N_d}$. This problem can be further reformulated as weighted minimal norm interpolation problem and standard arguments from linear algebra reveal that it allows for numerically stable solution if the singular values of the Fourier matrix $\boldsymbol{Y}$ are bounded away from zero.

We prove explicit bounds for the extremal eigenvalues of the interpolation problem under the simple but nevertheless sharp condition that the polynomial degree is bounded from below by a constant multiple of the inverse separation distance of the sampling nodes. The constant in this condition is estimated with respect to the spatial dimension. The proof of our result relies on three ingredients: an improvement of the packing argument [16, Theorem 2.3]; the construction of strongly localised polynomials on the sphere by using a smoothness-decay principle in Fourier analysis, and a simple eigenvalue estimate by the Gershgorin circle theorem. Here, the smoothness-decay principle relates the smoothness of Fourier coefficients to the localisation of the associated univariate trigonometric polynomial [14, 11], see [12, 9, 17, 8] for its recent application and generalisation.

The outline of this paper is as follows: Section 2 starts by introducing the necessary notation, including the spaces of spherical harmonics. We provide an improvement of the packing estimate [16, Theorem 2.3] in Lemma 2.1. In conjunction with the construction of localised zonal polynomials in Lemma 2.2, we prove in Theorem 2.4 the eigenvalue estimate for the interpolation problem. Section 3 presents numerical experiments, illustrating the eigenvalue estimate. Finally, we give our conclusions and discuss open problems and future work.

2 Main results

2.1 Prerequisites

The following fundamentals are taken from [15, 21] to keep the paper self contained. Let $d \in \mathbb{N}$, $\boldsymbol{x} \cdot \boldsymbol{y} = x_1 y_1 + \ldots + x_d y_d$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ denote the usual inner product, $\mathbb{S}^{d-1} = \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x} \cdot \boldsymbol{x} = 1\}$ be the unit sphere in the d dimensional Euclidean space $\mathbb{R}^d$. The canonical orthonormal basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d\}$ of $\mathbb{R}^d$ allows the representation

$$\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d)^\top = t\boldsymbol{e}_d + \sqrt{1-t^2}\,\tilde{\boldsymbol{\xi}}, \quad \tilde{\boldsymbol{\xi}} \in \mathbb{S}^{d-2},\ t \in [-1,1],$$

for every $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$ and $d \geq 2$. In particular, one obtains the local parametrisation

$$\xi_m = \begin{cases} \prod_{n=1}^{d-1} \sin\theta_n, & m = 1, \\ \cos\theta_{m-1} \prod_{n=m}^{d-1} \sin\theta_n, & m = 2, \ldots, d-1, \\ \cos\theta_{d-1}, & m = d, \end{cases}$$

of $\mathbb{S}^{d-1}$ in polar coordinates $\theta_1 \in [0, 2\pi)$ and $\theta_2, \ldots, \theta_{d-1} \in [0, \pi]$. The geodesic distance of $\boldsymbol{\xi}$, $\boldsymbol{\eta} \in \mathbb{S}^{d-1}$ is given by

$$\mathrm{dist}_d\left(\boldsymbol{\xi}, \boldsymbol{\eta}\right) = \arccos\left(\boldsymbol{\eta} \cdot \boldsymbol{\xi}\right).$$

Using polar coordinates, we denote by $\theta = \theta_{d-1} = \arccos(\boldsymbol{e}_d \cdot \boldsymbol{\xi}) = \arccos(\xi_d)$ the geodesic distance of $\boldsymbol{\xi}$ to the north pole $\boldsymbol{e}_d = (0, \ldots, 0, 1)^\top$.

Moreover, the surface element $\mathrm{d}\mu_d$ of $\mathbb{S}^{d-1}$ obeys $\mathrm{d}\mu_d = (1-t^2)^{(d-3)/2}\,\mathrm{d}t\,\mathrm{d}\mu_{d-1}$ and one easily shows that the surface of the sphere is

$$\omega_d = \int_{\mathbb{S}^{d-1}} \mathrm{d}\mu_d\left(\boldsymbol{\xi}\right) = 2\pi^{d/2}/\Gamma(d/2).$$

Now, let the space of square integrable functions $f : \mathbb{S}^{d-1} \to \mathbb{C}$ be decomposed into the mutual orthogonal spaces H_k^d of spherical harmonics of degree $k \in \mathbb{N}_0$ and let $\{Y_k^n : \mathbb{S}^{d-1} \to \mathbb{C} : n = 1, \ldots, N_d(k)\}$ denote an orthonormal basis for each of them. The dimension $N_d(k)$ of H_k^d obeys

$$N_d\left(k\right) = \begin{cases} \frac{(2k+d-2)\Gamma(k+d-2)}{\Gamma(k+1)\Gamma(d-1)}, & k \in \mathbb{N}, \\ 1, & k = 0. \end{cases}$$

In the present work, we consider the space $\Pi_N^d = \bigoplus_{k=0}^N H_k^d$ of spherical polynomials of degree at most N with dimension $N_d = |\Pi_N^d| = \sum_{k=0}^N N_d(k) \sim N^{d-1}$. For notational convenience, we set $I_d(N) = \{(k,n) : k = 0, \ldots, N; n = 1, \ldots, N_d(k)\}$, i.e., $\{Y_k^n : (k,n) \in I_d(N)\}$ constitutes an orthonormal basis for Π_N^d.

Moreover, let the Gegenbauer polynomials $C_k^{(d-2)/2} : [-1,1] \to \mathbb{R}$ be given by their three term recurrence relation

$$tC_k^{(d-2)/2}\left(t\right) = \alpha_k^{(d-2)/2} C_{k+1}^{(d-2)/2}\left(t\right) + \left(1 - \alpha_k^{(d-2)/2}\right) C_{k-1}^{(d-2)/2}\left(t\right)$$

for $k \in \mathbb{N}_0$ with $\alpha_k^{(d-2)/2} = (k+d-2)/(2k+d-2)$, $C_{-1}^{(d-2)/2}(t) = 0$, and $C_0^{(d-2)/2}(t) = 1$. In general the Gegenbauer polynomials are normalised by $C_k^{(d-2)/2}(1) = 1$ and form an orthogonal basis, i.e.,

$$\int_{-1}^{1} C_k^{(d-2)/2}\left(t\right) C_l^{(d-2)/2}\left(t\right) \left(1-t^2\right)^{(d-3)/2} \mathrm{d}t = \frac{\omega_d \delta_{k,l}}{\omega_{d-1} N_d\left(k\right)}, \tag{2.1}$$

where $\delta_{k,l}$ denotes the Kronecker delta $\delta_{k,k} = 1$, $\delta_{k,l} = 0$ for $k \neq l$. For $d = 2$ and another normalisation the Gegenbauer polynomials simplify to the Chebyshev polynomials $T_k(t) = \cos(k \arccos t)$, for $d = 3$ to the Legendre polynomials $P_k(t) = \frac{1}{2^k k!} \frac{\mathrm{d}^k}{\mathrm{d}t^k}(t^2-1)^k$.

The relation of the Gegenbauer polynomials to the spherical harmonics on $\mathbb{S}^{d-1}$ is given by the famous addition theorem

$$\sum_{n=1}^{N_d(k)} \overline{Y_k^n\left(\boldsymbol{\eta}\right)} Y_k^n\left(\boldsymbol{\xi}\right) = \frac{N_d\left(k\right)}{\omega_d} C_k^{(d-2)/2}\left(\boldsymbol{\eta} \cdot \boldsymbol{\xi}\right). \tag{2.2}$$

A function on the sphere $K : \mathbb{S}^{d-1} \to \mathbb{C}$ which values depend only on the geodesic distance from a centre $\boldsymbol{\xi}_0 \in \mathbb{S}^{d-1}$, i.e. $K(\boldsymbol{\xi}) = K_0(\arccos(\boldsymbol{\xi}_0 \cdot \boldsymbol{\xi}))$ for some $K_0 : [-1,1] \to \mathbb{C}$, is called zonal and might thus be expanded into Gegenbauer polynomials $C_k^{(d-2)/2}$.

2.2 An improved packing argument

For the purpose of interpolation at scattered nodes, the interesting "uniformity measure" of a sampling set $\mathcal{X} = \{\boldsymbol{\xi}_j \in \mathbb{S}^{d-1} : j = 0, \ldots, M-1\}$, $M \in \mathbb{N}$, is the separation distance

$$q_{\mathcal{X}} = \min_{0 \le j < l < M} \operatorname{dist}_d(\boldsymbol{\xi}_j, \boldsymbol{\xi}_l).$$

The sampling set $\mathcal{X}$ is called q-separated for some $0 < q \le 2\pi$, if $q_{\mathcal{X}} \ge q$. Following [16, Theorem 2.3], we prove how many q-separated nodes can be placed in a certain distance to the node $\boldsymbol{\xi}_0$ which is assumed without loss of generality to be the north pole $\boldsymbol{\xi}_0 = \boldsymbol{e}_d = (0, \ldots, 0, 1)^\top$.

For $0 < q \le \pi$, and $0 \le m < \lfloor \pi q^{-1} \rfloor$, we define the sets

$$S_{q,m} = \left\{ \boldsymbol{\xi} \in \mathbb{S}^{d-1} : mq \le \operatorname{dist}_d(\boldsymbol{\xi}, \boldsymbol{\xi}_0) < (m+1)\, q \right\}$$

and

$$S_{q,\lfloor \pi q^{-1} \rfloor} = \left\{ \boldsymbol{\xi} \in \mathbb{S}^{d-1} : \lfloor \pi q^{-1} \rfloor\, q \le \operatorname{dist}_d(\boldsymbol{\xi}, \boldsymbol{\xi}_0) \le \pi \right\}.$$

Their restrictions to the sampling set $\mathcal{X}$ will be denoted by $S_{\mathcal{X},q,m} = S_{q,m} \cap \mathcal{X}$. In contrast to the aforementioned result, the following estimate depends no longer on the particular separation distance.

Lemma 2.1. *Let $d \in \mathbb{N}$, $d \ge 2$, and the sampling set $\mathcal{X} \subset \mathbb{S}^{d-1}$ be q-separated for some $0 < q < \pi$, then*

$$|S_{\mathcal{X},q,m}| \le \frac{\pi^{d-2} 5^{d-1}}{2^{d-2}} m^{d-2}.$$

Proof. Analogously to [16, Theorem 2.3], we use that for each node in $S_{\mathcal{X},q,m}$ the centred cap around it of colatitude $q/2$ is contained in the larger ring $\tilde{S}_{q,m} = S_{q,m-\frac{1}{2}} \cup S_{q,m+\frac{1}{2}}$ and has no interior points common with the cap of another node, see also Figure 2.1.

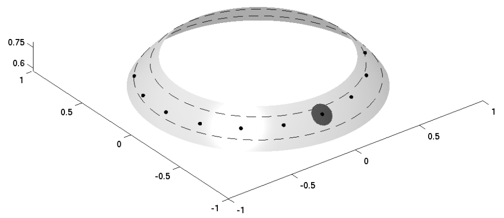


Figure 2.1: The set $S_{\mathcal{X},q,m}$, the ring $S_{q,m}$ (dashed), the larger ring $\tilde{S}_{q,m}$, and a spherical cap of colatitude $q/2$ centred at one node for generalised spiral nodes, cf. [20], on the sphere $\mathbb{S}^2$.

Using $2\theta/\pi \le \sin\theta$ for $0 \le \theta \le \pi/2$, we estimate the surface of the cap by

$$\int_0^{\frac{q}{2}} \sin^{d-2}\theta \, \mathrm{d}\theta \ge \frac{2^{d-2}}{\pi^{d-2}} \int_0^{\frac{q}{2}} \theta^{d-2} \mathrm{d}\theta = \frac{q^{d-1}}{2\,(d-1)\,\pi^{d-2}}.$$

Moreover, we apply $\sin\theta \leq \theta$ for $0 \leq \theta \leq \pi$ and estimate the surface of the m-th ring for $m = 1, \ldots, \lfloor \pi q^{-1} \rfloor - 2$ by

$$\begin{aligned}\int_{\left(m-\frac{1}{2}\right)q}^{\left(m+\frac{3}{2}\right)q} \sin^{d-2}\theta \mathrm{d}\theta &\leq \frac{\left(\left(m+\frac{3}{2}\right)q\right)^{d-1} - \left(\left(m-\frac{1}{2}\right)q\right)^{d-1}}{d-1} \\ &\leq \frac{q^{d-1}m^{d-2}}{d-1} \sum_{r=1}^{d-1} \binom{d-1}{r} \frac{3^r - (-1)^r}{2^r} \\ &\leq \frac{5^{d-1}q^{d-1}m^{d-2}}{2^{d-1}(d-1)}.\end{aligned}$$

Thus, we obtain for $m = 1, \ldots, \lfloor \pi q^{-1} \rfloor - 2$

$$|S_{\mathcal{X},q,m}| \leq \frac{\int_{\left(m-\frac{1}{2}\right)q}^{\left(m+\frac{3}{2}\right)q} \sin^{d-2}\theta \mathrm{d}\theta}{\int_0^{\frac{q}{2}} \sin^{d-2}\theta \mathrm{d}\theta} \leq \frac{\pi^{d-2}5^{d-1}}{2^{d-2}} m^{d-2}.$$

In conjunction with the estimate

$$\int_{\left(\lfloor \pi q^{-1} \rfloor - 1\right)q}^{\pi} \sin^{d-2}\theta \mathrm{d}\theta = \int_0^{\pi - \left(\lfloor \pi q^{-1} \rfloor - 1\right)q} \sin^{d-2}\theta \mathrm{d}\theta \leq \frac{2^{d-1}q^{d-1}}{d-1}$$

for the last two rings we obtain the assertion. □

2.3 Localised Chebyshev expansion

The simple fact that a zonal spherical polynomial of degree at most N is an univariate algebraic polynomial of degree at most N in the variable $t = \boldsymbol{\xi}_0 \cdot \boldsymbol{\xi}$ gives rise to the following explicit construction principle for localised polynomials on spheres of arbitrary dimension. We use the smoothness-decay principle in Fourier analysis to construct a localised kernel out of an admissible weight function of certain smoothness.

Let the normalised B-spline of order $\beta \in \mathbb{N}$ be defined by $g_\beta : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$, $g_\beta(z) = \beta N_\beta(\beta z + \frac{\beta}{2})$ with the cardinal B-spline given by

$$N_{\beta+1}(z) = \int_{z-1}^{z} N_\beta(\tau) \mathrm{d}\tau, \quad \beta \in \mathbb{N}, \qquad N_1(z) = \begin{cases} 1 & 0 < z < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, let for $N \in \mathbb{N}$ the B-spline kernel $B_{\beta,N} : [-1, 1] \to \mathbb{R}$ be given by

$$B_{\beta,N}(t) = \frac{1}{\|g_\beta\|_{1,N}} \sum_{l=0}^{N} (2 - \delta_{l,0}) g_\beta\left(\frac{l}{2(N+1)}\right) T_l(t), \tag{2.3}$$

with the normalisation constant $\|g_\beta\|_{1,N} = \sum_{l=-N}^{N} g_\beta(\frac{l}{2(N+1)})$. In contrast to [16] the order β of the B-spline and the degree N of the kernel $B_{\beta,N}$ are independent of each other. The involved localisation parameter β will be set to the spatial dimension d in subsequent considerations.

Lemma 2.2. *[9, Lemma 4.6] Let $N, \beta \in \mathbb{N}$, $N \geq \beta - 1$, $\beta \geq 2$. The B-spline kernel $B_{\beta,N}$ obeys for $t \in [-1, 1)$ the localisation property*

$$|B_{\beta,N}(t)| \leq \frac{\left(2^\beta - 1\right)\zeta(\beta)\,\beta^\beta}{2^{\beta-1} - \zeta(\beta)\,\pi^{-\beta}} \cdot |(N+1)\arccos(t)|^{-\beta}, \tag{2.4}$$

where $\zeta(\beta) = \sum_{m=1}^{\infty} m^{-\beta}$ denotes the Riemann zeta function. Moreover, the B-spline kernel is normalised by $B_{\beta,N}(1) = 1$.

We refer the interested reader to [9, Lemma 4.6] for a proof of this lemma and only note in passing that it relies on the Poisson summation formula to relate the smoothness of the coefficients in the Chebyshev expansion with the variation of an appropriate derivative of the B-spline. Of course, the B-spline samples $g_\beta(\frac{l}{2(N+1)})$ can be replaced by samples of other smooth functions g, see e.g. [11].

Remark 2.3. *Recently, localised kernels on the interval, the ball, and the sphere have been constructed out of admissible weight functions h in [12, 17, 8]. The common argument is that the Gegenbauer expansion*

$$K_N(t) = \sum_{k=0}^{N} h(k/N)\, C_k^{(d-2)/2}(t)$$

obeys an estimate similar to Lemma 2.2, i.e., $|K_N(t)| \leq C_\beta |h^{(\beta-1)}|_V |N \arccos t|^{-\beta}$, where $|h^{(\beta-1)}|_V$ denotes the variation of the $(\beta-1)$-th derivative of h. In contrast to the trigonometric case which is equivalent to the considered Chebyshev expansion [14, 11], the involved constant C_β is hard to access. This is mainly due to the subtle relation between the smoothness of the coefficients in the Gegenbauer expansion and derivatives of h, cf. [12, Lemma 4.4 - 4.6].

2.4 Regularity and conditioning of the spherical Fourier matrix

Let $d, M, N \in \mathbb{N}$ and a set of M nodes $\mathcal{X} \subset \mathbb{S}^{d-1}$ be given. The spherical Fourier matrix of size $M \times |N_d|$ is defined by

$$\boldsymbol{Y} = \left(Y_k^n(\boldsymbol{\xi}_j)\right)_{0 \leq j < M; (k,n) \in I_d(N)}.$$

We are now ready to prove that a polynomial degree N, bounded from below by a constant multiple of the inverse of the separation distance, suffices for the full rank of the spherical Fourier matrix $\boldsymbol{Y}$.

Theorem 2.4. *Let $N, d \in \mathbb{N}$, $N \geq d-1$, $d \geq 2$, and a q-separated sampling set $\mathcal{X} \subset \mathbb{S}^{d-1}$ be given. Moreover, let the weights w_k, $k = 0, \ldots, N$, be given by $\boldsymbol{w} = \boldsymbol{G}\boldsymbol{b}$, where the entries of $\boldsymbol{G} \in \mathbb{R}^{(N+1)\times(N+1)}$ and $\boldsymbol{b} \in \mathbb{R}^{N+1}$ are*

$$G_{k,l} = \omega_{d-1} \int_{-1}^{1} C_k^{(d-2)/2}(t)\, T_l(t) \left(1 - t^2\right)^{(d-3)/2} \mathrm{d}t, \quad b_l = \frac{(2 - \delta_{l,0})\, g_d\left(\frac{l}{2(N+1)}\right)}{\|g_d\|_{1,N}}. \tag{2.5}$$

Set up the diagonal matrix $\hat{\boldsymbol{W}} \in \mathbb{R}^{N_d \times N_d}$ *with entries* $\hat{w}_{(k,n),(k,n)} = w_k$ *for* $k = 0, \ldots, N$ *and* $n = 1, \ldots, N_d(k)$.

Then the entries of the matrix $\boldsymbol{K} = \boldsymbol{Y}\hat{\boldsymbol{W}}\boldsymbol{Y}^{\mathsf{H}}$, *where* $\boldsymbol{Y}^{\mathsf{H}}$ *denotes the adjoint spherical Fourier matrix, fulfil* $K_{j,l} = B_{d,N}(\boldsymbol{\xi}_l \cdot \boldsymbol{\xi}_i)$. *For* $N > C_d q^{-1}$ *the eigenvalues of this matrix are bounded by*

$$|\lambda(\boldsymbol{Y}\hat{\boldsymbol{W}}\boldsymbol{Y}^{\mathsf{H}}) - 1| < \left(\frac{C_d}{(N+1)\,q}\right)^d, \qquad C_d = \frac{5\pi d}{2}.$$

In particular, the nonequispaced Fourier matrix $\boldsymbol{Y}$ *has full rank* M *under the above conditions.*

Proof. We start by computing the entries $K_{j,l}$ of the matrix $\boldsymbol{K} = \boldsymbol{Y}\hat{\boldsymbol{W}}\boldsymbol{Y}^{\mathsf{H}}$. The addition theorem (2.2) yields

$$K_{j,l} = \sum_{k=0}^{N} \sum_{n=1}^{N_d(k)} Y_k^n\left(\boldsymbol{\xi}_j\right) w_k \overline{Y_k^n\left(\boldsymbol{\xi}_l\right)} = \sum_{k=0}^{N} w_k \frac{N_d\left(k\right)}{\omega_d} C_k^{(d-2)/2}\left(\boldsymbol{\xi}_l \cdot \boldsymbol{\xi}_j\right).$$

Moreover, the B-spline kernel $B_{d,N}$ is a polynomial of degree N and thus allows for the expansion, cf. (2.1),

$$B_{d,N}\left(t\right) = \sum_{k=0}^{N} \left(\frac{\omega_{d-1} N_d\left(k\right)}{\omega_d} \int_{-1}^{1} B_{d,N}\left(\tau\right) C_k^{(d-2)/2}\left(\tau\right)\left(1-\tau^2\right)^{(d-3)/2} \mathrm{d}\tau\right) C_k^{(d-2)/2}\left(t\right).$$

Setting $t = \boldsymbol{\xi}_l \cdot \boldsymbol{\xi}_j$, using the definitions of the weights w_k, and (2.3) indeed shows $K_{j,l} = B_{d,N}(\boldsymbol{\xi}_l \cdot \boldsymbol{\xi}_i)$.

Now we apply Gershgorin's circle theorem to estimate the extremal eigenvalues of $\boldsymbol{K}$. Without loss of generality we assume $\boldsymbol{\xi}_0 = \boldsymbol{e}_d = (0, \ldots, 0, 1)^{\top}$. Hence, we obtain

$$|\lambda\left(\boldsymbol{K}\right) - 1| \leq \sum_{j=1}^{M-1} \left|B_{d,N}\left(\boldsymbol{\xi}_0 \cdot \boldsymbol{\xi}_j\right)\right| \leq \sum_{m=1}^{\lfloor \pi q^{-1} \rfloor} |S_{\mathcal{X},q,m}| \max_{\boldsymbol{\xi} \in S_{q,m}} |B_{d,N}\left(\boldsymbol{\xi}_0 \cdot \boldsymbol{\xi}\right)|.$$

Hence, Lemma 2.1, Lemma 2.2, $\min_{\boldsymbol{\xi} \in S_{q,m}} \arccos(\boldsymbol{\xi}_0 \cdot \boldsymbol{\xi}) = mq$, and $\zeta(d) \leq \zeta(2) = \pi^2/6$ yield

$$\begin{aligned} |\lambda\left(\boldsymbol{K}\right) - 1| &\leq \sum_{m=1}^{\lfloor \pi q^{-1} \rfloor} \frac{\pi^{d-2} 5^{d-1}}{2^{d-2}} m^{d-2} \cdot \frac{\left(2^d - 1\right)\zeta\left(d\right) d^d}{2^{d-1} - \zeta\left(d\right)\pi^{-d}} \cdot \left(\left(N+1\right) mq\right)^{-d} \\ &< \left(\frac{5\pi d}{2\left(N+1\right) q}\right)^d. \end{aligned}$$

We conclude by the simple fact $\operatorname{rank}(\boldsymbol{Y}) \geq \operatorname{rank}(\boldsymbol{Y}\hat{\boldsymbol{W}}\boldsymbol{Y}^{\mathsf{H}}) = M$. ◻

Remark 2.5. *The above theorem gives a sufficient condition for unique and stable interpolation by spherical polynomials.*

1. *Regarding the necessity the following is known: The condition $N > C_d q^{-1}$ is optimal with respect to the separation distance q since, e.g., [10] assures that M nodes can be distributed on $\mathbb{S}^{d-1}$ with separation distance $q \geq c_{d,1} M^{-(d-1)}$. Furthermore, the dimension of Π_N^d fulfils $N_d \leq c_{d,2} N^{d-1}$ and we thus obtain for $N < c_{d,1}/(c_{d,2} q)$ the inequality $N_d < M$. This contradicts $\operatorname{rank}(\boldsymbol{Y}) = M$ for $\boldsymbol{Y} \in \mathbb{C}^{M \times N_d}$.*

2. *A related stability result for the oversampled case $N_d < M$ follows from so-called spherical Marcinkiewicz-Zygmund inequalities [13, 6]. If the mesh norm*

$$\delta_{\mathcal{X}} = 2 \max_{\boldsymbol{\xi} \in \mathbb{S}^{d-1}} \min_{j=0,\ldots,M-1} \operatorname{dist}_d(\boldsymbol{\xi}_j, \boldsymbol{\xi})$$

fulfils $N < \tilde{c}_d \delta_{\mathcal{X}}^{-1}$, then the corresponding Fourier matrix $\boldsymbol{Y}$ has full rank N_d. Note however that the constant $\tilde{c}_d$ is typically hard to access. In particular the order on which $\tilde{c}_d$ decreases to zero with increasing space dimension d is unknown.

3. *The linear growth of C_d is not expected to be optimal but has been obtained for trigonometric interpolation and trigonometric Marcinkiewicz-Zygmund inequalities in [3, 11] as well.*

Remark 2.6. *The precomputation of the weights w_k in (2.5) is a basis transform from a Chebyshev expansion to the appropriate Gegenbauer expansion. We obtain a sparse factorisation of the Gram matrix $\boldsymbol{G}$ by using the relation, cf. [1, (22.7.23)],*

$$C_k^{(d-2)/2} = \alpha_d(k)\, C_k^{d/2} + \beta_d(k)\, C_{k-2}^{d/2} \tag{2.6}$$

$\lfloor d/2 \rfloor - 1$ times. Hence, this computation can be done in $\mathcal{O}(N)$ floating point operations for even dimensions d. We need one additional fast polynomial transform [2, 4, 18] with complexity $\mathcal{O}(N \log^2 N)$ to compute Legendre coefficients out of the given Chebyshev coefficients for odd dimensions d.

The weights w_k in (2.5) are strictly positive for $d = 2$, since in this case, the Gram matrix $\boldsymbol{G}$ is a diagonal matrix with positive diagonal elements and for $d = 3$ the positivity is assured by [9, Lemma 4.6]. The same is true for $d = 4$: The evaluation of (2.5) yields as a special case of (2.6) that

$$G_{k,l} = \begin{cases} 2\pi^2, & k = l = 0, \\ \frac{\pi^2}{k+1}, & k = l > 0, \\ -\frac{\pi^2}{k+1}, & k + 2 = l \geq 2, \\ 0 & \textit{otherwise.} \end{cases}$$

In conjunction with the fact that $g_4(z)$ is strictly decreasing for $0 \leq z \leq \frac{1}{2}$, the positivity of the Gegenbauer coefficients w_k is guaranteed. Conditions for $d \geq 5$ can be obtained by the iterated application of (2.6).

3 Numerical experiments

In this section, we present two numerical examples using MATLAB in order to confirm our theoretical findings. We start with the introduction of particular node distributions on the spheres. Let $m, d \in \mathbb{N}$, we recursively define the following set of interpolation nodes:

$$\theta_m^j = \frac{\pi}{m} + \frac{2\pi j}{m-1} \in (0, \pi), \quad \mathcal{X}_m^{d-1} = \bigcup_{j=1}^{m/2} \{\theta_m^j\} \times \mathcal{X}^{d-2}_{[m \sin \theta_m^j]}, \quad \mathcal{X}_m^1 = \frac{\{0, \ldots, m-1\}}{2\pi m}, \tag{3.1}$$

where $[x]$ denotes the closest integer to x. By construction, this sampling set is quasi uniform in the sense that its separation distance fulfils

$$|\mathcal{X}_m^{d-1}|^{-1/(d-1)} \le c_{1,d}\, q_{\mathcal{X}_m^{d-1}} \le c_{2,d}\, m^{-1}.$$

Example 3.1. In this first example we aim to illustrate the estimate for the minimal eigenvalue of $\boldsymbol{Y}\hat{\boldsymbol{W}}\boldsymbol{Y}^{\mathsf{H}}$ for weights $\hat{\boldsymbol{W}}$ obtained from B-spline kernels of order $\beta = 1, 2, 3, 4$, a fixed polynomial degree $N = 50$, and spatial dimensions $d = 3, 4$. The sampling set is set up by (3.1), where we have chosen $m = 10, \ldots, 90$ for $d = 3$ and $m = 5, \ldots, 60$ for $d = 4$. Figure 3.1 shows the minimal eigenvalue of the interpolation matrix $\boldsymbol{Y}\hat{\boldsymbol{W}}\boldsymbol{Y}^{\mathsf{H}}$ with respect to the separation distance of the sampling set. Moreover, we included a plot of the estimate in Theorem 2.4 where we changed the proven and non-optimal constant $C_d = 5\pi d/2$ to $C_3 = C_4 = 4.5$.

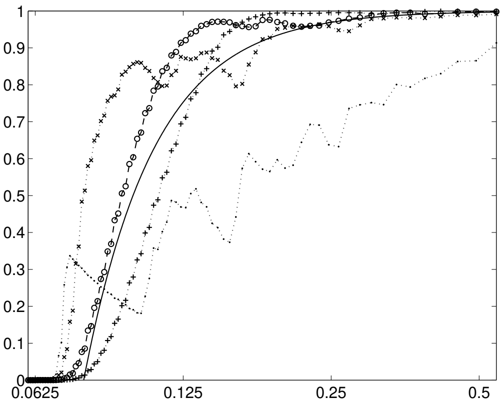
Clearly, the deviation of the minimal eigenvalue from 1 as the separation distance decreases is captured very good by the made estimate. However note that Theorem 2.4 does not give insight at which rate the minimal eigenvalue approaches zero.

Example 3.2. In our second example, we consider the condition number of $\boldsymbol{Y}\hat{\boldsymbol{W}}\boldsymbol{Y}^{\mathsf{H}}$ for weights $\hat{\boldsymbol{W}}$ obtained from B-spline kernels of order $\beta = 1, 2, 3, 4$, a fixed sampling set (3.1) with $m = 40$ for $d = 3$ and $m = 20$ for $d = 4$. The polynomial degree is varied as $N = 20, \ldots, 100$ for $d = 3$ and $N = 10, \ldots, 70$ for $d = 4$. Again, we changed the proven and non-optimal constant $C_d = 5\pi d/2$ to $C_3 = C_4 = 4.5$ in the plot of the estimated condition number.

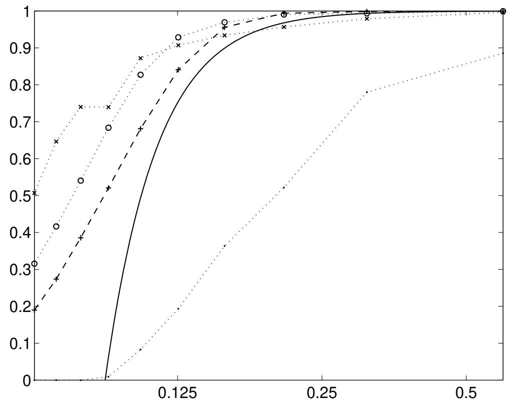
We see in Figure 3.2 that the estimate becomes effective if the polynomial degree is chosen large enough and hence the interpolation problem becomes well-conditioned.

4 Conclusions and open questions

We proved that arbitrary data at q separated nodes on S^{d-1} can be interpolated by spherical polynomials if the degree fulfils $N \ge 2.5\pi d q^{-1}$. This result is optimal with respect to the separation distance q but might be sharpened with respect to the spatial dimension d. We improved a previously established packing argument from [16]. The



(a) Spatial dimension $d = 3$.



(b) Spatial dimension $d = 4$.

Figure 3.1: Minimal eigenvalue of the interpolation matrix $\boldsymbol{Y}\hat{\boldsymbol{W}}\boldsymbol{Y}^{\mathsf{H}}$ with respect to the separation distance $q_{\mathcal{X}}$ for a fixed polynomial degree $N = 50$. The order of the B-spline kernel and hence the localisation order is $\beta = 1$ $(\cdot)$, $\beta = 2$ $(\times)$, $\beta = 3$ $(\circ)$, and $\beta = 4$ $(+)$. Moreover, we plotted the estimate $1 - (\frac{C_d}{(N+1)q})^d$ (solid) and additionally the minimal eigenvalue for $\beta = d$ (dashed).

construction of localised polynomials for the proof of the eigenvalue estimate uses techniques from [9]. In particular, we clarified that the localisation parameter should be chosen in accordance with the spatial dimension d instead of the polynomial degree N as in [16].

It remains to provide good conditions for the positivity of the constructed weights w_k for $d \geq 5$. Lower bounds for these weights will be needed for eigenvalue estimates of minimal norm interpolation problems when using e.g. Sobolev norms, see also [16, Theorem 2.7].

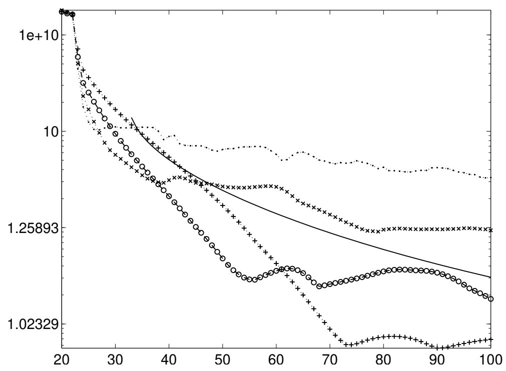
References

[1] M. Abramowitz and I. A. Stegun, editors. *Handbook of Mathematical Functions*. National Bureau of Standards, Washington, D.C., 1972.

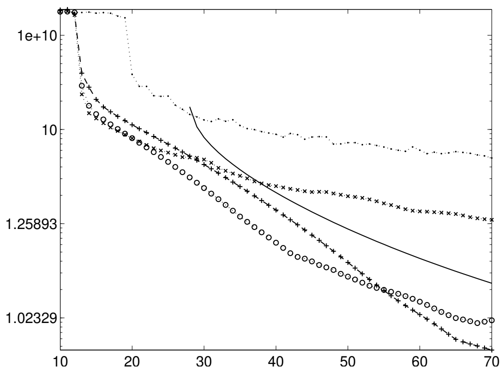
[2] B. Alpert and V. Rokhlin. A fast algorithm for the evaluation of Legendre expansions. SIAM *J. Sci. Stat. Comput.*, 12:158 – 179, 1991.

[3] R. F. Bass and K. Gröchenig. Random sampling of multivariate trigonometric polynomials. SIAM *J. Math. Anal.*, 36:773 – 795, 2004.

[4] J. R. Driscoll, D. Healy, and D. Rockmore. Fast discrete polynomial transforms with applications to data analysis for distance transitive graphs. SIAM *J. Comput.*, 26:1066 – 1099, 1996.



(a) Spatial dimension $d = 3$.



(b) Spatial dimension $d = 4$.

Figure 3.2: Condition number of the interpolation matrix $\boldsymbol{Y}\hat{\boldsymbol{W}}\boldsymbol{Y}^{\mathrm{H}}$ with respect to the polynomial degree N for a fixed sampling set (3.1) with $m = 40$ for $d = 3$ and $m = 20$ for $d = 4$. The order of the B-spline kernel and hence the localisation order is $\beta = 1$ $(\cdot)$, $\beta = 2$ $(\times)$, $\beta = 3$ $(\circ)$, and $\beta = 4$ $(+)$. Moreover, we plotted the estimate $(1 + (\frac{C_d}{(N+1)q})^d)/(1 - (\frac{C_d}{(N+1)q})^d)$ (solid) and additionally the condition number for $\beta = d$ (dashed).

[5] G. E. Fasshauer and L. L. Schumaker. Scattered data fitting on the sphere. In M. Dahlen, T. Lyche, and L. L. Schumaker, editors, *Mathematical Methods for Curves and Surfaces II*, pages 117 – 166, Nashville, 1998. Vanderbilt University Press.

[6] F. Filbir and W. Themistoclakis. Polynomial approximation on the sphere using scattered data. Preprint 06-12, GSF München, 2006.

[7] W. Freeden, T. Gervens, and M. Schreiner. *Constructive Approximation on the Sphere*. Oxford University Press, Oxford, 1998.

[8] Q. T. L. Gia and H. N. Mhaskar. Quadrature formulas and localized linear polynomial operators on the sphere. *Preprint*, 2007.

[9] J. Keiner, S. Kunis, and D. Potts. Efficient reconstruction of functions on the sphere from scattered data. *J. Fourier Anal. Appl.*, 2007. accepted.

[10] A. B. J. Kuijlaars, E. B. Saff, and X. Sun. On separation of minimal Riesz energy points on spheres in euclidean spaces. *J. Comput. Appl. Math.*, 199:172 – 180, 2007.

[11] S. Kunis and D. Potts. Stability results for scattered data interpolation by trigonometric polynomials. SIAM *J. Sci. Comput.*, 2007. accepted.

[12] H. N. Mhaskar. Polynomial operators and local smoothness classes on the unit interval. *J. Approx. Theory*, 131:243 – 267, 2004.

[13] H. N. Mhaskar, F. J. Narcowich, and J. D. Ward. Spherical Marcinkiewicz-Zygmund inequalities and positive quadrature. *Math. Comput.*, 70:1113 – 1130, 2001. Corrigendum on the positivity of the quadrature weights in 71:453 – 454, 2002.

[14] H. N. Mhaskar and J. Prestin. On the detection of singularities of a periodic function. *Adv. Comput. Math.*, 12:95 – 131, 2000.

[15] C. Müller. *Spherical Harmonics.* Springer, Aachen, 1966.

[16] F. J. Narcowich, N. Sivakumar, and J. D. Ward. Stability results for scattered-data interpolation on euclidean spheres. *Adv. Comput. Math.*, 8:137 – 163, 1998.

[17] P. Petrushev and Y. Xu. Localized polynomial frames on the ball. *Preprint arXiv:math/0611145*, 2006.

[18] D. Potts, G. Steidl, and M. Tasche. Fast algorithms for discrete polynomial transforms. *Math. Comput.*, 67:1577 – 1590, 1998.

[19] M. Reimer. *Multivariate Polynomial Approximation.* Birkhäuser Verlag, Basel, 2003.

[20] E. B. Saff and A. B. J. Kuijlaars. Distributing many points on a sphere. *Math. Intelligencer*, 19:5 – 11, 1997.

[21] G. Szegő. *Orthogonal Polynomials.* Amer. Math. Soc., Providence, RI, 4th edition, 1975.

[22] H. Wendland. *Scattered Data Approximation.* Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, Cambridge, 2005.

[23] X. S. Z. Luo and J. Levesley. Norm estimates of interpolation matrices and their inverses associated with strictly positive definite functions. *Proc. Amer. Math. Soc.*, 127(7):2127 – 2134, 1999.