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C^*-algebras and asymptotic spectral theory

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Abstract. The presented material is a slightly polished and extended version of lectures given at Lisbon, WOAT 2006. Three basic topics of numerical functional analysis are discussed: stability, fractality, and Fredholmness. It is further shown that these notions are corner stones in order to understand a few topics in asymptotic spectral theory: asymptotic behavior of singular values, ε-pseudospectra, norms. Four important examples are discussed: Finite sections of quasidiagonal operators, Toeplitz operators, band-dominated operators with almost periodic coefficients, and general band-dominated operators. The elementary theory of C^*-algebras serves as the natural background of these topics.

1. Introduction

One goal of functional analysis is to solve equations with "infinitely" many variables, and that of linear algebra to solve equations in finitely many variables. Numerical analysis builds a bridge between these fields. Functional numerical analysis is concerned with the theoretical foundation of numerical analysis.

Given a bounded linear operator A acting on some Hilbert space H, that is $A \in \mathcal{B}(H)$, consider the equation

$$Ag = h\,, \tag{1.1}$$

where $h \in H$ is given and g is to find if this equation is supposed to be uniquely solvable.

Even if the operator A is continuously invertible (and this will be assumed in what follows), it is as a rule impossible to compute the solution $A^{-1}h$. Then one tries to solve (1) approximatly. For, one chooses a sequence $(h_n) \subset H$ of elements

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which approximates the right hand side y, and a sequence (A_n) of operators which approximates the operator A, and one replaces (1) by the approximation equations

$$A_n g_n = h_n\,,\ n = 1, 2, \ldots \tag{1.2}$$

the solutions g_n of which are sought in H (or in certain subspaces H_n of H) again. Approximation of h by h_n means that $\|h - h_n\|_H \to 0$ as $n \to \infty$.

It is tempting to suppose that the operators A_n also approximate A in the norm, but this assumption does not work in practice. The point is that usually A acts on an infinite-dimensional space, whereas one will, of course, try to choose the A_n as acting on spaces of finite dimension, i.e., as finite matrices. But the only operators which can be approximated in norm by finite rank operators, are the compact ones.

The kind of approximation which fits much better to the purpose of numerical analysis is that of pointwise or strong convergence: the sequence (A_n) converges strongly to the operator A if $\|Ah - A_nh\|_H \to 0$ for every $h \in H$ (notation: s-lim $A_n = A$).

We write s^*-lim $A_n = A$, if s-lim $A_n = A$ and s-lim $A_n^* = A^*$.

Basic question: suppose that A_n is invertible for all $n \geq n_0$. Does the sequence (g_n) of solutions of (2) converge to the solution g of (1)?

The answer is NO!

Example 1: $l^2 := \{(x_0, \ldots, x_n, \ldots) : \sum_{k \in \mathbb{Z}_+} |x_k|^2 < \infty\}$, $A_\varepsilon(\varepsilon = (\varepsilon_n)\,,\ \varepsilon_n > 0$ and $\lim \varepsilon_n = 0)$ is given by the infinite matrix

$$\begin{pmatrix} \varepsilon_0 & 1 & & & & & \\ 1 & \varepsilon_0 & 0 & & & & \\ & 0 & \varepsilon_1 & 1 & & & \\ & & 1 & \varepsilon_1 & 0 & & \\ & & & 0 & \varepsilon_2 & 1 & \\ & & & & 1 & \varepsilon_2 & \ddots \\ & & & & \ddots & & \ddots \end{pmatrix}$$

(with respect to the standard basis of l^2).

Consider $P_n A_\varepsilon P_n\,,\ P_n : l^2 \to l^2\,, (x_0, \ldots, x_n, x_{n+1}, \ldots) \mapsto (x_0, \ldots, x_n, 0, 0, \ldots)$.

Then:
$\|(P_n A_\varepsilon P_n)^{-1} P_n\| \leq 2$, if n is odd,
$\|(P_n A_\varepsilon P_n)^{-1} P_n\|$ are at least ε_n^{-1}, if n is even
$\Rightarrow$ if n is even then $\|(P_n A_\varepsilon P_n)^{-1} P_n\| \to \infty$, that is there is an $h \in H$ such that

$$g_n = (P_n A_\varepsilon P_n)^{-1} P_n h \nrightarrow A^{-1} h = g\,.$$

Reason: $\big((P_nA_\varepsilon P_n)^{-1}P_n\big)$ is not uniformly bounded.

Back to the general situation: Suppose that $(A_n^{-1}P_n)$ is uniformly bounded ($n \geq n_0$) ($(P_n) \subset \mathcal{B}(H)$ is a sequence of orthogonal projections such that s-lim $P_n = I$, A_n: im $P_n \to$ imP_n invertible and bounded) and A is invertible. Then

$$\|A_n^{-1}P_nx - A^{-1}x\| \to 0 \text{ for every } x \in H:$$

$$\begin{aligned} \|A_n^{-1}P_nx - A^{-1}x\| &\leq A_n^{-1}P_nx - P_nA_x^{-1}\| + \|P_nA^{-1}x - A^{-1}x\| \\ &\leq \|A_n^{-1}P_n\|\,\|x - A_nP_nA^{-1}x\| + \\ &\quad \|P_nA^{-1}x - A^{-1}x\| \to 0\,. \end{aligned}$$

Remark: Suppose $(A_n^{-1}P_n)_{n\geq n_0}$ is uniformly bounded and s^*-lim $A_nP_n = A$. $\Rightarrow$ A is invertible.

<u>Indeed:</u>

$$\begin{array}{ll} \|A_nP_nx & \| \geq C\|P_nx\| \quad (n \geq n_0, C > 0) \\ \quad\downarrow & \\ \|Ax & \| \geq C\|x\| \Rightarrow \operatorname{im} A = \overline{\operatorname{im} A}\,, \ker A = \{0\} \end{array}$$

and

$$\begin{array}{ll} \|A_n^*P_nx & \| \geq C\|P_nx\| \\ \quad\downarrow & \\ \|A^*x & \| \geq C\|x\| \Rightarrow \operatorname{im} A^* = \overline{\operatorname{im} A^*}\,, \ker A^* = \{0\} \end{array}$$

Definition 1: A sequence of operators $A_n \in \mathcal{B}(\text{im}P_n)$ is called stable if there exists a number n_0 such that the operators A_n are invertible for every $n \geq n_0$ and if the norms of their inverses are uniformly bounded:

$$\sup_{n\geq n_0} \|A_n^{-1}P_n\| < \infty\,.$$

The above discussion shows the crusial role of stability in analysis. How to prove stability? No general idea. In most cases it is very complicated.

<u>Easy cases:</u>

- $A = B + iS$, B positive and S selfadjoint,
- $A = I + T$, T compact,

  and $A_n = P_nAP_n$, where (P_n) is a sequence of orthogonal projections with s-lim $P_n = I$.

Exercise: prove it!

We will show that the stability problem can frequently be tackled by the help of C^*-algebra techniques. Recall that a complex Banach algebra is called C^*-algebra if there is an involution $a \mapsto a^*$ such that $\|aa^*\| = \|a\|^2$. Given two C^*-algebras $\mathcal{A}$

and $\mathcal{B}$, a $\star$-homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ is a continuous homomorphism such that $\varphi(a^*) = \varphi(a)^*$ for all $a \in \mathcal{A}$.

2. Algebraization of stability

Let H be a (separable) Hilbert space and (L_n) be a sequence of orthoprojections on H with $s\text{-lim } L_n = I$.

Definition 2: Let $\mathcal{F}$ be the set of all sequences $(A_n)_{n=0}^{\infty}$ of operators $A_n \in \mathcal{B}$ (im L_n) which are uniformly bounded:

$$\sup_{n \geq 0} \|A_n L_n\| < \infty .$$

The natural operations $(A_n)+(B_n) := (A_n + B_n)$, $(A_n)(B_n) := (A_n B_n)$, $\lambda(A_n) := (\lambda A_n)$, $(A_n)^* := (A_n^*)$ make $\mathcal{F}$ to an algebra with involution.

Proposition 1: $\mathcal{F}$ is a C^*-algebra (prove it).
We are mainly interested in the asymptotic behavior of the sequences belonging to $\mathcal{F}$. This means that sequences which differ in a finite number of entries only will have the same asymptotic behavior, and therefore can be identified. For this goal we introduce the set G of all sequences (G_n) in $\mathcal{F}$ with $\lim_{n\to\infty} \|G_n L_n\| = 0$.

Proposition 2: G is a closed ideal in $\mathcal{F}$ (prove it).
The following theorem reveals a perfect frame to study stability problems in an algebraic way.

Theorem 1: (A. Kozak) A sequence $(A_n) \in \mathcal{F}$ is stable $\Leftrightarrow$ the coset $(A_n) + G$ is invertible in the quotient algebra $\mathcal{F}/G$.

Proof: $\Rightarrow$: If (A_n) is stable, then $(A_n^{-1})_{n \geq n_0}$ is bounded for some sufficiently large n_0 by definition. We make $(A_n^{-1})_{n\geq n_0}$ to a bounded sequence $(B_0, B_1, \ldots, B_{n_0-1}, A_{n_0}^{-1}, A_{n_0+1}^{-1}, \ldots)$ in $\mathcal{F}$ by freely choosing operators $B_i \in \mathcal{B}(\text{im} L_i)$. It is evident that this sequence is an inverse of (A_n) modulo G.
$\Leftarrow$ Let conversely, $(A_n)+G$ be invertible in $\mathcal{F}/G$. Then there are sequences $(B_n) \in \mathcal{F}$ as well as (G_n) and (H_n) in G such that $A_n B_n = L_n + G_n$, $G_n A_n = L_n + H_n$. If n is large enough, then $\|G_n\| < \frac{1}{2}$, $\|H_n\| < \frac{1}{2}$, and a Neumann series argument yields the invertibility of $L_n + G_n$ and $L_n + H_n$ as well as the uniform boundedness of this inverses by 2. Hence, $A_n B_n (L_n + G_n)^{-1}$, $(L_n + H_n)^{-1} B_n A_n$ are uniformly bounded. Thus, the operators A_n are invertible for all sufficiently large n, and their inverses are uniformly bounded. ∎

Proposition 3: For all $(A_n) \in \mathcal{F}$,

$$\|(A_n + G)\|_{\mathcal{F}/G} = \lim_{n\to\infty} \sup \|A_n L_n\| \,. \tag{2.1}$$

Proof: Exercise.

Formula (3) gives raise to ask if there are interesting sequences in $\mathcal{F}$ for which lim sup in (3) can be replaced by lim. This question is important in order to prove that the condition numbers of a stable sequence converge. Recall, that the condition number (cond A) for an invertible matrix (operator) A is defined by

$$\text{cond} A := \|A\| \, \|A^{-1}\|$$

(for computational purposes: cond A should be small).

The right tool to study this and related questions is another fundamental notion of numerical analysis – that of a fractal sequence.

It is not important in this place that the elements of the sequences under consideration are operators. So we will use slightly generalized definitions of the C^*-algebras $\mathcal{F}$ and G, namely, given unital C^*-algebras $\mathcal{C}_n$, $n = 0, 1, 2, \ldots$, with identity elements e_n, let $\mathcal{F}$ stand for the set of all bounded sequences $(c_0, c_1, \ldots)$ with $c_n \in \mathcal{C}_n$, and let G refer to the set of all sequences $(c_0, c_1, \cdots)$ in $\mathcal{F}$ with $\|c_n\| \to 0$ as $n \to \infty$. Defining elementwise algebraic operations and an elementwise involution, and taking the supremum norm, we make $\mathcal{F}$ to a C^*-algebra and G to a closed ideal of $\mathcal{F}$. Thus, is the $\mathcal{F}$-product of the C^*-algebras $\mathcal{C}_n$, and G-their restricted product.

Given a strongly monotonically increasing sequence $\eta : \mathbb{Z}_+ \to \mathbb{Z}_+$, let $\mathcal{F}_\eta$ and G_η denote the product and the restricted product of the C^*-algebras $\mathcal{C}_{\eta(0)}, \mathcal{C}_{\eta(1)}, \ldots$, respectively, and let R_η stand for the restriction mapping $R_\eta : \mathcal{F} \to \mathcal{F}_\eta$, $(a_n) \mapsto (a_{\eta(n)})$. The mapping R_η is a $*$-homomorphism from $\mathcal{F}$ onto $\mathcal{F}_\eta$. Further, given a C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$, let $\mathcal{A}_\eta$ refer to the image of $\mathcal{A}$ under R_η. By the first isomorphy theorem for C^*-algebras ([1], Theorem 1.45), $\mathcal{A}_\eta$ actually is a C^*-algebra.

Definition 3: Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{F}$.

(a) A $*$-homomorphism $W : \mathcal{A} \to \mathcal{B}$ of $\mathcal{A}$ into a C^*-algebra $\mathcal{B}$ is fractal if for every strongly monotonically increasing sequence η, there is a $*$-homomorphism $W_\eta : \mathcal{A}_\eta \to \mathcal{B}$ such that $W = W_\eta R_\eta$.

(b) The algebra $\mathcal{A}$ is fractal, if the canonical homomorphism $\|i : \mathcal{A} \to \mathcal{A}/\mathcal{A} \cap G$ is fractal.

(c) A sequence $(a_n) \in \mathcal{F}$ is fractal, if the smallest C^*-subalgebra of $\mathcal{F}$ containing (a_n), is fractal.

Roughly spoken: given a subsequence $(a_{\eta(n)})$ of a sequence (a_n) which belongs to a fractal algebra $\mathcal{A}$, it is possible to reconstruct the original sequence (a_n) from its subsequence modulo sequences in $\mathcal{A} \cap G$.

Consequences:

$$\bullet\ (a_{\eta(n)}) \in G_\eta \Rightarrow (a_n) \in G \tag{2.2}$$

([7], Theorem 1.66)

$$\bullet\ (a_{\eta(n)}) \text{ stable } \Rightarrow (a_n) \text{ stable (see Theorem 4 } (\downarrow)). \tag{2.3}$$

Theorem 2: Let $\mathcal{A}$ be a fractal C^*-subalgebra of $\mathcal{F}$.

If $(a_n) \in \mathcal{A}$, then the limit $\lim \|a_n\|$ exists and is equal to $\|(a_n)+G\|$. ([7], Theorem 1.7.1)

Example 2: Consider $l^2(\mathbb{Z}_+) := \{(a_n)_{n\in\mathbb{Z}_+} : \sum\limits_{n\in\mathbb{Z}_+} |a_n|^2 < \infty\}$ and the bounded linear operators $P_n, R_n : l^2(\mathbb{Z}_+) \to l^2(\mathbb{Z}_+)$ given by

$$\begin{aligned}(a_k) &\mapsto (a_0, a_1, \dots, a_n, 0, 0, 0, \dots)\,,\\ (a_k) &\mapsto (a_n, a_{n-1}, \dots a_0, 0, 0, 0, \cdots)\,,\end{aligned}$$

respectively.

Let $\mathcal{C}_n = \mathcal{B}(\mathrm{im}P_n)$, and let $\mathcal{F}^W$ refer to the set of all sequences $(A_n) \in \mathcal{F}$ for which the strong limits $W(A_n) := s - \lim A_n P_n$ and $\tilde{W}(A_n) := s - \lim R_n A_n R_n$ as well as the strong limits $W(A_n^*)$ and $\tilde{W}(A_n^*)$ exist. The set $\mathcal{F}^W$ actually forms a C^*-subalgebra of $\mathcal{F}$ (prove it or compare the proof of Theorem 1.18 (a) in [7]).

The $*$-homomorphism $W, \tilde{W} : \mathcal{F}^W \to \mathcal{B}(l^2(\mathbb{Z}_+))$ turn out to be fractal: given a strongly monotonically increasing sequence η, we can define $W_\eta, \tilde{W}_\eta : \mathcal{F}_\eta \to \mathcal{F}(l^2(\mathbb{Z}_+))$ via

$$W_\eta(A_{\eta(n)}) := s - \lim A_{\eta(n)} P_{\eta(n)}$$

and

$$\tilde{W}_\eta(A_{\eta(n)}) := s - \lim R_{\eta(n)} A_{\eta(n)} R_{\eta(n)}\,.$$

Then, obviously, $W = W_\eta R_\eta\,,\ \tilde{W} = \tilde{W}_\eta R_\eta$.

The algebra $\mathcal{F}^W$ is not fractal: consider the sequence $(A_n) \in \mathcal{F}$, where $A_{2n+1} = 0$ and $A_{2n} = \mathrm{diag}\,(0, \dots, 0, 1, 0, \dots, 0)$, where the 1 stands in the center of this diagonal matrix. It is easily seen, that $(A_n) \in \mathcal{F}^W$, $W(A_n) = 0$, $\tilde{W}(A_n) = 0$, but $(A_n) \notin G (\subset \mathcal{F}^W)$.

For the special choise $\eta(n) = 2n+1$ one obtains $R_\eta(A_n) = (A_{2n+1}) \in G_\eta$. By (4) $\mathcal{F}^W$ cannot be fractal. One the other hand, $\mathcal{F}^W$ contains interesting fractal subalgebras as we will see later on.

3. Asymptotic behavior

Given a sequence $(A_n) \in \mathcal{F}$ one can ask how the spectra (ε-pseudospectra) of the entries develop.

Let $(M_n)_{n=1}^{\infty}$ be a set sequence with values in the set of all subsets of the complex plane. For instance, if $(A_n) \in \mathcal{F}$, then the mapping $n \to \text{sp}\, A_n$ is a set sequence in this sense.

Definition 4: (a) Let $(M_n)_{n=1}^{\infty}$ be a set sequence. The partial limiting set or limes superior lim sup M_n (resp. the uniform limiting set or limes inferior lim inf M_n) of the sequence (M_n) consists of all points $m \in \mathbb{C}$ which are a partial limit (resp. limit) of a sequence (m_n) of points $m_n \in M_n$ (partial limit of a sequence (m_n) is by definition a limit of some subsequence of (m_n)).

Observe that the partial limiting set lim sup M_n is non-empty if infinitely many of the M_n are non-empty and if $\bigcup\limits_n M_n$ is bounded, whereas the uniform limiting set can be empty even under these restrictions as the trivial example $M_n = \{(-1)^n\}$ shows.

Let $\mathbb{C}^C$ denote the set of all non-empty and compact subsets of $\mathbb{C}$. The Hausdorff distance of two elements A and B of $\mathbb{C}^C$ is defined by

$$h(A, B) := \max \left\{ \max_{a \in A} \text{dist}\,(a, B), \max_{b \in B} \text{dist}\,(b, A) \right\} ,$$

where dist $(a, B) = \min\limits_{b \in B} |a - b|$. The function h is actually a metric on $\mathbb{C}^C$. We denote limits with respect to this metric by h-lim.

Proposition 4: Let (M_n) be a set sequence taking values in $\mathbb{C}^C$. Then lim sup M_n and lim inf M_n coincide if and only if the sequence (M_n) is h-convergent. In that case

$$\limsup M_n = \liminf M_n = h - \lim M_n$$

([7], Proposition 3.6).

Example 3: Let $V : l^2(\mathbb{Z}_+) \to l^2(\mathbb{Z}_+)$ be the shift operator acting by

$$(a_0, a_1, a_2, \dots) \mapsto (0, a_0, a_1, a_2, \dots)$$

and consider (P_nVP_n). It is easy to see that the matrix representation of P_nVP_n with respect to the standard basis of im P_n equals

$$\begin{pmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ & 1 & 0 & & 0 & \\ 0 & & 1 & 0 & & \\ & & & 1 & 0 & \\ & & & & 1 & 0 \end{pmatrix}$$

$\Rightarrow$ sp $P_nVP_n = \{0\}$ for all n and lim inf sp $P_nVP_n = \limsup \text{sp}\, P_nVP_n = \{0\}$, but sp $V = \{z \in \mathbb{C} : \|z\| \leq 1\} \subset \text{sp}_{\mathcal{F}/G}((P_nVP_n) + G)$.

What is the reason for this unpleasant fact? One can prove that for $(a_n) \in \mathcal{F}$ a point $s \in \mathbb{C}$ belongs to the partial limiting set $\limsup \text{sp}\, a_n$ if and only if the sequence $(a_n - se_n)$ is not spectrally stable (Theorem 3.17 in [7]). (A sequence (a_n) is spectrally stable if its entries a_n are invertible for sufficiently large n and if the spectral radii $\rho(a_n^{-1})$ of their inverses are uniformly bounded.) Spectral stability is a very involved notion and not much is known. We accomplish this discussion with

Theorem 3: Let $\mathcal{C}_n = \mathbb{C}^{n \times n}$ and $(A_n) \in \mathcal{F}$. Then

$$\bigcup_{(C_n) \in G} \limsup \text{sp}\,(A_n + C_n) = \text{sp}_{\mathcal{F}/G}((A_n) + G)$$

([7], Theorem 3.19).

One conclusion: Spectral stability is very sensitive with respect to perturbations from G (contrary to stability).

These difficulties disappear if we restrict our attention to sequences for which stability and spectral stability coincide.

Corollary 1: If $(a_n) \in \mathcal{F}$ is a sequence of normal elements, then

$$\limsup \text{sp}\, a_n = \text{sp}_{\mathcal{F}/G}((a_n) + G)$$

([7], Corollary 3.18).

For fractal algebras we get refinements.

Theorem 4: Let $\mathcal{A}$ be a fractal C^*-subalgebra of $\mathcal{F}$ which contains the identity.

(a) A sequence $(a_n) \in \mathcal{A}$ is stable if and only if it possesses a stable (infinite) subsequence.
(b) If $(a_n) \in \mathcal{A}$ is normal, then lim sup sp $a_n = \liminf \text{sp}\, a_n = h - \text{limsp} a_n$.

(c) If $(a_n) \in \mathcal{A}$ is normal, then the limit $\lim \rho(a_n)$ exists and is equal to $\rho((a_n)+G)$ (ρ-spectral radius).
([7], Theorem 3.20)

Let us shortly discuss limiting sets of singular values (because of their importance in numerical analysis). Let B be a unital C^*-algebra and $a \in B$. The set $\sum(a)$ of the singular values of a is defined to be $\{\lambda \in \mathbb{R}^+ : \lambda^2 \in \mathrm{sp}\,(a^*a)\}$. Since the determination of the singular values is equivalent to the determination of the spectrum of a self-adjoint element, the previous results have the following evident analogues for singular value sets.

Theorem 5: If $(a_n) \in \mathcal{F}$, then $\limsup \sum(a_n) = \sum((a_n)+G)$.

Theorem 6: If $\mathcal{A}$ is a fractal C^*-subalgebra of $\mathcal{F}$ containing the identity and if $(a_n) \in \mathcal{A}$, then

$$\limsup \sum(a_n) = \liminf \sum(a_n) = h - \lim \sum(a_n)\,.$$

The last topic in this section is ε-pseudospectra.

A computer working with finite accuracy cannot distinguish between a noninvertible matrix and an invertible matrix the inverse of which has a very large norm. This suggests the following definition reflecting finite accuracy.

Definition 5: Let B be a C^*-algebra with identity e and let ε be a positive constant. An element $a \in B$ is ε-invertible if it is invertible and $\|a^{-1}\| < \frac{1}{\varepsilon}$. The ε-pseudospectrum $\mathrm{sp}_\varepsilon(a)$ of a consists of all $\lambda \in \mathbb{C}$ for which $a - \lambda e$ is not ε-invertible.

It is easily seen that ε-invertible elements of a C^*-algebra form an open set, and that ε-pseudospectra are compact and non-empty subsets of $\mathbb{C}$.

The following theorem provides an equivalent description of the ε-pseudospectrum which offers a way for numerical computations at least for (finite) matrices.

Theorem 7: Let B be a unital C^*-algebra and $\varepsilon > 0$. Then, for every $a \in B$, the ε-pseudospectrum is equal to

$$\mathrm{sp}_\varepsilon(a) = \bigcup_{\substack{p \in B \\ \|p\| < \varepsilon}} \mathrm{sp}\,(a+p)$$

([7], Theorem 3.27).

Let us still remark that (unital) C^*-algebras are also inverse closed with respect to ε-invertibility. What about limiting sets of ε-pseudospectra?

Theorem 8: Let $(a_n) \in \mathcal{F}$ and $\varepsilon > 0$. Then

$$\limsup \operatorname{sp}_\varepsilon^{\mathcal{C}_n}(a_n) = \operatorname{sp}_\varepsilon^{\mathcal{F}/G}((a_n) + G)$$

([7], Theorem 3.31).

The proof of Theorem 8 is based on the following result.

Proposition 5: (Daniluk) Let B be a C^*-algebra with identity e, let $a \in B$, and suppose $a - \lambda e$ is invertible for all λ in some open subset $\mathcal{U}$ of the complex plane. If $\|(a-\lambda e)^{-1}\| \le C$ for all $\lambda \in \mathcal{U}$, then $\|(a-\lambda e)^{-1}\| < C$ or all $\lambda \in \mathcal{U}$. ([3], Theorem 3.14)

In other words: the analytic function $U \to B, \lambda \mapsto (a-\lambda e)^{-1}$ satisfies the maximum principle. This is a surprising fact since – in contrast to complex-valued analytic functions – the maximum principle fails in general for operator-valued analytic functions (consider $\mathbb{C} \to \mathbb{C}^{2\times 2}, \lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$).

It is an open question for which Banach algebras Daniluk's result is true (one particular answer is in [3], Theorem 7.15).

In case $\mathcal{A}$ is a fractal C^*-subalgebra of $\mathcal{F}$ behave the following refinement of Theorem 8:

$$h - \lim \operatorname{sp}_\varepsilon^{\mathcal{C}_n}(a_n) = \operatorname{sp}_\varepsilon^{\mathcal{F}/G}((a_n) + G)\,.$$

4. First applications

I. Quasidiagonal operators and their finite sections.

Recall that a bounded linear operator T on a separable (complex) Hilbert space is said to be quasidiagonal if there exists a sequence $(P_n)_{n\in\mathbb{N}}$ of finite rank orthogonal projections such that s-lim $P_n = I$ and which asymptotically commute with T, that is

$$\|[T, P_n]\| := \|TP_n - P_nT\| \to 0 \text{ as } n \to \infty\,.$$

In particular, every selfadjoint or even normal operator is quasidiagonal as well as their perturbations by compact oeprators. However it is by no means trivial to single out a related sequence (P_n). For instance, for multiplication operators in periodic Sobolev spaces H^λ related sequences can explicitely be given: these are orthogonal projections on some spline spaces.

Let T be quasidiagonal with respect to $(P_n) = (P_n)_{n\in\mathbb{N}}$. Consider the C^*-subalgebra $\mathcal{F}^l$ of $\mathcal{F}$, the last one defined by help of (P_n), consisting of all sequences of $\mathcal{F}$ for which s^*-lim A_nP_n exist. It is not hard to prove that $J := \{(P_nKP_n) + (C_n) :$ K-compact, $(C_n) \in G\}$ forms a two-sided closed ideal in $\mathcal{F}^l$ (but not in $\mathcal{F}$!)

Let $C_{(P_n)}(T)$ denote the smallest C^*-subalgebra of $\mathcal{F}^l$ containing the sequences (P_nTP_n), (P_n), and the ideal J.

Proposition 6: The quotient algebra $C_{(P_n)}(T)/G$ is isometrically isomorphic to the smallest C^*-subalgebra $C(T)$ of $\mathcal{B}(H)$ containing T, I, and all compact operators. This isomorphism is given by the quotient map induced via s-lim A_nP_n $((A_n) \in C_{(P_n)}(T))$.

Sketch of the proof: Suppose s-lim $A_nP_n =: A$ is invertible. then $A^{-1} \in C(T)$, and since every element in $C(T)$ is quasidiagonal, A^{-1} also owns this property, and

$$\begin{aligned}&\|P_n - P_nAP_nA^{-1}P_n\| = \|P_nAA^{-1}P_n - P_nAP_nA^{-1}P_n\|\\ &= \|P_n(P_nA - AP_n)A^{-1}P_n\| \to 0\,.\end{aligned}$$

Hence, (P_nAP_n) is stable. This means that $(P_nAP_n) + G$ is invertible if and only if s-lim P_nAP_n is invertible. Now it is sufficient to prove that $P_nAP_n - A_n \in G$. For, it is sufficient to show this for the special case $A_n = P_nB_1P_nB_2P_n$. We have $\|P_nB_1B_2P_n - P_nB_1P_nB_2P_n\| = \|P_n(P_nB_1 - B_1P_n)B_2P_n\| \to 0$ as $n \to \infty$. #

Corollary 2: A sequence $(A_n) \in C_{(P_n)}(T)$ is stable if and only if s-lim A_n is invertible. Moreover, $C_{(P_n)}(T)$ is fractal.

Now it is evident that the theory of Section 3 applies.

Proposition 7: Let $(A_n) \in C_{(P_n)}(T)$.

(a) $\lim \|A_n\| = \|s - \lim A_nP_n\|$.
(b) $\liminf \operatorname{sp}_\varepsilon A_n = \limsup \operatorname{sp}_\varepsilon A_n = \operatorname{sp}_\varepsilon(s - \lim A_n)$ $(\varepsilon > 0)$.
(c) If (A_n) is normal then

$$\liminf \operatorname{sp} A_n = \limsup \operatorname{sp} A_n = \operatorname{sp}(s - \lim A_n)\,.$$

(d) $\liminf \sum(A_n) = \limsup \sum(A_n) = \sum(s - \lim A_n)$.

In the papers [5], [6] Nathaniel Brown proposed further refinements into two directions: speed of convergence and how to choose the sequence (P_n) of orthoprojections in some special cases such as quasidiagonal unilateral band operators, bilateral band operators or operators in irrational rotation algebras.

Remark. If $(a_n) \in \mathcal{F}^l$ is stable and s^*-lim $a_n = A$, $A + K$ invertible and compact, then $(a_n + P_nkP_n)$ is stable (this sequence equals $(a_n)(I + P_na_n^{-1}P_nKP_n)$ for n large enough.

II. Toeplitz operators and their finite sections

Let $a \in L^\infty(\mathbb{T})$ and denote by a_k the k-th Fourier coefficient of a:

$$a_k = \frac{1}{2\|i} \int_0^{2\|i} a(e^{i\theta})e^{-ik\theta}\,d\theta\,,\ k \in \mathbb{Z}\,.$$

Then the Laurent operator $L(a)$ on $l^2(\mathbb{Z})$, the Toeplitz operator $T(a)$ on $l^2(\mathbb{Z}_+)$, and the Hankel operator $H(a)$ on $l^2(\mathbb{Z}_+)$ are given via their matrix representation with respect to the standard bases of $l^2(\mathbb{Z})$ and $l^2(\mathbb{Z}_+)$ by

$$\begin{aligned} L(a) &= (a_{k-j})_{k,j=-\infty}^{\infty}\,, \\ T(a) &= (a_{k-j})_{k,j=0}^{\infty}\,,\ H(a) = \begin{pmatrix} a_1 & a_2 & a_3 & \iddots & \\ a_2 & a_3 & \iddots & & \iddots \\ a_3 & \iddots & & \iddots & \\ \iddots & & \iddots & & \end{pmatrix} \end{aligned}$$

Here is a list of elementary properties of these operators (see any textbook on Toeplitz operators).

- If $a \in L^\infty(\mathbb{T})$, then Laurent operator $L(a)$ is bounded on $l^2(\mathbb{Z})$.
- (Brown/Halmos) If $a \in L^\infty(\mathbb{T})$, then the Toeplitz operator $T(a)$ is bounded on $l^2(\mathbb{Z}_+)$, and $\|T(a)\| = \|a\|_\infty$.
- (Nehari) If $a \in L^\infty(\mathbb{T})$, then $H(a)$ is bounded on $l^2(\mathbb{Z}_+)$, and $\|H(a)\| = \text{dist}_{L^\infty(\mathbb{T})}(a, \overline{H^\infty})$.
- $T(ab) = T(a)T(b) + H(a)H(\tilde{b})$, where $\tilde{b}(t) := b(\frac{1}{t})$.
- $T(a)^* = T(\overline{a})$.
- (Coburn) Let $a \in L^\infty(\mathbb{T}) \setminus \{0\}$. Then at least one of the spaces $\ker T(a)$ and $l^2/\text{im}T(a)$ consists of the zero element only.

Proposition 8:

(i) Let $a \in C(\mathbb{T})$. The Toeplitz operator $T(a)$ is Fredholm on $l^2 = l^2(\mathbb{Z}_+)$ if and only $0 \notin a(\mathbb{T})$. In this case, $\text{ind}\, T(a) = -\text{wind}\, a$, where wind a refers to the winding number of the curve $a(\mathbb{T})$, provided with the orientation inherited by the usual counter-clockwise orientation of the unit circle, around the origin.

(ii) Let $a \in C(\mathbb{T})$. The Toeplitz operator is invertible on l^2 if and only if $0 \notin a(\mathbb{T})$ and wind $a = 0$.

(iii) Let $a \in C(\mathbb{T})$. Then $H(a)$ is compact on l^2.

(iv) The smallest C^*-subalgebra $\mathcal{T}(C)$ of $\mathcal{B}(l^2)$ containing all Toeplitz operators with continuous generating functions, decomposes as

$$\mathcal{T}(C) = \{T(a) : a \in C(\mathbb{T})\} \dot{+} \mathcal{K}(l^2)\,,$$

where $\mathcal{K}(l^2)$ stands for the (closed) ideal of all compact operators. ([3], Chapter 1)

Now let us turn to the finite section method for Toeplitz operators (with continuous generating function). The first question is about the stability of the sequence $(P_nT(a)P_n)$, where $P_n : l^2 \to l^2$ is the projection defined by

$$(a_0, a_1, \dots, a_n, a_{n+1}, \dots) \mapsto (a_0, \dots, a_n, 0, 0, \dots).$$

This problem was investigated by many people.

G. Baxter, 63' : $(P_nT(a)P_n)$ stable in l^1 if and only if $T(a)$ is invertible ($a \in W, 0 \notin a(\mathbb{T})$, wind $a = 0$).

I. Gohberg, I. Feldmann, 65' : $(P_nT(a)P_n)$ stable in l^2 if and only if $T(a)$ is invertible.

Later on related results for classes of discontinuous generating functions were achieved: $QC, C + H^\infty$, PC, PQC.

Treil, 87' : There are generating functions a with only one point of discontinuity such that $T(a)$ is invertible but $\{P_nT(a)P_n)$ is not stable.

Recall the definition of the algebra $\mathcal{F}^W$ (Example 2):
A sequence $(A_n) \in \mathcal{F}$ belongs to $\mathcal{F}^W$, if and only if the strong limits $W(A_n) := s-\lim A_nP_n$, $\tilde{W}(A_n) = s-\lim R_nA_nR_n$ as well as $W(A_n^*)$, $\tilde{W}(A_n^*)$ exist. Because of $R_nT(a)R_n = P_nT(\tilde{a})P_n$ ($a \in L^\infty(\mathbb{T})$) it is easy to see that $(P_nT(a)P_n) \in \mathcal{F}^W$. Moreover, R_nKR_n tends strongly to zero for every compact operator due to the week convergence of (R_n) to zero.

Hence, the smallest C^*-subalgebra $S(C)$ in $\mathcal{F}$ containing all sequences $(P_nT(a)P_n)$, $a \in C(\mathbb{T})$, is actually contained in $\mathcal{F}^W$. Our next goal is to describe the structure of the algebra $S(C)$. For, we need Widom's identity

$$\begin{aligned} P_nT(ab)P_n = {} & P_nT(a)P_nT(b)P_n + P_nH(a)H(\tilde{b})P_n + \\ & + R_nH(\tilde{a})H(b)R_n \end{aligned}$$

(prove it).

The collection

$$\mathcal{J}_W := \{(P_nKP_n + R_nLR_n + C_n) : K, L \text{ compact}, (C_n) \in G\}$$

forms a closed two-sided ideal in $\mathcal{F}^W$.

Theorem 9: $\mathcal{J}_W \subset S(C)$. Moreover, each element $(A_n) \in S(C)$ can uniquely be written as

$$A_n = P_nT(a)P_n + P_nKP_n + R_nLR_n + C_n,$$

where K, L are compact operators, $(C_n) \in G$. ([7], Theorem 1.5.3).

Using this representation, it is evident that

$$W(A_n) = T(a) + K\,,\ \tilde{W}(A_n) = T(\tilde{a}) + L\,,$$

and $\ker W \cap \ker \tilde{W} = G$.

Now take the $*$-homomorphisms $W, \tilde{W} : S(C) \to \mathcal{B}(l^2)$ and glue them together to obtain a $*$-homomorphism $\mathrm{smb}^0 : S(C) \to \mathcal{B}(l^2) \times \mathcal{B}(l^2)$,

$$(A_n) \mapsto (W(A_n), \tilde{W}(A_n))\,.$$

Furthermore, it is clear that $\ker \mathrm{smb}^0 = G$. Thus, the quotient homomorphism

$$\mathrm{smb} : S(C)/G \to \mathcal{B}(l^2) \times \mathcal{B}(l^2)$$

is correctly defined and is injectiv. Notice that an injective $\star$-homomorphism is isometric.

Theorem 10:

(i) The map smb is a $*$-isomorphismus from $S(C)/G$ onto the C^*-subalgebra of $\mathcal{B}(l^2) \times \mathcal{B}(l^2)$ which consists of all pairs $(W(A_n), \tilde{W}(A_n))$ with (A_n) running through $S(C)$.

(ii) $(A_n) \in S(C)$ is stable if and only if $W(A_n)$ and $\tilde{W}(A_n)$ are invertible operators.

(iii) $S(C)$ is fractal.

Now it is clear that the theory of Section 3 applies.

Theorem 11:

(a) $\lim \|A_n\| = \max\{\|W(A_n)\|\,,\ \|\tilde{W}(A_n)\|\}$.

(b) $\liminf \mathrm{sp}_\varepsilon A_n = \limsup \mathrm{sp}_\varepsilon A_n = \mathrm{sp}_\varepsilon W(A_n) \mathcal{U}p \mathrm{sp}_\varepsilon \tilde{W}(A_n)$. If $(A_n) = (P_n T(a) P_n)$, then $\mathrm{sp}_\varepsilon W(A_n) = \mathrm{sp}_\varepsilon T(a) = \mathrm{sp}_\varepsilon T(\tilde{a}) = \mathrm{sp}_\varepsilon \tilde{W}(A_n)$.

(c) If (A_n) is normal then $\liminf \mathrm{sp} A_n = \limsup \mathrm{sp}\, A_n = \mathrm{sp}\, W(A_n) \cup \mathrm{sp}\, \tilde{W}(A_n)$.

(d) $\liminf \sum(A_n) = \limsup \sum(A_n) = \sum(W(A_n)) \cup \sum(\tilde{W}(A_n))$.

5. Fredholm sequences

Now we are going to introduce a third fundamental notion, namely that one of Fredholm sequences. First we introduce Fredholm sequences in some restricted form and finally in full generality.

We introduce C^*-subalgebras of $\mathcal{F}$ which are generalizations of the algebras $\mathcal{F}^l$ and $\mathcal{F}^w$ and which give raise to consider Fredholm sequences. Let H be an infinite Hilbert space and (L_n) be a sequence of orthogonal projections such that

$L_n \to I$ strongly as $n \to \infty$. The related C^*-algebra of all bounded sequences is again denoted by $\mathcal{F}$. We shall assume that all projection operators are finite rank operators.

Let T be a (possibly infinite) index set and suppose that, for every $t \in T$, we are given an infinite dimensional Hilbert space H^t with identity operator I^t as well as a sequence (E_n^t) of partial isometries $E_n^t \colon H^t \to H$ such that

- the initial projections L_n^t of E_n^t converge strongly to I^t as $n \to \infty$,
- the range projection of E_n^t is L_n,
- the separation condition
$$(E_n^s)^* \, E_n^t \to 0 \text{ weekly as } n \to \infty \tag{5.1}$$
holds for every $s, t \in T$ with $s \neq t$. (Recall that an operator $E : H' \to H''$ is a partial isometry if $EE^*E = E$ and that E^*E and EE^* are orthogonal projections which are called the initial and the range projections of E, respectively). For brevity, write E_{-n}^t instead of $(E_n^t)^*$, and set $H_n := \operatorname{im} L_n$ and $H_n^t := \operatorname{im} L_n^t$.

Let $\mathcal{F}^T$ stand for the set of all sequences $(A_n) \in \mathcal{F}$ for which the strong limits

$$s - \lim_{n\to\infty} E_{-n}^t A_n E_n^t \text{ and } s - \lim (E_{-n}^t A_n E_n^t)^*$$

exist for every $t \in T$, and define mappings $W^t : \mathcal{F}^T \to \mathcal{B}(H^t)$ by $W^t(A_n) := s\!-\lim_{n\to\infty} E_{-n}^t A_n E_n^t$. It is easily seen that $\mathcal{F}^T$ is a C^*-subalgebra of $\mathcal{F}$ which contains the identity, and that the W^t are $*$-homomorphisms.

The separation condition (5) ensures that, for every $t \in T$ and every compact operator $K^t \in \mathcal{K}(H^t)$, the sequence $(E_n^t K^t E_{-n}^t)$ belongs to the algebra $\mathcal{F}^T$, and that for all $s \in T$

$$W^s(E_n^t K^t E_{-n}^t) = \begin{cases} K^t & \text{if} \quad s = t \\ 0 & \text{if} \quad s \neq t \end{cases} . \tag{5.2}$$

Conversely, (6) implies (5). Moreover, the ideal G belongs to $\mathcal{F}^T$. So we can introduce the smallest closed ideal J^T which contains all sequences $(E_n^t K^t E_{-n}^t)$ with $t \in T$ and $K^t \in \mathcal{K}(H^t)$ as well as all sequences $(G_n) \in G$.

Remark: The algebra $\mathcal{F}^W$ provides an example of this type. Indeed T consists only of two points, say 1 and 2. Then $W^1 = W, W^2 = \tilde{W}$, and

$$\begin{aligned} J_1 &= \{(P_n K P_n + C_n) : K \text{ compact}, \, (C_n) \in G\}, \\ J_2 &= \{(R_n L R_n + C_n) : L \text{ compact}, \, (C_n) \in G\}. \end{aligned}$$

The separation condition (5) is obviously is fulfilled (recall that R_n tends weekly to zero).

There are examples which show that indeed infinite index sets T are needed ([7], 4.5.1 – 4.5.2, for instance).

Theorem 12:

(a) A sequence $(A_n) \in \mathcal{F}^T$ is stable if and only if the operators $W^t(A_n)$ are invertible in $\mathcal{B}(H^t)$ for every $t \in T$ and if the coset $(A_n) + \mathcal{I}^T$ is invertible in the quotient algebra $\mathcal{F}^T/\mathcal{I}^T$.
(b) If $(A_n) \in \mathcal{F}^T$ is a sequence with invertible coset $(A_n) + \mathcal{I}^T$, then all operators $W^T(A_n)$ are Fredholm on H^t, and the number of the non-invertible operators among the $W^t(A_n)$ is finite. ([7], Theorem 6.1)

Notice that this theorem can be used to give a different proof of Theorem 10, (ii).

Definition 6:

(a) A sequence $(A_n) \in \mathcal{F}^T$ is called Fredholm if the coset $(A_n) + \mathcal{I}^T$ is invertible.
(b) If the sequence $(A_n) \in \mathcal{F}^T$ is Fredholm then its nullity $\alpha(A_n)$, deficiency $\beta(A_n)$ and index ind (A_n) are defined by $\alpha(A_n) := \sum_{t\in T} \dim\ker W^t(A_n)$, $\beta(A_n) := \sum_{t\in T} \dim \operatorname{coker} W^t(A_n)$ and
ind $(A_n) := \alpha(A_n) - \beta(A_n)$.

It is a triviality to carry over the well-known properties of Fredholm operators to Fredholm sequences.

Remark: As we will see later on, this notion of Fredholm sequence depends on the underlying algebra $\mathcal{F}^T$. We shall also see that Fredholmness of a sequence in the sense of Definition 6 implies its Fredholmness in a general sense which has still to be defined.

Let $(A_n) \in \mathcal{F}$ be arbitrary. We order the singular values of A_n as follows ($l_n = \operatorname{rank} L_n$):

$$0 \le s_1(A_n) \le \cdots \le s_{l_n}(A_n) (= \|A_n\|) \, .$$

For the sake of convenience let us also put $s_0(A_n) = 0$. Recall that usually the singular values are ordered in the reverse manner.

Definition 7: We say that $(A_n) \in \mathcal{F}$ has the k-splitting property if there is a $k \in \mathbb{Z}_+$ such that

$$\lim_{n\to\infty} s_k(A_n) = 0 \, ,$$

while the remaining $l_n - k$ singular values stay away from zero, that is

$$s_{k+1}(A_n) \ge \delta > 0$$

for n large enough. The number k is also called the splitting number.

Notice if (A_n) has the 0-splitting property then (A_n) is stable (hint: if $s_1(A_n) \neq 0$ then A_n is invertible and $\|A^{-1}\| = s_1(A_n)^{-1}$).

Theorem 13: Let $(A_n) \in \mathcal{F}^T$ be Fredholm. Then

(a) (A_n) is subject to the k-splitting property with $k = \alpha(A_n)$.

(b) $s_{\alpha(A_n)}(A_n) \leq \|A_n(\sum\limits_{t\in T} E_n^t P^{H^t}_{\ker W^t(A_n)} E_{-n}^t)\|$.

If for $(A_n) \in \mathcal{F}^T$ there is at least one $t_1 \in T$ such that $W^{t_1}(A_n)$ is not Fredholm, then

(c) $\lim\limits_{n\to\infty} s_l(A_n) = 0$ for all $l \in \mathbb{Z}_+$.

Assertions (a) and (c) can be proved slightly modifying the idea of the proof of Theorem 6.11 and using Theorem 6.67 in [7]. A complete proof of Theorem 13 is contained in [16]. We present here the proof of Theorem 13, (a) and (b) for the special case $\mathcal{F}^l$.

Proof: We shall make use of the following alternative description of the singular values (as approximation numbers):

$$s_j(A_n) := \min_{B\in\mathcal{F}^{l_n}_{l_n-j}} \|A_n - B\|\,,$$

where $\mathcal{F}^{l_n}_m$ denotes the collection of all $l_n \times l_n$-matrices of rank at most m. Let R_n be the orthoprojection onto im $(P_n P_{\ker A} P_n)$, where $A = s - \lim A_n P_n$. It is easy to check that

$$\operatorname{im} R_n = \operatorname{im} P_n P_{\ker A} P_n\,,$$

rank R_n = rank $P_n P_{\ker A} P_n$ = rank $P_{\ker A}$ = dim ker $A =: k$ for n large enough, and

$$\|R_n - P_n P_{\ker A} P_n\| \to 0 \text{ as } n \to \infty\,.$$

Consequently, $||A_n R_n|| \to 0$ as $n \to \infty$, and $(A_n R_n) \in G$. Consider the sequence $(B_n) \in \mathcal{F}^l$, $B_n := A_n^* A_n (P_n - R_n) + P_n P_{\ker A} P_n$. Obviously, this sequence is also Fredholm and s-lim $B_n P_n = A^*A + P_{\ker A}$ is invertible. Then (B_n) is stable by Theorem 12, (a). Since rank $(P_n - R_n) = l_n - k$ we get for n large enough

$$\begin{aligned} s_k(A_n) &\leq ||\left(A_n - A_n A_n^* A_n (P_n - R_n) B_n^{-1}\right) P_n|| \\ &\leq ||\left(A_n B_n - A_n A_n^* A_n (P_n - R_n)\right) P_n||\,||B_n^{-1} P_n|| \\ &\leq ||B_n^{-1} P_n||\,||A_n P_n P_{\ker A} P_n||\,. \end{aligned}$$

Since (B_n) is stable, there exists for n large enough a constant C with $||B_n^{-1}P_n|| \leq C$. Thus we have

$$s_k(A_n) \leq C||A_n P_n P_{\ker A} P_n|| \to 0 \text{ as } n \to \infty\,.$$

Now consider $s_{k+1}(A_n)$. By using the well-known inequality $s_{k+1}(A_n^* A_n) \leq s_{k+1}(A_n)||A_n^*||$ and that $||A_n^*||$ is bounded (recall that $A_n^* P_n$ converges strongly to $A^* \neq 0$) it has to be shown that $s_{k+1}(A^*A_n)$ is bounded away from zero (n large enough). We have

$$\begin{aligned} s_{k+1}\left(A_n^*A_n\right) &= \min_{B\in\mathcal{F}^{l_n}_{l_n-k-1}} \|\left(A_n^*A_n - B\right)P_n\| = \\ = &\min_{B\in\mathcal{F}^{l_n}_{l_n-k-1}} \|\left(\left(A_n^*A_n + P_nP_{\ker A}P_n\right) - B - P_nP_{\ker A}P_n\right)P_n\| \\ \geq &\min_{B\in\mathcal{F}^{l_n}_{l_n-1}} \|\left(\left(A_n^*A_n + P_nP_{\ker A}P_n\right) - B\right)P_n\| = \\ = &\; s_1\left(A_n^*A_n + P_nP_{\ker A}P_n\right) \geq \delta > 0 \end{aligned}$$

for n large enough since $(A_n^*A_n + P_nP_{\ker A}P_n)$ is stable, and we are done. ■

Corollary 3: If $(A_n) \in \mathcal{F}^T$ is Fredholm, then

$$\operatorname{ind}\,(A_n) = 0\,.$$

Proof: One has only to use that the matrices $A_n^*A_n$ and $A_nA_n^*$ are unitarily equivalent. This shows that the splitting numbers of (A_n) and (A_n^*) coincide.

Example 4: The sequence (P_nVP_n) belongs to both algebras $\mathcal{F}^l$ and $\mathcal{F}^W(\uparrow)$. This sequence is Fredholm in $\mathcal{F}^W$ but not in $\mathcal{F}^l$. If it would be Fredholm in $\mathcal{F}^l$ then $\operatorname{ind} V = 0$; but $\operatorname{ind} V = -1$.
$(V = T(t))$

Theorem 13 has remarkable applications. Let us mention some simple results:

- If $T(a)$ $(a \in C(\mathbb{T}))$ is Fredholm, then the Moore-Penrose inverses $P_nT(a)P_n^+$ converge strongly to $T(a)^+$ if and only if

  $$\dim\ker P_nT(a)P_n = \alpha(P_nT(a)P_n)$$

  for n large enough.
  A deeper study of this problem is presented in [3], Chapter 4.
- Let $T(a)$ $(a \in C(\mathbb{T}))$ be Fredholm and K be compact. Since $(P_n(T(a) + K)P_n)$ is subject to the splitting property with splitting number $\alpha(P_n(T(a)+K)P_n) = \dim\ker(T(a)+K) + \dim\ker T(\tilde{a})$ and $\dim\ker T(\tilde{a})$ is known by Coburns Theorem, $\dim\ker(T(a)+K)$ can be found numerically (in principle).
- If $T(a)$ $(a \in C(\mathbb{T}))$ is Fredholm and a is smooth then $s_\alpha(P_nT(a)P_n)$ tends fast to zero.
- It was mentioned before that multiplication operators M_a with continuous functions a are quasidiagonal in $L^2(\mathbb{T})$. The corresponding sequence of finite dimensional projections can be taken as orthogonal projcetions on some spline spaces. To be more precise let $\mathbb{T} := \{|z| = 1\}$ be parametrized be $\varphi : [0,1] \to \mathbb{T}$, $\varphi(t) = e^{2\pi it}$. A sequence of partitions $(\Delta_K)_{k\in\mathbb{N}}$, $\Delta_K : \{\sigma_0^k, \ldots, \sigma_{n_k}^k\}$, $0 = \sigma_0^k < \sigma_1^k < \cdots < \sigma_{n_k}^k = 1$, is said to be admissible if $h_{\Delta_K} := \max(\sigma_{j+1}^k - \sigma_j) \to 0$ as k tends to infinity. We denote by $\tilde{S}^\delta(\Delta_K)$ the space of all $\psi \in C(\mathbb{T})$ such that $\psi \circ \varphi$ is $(\delta - 1)$ times continuously differentiable and the restriction of

$\psi \circ \varphi$ to each intervall $(\sigma_j^k, \sigma_{j+1}^k)$ is a polynom of degree $\leq \delta$ (smoothest splines). Let P_{Δ_K} denote the orthogonal projections of $L^2(\mathbb{T})$ onto $\tilde{S}^\delta(\Delta_K)$.

Then (see S. Prdorf, B. Silbermann: Numerical Analysis for Integral and related Operator Equations, Akademie Verlag, Berlin 1991, Section 2.14)

$$\|(I - P_{\Delta_K}) f P_{\Delta_K}\| \to 0\,,\ \|P_{\Delta_K} f (I - P_{\Delta_K})\| \to 0$$

as $k \to \infty$, where

- $\|\cdot\|$ stands for the operator norm in $L^2(\mathbb{T})$,
- f is continuous,
- (Δ_K) is admissible.

Consider the singular integral operator A with continuous coefficients:

$$Ag = ag + \frac{b}{\pi i} \int_{\mathbb{T}} \frac{g(\tau)}{\tau - t} d\tau$$

(integral understood in the sense of Cauchy's principal value).

A is called strongly (locally) elliptic, if there is a continuous function c on $\mathbb{T}$, a linear operator T with $\|T\| < 1$ and a compact operator K such that

$$A = c(I + T) + K\,,\ c(t) \neq 0 \ \text{for all}\ t \in \mathbb{T}\,.$$

$\Rightarrow A$ is Fredholm with index 0 (even invertible).

Well-known: A is strongly elliptic $\Leftrightarrow$

$$a(t) + \lambda b(t) \neq 0\ \forall t \in \mathbb{T}\ \text{und}\ \forall \lambda \in [-1, 1]\,.$$

If A is strongly elliptic and (Δ_K) admissible then

$$(P_{\Delta_K} A P_{\Delta_K})\ \text{is stable (use}\ P_{\Delta_K} A P_{\Delta_K} = P_{\Delta_K} M_c P_{\Delta_K} (I + T)$$
$$P_{\Delta_K} + P_{\Delta_K} k P_{\Delta_K} + C_{\Delta_K}\,, \|C_{\Delta_K}\| \to 0)\,.$$

If a and b are merely continuous $N \times N$-matrix functions, then A is strongly elliptic if and only if

$$\det(a(t) + \lambda b(t)) \neq 0\ \text{for}\ \forall t \in \mathbb{T}\ \text{and}\ \forall \lambda \in [-1, 1]$$

(see S. Prdorf, B. Silbermann: Numerical Analysis for Integral and related Operator Equations, Akademie Verlag, Berlin 1991, Section 13.31).

In this case A is Fredholm of index 0, but might be not invertible. In any case, $(P_{\Delta_K} A P_{\Delta_K})$ is Fredholm and $\alpha(P_{\Delta_K} A P_{\Delta_K}) = \dim \ker A$ $((P_{\Delta_K})$ admissible).

Now we turn to general Fredholm sequences.

Definition 8: Let $\mathcal{B}$ be a unital C^*-algebra. An element $k \in \mathcal{B}$ is of central rank one if, for every $b \in \mathcal{B}$, there is an element $\mu(b)$ belonging to the center of $\mathcal{B}$ such that $kbk = \mu(b)k$. An element of $\mathcal{B}$ is of finite central rank if it is the sum of a finite number of elements of central rank one, and it is centrally compact if it lies in the closure of the set of all elements of finite central rank.

We denote the set of all centrally compact elements in $\mathcal{B}$ by $\mathcal{J}(B)$. It is easy to check that $\mathcal{J}(\mathcal{B})$ forms a closed two-sided ideal in $\mathcal{B}$.

Proposition 9: A sequence $(A_n) \in \mathcal{F}$ is centrally compact if and only if, for every $\varepsilon > 0$, there is a sequence $(K_n) \in \mathcal{F}$ such that

$$\sup_n \|A_n - K_n\| < \varepsilon \text{ and } \sup \dim \operatorname{im} K_n < \infty .$$

([7], Proposition 6.33)

Definition 9: A sequence $(A_n) \in \mathcal{F}$ is a Fredholm sequence if it is invertible modulo the ideal $J(\mathcal{F})$ of the centrally compact sequences.

Theorem 14: A sequence $(A_n) \in \mathcal{F}$ is Fredholm if and only if there is a $l \in \mathbb{Z}_+$ such that

$$\lim_{n\to\infty} \inf s_{l+1}(A_n) > 0 .$$

([7], Theorem 6.35)

Conclusion: It $(A_n) \in \mathcal{F}^T$ is Fredholm then it is also Fredholm in the sense of Definition 9.

6. Applications continued: Around Finite Sections of Operators with Almost Periodic Diagonals

This material is based on [9].

6.1. Example:

The *Almost Mathieu Operator* is the operator

$$H_{\alpha,x,\theta} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}) ,$$

given by $(x = (x_n)_{n\in\mathbb{Z}})$

$$(H_{\alpha,\lambda,\theta} x)_n := x_{n+1} + x_{n-1} + \lambda x_n \cos 2\pi(n\alpha + \theta)$$

$\Rightarrow H_{\alpha,\lambda,\theta}$ is a band operator with almost periodic coefficients;

this means: $a \in l^\infty(\mathbb{Z})$ is called almost periodic, if the set $\{U_m a\}_{m\in\mathbb{Z}}$ is relatively compact,

$$(U_m a)(n) = a(n - m) .$$

Thus, $H_{\alpha,\lambda,\theta} = U_{-1} + U_1 + aI$,

$$a(n) = \lambda \cos 2\pi(n\alpha + \theta) .$$

Only recently the long-standing Ten Martini problem was solved, see [1], [8], and for a introduction to the topic [2].

Basic: commutation relation $UV = e^{2\pi i\alpha}VU$ and $\tilde{H}_{\alpha,\lambda,\theta} := U + U^* + \frac{\lambda}{2}(e^{2\pi i\theta}V + e^{-2\pi i\theta}V^*)$.

Model case: $U = U_1$ and $V = V_1, (V_1x)(n) = e^{2\pi i\alpha}x_n$.

The result says (in a somewhat incomplete form) that

- If α is rational, $\alpha = \frac{p}{q}$ and p, q relatively prime with $q > 0$, then the spectrum of $H_{\alpha,\lambda,\theta}$ is the union of exactly q closed and pairwise disjoint intervals, $\theta\frac{p}{\pi} \notin \mathbb{Z}$.
- If $\alpha \in [0, 1)$ is irrational, then the spectrum is a Cantor type set (means: nowhere dense, closed, and does not contain isolated points).

This result is a qualitative one! It does not allow to say that a given number μ belongs to the spectrum (or not). There is (at least in present time) only one way to tackle this problem, namely the use of approximation methods. For, introduce the projection operators P_n and $\tilde{P}_n$:

$$\begin{aligned} \tilde{P}_n x &= \{\dots, 0, x_{-n+1}, \dots, x_{n-1}, 0, \dots\} \\ P_n x &= \{\dots, 0, x_0, \dots, x_{n-1}, 0, \dots\} \,. \end{aligned}$$

First idea: consider the operators (matrices) $(\tilde{P}_n H_{\alpha,\lambda,\theta} \tilde{P}_n)$ (restricted to im $\tilde{P}_n$ and with respect to the standard basis) and compute the eigenvalues using Matlab or something else. Then the question arises, is this spectrum some how related to the spectrum of the AMO $H_{\alpha,\lambda,\theta}$? The following is devoted to some theory around this problem. However, we will merely make use of the projections P_n.

6.2. Band-dominated operators with almost periodic diagonals and related Toeplitz-like operators

Band operator with almost periodic diagonals:

$$A = \sum_{-k}^{k} a_k U_k,\ A : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}), a_k \text{ almost periodic },$$

that is

$$a_k \in AP(\mathbb{Z})\,.$$

Band-dominated operator with almost periodic diagonals: norm limits of band operator with almost periodic diagonals. The collection of all such operators is denoted by $\mathcal{A}_{AP}(\mathbb{Z})$ and actually forms a C^*-algebra.

Simple example: Laurent operator $L(a)$ with continuous generating function $a \in C(\mathbb{T})$. Matrix-representation of $a \in L^\infty(\mathbb{T})$ (with respect to the standard basis):

$$(a_{i-j})_{i,j\in\mathbb{Z}} ,$$

where

$$a_j := \frac{1}{2\pi} \int_0^{2\pi} a(e^{it}) e^{-ijt}\, dt .$$

Toeplitz-like operators: Clearly, $l^2(\mathbb{Z}^+)$ can be thought of as a subspace of $l^2(\mathbb{Z})$. Let P denote the orthogonal projection onto $l^2(\mathbb{Z}^+)$, $Q := I - P$. Consider $T(A) : l^2(\mathbb{Z}^+) \to l^2(\mathbb{Z}^+)$, $A \in \mathcal{A}_{AP}(\mathbb{Z})$,

$$T(A) := PAP|_{\mathrm{im} P} .$$

If $A = L(a), a \in C(\mathbb{T})$, then $T(L(a))$ is denoted simply by $T(a)$, and this is a familiar Toeplitz operator. Introduce $J : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ (flip operator) $x_n \mapsto (x_{-n-1}))$ and $H(A) := PAQJ$, $\tilde{A} := JAJ$.

Then one has

$$T(AB) = T(A)T(B) + H(A)H(\tilde{B}) \quad (A, B \in \mathcal{A}_{AP}(\mathbb{Z})) ,$$

which reminds the basic identity relating Toeplitz and Hankel operators and it is this identity for $A = L(a), B = L(b)$, $a, b \in C(\mathbb{T})$.

Notice: $H(A), H(\tilde{B})$ are compact operators!
Let $\mathcal{A}_{AP}(\mathbb{Z}^+)$ denote the smallest C^*-subalgebra of $\mathcal{B}(l^2(\mathbb{Z}))$ containing all operators $T(A), A \in \mathcal{A}_{AP}(\mathbb{Z})$.
$\Rightarrow$

- $\|T(A)\| = \|A\|$
- $\mathcal{A}_{AP}(\mathbb{Z}^+) = \{T(A) : A \in \mathcal{A}_{AP}(\mathbb{Z})\} \dot{+} \mathcal{K}(l^2(\mathbb{Z}^+)$ (6.1)

The first identity is based on a remarkable fact which plays an important role in what follows. Let us have a closer look. Let $\mathcal{H}$ refer to the set of all sequences $h : \mathbb{Z}^+ \to \mathbb{Z}$ which tend to $+\infty$ or $-\infty$.

Definition 10: An operator $A_h \in \mathcal{B}(l^2(\mathbb{Z}))$ is called a norm limit operator of the operator $A \in \mathcal{B}(l^2(\mathbb{Z}))$ with respect to the sequence $h \in \mathcal{H}$ if

$$U_{-h(k)} A U_{h(k)} \to A_h \quad \text{as} \quad k \to \infty$$

in norm. The set of all norm limit operators is called the norm operator spectrum $\sigma_{op}(A)$.

Theorem 15: $A \in \mathcal{A}_{AP}(\mathbb{Z}^+)$ is Fredholm $\Leftrightarrow$ each $A_h \in \sigma_{op}(A)$ is invertible.

Lemma 1: If $A \in \mathcal{A}_{AP}(\mathbb{Z})$ then $A \in \sigma_{op}(A)$.

Definition 11: A monotincally increasing sequence $h : \mathbb{Z}^+ \to \mathbb{Z}^+$ is called distinguished if A_h exists and *equals* A.

Notice: Distinguished sequences exist!

The first assertion in (7) is now easy to prove: Consider

$$U_{-h(k)}PAPU_{h(k)} = \underbrace{U_{-h(k)}PU_{h(k)}}_{\substack{\downarrow \text{ strongly} \\ I}} \underbrace{U_{-h(k)}AU_{h(k)}}_{\substack{\downarrow \text{ in norm} \\ A}} \underbrace{U_{-h(k)}PU_{h(k)}}_{\substack{\downarrow \text{ strongly} \\ I}}$$

Banach-Steinhaus $\Rightarrow$ result.

Further conclusion: ess sp $A = \mathrm{sp}A$ for $A \in \mathcal{A}_{PA}(\mathbb{Z})$ and $T(A)$ is Fredholm if and only if A is invertible (ess $sp\, A = \mathrm{sp}(A + \mathcal{K}(l^2(\mathbb{Z})))$.

Example: Almost Mathieu operators.

We have

$$\begin{aligned} & U_{-k}H_{\alpha,\lambda,\theta}U_k = U_{-1} + U_1 + a_k I\,, \\ a_k(n) &= a(n+k) = \lambda\cos 2\pi((n+k)\alpha+\theta) \\ &= \lambda(\cos 2\pi(n\alpha+\theta)\cos 2\pi(k\alpha) - \sin 2\pi(n\alpha+\theta)\sin 2\pi(k\alpha))\,. \end{aligned}$$

Let $\alpha \in (0,1)$ be irrational.

We write α as a continued fraction with n-th approximant $\frac{p_n}{q_n}$ such that

$$\alpha = \lim_{n\to\infty} \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots\, b_{n-1} + \cfrac{1}{b_n}}}}$$

with uniquely determined positive integers.

Write this continued fraction as p_n/q_n with positive and relatively prime integers p_n, q_n. These integers satisfy the recursions

$$p_n = b_n p_{n-1} + p_{n-2},\, q_n = b_n q_{n-1} + q_{n-2}$$

with $p_0 = 0, p_1 = 1, q_0 = 1$ and $q_1 = b_1$, and one has for all $n \geq 1$

$$\begin{aligned} & \left|\alpha - \tfrac{p_n}{q_n}\right| < \tfrac{1}{q_n^2}\,, \\ \Rightarrow\quad & |\alpha q_n - p_n| \leq \tfrac{1}{q_n} \to 0\,. \end{aligned}$$

Now it is not hard to see that (q_n) is a distinguished sequence for $H_{\alpha,\lambda,\theta}$ (note: (q_n) depends only on α).

6.3. Distinguished sequences and finite sections

In what follows we fix a strongly monotonically increasing sequence $h : \mathbb{Z}^+ \to \mathbb{Z}^+$ and define

$$\mathcal{A}_{AP,h}(\mathbb{Z}) := \{A \in \mathcal{A}_{AP}(\mathbb{Z}) : A_h \text{ exists and } A_h = A\}.$$

It is easy to check that $\mathcal{A}_{AP,h}(\mathbb{Z})$ is a C^*-subalgebra of $\mathcal{B}(l^2(\mathbb{Z}))$ which is moreover shift invariant, i.e., $U_{-k}AU_k$ again lies in $\mathcal{A}_{AP,h}(\mathbb{Z}^+)$ whenever A does. Let $\mathcal{A}_{AP,h}(\mathbb{Z}^+)$ refer to the smallest closed subalgebra of $\mathcal{B}(l^2(\mathbb{Z}^+))$ which contains all operators $T(A)$ with $A \in \mathcal{A}_{AP,h}(\mathbb{Z})$.

For instance, all Toeplitz operators with continuous generating functions lie in this algebra.

$\Rightarrow$

- $\mathcal{K}(l^2(\mathbb{Z}^+)) \subset \mathcal{A}_{AP,h}(\mathbb{Z}^+)$ and
- $\mathcal{A}_{AP,h}(\mathbb{Z}^+) = \{T(A) : A \in \mathcal{A}_{AP,h}(\mathbb{Z})\} \dot{+} \mathcal{K}(l^2(\mathbb{Z}^+))$.

Let us turn over to finite sections. For, let $\mathcal{F}_h$ denote the set of all bounded sequences (A_n) of matrices $A_n \in \mathbb{C}^{h(n)\times h(n)}$. Provided with pointwise defined operations and the supremum norm, $\mathcal{F}_h$ becomes a C^*-algebra ($\|A_n\|$ – norm of the operator defined by A_n on im $P_{h(n)}$).

Finally, we let $S_{AP,h}(\mathbb{Z}^+)$ denote the smallest closed subalgebra of $\mathcal{F}_h$ which contains all sequences $(P_{h(n)}T(A)P_{h(n)})$ with operators $A \in \mathcal{A}_{AP,h}(\mathbb{Z})$.

Define $R_n : l^2(\mathbb{Z}^+) \to l^2(\mathbb{Z}^+), (x_n)_{n\geq 0} \mapsto (x_n, x_{n-1}, \ldots, x_0, 0, 0, \ldots)$.

Theorem 16: The C^*-algebra $S_{AP,h}(\mathbb{Z}^+)$ consists exactly of all sequences of the form

$$(P_{h(n)}T(A)P_{h(n)} + P_{h(n)}KP_{h(n)} + R_{h(n)}LR_{h(n)} + C_{h(n)}) \tag{6.2}$$

with $A \in \mathcal{A}_{AP,h}(\mathbb{Z})$, $K, L \in \mathcal{K}(l^2(\mathbb{Z}^+))$, $\|C_{h(n)}\| \to 0$ as $n \to \infty$, and each sequence in $S_{AP,h}(\mathbb{Z}^+)$ can be written in the form (2) in a unique way.

Define mappings $W, \tilde{W} : S_{AP,h}(\mathbb{Z}^+) \to \mathcal{A}_{AP,h}(\mathbb{Z}^+)$ by

$$\begin{aligned} W(A_n) &= s - \lim P_{h(n)}A_nP_{h(n)}, \\ \tilde{W}(A_n) &= s - \lim R_{h(n)}A_nR_{h(n)}. \end{aligned}$$

These mappings are well-defined C^*-homomorphisms. Their importance is given by the following stability theorem

Theorem 17: A sequence $(A_n) \in S_{AP,h}(\mathbb{Z}^+)$ is stable $\Leftrightarrow$ the operators $W(A_n), \tilde{W}(A_n)$ are invertible, that is, if

$$A_n = P_{h(n)}T(A)P_{h(n)} + P_{h(n)}KP_{h(n)} + R_{h(n)}LR_{h(n)} + C_{h(n)},$$
$$(K, L, C_{h(n)} \text{ as above})$$

then (A_n) is stable $\Leftrightarrow$ $T(A) + K$, $T(\tilde{A}) + L$ are invertible. Moreover, $S_{AP,h}(\mathbb{Z}_+)$ is fractal.

The proof is basically the same as in the Toeplitz case.

6.4. Spectral approximations

The last theorem in Section 6.3 is one of the keys to study spectral approximations.

Theorem 18: Let $\mathbf{A} := (A_n) \in S_{AP,h}(\mathbb{Z}^+)$ be a self-adjoint sequence. Then the spectra $sp\, A_n$ converges in the Hausdorff metric to $sp\, W(\mathbf{A}) \cup sp\, \tilde{W}(\mathbf{A})$.

Theorem 19: Let $\mathbf{A} := (A_n) \in S_{AP,h}(\mathbb{Z}^+)$. Then the set of the singular values $\sum(A_n)$ converges in the Hausdorff metric to $\sum(W(A)) \cup \sum(\tilde{W}(A))$.

(Hausdorff metric: $A, B \subset \mathbb{C}$ compact,
$\Rightarrow h(A, B) = \max\{ \max_{a \in A} \text{dist}\,(a, B) \max_{b \in B} \text{dist}\,(b, A) \; \} \;)$

Theorem 20: A sequence $\mathbf{A} = (A_n) \in S_{AP,h}(\mathbb{Z}^+)$ is Fredholm if and only if its strong limit $W(\mathbf{A})$ is a Fredholm operator. In this case $\tilde{W}(\mathbf{A})$ is a Fredholm operator too, and

$$\alpha(\mathbf{A}) = \dim\ker W(\mathbf{A}) + \dim\ker \tilde{\mathbf{W}}(\mathbf{A})\,;$$

moreover, $\lim_{n \to \infty} s_\alpha(A_n) = 0$.

These theorems can be completed by results concerning ε-pseudospectra and the so-called *Arveson's dichotomy* (the last for self-adjoint sequences).

Arveson's dichotomy: Given a self-adjoint sequence $\mathbf{A} := (A_n) \in S_{AP,h}(\mathbb{Z}^+)$ and an open interval $U \subset \mathbb{R}$, let $N_n(U)$ refer to the number of eigenvalues of A_n in U, counted with respect to their multiplicity. A point $\lambda \in \mathbb{R}$ is called essential for $\mathbf{A}$, if for every open interval U containing λ,

$$\lim_{n \to \infty} N_n(U) = \infty\,,$$

and $\lambda \in \mathbb{R}$ is called a transient point for $\mathbf{A}$ is there is an open interval U containing λ such that

$$\sup_n N_n(U) < \infty\,.$$

Theorem 21: Let $\mathbf{A} := (A_n) \in S_{AP,h}(\mathbb{Z}^+)$ be self adjoint, s-lim $A_n = T(A) + K$. Then every point $\lambda \in sp\, A$ is essential, and every point $\lambda \in \mathbb{R} \setminus sp\, A$ is transient for $\mathbf{A}$. Moreover, for every point $\lambda \in \mathbb{R} \setminus sp\, A$, the sequence $\mathbf{A} - \lambda\mathbf{P}(\mathbf{P} := (P_{h(n)}) \in S_{AP,h}(\mathbb{Z}^+))$ is Fredholm and there is an open interval $U \subset \mathbb{R}$ containing λ such

that $\sup\limits_n N_n(U) = \alpha(\mathbf{A} - \lambda\mathbf{P})$.
([7], Theorem 7.12)

The first assertion of Theorem 21 implies in particular that each real number is either essential or transient for $\mathbf{A}$. This property is usually referred to as the Arveson's dichotomy of that sequence.

A deep study of the finite sections for general band-dominated operators in $l^2(\mathbb{Z})$ is carried out in [14]. Let us mention also the recent book [4], where spectral properties of banded Toeplitz matrices are studied.

6.5. Test calculations

Here we shall demonstrate how the theory can be used to determine numerically the spectrum of the Almost Mathieu operator for some choices of the parameters α, λ and θ (using Matlab).

For each of the triples

$$\left(\frac{2}{5}, 2, 0\right), \left(\frac{2}{5}, 2, \frac{1}{2}\right), \left(\frac{2}{7}, 2, 0\right), \left(\frac{\sqrt{2}}{2}, 2, \frac{1}{2}\right), \left(\frac{\sqrt{5}-1}{2}, 2, \frac{1}{2}\right),$$

in place of $(\alpha, \lambda, \theta)$, we choose a distinguished sequence of the corresponding Almost Mathieu operator which depends only on

$$\alpha_j \in \left\{\frac{2}{5}, \frac{2}{7}, \frac{\sqrt{2}}{2}, \frac{\sqrt{5}-1}{2}\right\},$$

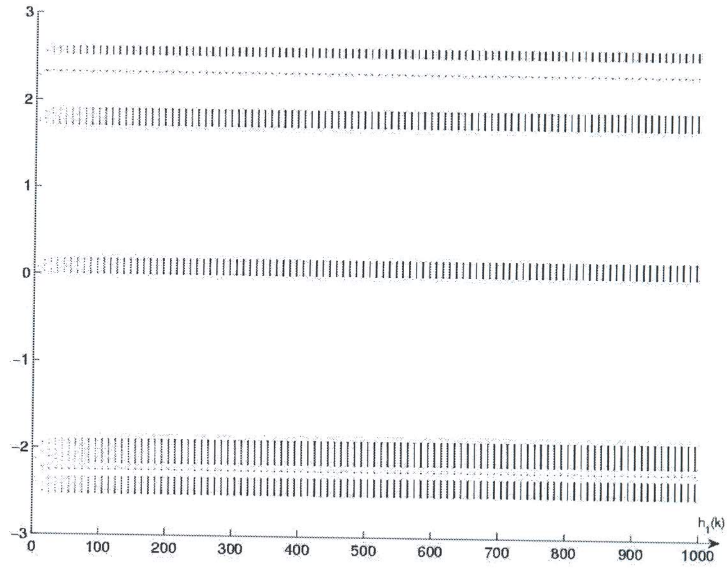
namely

$$\begin{aligned}
\alpha_1 &= \tfrac{2}{5} : h_1(k) = 5k\,, \\
\alpha_2 &= \tfrac{2}{7} : h_2(k) = 7k\,, \\
\alpha_3 &= \tfrac{\sqrt{2}}{5} : h_3(k) = \tfrac{1}{2}\left((1+\sqrt{2})^k + (1-\sqrt{2})^k\right)\,, \\
\alpha_4 &= \tfrac{\sqrt{5}-1}{2} : h_4(k) = \tfrac{5+\sqrt{5}}{10}\left(\tfrac{1+\sqrt{5}}{2}\right)^k + \tfrac{5-\sqrt{5}}{10}\left(\tfrac{1-\sqrt{5}}{2}\right)^k\,.
\end{aligned}$$

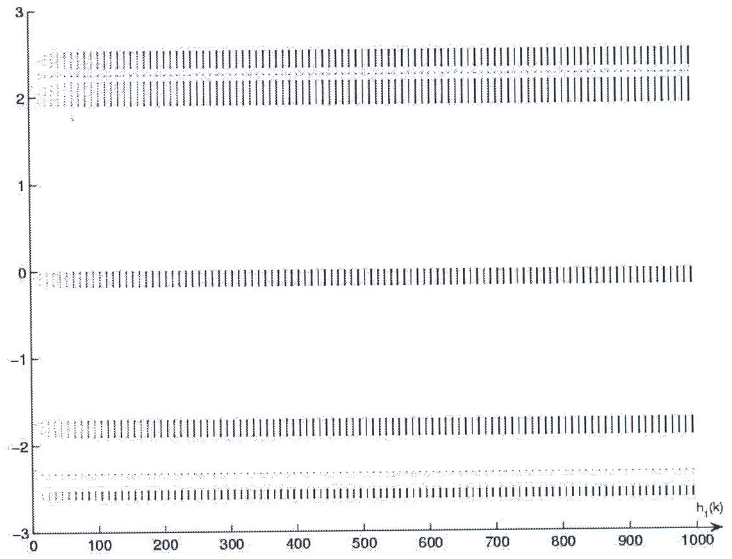
For irrational α_k, this choice has been done via continued fractions. Notice that the sequences h_3 and h_4 are rapidly growing. For instance, $h_3(13) = 47321$, $h_4(23) = 46368$. The results are plotted in pictures 1 – 8.

The results for α_4, $\lambda = 2$, $\theta = 0{,}5$ *and* $h_5(k) := 2k$ (non-distinguished!) are plotted in picture 9.

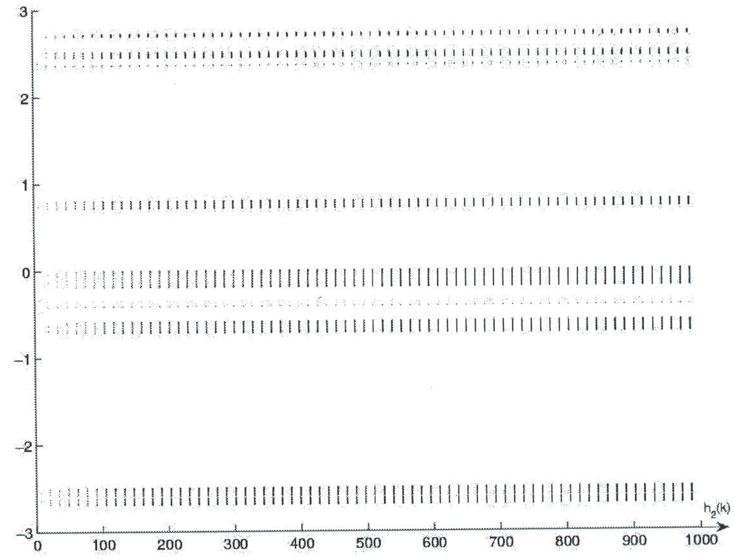
The computations clearly indicate the advantage of distinguished sequences over non-distinguished (compare the pictures 7 and 9). For irrational α the Cantor-like structure of the spectrum is also somehow reflected in the computations (see the pictures 5 and 6). The computations also show that the speed of converges is very high. There is only a guess why it should be, but not a proof.



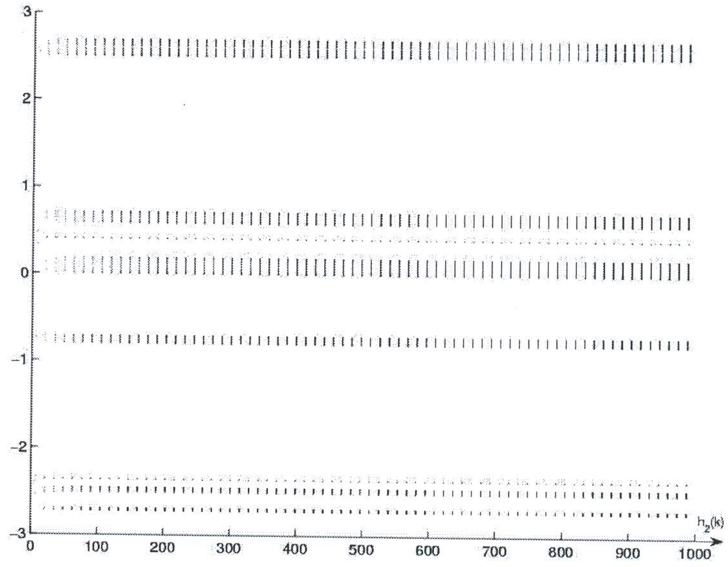
Picture 1: Eigenvalues of $P_{h_1}(k)H_{\alpha,\lambda,\theta}P_{h_1}(k)$ with $\alpha = 2/5, \lambda = 2, \theta = 0$.



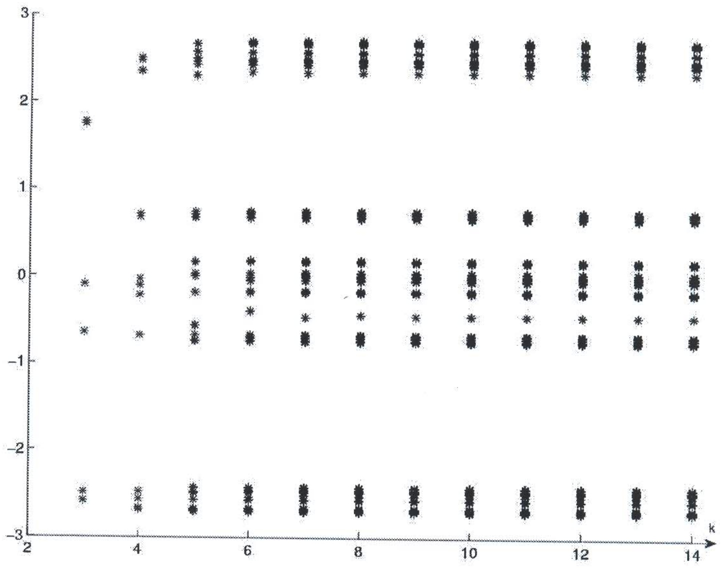
Picture 2: Eigenvalues of $P_{h_1}(k)H_{\alpha,\lambda,\theta}P_{h_1}(k)$ with $\alpha = 2/5, \lambda = 2, \theta = 1/2$.



Picture 3: Eigenvalues of $P_{h_2}(k)H_{\alpha,\lambda,\theta}P_{h_2}(k)$ with $\alpha = 2/7, \lambda = 2, \theta = 0$.



Picture 4: Eigenvalues of $P_{h_2}(k)H_{\alpha,\lambda,\theta}P_{h_2}(k)$ with $\alpha = 2/7, \lambda = 2, \theta = 1/2$.



Picture 5: Eigenvalues of $P_{h_3}(k)H_{\alpha,\lambda,\theta}P_{h_3}(k)$ with $\alpha = \sqrt{2}/2, \lambda = 2, \theta = 0$.