Orthogonal Symmetric Toeplitz Matrices

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In Memory of Georgii Litvinchuk (1931 - 2006)

Abstract. We show that the number of orthogonal and symmetric Toeplitz matrices of a given order is finite and determine all these matrices. In this way we also obtain a description of the set of all symmetric Toeplitz matrices whose spectrum is a prescribed doubleton.

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1. Introduction and main result

Georgii Semyonovich Litvinchuk’s favorites included singular integral equations with Carleman shifts, that is, with diffeomorphisms $\alpha$ for which the $m$th iterate $\alpha^m$ is the identity map. My favorites are Toeplitz matrices and hence, when receiving the invitation to contribute to this volume, I thought it might perhaps be an interesting problem to look for all Toeplitz matrices $A$ whose $m$th power $A^m$ is equal to the identity matrix $I$.

The spectrum of an infinite Toeplitz matrix $A$ that generates a bounded operator on $\ell^2$ is always connected [6]. Consequently, such a matrix satisfies $A^m = I$ if and only if $A = e^{2\pi i j/m} I$ for some $j \in \{0, 1, \ldots, m-1\}$, and thus the question is not interesting for infinite matrices.

So let us consider finite matrices. We denote by $\mathbb{R}^{n \times n}$ the set of all real $n \times n$ matrices and define $O_n$, $S_n$, $T_n$ as the subsets of $\mathbb{R}^{n \times n}$ constituted by the orthogonal, symmetric, and Toeplitz matrices, respectively. We also put

$$OS_n = O_n \cap S_n, \quad OT_n = O_n \cap T_n, \quad ST_n = S_n \cap T_n, \quad OST_n = O_n \cap S_n \cap T_n.$$
The number of elements of a finite set $E$ will be denoted by $|E|$. Throughout the paper we assume that $n \geq 2$. The following three simple propositions will be proved in Section 2.

**Proposition 1.1.** For $m \geq 2$, there are uncountably many matrices $A \in T_n$ such that $A^m = I$ and $A^k \neq I$ for $1 \leq k \leq m - 1$.

The infinite set we encounter in Proposition 1.1 seems to have no nice description. Well, letting $\omega_m := e^{2\pi i/m}$ and $T_m := \{1, \omega_m, \ldots, \omega_m^{m-1}\}$, one could describe the set as the set of all diagonalizable Toeplitz matrices with eigenvalues in $T_m \setminus \cup_{1 \leq k \leq m-1} T_k$, but this is not what I understand by nice. However, the problem becomes charming when restricting the search to symmetric matrices.

**Proposition 1.2.** (a) For $m \geq 3$, there is no $A \in S_n$ such that $A^m = I$ and $A^k \neq I$ for $1 \leq k \leq m - 1$.

(b) The set of all $A \in S_n$ for which $A^2 = I$ but $A \neq I$ coincides with $OS_n \setminus \{I\}$.

This result shows that if $A \in ST_n$, then $A^m = I$ can only happen if $A$ is in $OST_n$, in which case $A^2 = I$. We are thus led to the set $OST_n = T_n \cap OS_n = ST_n \cap OT_n$.

**Proposition 1.3.** The sets $T_n$, $OS_n$, $ST_n$, $OT_n$ are all uncountably infinite.

In spite of this proposition, the following fact, which is our main result, is quite remarkable.

**Theorem 1.4.** The set $OST_n$ is finite and

$$|OST_n| = \begin{cases} 3 \cdot 2^{n/2} - 2 & \text{if } n \text{ is even}, \\ 2\sqrt{2} \cdot 2^{n/2} - 2 & \text{if } n \text{ is odd}. \end{cases}$$

Our proof of Theorem 1.4 is constructive and will yield all matrices in $OST_n$. Here is a simple consequence of Theorem 1.4 that concerns the inverse eigenvalue problem for Toeplitz matrices.

**Corollary 1.5.** Let $\alpha$ and $\beta$ be two prescribed distinct real numbers. The number of all matrices in $ST_n$ which have both $\alpha$ and $\beta$ as eigenvalues and no other eigenvalues is $3 \cdot 2^{n/2} - 2$ if $n$ is even and $2\sqrt{2} \cdot 2^{n/2} - 4$ if $n$ is odd.

The paper is organized as follows. Section 2 contains the proofs of Propositions 1.1 to 1.3. In Section 3 we show that every matrix in $OST_n$ is a circulant or a skew circulant. This is enough to conclude that $OST_n$ is finite. Theorem 1.4 is proved in Section 4, examples revealing the structure of the matrices in $OST_n$ are given in Section 5, and Corollary 1.5 is the subject of Section 6.
2. Proofs of the surrounding results

We denote by \( \text{circ} (a_1, \ldots, a_n) \) the \( n \times n \) circulant matrix whose first column is \( (a_1 \ldots a_n)^\top \) and by \( F_n \) the \( n \times n \) Fourier matrix,

\[
F_n = (\omega_n^{(j-1)(k-1)})_{j,k=1}^n = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & \omega_n & \omega_n^2 & \ldots & \omega_n^{n-1} \\
: & : & : & \ldots & : \\
1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \ldots & \omega_n^{(n-1)(n-1)}
\end{pmatrix},
\]

where \( \omega_n := e^{2\pi i/n} \). It is well known [2] that the eigenvalues of the circulant \( \text{circ} (a_1, \ldots, a_n) \) are \( a(1), a(\omega_n), \ldots, a(\omega_n^{n-1}) \) where

\[
a(z) := a_1 + a_2 z + \ldots + a_n z^{n-1}.
\]

The equations \( a(\omega_n^{j-1}) = \mu_j \) \( (j = 1, \ldots, n) \) may be written in the form

\[
F_n \begin{pmatrix}
a_1 \\
: \\
\vdots \\
a_n
\end{pmatrix} = \begin{pmatrix}
\mu_1 \\
\vdots \\
\mu_n
\end{pmatrix}
\]

or, equivalently,

\[
n \begin{pmatrix}
a_1 \\
: \\
\vdots \\
a_n
\end{pmatrix} = F_n^* \begin{pmatrix}
\mu_1 \\
\vdots \\
\mu_n
\end{pmatrix}.
\]

**Lemma 2.1.** Let \( \alpha, \beta \in \mathbb{R} \) and \( \lambda_1, \ldots, \lambda_k \in \mathbb{C} \). If \( n = 2k + 2 \), there exists a circulant \( C \in \mathbb{R}^{n \times n} \) having the eigenvalues \( \alpha, \beta, \lambda_1, \ldots, \lambda_k, \bar{\lambda}_1, \ldots, \bar{\lambda}_k \), and if \( n = 2k + 1 \), there exists a circulant \( C \in \mathbb{R}^{n \times n} \) whose eigenvalues are \( \alpha, \lambda_1, \ldots, \lambda_k, \bar{\lambda}_1, \ldots, \bar{\lambda}_k \).

**Proof.** Let first \( n = 2k + 2 \). We define \( a_1, \ldots, a_n \) by (2) with

\[
(\mu_1, \ldots, \mu_n) = (\alpha, \lambda_1, \ldots, \lambda_k, \beta, \bar{\lambda}_1, \ldots, \bar{\lambda}_k).
\]

It remains to show that \( a_1, \ldots, a_n \) are real. But, for \( j = 0, 1, \ldots, n-1 \),

\[
na_{j+1} = \alpha + \bar{\omega}_n^{(j+1)}\beta + \sum_{p=1}^k (\bar{\omega}_n^p \lambda_p + \omega_n^{(n-p+1)} \bar{\lambda}_p) = \alpha - \beta + \sum_{p=1}^k 2 \Re(\bar{\omega}_n^p \lambda_p)
\]

and this is a real number. If \( n = 2k + 1 \), we take

\[
(\mu_1, \ldots, \mu_n) = (\alpha, \lambda_1, \ldots, \lambda_k, \bar{\lambda}_1, \ldots, \bar{\lambda}_k)
\]

and define \( a_1, \ldots, a_n \) by (2). The same argument as before shows that these are real numbers. \( \square \)

**Proof of Proposition 1.1.** If \( n \geq 3 \), Lemma 2.1 yields the existence of a circulant \( C \in \mathbb{R}^{n \times n} \) with the eigenvalues \( \omega_n, \bar{\omega}_n, 1, \ldots, 1 \). It follows that \( C^m = I \) and \( C^k \neq I \) for \( 1 \leq k \leq m-1 \). For \( \mu \in \mathbb{R} \setminus \{0\} \), put \( D_\mu = \text{diag} (1, \mu, \ldots, \mu^{n-1}) \). Then \( A = D_\mu C D_\mu^{-1} \) is a matrix in \( T_n \) for which \( A^m = I \) and \( A^k \neq I \) for \( 1 \leq k \leq m-1 \).
As different \( \mu \)'s produce different \( A \)'s we get the assertion. We are left with the case \( n = 2 \). The eigenvalues of

\[
A = \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in T_2
\]

are \( a \pm \sqrt{bc} \). Letting \( a, b, c \) be any real numbers such that \( a = \cos(2\pi/m) \) and \( bc = -\sin^2(2\pi/m) \), we obtain uncountably many matrices \( A \) with the eigenvalues \( \omega_m \) and \( \overline{\omega}_m \) and thus with the desired property that \( A^m = I \) and \( A^k \neq I \) for \( 1 \leq k \leq m - 1 \).

**Proof of Proposition 1.2.** (a) Let \( m \geq 3 \) and \( A \in S_n \). Then \( A = U^T DU \) with an orthogonal matrix \( U \) and a diagonal matrix \( D \) which contains all eigenvalues of \( A \). If \( A^m = I \) then \( \text{sp} A \), the set of the eigenvalues of \( A \), is contained in \( \mathbb{T}_m \cap \mathbb{R} \). Since \( \mathbb{T}_m \cap \mathbb{R} = \{1\} \) for odd \( m \), we see that \( A = I \) in this case, which contradicts the requirement that \( A^k \neq I \) for \( 1 \leq k \leq m - 1 \). If \( m \) is even, we have \( \mathbb{T}_m \cap \mathbb{R} = \{-1, 1\} \) and hence \( A^2 = I \). As necessarily \( m \geq 4 \), this is again a contradiction to the requirement that \( A^k \neq I \) for \( 1 \leq k \leq m - 1 \).

(b) Let \( A \in S_n \setminus \{I\} \) and \( A^2 = I \). Write \( A = U^T DU \) as above. The entries of \( D \) are all \(-1\) or \(1\) and hence \( D \) is orthogonal. This implies that \( A \) is orthogonal, too. Conversely, if \( A \in OS_n \setminus \{I\} \) then \( \text{sp} A \subset \mathbb{T} \cap \mathbb{R} = \{-1, 1\} \), which shows that each diagonal entry of \( D \) is \(-1\) or \(1\). It follows that \( A^2 = I \).

**Proof of Proposition 1.3.** The assertion is trivial for \( T_n \) and \( ST_n \). Since each matrix of the form \( U^T DU \) with \( U \in O_n \) and a diagonal matrix with \( \text{sp} D \subset \{-1, 1\} \) is in \( OS_n \), we see that \( OS_n \) is uncountably infinite. Lemma 2.1 tells us that if \( \tau \in \mathbb{T} \) is any given number, then there is a circulant \( C \in \mathbb{R}^{n \times n} \) (\( n \geq 3 \)) having the eigenvalues \( \tau, \tau^2, 1, \ldots, 1 \). Since \( C \in OT_n \), we conclude that \( OT_n \) is uncountably infinite for \( n \geq 3 \). Finally, the matrix

\[
\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in T_2
\]

is orthogonal whenever \( a^2 + b^2 = 1 \). This shows that \( OT_2 \) is an uncountable set. \( \square \)

**Remark 2.2.** Let \( UHT_n \) be the set of all unitary Hermitian Toeplitz matrices in \( \mathbb{C}^{n \times n} \). The symmetric Toeplitz matrix \( B \) with the top row

\[
\begin{pmatrix} a/n & -\frac{2}{n} & \cdots & -\frac{2}{n} \end{pmatrix}
\]

belongs to \( OST_n \). Let \( D_\mu = \text{diag}(1, \mu, \ldots, \mu^{n-1}) \). If \( \mu \in \mathbb{T} \), then \( D_\mu \) is unitary and hence \( D_\mu BD_\mu^{-1} \) is in \( UHT_n \). This shows that \( UHT_n \) is uncountably infinite. The matrices in \( OST_2 \) are

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},
\]

and it is easily seen that \( UHT_2 \) equals

\[
\left\{ \begin{pmatrix} 0 & \mu \\ \overline{\mu} & 0 \end{pmatrix} : \mu \in \mathbb{T} \right\} \cup \left\{ \begin{pmatrix} a & i\sqrt{1-a^2} \\ -i\sqrt{1-a^2} & a \end{pmatrix} : a \in [-1,1] \setminus \{0\} \right\}.
\]
This reveals that not every matrix in $UHT_2$ is of the form $D_\mu AD_\mu^{-1}$ with $A \in OST_n$ and $\mu \in T$.

3. The set $OST_n$ is finite

We denote by $T(a_1, \ldots, a_n)$ the symmetric Toeplitz matrix whose top row is $(a_1 \ldots a_n)$. Let $A = T(a_1, \ldots, a_n)$ be a matrix in $OST_n$.

Suppose first that $n = 2k + 1$ is odd. Consider, for example,

$$T(a, b, c, d, e) = \begin{pmatrix}
  a & b & c & d & e \\
  b & a & b & c & d \\
  c & b & a & b & c \\
  d & c & b & a & b \\
  e & d & c & b & a \\
\end{pmatrix}.$$ 

Since each row has $\ell^2$ norm 1, it follows that $c^2 = b^2$ and $d^2 = c^2$. Thus, $d = \delta c$ and $e = \varepsilon b$ with $\varepsilon, \delta \in \{-1, 1\}$ and therefore

$$T(a, b, c, d, e) = T(a, b, c, \delta c, \varepsilon b).$$

In the general case we see in this way that $T(a_1, \ldots, a_n)$ must be of the form

$$A = T(a_1, \ldots, a_n) = T(a, b_1, \ldots, b_k, \varepsilon_k b_k, \ldots, \varepsilon_1 b_1)$$

with $\varepsilon_j \in \{-1, 1\}$. If $\varepsilon_1 = \ldots = \varepsilon_k = 1$, then (3) is a circulant. For instance,

$$T(a, b, c, c, b) = \begin{pmatrix}
  a & b & c & c & b \\
  b & a & b & c & c \\
  c & b & a & b & c \\
  c & c & b & a & b \\
  b & c & c & b & a \\
\end{pmatrix}$$

is a circulant. If $\varepsilon_j = -1$ and $b_j = 0$, we change $\varepsilon_j$ to 1. This does not change $A$. Thus, we may assume that $b_j \neq 0$ whenever $\varepsilon_j = -1$.

Lemma 3.1. If $-1 \in \{\varepsilon_1, \ldots, \varepsilon_k\}$, then $b_j = 0$ whenever $\varepsilon_j = 1$.

Proof. We denote the rows of $A$ by $r_1, \ldots, r_{2k+1}$. Since $A$ is orthogonal, the scalar product $(r_i, r_j)$ is zero for $i \neq j$. Let $\varepsilon_i = -1$ and thus $b_i \neq 0$ for some $i$.

Step 1. We show that if $i \geq 2$ and $\varepsilon_{i-1} = 1$, then $b_{i-1} = 0$. Let us first consider the example

$$A = T(a, b, c, d, e, f, -f, e, d, c, -b)$$

in which $\varepsilon_1 = -1, \varepsilon_2 = 1, \varepsilon_3 = 1, \varepsilon_4 = 1, \varepsilon_5 = -1, f \neq 0, b \neq 0$. We have $i = 5$ and we want to show that $e = 0$. The rows 4 to 6 of (4) are

$$\begin{pmatrix}
  r_4 \\
  r_5 \\
  r_6
\end{pmatrix} = \begin{pmatrix}
  d & c & b & a & b & c & d & e & f & -f & e \\
  e & d & c & b & a & b & c & d & e & f & -f \\
  f & e & d & c & b & a & b & c & d & e & f
\end{pmatrix}.$$
Thus,

\[
0 = (r_4, r_5) = (de + \ldots + (-f)f) + e(-f),
\]

\[
0 = (r_5, r_6) = (de + \ldots + (-f)f) + ef,
\]

which gives \( e = 0 \) as desired because \( f \neq 0 \). In the general case we have

\[
0 = (r_{i-1}, r_i) = \Sigma + b_{i-1}(-b_i), \quad 0 = (r_i, r_{i+1}) = \Sigma + b_{i-1}b_i,
\]

which implies that \( b_{i-1} = 0 \). Analogously one can show that \( b_{i+1} = 0 \) if \( i \leq k - 2 \) and \( \varepsilon_{i+1} = 1 \).

**Step 2.** Suppose \( i \geq 3 \) and consider \( b_{i-2} \). If \( \varepsilon_{i-2} = -1 \), we have nothing to prove. So assume that \( \varepsilon_{i-2} = 1 \). We want to show that \( b_{i-2} = 0 \). If \( \varepsilon_{i-1} = -1 \), then \( b_{i-2} = 0 \) by Step 1. \( b_{i-1} = 0 \) from Step 1 we know that \( b_{i-1} = 0 \).

Example (4) for \( i = 5 \) illustrates just this situation. We have (with \( e = 0 \))

\[
\begin{pmatrix}
  r_3 \\
  r_4 \\
  r_5 \\
  r_6
\end{pmatrix} =
\begin{pmatrix}
  c & b & a & b & c & d & 0 & f & -f & 0 & d \\
  d & c & b & a & b & c & d & 0 & f & -f & 0 \\
  0 & d & c & b & a & b & c & d & 0 & f & -f \\
  f & 0 & d & c & b & a & b & c & d & 0 & f
\end{pmatrix}
\]

and hence

\[
0 = (r_3, r_5) = (bd + \ldots + fd) + d(-f),
\]

\[
0 = (r_4, r_6) = (bd + \ldots + fd) + df,
\]

which yields \( d = 0 \) as desired. In the general case,

\[
0 = (r_{i-2}, r_i) = \Sigma + b_{i-2}(-b_i), \quad 0 = (r_i, r_{i+1}) = \Sigma + b_{i-2}b_i
\]

and thus \( b_{i-2} = 0 \). Similarly one gets \( b_{i+2} = 0 \) if \( i \leq k - 3 \) and \( \varepsilon_{i+2} = 1 \).

**Step 3.** Continuing as above we see that \( b_{i\pm \ell} = 0 \) whenever \( i \pm \ell \in \{1, \ldots, k\} \) and \( \varepsilon_{i\pm \ell} = -1 \).

Lemma 3.1 implies that if at least one of the numbers \( \varepsilon_1, \ldots, \varepsilon_k \) is \( -1 \), then \( A \) is a skew circulant, that is, a matrix that results from a circulant by multiplying all entries below the main diagonal by \(-1\). For example,

\[
T(a, 0, c, -c, 0) =
\begin{pmatrix}
  a & 0 & c & -c & 0 \\
  0 & a & 0 & c & -c \\
  c & 0 & a & 0 & c \\
  -c & c & 0 & a & 0 \\
  0 & -c & c & 0 & a
\end{pmatrix}
\]

is a skew circulant. Let \( \text{scirc} (a_1, \ldots, a_n) \) be the skew circulant whose first column is \( (a_1 \ldots a_n)^\top \). Thus,

\[
\text{scirc} (a_1, a_2, a_3) =
\begin{pmatrix}
  a_1 & -a_3 & -a_2 \\
  a_2 & a_1 & -a_3 \\
  a_3 & a_2 & a_1
\end{pmatrix}
\].
The eigenvalues of $\text{scirc}(a_1, \ldots, a_n)$ are $a(\sigma_n)$, $a(\sigma_n \omega_n)$, \ldots, $a(\sigma_n \omega_n^{n-1})$ where $\sigma_n = e^{\pi i/n}$ and $\omega_n$ and $a(z)$ are as above (see [2]). The equations $a(\sigma_n \omega_n^{j-1}) = \mu_j$ $(j = 1, \ldots, n)$ now take the form

$$F_n \begin{pmatrix} a_1 \\ \sigma_n a_2 \\ \vdots \\ \sigma_n^{n-1} a_n \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix},$$

(5)

which is equivalent to

$$F_n \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = F_n^* \begin{pmatrix} \mu_1 \\ \sigma_n \mu_2 \\ \vdots \\ \sigma_n^{n-1} \mu_n \end{pmatrix},$$

(6)

At this point we are in a position to prove the following weakened version of Theorem 1.4.

**Proposition 3.2.** If $n$ is odd, then $\text{OST}_n$ is a finite set and $|\text{OST}_n| \leq 2^{n+1} - 2$.

**Proof.** We have seen that a matrix $A$ in $\text{OST}_n$ is a circulant or a skew circulant. The eigenvalues of $A$ belong to $\{-1, 1\}$. Thus, if $A = \text{circ}(a_1, \ldots, a_n)$, then $a_1, \ldots, a_n$ are given by (2) with each $\mu_j$ being $-1$ or $1$, and if $A = \text{scirc}(a_1, \ldots, a_n)$, then $a_1, \ldots, a_n$ are determined by (6) with each $\mu_j$ in $\{-1, 1\}$. The two matrices $I$ and $-I$ are counted both as a circulant and as a skew circulant. This gives at most $2 \cdot 2^n - 2$ solutions. (That $|\text{OST}_n|$ will actually turn out to be much smaller than $2^{n+1} - 2$ is due to the circumstance that not every right-hand side of (2) and (6) with $\mu_j \in \{-1, 1\}$ gives a left-hand side with real numbers.)

Now suppose $n = 2k + 2 \geq 2$ is even. The case $k = 0$ was disposed of in Remark 2.2, where we observed that $\text{OST}_2$ consists of exactly four matrices. So let $k \geq 1$. We denote the rows of $A$ by $r_1, \ldots, r_{2k+2}$. Consideration of the scalar products $(r_j, r_j) = 1$ shows that $A$ is of the form

$$A = T(a_1, \ldots, a_n) = T(a, b_1, \ldots, b_k, c, b_k, \ldots, b_1)$$

with $\varepsilon_j \in \{-1, 1\}$. Again we will assume without loss of generality that $\varepsilon_j = 1$ if $b_j = 0$. If all $\varepsilon_j$ are 1, then $A$ is a circulant. For example,

$$T(a, b, d, c, d, b) = \begin{pmatrix} a & b & d & c & d & b \\ b & a & b & d & c & d \\ d & b & a & b & d & c \\ c & d & b & a & b & d \\ d & c & b & a & b & d \\ b & d & c & d & b & a \end{pmatrix}$$

is a circulant. Thus, assume there is a $-1$ among $\varepsilon_1, \ldots, \varepsilon_k$.

**Lemma 3.3.** Let $-1 \in \{\varepsilon_1, \ldots, \varepsilon_k\}$. Then $c = 0$ and $b_j = 0$ whenever $\varepsilon_j = 1$. 
Proof. It can be shown as in the proof of Lemma 3.1 that \( b_j = 0 \) if \( \varepsilon_j = 1 \). Let us prove that \( c = 0 \). Again we first do an example. If

\[
A = T(a, 0, d, c, -d, 0) = \left( \begin{array}{cccccc}
  a & 0 & d & c & -d & 0 \\
  0 & a & 0 & d & c & -d \\
  d & 0 & a & 0 & d & c \\
  c & d & 0 & a & 0 & d \\
 -d & c & d & 0 & a & 0 \\
 b & -d & c & d & 0 & a \\
\end{array} \right),
\]

with \( d \neq 0 \), then \( 0 = (r_3, r_4) = 2cd = 0 \) gives \( c = 0 \). In the general case the argument is as follows. If \( \varepsilon_k = -1 \) and hence \( b_k \neq 0 \), then

\[
(r_k, r_{k+1}) = (r_{k+1}, r_{k+2}) - 2cb_k,
\]

and since \( (r_k, r_{k+1}) = (r_{k+1}, r_{k+2}) = 0 \), it follows that \( c = 0 \). So let \( \varepsilon_k = 1 \) and thus \( b_k = 0 \). If \( \varepsilon_{k-1} = -1 \), then \( b_{k-1} \neq 0 \) and

\[
(r_{k-1}, r_{k+1}) = (r_k, r_{k+2}) - 2cb_{k-1},
\]

which gives \( c = 0 \) as before. If \( \varepsilon_k = 1 \), \( b_k = 0 \), \( \varepsilon_{k-1} = 1 \), \( b_{k-1} = 0 \), \( \varepsilon_{k-2} = -1 \), \( b_{k-2} \neq 0 \) we have

\[
(r_{k-2}, r_{k+1}) = (r_{k-1}, r_{k+2}) - 2cb_{k-2}
\]

and so on. Eventually we get \( c = 0 \). □

Lemma 3.3 reveals that if \(-1 \in \{\varepsilon_1, \ldots, \varepsilon_k\}\), then \( A \) is a skew circulant. For example,

\[
T(a, b, d, 0, -d, -b) = \left( \begin{array}{cccccc}
  a & b & d & 0 & -d & -b \\
  b & a & b & d & 0 & -d \\
  d & b & a & b & d & 0 \\
  0 & d & b & a & b & d \\
 -d & 0 & d & b & a & b \\
 -b & -d & 0 & d & b & a \\
\end{array} \right)
\]

is a skew circulant.

**Proposition 3.4.** If \( n \) is even, then \( \text{OST}_n \) is finite and \( |\text{OST}_n| \leq 2^{n+1} - 2 \).

**Proof.** Proceed as in the proof of Proposition 3.2. □

### 4. The matrices in \( \text{OST}_n \)

In this section we prove Theorem 1.4.

Let first \( n = 2k + 1 \geq 3 \) be odd. By the results of Section 3, a matrix \( A \) is in \( \text{OST}_n \) if and only if \( \text{sp} A \in \{-1, 1\} \) and \( A \) is a circulant of the form

\[
A = \circ(a_1, \ldots, a_n) = \circ(a, b_1, \ldots, b_k, b_k, \ldots, b_1)
\]

or a skew circulant of the form

\[
A = \scirc(a_1, \ldots, a_n) = \scirc(a, b_1, \ldots, b_k, -b_k, \ldots, -b_1)
\]
Suppose \( A \) is the circulant (7). We denote the right-hand side of (1) by
\[
(\gamma \varepsilon_1 \ldots \varepsilon_k \delta_k \ldots \delta_1)^T.
\]
The numbers \( \gamma, \varepsilon_j, \delta_j \) are all \(-1 \) or \( 1 \). Abbreviating \( \omega_n \) to \( \omega \) we then get from (1) that
\[
\varepsilon_j = a + \sum_{p=1}^{k} b_p (\omega^{pj} + \omega^{(n-p)j}) = a + \sum_{k=1}^{p} b_p (\omega^{pj} + \omega^{-pj}) ,
\]
\[
\delta_j = a + \sum_{k=1}^{p} b_p (\omega^{p(n-j)} + \omega^{(n-p)(n-j)}) = a + \sum_{k=1}^{p} b_p (\omega^{-pj} + \omega^{pj}) ,
\]
whence \( \varepsilon_j = \delta_j \) for \( 1 \leq j \leq k \). Conversely, choose \( \mu_1, \ldots, \mu_n \) in \( \{-1, 1\} \), insert
\[
(\mu_1, \ldots, \mu_n) = (\gamma, \varepsilon_1, \ldots, \varepsilon_k, \delta_k, \ldots, \delta_1)
\]
in (2), denote the column on the left of (2) by
\[
(a \ b_1 \ldots \ b_k \ c_k \ldots \ c_1)^T,
\]
and pass to complex conjugates. Clearly, \( a = \gamma + 2 \sum \varepsilon_j \) is real. Furthermore,
\[
n\bar{b}_j = \gamma + \sum_{p=1}^{k} \varepsilon_p (\omega^{pj} + \omega^{j(n-p)}) = \gamma + \sum_{p=1}^{k} \varepsilon_p (\omega^{pj} + \omega^{-pj}) ,
\]
\[
n\bar{c}_j = \gamma + \sum_{p=1}^{k} \varepsilon_p (\omega^{p(n-j)} + \omega^{j(n-j)(n-p)}) = \gamma + \sum_{p=1}^{k} \varepsilon_p (\omega^{-pj} + \omega^{pj}) .
\]
It follows that \( b_j = c_j \) for \( 1 \leq j \leq k \), and since \( \omega^{pj} + \omega^{-pj} \) is real, the numbers \( b_j \) are also real. As there are exactly \( 2^{k+1} \) choices of \( \gamma, \varepsilon_1, \ldots, \varepsilon_k \) in \( \{-1, 1\}^k \) and as different right-hand sides of (1) give different \( (a_1, \ldots, a_n) \), we conclude that \( OSTS_n \) contains exactly \( 2^{k+1} \) matrices of the form (7).

Now suppose \( A \) is of the form (8). Put \( \sigma := \sigma_n \) and notice that
\[
(1, \sigma, \ldots, \sigma^n) = (1, \sigma, \ldots, \sigma^k, -\sigma^k + 1, \ldots, -\sigma^{2k}).
\]
We denote the columns on the left and right of (5) by
\[
(a \ \sigma b_1 \ldots \sigma^k b_k - \sigma^k + 1 b_k \ldots - \sigma^{2k} b_1)^T \quad \text{and} \quad (\varepsilon_1 \ldots \varepsilon_k \gamma \delta_k \ldots \delta_1)^T,
\]
respectively. The numbers \( \gamma, \varepsilon_j, \delta_j \) are all in \( \{-1, 1\} \). From (5) we infer that
\[
\varepsilon_j = a + \sum_{p=1}^{k} b_p (\omega^{(j-1)p}\sigma^p - \omega^{(j-1)(n-p)}\sigma^{2k+1-p}) ,
\]
\[
\delta_j = a + \sum_{p=1}^{k} b_p (\omega^{(n-j)p}\sigma^p - \omega^{(n-j)(n-p)}\sigma^{2k+1-p}) .
\]
Since $\omega^n = 1$, $\sigma^{2k+1} = -1$, $\omega = \sigma^2$, we have
\[
\omega^{(j-1)p} \sigma^p = \sigma^{2j-p-2}, \quad -\omega^{(j-1)(n-p)} \sigma^{2k+1-p} = \omega^{-p(j-1)} \sigma^{-p} = \sigma^{p-2j+p}, \\
\omega^{(n-j)p} \sigma^p = \omega^{-j(p-2)}, \quad -\omega^{(n-j)(n-p)} \sigma^{2k+1-p} = \omega^{j(p-n)} \sigma^{-p} = \sigma^{2p-2j-p},
\]
which shows that $\varepsilon_j = \delta_j$ for $1 \leq j \leq k$. Conversely, choose $\gamma, \varepsilon_1, \ldots, \varepsilon_k$ in \{-1, 1\}, insert
\[
(\mu_1, \ldots, \mu_n) = (\varepsilon_1, \ldots, \varepsilon_k, \gamma, \varepsilon_k, \ldots, \varepsilon_1),
\]
in (6), write $(a_1, \ldots, a_n)$ as $(a, b_1, \ldots, b_k, c, c, \ldots, c)$, and pass to complex conjugates. It follows that $a = \gamma + 2 \sum \varepsilon_k$ is real and that
\[
\tilde{n}b_j = \sigma^j \omega^j \gamma + \sum_{p=1}^{k} \varepsilon_p \left( \sigma^j \omega^{j(p-1)} + \sigma^j \omega^{j(n-p)} \right), \\
\tilde{n}a_j = \sigma^{n-j} \omega^{n-j} \gamma + \sum_{p=1}^{k} \varepsilon_p \left( \sigma^{n-j} \omega^{n-j(p-1)} + \sigma^{n-j} \omega^{n-j(n-p)} \right).
\]
Taking into account that $\omega^n = 1$, $\sigma^{2k+1} = -1$, $\omega = \sigma^2$, we see as above that $c_j = -b_j$ for $1 \leq j \leq k$ and that $b_1, \ldots, b_k$ are real. Thus, what we obtained is a real matrix of the form (8). This proves that there are exactly $2^{k+1}$ matrices of the form (8) in $OST_n$.

The matrix $A$ is both of the form (7) and of the form (8) if and only if $b_1 = \ldots = b_k = 0$, that is, if and only if $A = I$ or $A = -I$. Thus, in summary the number of matrices in $OST_n$ is
\[
2 \cdot 2^{k+1} - 2 = 2 \cdot 2^{(n+1)/2} - 2 = 2\sqrt{2} \cdot 2^{n/2} - 2.
\]
In addition, we have proved the following.

**Theorem 4.1.** Let $n = 2k + 1 \geq 3$. The set $OST_n$ consists of the $2^{k+1}$ circulants $T(a_1, \ldots, a_n)$ which are given by (2) with $(\mu_1, \ldots, \mu_n) = (\gamma, \varepsilon_1, \ldots, \varepsilon_k, \gamma, \varepsilon_k, \ldots, \varepsilon_1)$ and
\[
(\gamma, \varepsilon_1, \ldots, \varepsilon_k) \in \{-1, 1\}^{k+1}
\]
and the $2^{k+1} - 2$ skew circulants $T(a_1, \ldots, a_n)$ which result from (6) with the choice $(\mu_1, \ldots, \mu_n) = (\varepsilon_1, \ldots, \varepsilon_k, \gamma, \varepsilon_k, \ldots, \varepsilon_1)$ and
\[
(\varepsilon_1, \ldots, \varepsilon_k, \gamma) \in \{-1, 1\}^{k+1} \setminus \{(1, 1, \ldots, 1), (-1, -1, \ldots, -1)\}.
\]

Now suppose $n = 2k + 2 \geq 4$ is even. We know from Section 3 that $A$ belongs to $OST_n$ if and only if $sp A \subset \{-1, 1\}$ and $A$ is a circulant of the form
\[
A = \text{circ} (a_1, \ldots, a_n) = \text{circ} (a, b_1, \ldots, b_k, c, b_k, \ldots, b_1)
\]
(9)
or a skew circulant of the form
\[
A = \text{scirc} (a_1, \ldots, a_n) = \text{scirc} (a, b_1, \ldots, b_k, 0, -b_k, \ldots, -b_1)
\]
(10)
Proceeding as in the case where $n$ is odd, we see that if $A$ is given by (9), then the right-hand side of (1) is
\[
(\mu_1, \ldots, \mu_n) = (\gamma, \varepsilon_1, \ldots, \varepsilon_k, \beta, \varepsilon_k, \ldots, \varepsilon_1)
\]
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with \( \gamma, \beta, \epsilon_j \in \{-1, 1\} \) and that different such right-hand sides produce different real matrices of the form (9) via (2). Analogously, a matrix of the form (10) yields

\[
(\mu_1, \ldots, \mu_n) = (\epsilon_1, \ldots, \epsilon_k, \gamma, \beta, \epsilon_k, \ldots, \epsilon_1)
\]

with \( \gamma, \beta, \epsilon_j \in \{-1, 1\} \) in (5). If we insert such a right-hand side in (6), we obtain a left-hand side with

\[
(a_1, \ldots, a_n) = (a, b_1, \ldots, b_k, d, c_k, \ldots, c_1).
\]

The numbers \( b_1, \ldots, b_k \) and \( c_1, \ldots, c_k \) are real if and only if \( \gamma = \beta \). (This is not obvious but requires a computation, which, however, is straightforward.) If \( \gamma = \beta \), then \( c_j = -b_j \) for all \( j \) and \( d = 0 \), that is, we get indeed a real matrix of the form (10). Different choices of (11) with \( \gamma = \beta \) lead to different left-hand sides of (9), and \( A = I \) and \( A = -I \) are the only matrices that are both of the form (9) and the form (10). Thus, we arrive at the conclusion that the number of matrices in \( \text{OST}_n \) is

\[
2^{k+2} + 2^{k+1} - 2 = 3 \cdot 2^{k+1} - 2 = 3 \cdot 2^{n/2} - 2
\]

and that, moreover, \( \text{OST}_n \) can be described as follows.

**Theorem 4.2.** Let \( n = 2k + 2 \geq 4 \). The set \( \text{OST}_n \) consists of the \( 2^{k+2} \) circulants \( T(a_1, \ldots, a_n) \) which result from (2) with \( (\mu_1, \ldots, \mu_n) = (\gamma, \epsilon_1, \ldots, \epsilon_k, \beta, \epsilon_k, \ldots, \epsilon_1) \) and

\[
(\gamma, \epsilon_1, \ldots, \epsilon_k, \beta) \in \{-1, 1\}^{k+2}
\]

and the \( 2^{k+1} - 2 \) skew circulants \( T(a_1, \ldots, a_n) \) which are obtained from (6) with \( (\mu_1, \ldots, \mu_n) = (\epsilon_1, \ldots, \epsilon_k, \beta, \beta, \epsilon_k, \ldots, \epsilon_1) \) and

\[
(\epsilon_1, \ldots, \epsilon_k, \beta) \in \{-1, 1\}^{k+1} \setminus \{(1, 1, \ldots, 1), (-1, -1, \ldots, -1)\}.
\]

### 5. Examples

**\( \text{OST}_7 \).** By Theorem 1.4, \( |\text{OST}_7| = 30 \). Theorems 4.1 and 4.2 in conjunction with some obvious simplifications provide us with all matrices in \( \text{OST}_7 \). Formula (2) with

\[
(\mu_1, \ldots, \mu_7) = (\gamma, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_3, \epsilon_2, \epsilon_1)
\]

and \( (\gamma, \epsilon_1, \epsilon_2, \epsilon_3) \in \{-1, 1\}^4 \) yields the 16 matrices

\[
\text{circ} (a, b_1, b_2, b_3, b_3, b_2, b_1) = T(a, b_1, b_2, b_3, b_3, b_2, b_1)
\]

with

\[
\begin{align*}
7a &= \gamma + 2\epsilon_1 + 2\epsilon_2 + 2\epsilon_3, \\
7b_1 &= \gamma + 2\epsilon_1 \cos \frac{2\pi}{7} + 2\epsilon_2 \cos \frac{4\pi}{7} + 2\epsilon_3 \cos \frac{6\pi}{7}, \\
7b_2 &= \gamma + 2\epsilon_1 \cos \frac{4\pi}{7} + 2\epsilon_2 \cos \frac{6\pi}{7} + 2\epsilon_3 \cos \frac{2\pi}{7}, \\
7b_3 &= \gamma + 2\epsilon_1 \cos \frac{6\pi}{7} + 2\epsilon_2 \cos \frac{2\pi}{7} + 2\epsilon_3 \cos \frac{4\pi}{7}.
\end{align*}
\]
The remaining 14 matrices are delivered by (6) with
\[(\mu_1, \ldots, \mu_7) = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma, \varepsilon_3, \varepsilon_2, \varepsilon_1)\]
and \((\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma) \in \{-1, 1\}^4 \setminus \{(1, 1, 1, 1), (-1, -1, -1, -1)\}\). These matrices are
\[\text{scirc}(a, b_1, b_2, b_3, -b_3, -b_2, -b_1) = T(a, b_1, b_2, b_3, -b_3, -b_2, -b_1)\]
with
\[
\begin{align*}
7a &= \gamma + 2\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3, \\
7b_1 &= -\gamma + 2\varepsilon_1 \cos \frac{\pi}{7} + 2\varepsilon_2 \cos \frac{3\pi}{7} - 2\varepsilon_3 \cos \frac{2\pi}{7}, \\
7b_2 &= \gamma + 2\varepsilon_1 \cos \frac{2\pi}{7} - 2\varepsilon_2 \cos \frac{\pi}{7} - 2\varepsilon_3 \cos \frac{3\pi}{7}, \\
7b_3 &= -\gamma + 2\varepsilon_1 \cos \frac{3\pi}{7} - 2\varepsilon_2 \cos \frac{2\pi}{7} + 2\varepsilon_3 \cos \frac{\pi}{7}.
\end{align*}
\]
This “parametrization” of \(OST_7\) indicates that our approach perfectly fits with
the problem and that any attempt to tackle the problem straightforwardly seems
be a hopeless venture. For instance, a matrix \(A\) of the form \(T(a, b, c, d, d, c, b)\)
is orthogonal if and only if
\[
\begin{align*}
a^2 + 2b^2 + 2c^2 + 2d^2 &= 1, \\
2ab + 2bc + 2cd + d^2 &= 0, \\
2ac + b^2 + 2bd + 2cd &= 0, \\
2ad + 2bc + 2bd + c^2 &= 0.
\end{align*}
\]
(Incidentally, the same system is produced when applying the results of Gu and
Patton [4] to the equation \(A^\top A - I = 0\).) Thus, we arrived at four equations with
four unknowns. This makes us expect that the set of solutions is finite. However,
proving that the system has indeed only finitely many solution and even finding
the solutions is another story. The skeptical reader is invited to try and solve this
system “directly” and to find the 16 solutions listed above.

**OST\(_2\).** We have \(|OST_2| = 4\) and \(OST_2 = \{T(\varepsilon, 0), T(0, \varepsilon) : \varepsilon \in \{-1, 1\}\}\).

**OST\(_3\).** The 6 matrices in the set \(OST_3\) are \(T(\varepsilon, 0, 0)\) and \(T(\frac{\varepsilon}{3}, \frac{\delta}{3}, -\frac{\delta}{3})\) with
\(\varepsilon, \delta \in \{-1, 1\}\).

**OST\(_4\).** The set \(OST_4\) consists of the 10 matrices
\[
T(\varepsilon, 0, 0, 0), \quad T(0, 0, \varepsilon, 0), \quad T\left(\frac{\varepsilon \delta}{2}, \frac{\varepsilon}{2}, -\frac{\delta}{2}, \frac{\varepsilon}{2}\right), \quad T\left(0, \frac{\varepsilon}{\sqrt{2}}, 0, -\frac{\varepsilon}{\sqrt{2}}\right)
\]
with \(\varepsilon, \delta \in \{-1, 1\}\).
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OST₅. A list of the 14 matrices in OST₅ is

\[
T(\varepsilon, 0, 0, 0, 0), \quad T\left(\frac{3\varepsilon}{2}, \frac{3\varepsilon}{2}, -\frac{2\varepsilon}{5}, -\frac{2\varepsilon}{5}, -\frac{2\varepsilon}{5}\right), \\
T\left(\frac{1}{5}, 1 + \frac{\gamma\sqrt{5}}{5}, \frac{1 - \frac{\gamma\sqrt{5}}{5}}{\varepsilon}, \frac{1 - \frac{\gamma\sqrt{5}}{5}}{\varepsilon}, \frac{1 + \gamma\sqrt{5}}{\varepsilon}\right)
\]

with \(\varepsilon, \delta, \gamma \in \{-1, 1\}\).

OST₆. The set OST₆ is constituted by the 22 matrices

\[
T(\varepsilon, 0, 0, 0, 0, 0), \quad T(0, 0, 0, 0, 0, 0), \\
T\left(0, \frac{2\varepsilon}{3}, 0, 0, -\frac{2\varepsilon}{3}, 0\right), \quad T\left(\frac{2\varepsilon}{3}, 0, \frac{2\varepsilon}{3}, 0, \frac{2\varepsilon}{3}, 0\right), \\
T\left(\frac{\varepsilon}{3}, \frac{\delta}{3}, \frac{\varepsilon}{3}, \frac{\delta}{3}, -\frac{2\delta}{3}, \frac{\delta}{3}\right), \quad T\left(-\frac{2\delta}{3}, \frac{\delta}{3}, -\frac{2\delta}{3}, \frac{\delta}{3}, \frac{\delta}{3}, \frac{\delta}{3}\right), \\
T\left(\frac{\varepsilon}{3}, \frac{\delta}{\sqrt{3}}, \frac{\varepsilon}{3}, \frac{\delta}{\sqrt{3}}, 0, \frac{\delta}{\sqrt{3}}, \frac{\delta}{\sqrt{3}}, 0\right)
\]

with \(\varepsilon, \delta \in \{-1, 1\}\).

6. An inverse eigenvalue problem

The inverse eigenvalue problem for symmetric Toeplitz matrices consists in finding all \(A \in ST_n\) for which \(\text{sp} A\) is a prescribed set of at most \(n\) points on \(\mathbb{R}\). The pioneering work on this problem is due to Delsarte and Genin [3], who in particular solved the problem for \(n \leq 4\). Henry Landau [5] proved that if \(E \subseteq \mathbb{R}\) is any set with \(|E| \leq n\), then there exists a matrix \(A \in ST_n\) with \(\text{sp} A = E\). In connection with the problem studied in the present paper, we mention the paper [1] by Chu and Erbrecht, which deals with the question of finding all matrices in \(ST_n\) that have two prescribed pairs of eigenvalues and eigenvectors. The following result in conjunction with Theorems 4.1 and 4.2 provides us with all matrices \(A \in ST_n\) for which \(\text{sp} A\) is a given doubleton.

Proposition. Let \(\alpha < \beta\) be two real numbers. The set of all matrices \(A \in ST_n\) with \(\text{sp} A = \{\alpha, \beta\}\) is

\[
\left\{ \frac{\alpha + \beta}{2} + \frac{\beta - \alpha}{2} OST_n' \right\}
\]

where \(OST_n' := OST_n \setminus \{I, -I\}\).

Proof. Let \(f(\lambda) := \frac{\lambda + \beta}{2} + \frac{\beta - \alpha}{2} \lambda\) and \(f^{-1}(\lambda) := \frac{2}{\beta - \alpha} \left(\lambda - \frac{\alpha + \beta}{2}\right)\). If \(A \in ST_n\) and \(\text{sp} A = \{\alpha, \beta\}\), then \(f^{-1}(A) \in ST_n\) and \(\text{sp} f^{-1}(A) = \{f^{-1}(\alpha), f^{-1}(\beta)\} = \{-1, 1\}\). It follows that \(f^{-1}(A) \in OST_n'\) and hence \(A \in f(OST_n')\). Conversely, if \(A\) is in \(f(OST_n')\) then \(A\) is in \(ST_n\) and \(\text{sp} A = \{f(-1), f(1)\} = \{\alpha, \beta\}\). \(\Box\)

Clearly this proposition and Theorem 1.4 yield Corollary 1.5.
References


