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Sequential optimality conditions in convex programming via perturbation approach

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Abstract. In this paper a necessary and sufficient sequential optimality condition without a constraint qualification for a general convex optimization problem is given in terms of the ε-subdifferential. Further, a sequential characterization of optimal solutions involving the convex subdifferential is derived using a version of the Brøndsted-Rockafellar Theorem. We prove that some results from the literature concerning sequential generalizations of the Pshenichnyi-Rockafellar Lemma are obtained as particular cases of our results. Moreover, by this general approach we succeed to improve some sequential Lagrange multiplier conditions given in the past.

Key Words. convex programming, conjugate function, ε-subdifferential, sequential optimality conditions

AMS subject classification. 90C25, 90C46, 47A55, 42A50

1 Introduction

Consider the convex optimization problem

$$(P_0) \quad \inf_{x \in C} f(x),$$

where $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is a proper, convex and lower semicontinuous function, X is a locally convex vector space and C is a convex subset of X. The Pshenichnyi-Rockafellar Lemma ([13], [14], [16]) gives a necessary and sufficient optimality condition for the problem (P_0), whenever a constraint qualification is

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fulfilled: in case $\text{dom}(f) \cap \text{int}(C) \neq \emptyset$ (or f is continuous at some $x_0 \in C \cap \text{dom}(f)$), an element $a \in \text{dom}(f) \cap C$ is an optimal solution of (P_0) if and only if $0 \in \partial f(a) + N_C(a)$, where $\partial f(a)$ is the convex subdifferential of f at a and $N_C(a)$ is the normal cone of C at a. This is a very important result in convex optimization with many applications. Nevertheless, it has some disadvantages. First of all, a can be a minimizer of f on C even if $0 \notin \partial f(a) + N_C(a)$ (because for instance the set $\partial f(a)$ could be empty; see [10] for such an example). Moreover, the constraint qualification does not always hold even in the finite dimensional case.

Consider now the convex optimization problem

$$(P_K) \quad \inf_{\substack{x \in C \\ g(x) \in -K}} f(x),$$

where C is a convex subset of a locally convex vector space X, K is a closed convex cone of another locally convex vector space Y, $f : X \to \overline{\mathbb{R}}$ is a proper and convex function and $g : X \to Y$ is a continuous K-convex function. If f is continuous at some $x_0 \in C \cap g^{-1}(-K) \cap \text{dom}(f)$ and a constraint qualification is satisfied, then $a \in C \cap g^{-1}(-K) \cap \text{dom}(f)$ solves the problem (P_K) if and only if $\exists \lambda \in K^*, \exists u \in \partial f(a), \exists v \in \partial(\lambda g)(a), \exists \omega \in N_C(a)$ such that $u + v + \omega = 0$ and $(\lambda g)(a) = 0$, where K^* is the dual cone of K. The same disadvantages arise with this result as for the one mentioned above for (P_0).

Trying to eliminate these drawbacks, many mathematicians have given optimality conditions that do not require any constraint qualification. Concerning the problem (P_0), a nice generalization of the Pshenichnyi-Rockafellar Lemma, stated in terms of a sequence of ε-subdifferentials and ε-normal cones was recently given in [10], providing a necessary and sufficient optimality condition without constraint qualifications.

For the problem (P_K), various modified Lagrange multiplier conditions without constraint qualifications have been given in the literature ([1], [2], [3], [4], [6], [8], [12]). In [15] Thibault gave a sequential form of the Lagrange multiplier condition for (P_K), in the case where K is a closed convex normal cone. Also, in [9] and [11], the authors introduced some sequential optimality conditions regarding (P_K).

The purpose of this paper is to give sequential optimality conditions without any constraint qualification for the general convex optimization problem

$$(P_\phi) \quad \inf_{x \in X} \phi(x, 0),$$

where $\phi : X \times Y \to \overline{\mathbb{R}}$, the so-called *perturbation function*, is proper, convex and lower semicontinuous, X and Y are locally convex spaces (see [7] or [16] for more details on the perturbation theory). More precisely, we show that an element $a \in \text{dom}(\phi(\cdot, 0))$ is a minimizer of $\phi(\cdot, 0)$ on X if and only if there exist sequences $\{\varepsilon_n\} \downarrow 0$ and $(x_n^*, y_n^*) \in \partial_{\varepsilon_n} \phi(a, 0)$ such that $x_n^* \to 0$ $(n \to +\infty)$, where

$\partial_{\varepsilon_n}\phi(a, 0)$ is the ε_n-subdifferential of ϕ at $(a, 0)$. This sequential characterization is obtained using the formula for the epigraph of a conjugate function written in terms of the ε-subdifferential. Combining the above condition with a version of the Brøndsted-Rockafellar Theorem, we obtain another qualification free sequential characterization of optimal solutions involving only subdifferentials.

The perturbation function ϕ plays a determinant role in the duality theory as it can be used for constructing a dual problem to a given primal optimization problem. For a particular choice of the perturbation function the dual problem will be defined by using the conjugate of ϕ.

We extend the general approach to the problems (P_0) and (P_K) which are particular cases of the more general problem (P_ϕ). For a particular choice of the function ϕ, we derive sequential optimality conditions for the optimization problem with the objective function being the sum of a proper convex and lower semicontinuous function with the composition of another proper convex and lower semicontinuous function with a continuous linear operator. The sequential generalizations of the Pshenichnyi-Rockafellar Lemma given by Jeyakumar and Wu for (P_0) (see Theorem 3.3 and Corollary 3.5 in [10]) follow as particular cases.

For an appropriate choice of the function ϕ, we also get the sequential Lagrange multiplier condition regarding the optimization problem (P_K) given by Thibault (see Theorem 4.1 in [15]). Moreover, this sequential necessary and sufficient optimality condition is established under weaker hypothesis than in [15], since we do not ask the cone K to be normal.

The paper is organized as follows. In the next section we introduce the necessary tools from convex analysis which will be used later in the paper. In section 3 we give the announced qualification free sequential necessary and sufficient optimality conditions for the convex optimization problem (P_ϕ). Finally, in section 4 we treat some particular cases of the main results, obtaining amongst others sequential optimality conditions for both convex optimization problems (P_0) and (P_K).

2 Preliminaries

Consider X a real locally convex vector space and X^* its continuous dual space endowed with an arbitrary locally convex topology τ giving X as dual. The most prominent examples of such a topology are the weak* topology $\omega(X^*, X)$ or the strong topology when X is a reflexive Banach space. We denote by $\langle x^*, x\rangle$ the value of the linear continuous functional $x^* \in X^*$ at $x \in X$. Consider the *identity function* on X, $\mathrm{id}_X : X \to X, \mathrm{id}_X(x) = x, \forall x \in X$. For $C \subseteq X$ we denote by $\mathrm{cl}(C)$ its *closure*. The *indicator function* of C, denoted by δ_C, is defined as $\delta_C : X \to \overline{\mathbb{R}}$,

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. For a function $f : X \to \overline{\mathbb{R}}$ we denote by $\text{dom}(f) = \{x \in X : f(x) < +\infty\}$ its *domain* and by $\text{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$ its *epigraph*. We call f *proper* if $\text{dom}(f) \neq \emptyset$ and $f(x) > -\infty, \forall x \in X$. By $\text{cl}(f)$ we denote the *lower semicontinuous hull* of f, namely the function of which epigraph is the closure of $\text{epi}(f)$ in $X \times \mathbb{R}$, that is $\text{epi}(\text{cl}(f)) = \text{cl}(\text{epi}(f))$. For $x \in X$ such that $f(x) \in \mathbb{R}$ we define the *ε-sudifferential* of f at x, where $\varepsilon \geq 0$, by

$$\partial_\varepsilon f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x\rangle - \varepsilon, \forall y \in X\}.$$

The set $\partial f(x) := \partial_0 f(x)$ is then the classical *subdifferential* of f at x. If f is proper then for $a \in \text{dom}(f)$ we have the following relation

$$\inf_{x \in X} f(x) = f(a) \Leftrightarrow 0 \in \partial f(a).$$

The *normal cone* of a closed set C at $x \in X$ is defined by $N_C(x) := \partial(\delta_C)(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0, \forall y \in C\}$ when $x \in C$, and $N_C(x) := \emptyset$ when $x \notin C$. For $\varepsilon \geq 0$, the *ε-normal cone* of C at $x \in X$ is defined as $N_C^\varepsilon(x) := \partial_\varepsilon(\delta_C)(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq \varepsilon, \forall y \in C\}$ if $x \in C$, and $N_C^\varepsilon(x) := \emptyset$ if $x \notin C$.

The *Fenchel-Moreau conjugate* of f is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) = \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\}, \forall x^* \in X^*.$$

We have the so called Young-Fenchel inequality

$$f^*(x^*) + f(x) \geq \langle x^*, x\rangle, \forall x \in X, \forall x^* \in X^*.$$

We mention here some important properties of conjugate functions. If f is proper, then f is convex and lower semicontinuous if and only if $f^{**} = f$ (see [7], [16]). Also, if f is convex, $\text{dom}(f) \neq \emptyset$ and $\text{cl}(f)$ is proper, then $f^{**} = \text{cl}(f)$ (Theorem 2.3.4 in [16]).

The following characterizations of the subdifferential and ε-sudifferential of a proper function f, by means of conjugate functions will be useful in the paper (see [7], [16]):

$$x^* \in \partial f(x) \Leftrightarrow f(x) + f^*(x^*) = \langle x^*, x\rangle$$

and

$$x^* \in \partial_\varepsilon f(x) \Leftrightarrow f(x) + f^*(x^*) \leq \langle x^*, x\rangle + \varepsilon.$$

In case $f : X \to \overline{\mathbb{R}}$ is a proper function and $a \in \text{dom}(f)$ the epigraph of f^* can be represented as follows

$$\text{epi}(f^*) = \bigcup_{\varepsilon \geq 0} \big\{(x^*, \langle x^*, a\rangle + \varepsilon - f(a)) : x^* \in \partial_\varepsilon f(a)\big\}. \tag{1}$$

This formula, which is an easy consequence of the definitions above, describes the epigraph of a conjugate function in terms of the ε-subdifferential of the function and will play an important role in the proof of the main results. It was stated in [8], where the function f was considered convex and lower semicontinuous, but the formula is valid even without these hypotheses.

The following version of the Brøndsted-Rockafellar Theorem ([5]) was proved in [15] and will be used for providing sequential optimality conditions written in terms of the subdifferentials of the functions involved.

Theorem 2.1 (Brøndsted-Rockafellar Theorem [5], [15]). *Let X be a Banach space, $f : X \to \overline{\mathbb{R}}$ be a proper convex and lower semicontinuous function and $a \in \operatorname{dom}(f)$. Then for every $\varepsilon > 0$ and for every $x^* \in \partial_\varepsilon f(a)$ there exist $x_\varepsilon \in X$ and $x_\varepsilon^* \in \partial f(x_\varepsilon)$ such that*

$$\|x_\varepsilon - a\| \leq \sqrt{\varepsilon},\ \|x_\varepsilon^* - x^*\|_* \leq \sqrt{\varepsilon} \text{ and } |f(x_\varepsilon) - \langle x_\varepsilon^*, x_\varepsilon - a\rangle - f(a)| \leq 2\varepsilon.$$

3 Sequential optimality conditions

Let us consider $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ two Banach spaces and $(X^*, \|\cdot\|_*)$, $(Y^*, \|\cdot\|_*)$ their continuous dual spaces. Although the spaces X, Y and X^*, Y^*, respectively, are endowed with different norms, we use the same notations for them as there will be no danger of confusion. Let $\{x_n^* : n \in \mathbb{N}\}$ be a sequence in X^*. We write $x_n^* \xrightarrow{\omega^*} 0$ ($x_n^* \xrightarrow{\|\cdot\|_*} 0$) for the case when x_n^* converges to 0 in the weak* (strong) topology. We make the following convention: if in a certain property we write $x_n^* \to 0$ ($n \to +\infty$), we understand that the property holds no matter which of the two topologies (weak* or strong) is used. The following property will be frequently used in the paper:

$$\text{if } x_n^* \to 0 \text{ and } x_n \to a\ (n \to +\infty),\ \text{then } \langle x_n^*, x_n\rangle \to 0\ (n \to +\infty),$$

where $\{x_n\} \subseteq X$, $\forall n \in \mathbb{N}$, $a \in X$ and $x_n \to a$ ($n \to +\infty$) means $\|x_n - a\| \to 0$ ($n \to +\infty$), that is the convergence in the topology induced by the norm on X. On $X \times Y$ we use the norm $\|(x, y)\| = \sqrt{\|x\|^2 + \|y\|^2}$, for $(x, y) \in X \times Y$. Similarly we define the norm on $X^* \times Y^*$.

Let $\phi : X \times Y \to \overline{\mathbb{R}}$ be a given function. In this section we give sequential optimality conditions for the general optimization problem

$$(P_\phi) \quad \inf_{x \in X} \phi(x, 0).$$

To this end we consider the *infimal value* function $\eta : X^* \to \overline{\mathbb{R}}$ of the conjugate ϕ^* defined by $\eta(x^*) = \inf_{y^* \in Y^*} \phi^*(x^*, y^*)$, for $x^* \in X^*$. Let us notice that, since ϕ^* is a convex function on $X^* \times Y^*$, η is a convex function on X^*.

Lemma 3.1 *Let* $\phi : X \times Y \to \overline{\mathbb{R}}$ *be a proper convex and lower semicontinuous function such that* $\inf_{x\in X} \phi(x, 0) < +\infty$. *Then* $a \in \mathrm{dom}(\phi(\cdot, 0))$ *is a minimizer of* $\phi(\cdot, 0)$ *on* X *if and only if* $(0, -\eta^*(a)) \in \mathrm{cl}(\mathrm{epi}(\eta))$ *(the closure is taken in* $(X^*, \omega(X^*, X)) \times \mathbb{R}$*).*

Proof. One can see that $\mathrm{dom}(\eta) \neq \emptyset$ and $\eta^*(x) = (\phi^*)^*(x, 0) = \phi(x, 0)$, $\forall x \in X$. We get that η^* is proper, hence $\mathrm{cl}(\eta)$ is also proper and $\eta^{**} = \mathrm{cl}(\eta)$. Then a is a minimizer of $\phi(\cdot, 0)$ on $X \Leftrightarrow a$ is a minimizer of η^* on $X \Leftrightarrow 0 \in \partial(\eta^*)(a) \Leftrightarrow \eta^{**}(0) + \eta^*(a) = 0 \Leftrightarrow \eta^{**}(0) + \eta^*(a) \leq 0$ (since the opposite inequality is always true). This is the same with $\mathrm{cl}(\eta)(0) = \eta^{**}(0) \leq -\eta^*(a) \Leftrightarrow (0, -\eta^*(a)) \in \mathrm{epi}(\mathrm{cl}(\eta)) = \mathrm{cl}(\mathrm{epi}(\eta))$. □

Using Lemma 3.1 and formula (1), we can give now a general optimality condition involving ε-subdifferentials.

Theorem 3.2 *Let* $\phi : X \times Y \to \overline{\mathbb{R}}$ *be a proper convex and lower semicontinuous function such that* $\inf_{x\in X} \phi(x, 0) < +\infty$. *The following statements are equivalent:*

(a) $a \in \mathrm{dom}(\phi(\cdot, 0))$ *is a minimizer of* $\phi(\cdot, 0)$ *on* X*;*

(b) there exist sequences $\{\varepsilon_n\} \downarrow 0$ *and* $(x_n^*, y_n^*) \in \partial_{\varepsilon_n}\phi(a, 0)$ *such that* $x_n^* \xrightarrow{\|\cdot\|_*} 0$ $(n \to +\infty)$*;*

(c) there exist sequences $\{\varepsilon_n\} \downarrow 0$ *and* $(x_n^*, y_n^*) \in \partial_{\varepsilon_n}\phi(a, 0)$ *such that* $x_n^* \xrightarrow{\omega^*} 0$ $(n \to +\infty)$*.*

Proof. $(a) \Rightarrow (b)$ Suppose that $a \in \mathrm{dom}(\phi(\cdot, 0))$ is a minimizer of $\phi(\cdot, 0)$ on X. Applying the previous lemma, we have $(0, -\eta^*(a)) \in \mathrm{cl}(\mathrm{epi}(\eta))$. Since η is a convex function, the closure in $(X^*, \omega(X^*, X)) \times \mathbb{R}$ and the closure in $(X^*, \|\cdot\|_*) \times \mathbb{R}$ of the set $\mathrm{epi}(\eta)$ coincide. Hence $\exists (x_n^*, r_n) \in X^* \times \mathbb{R}$ such that $\eta(x_n^*) \leq r_n, x_n^* \xrightarrow{\|\cdot\|_*} 0$ and $r_n \to -\eta^*(a)$ $(n \to +\infty)$. The inequality $\eta(x_n^*) \leq r_n$ yields $\inf_{y^*\in Y^*} \phi^*(x_n^*, y^*) < r_n + 1/n$, $\forall n \in \mathbb{N}$, so there exists a sequence $\{y_n^*\} \subseteq Y^*$ such that $\phi^*(x_n^*, y_n^*) < r_n + 1/n$, $\forall n \in \mathbb{N}$, thus $(x_n^*, y_n^*, r_n + 1/n) \in \mathrm{epi}(\phi^*)$, $\forall n \in \mathbb{N}$. As $(a, 0) \in \mathrm{dom}(\phi)$, we get by (1)

$$\mathrm{epi}(\phi^*) = \bigcup_{\varepsilon\geq 0} \{(x^*, y^*, \langle x^*, a\rangle + \varepsilon - \phi(a, 0)) : (x^*, y^*) \in \partial_\varepsilon\phi(a, 0)\}.$$

Since $(x_n^*, y_n^*, r_n + 1/n) \in \mathrm{epi}(\phi^*)$, $\forall n \in \mathbb{N}$, there exists a sequence $\{\varepsilon_n\} \subseteq \mathbb{R}_+$ such that $r_n + 1/n = \langle x_n^*, a\rangle + \varepsilon_n - \phi(a, 0)$, $(x_n^*, y_n^*) \in \partial_{\varepsilon_n}\phi(a, 0)$, $x_n^* \xrightarrow{\|\cdot\|_*} 0$ and $r_n \to -\eta^*(a)$ $(n \to +\infty)$. From the last equality we conclude that $\varepsilon_n \to 0$ $(n \to +\infty)$.

The implication $(b) \Rightarrow (c)$ follows since $x_n^* \xrightarrow{\|\cdot\|_*} 0$ $(n \to +\infty)$ implies $x_n^* \xrightarrow{\omega^*} 0$ $(n \to +\infty)$.

$(c) \Rightarrow (a)$ If there exist sequences $\{\varepsilon_n\} \downarrow 0$ and $(x_n^*, y_n^*) \in \partial_{\varepsilon_n}\phi(a, 0)$ such that $x_n^* \xrightarrow{\omega^*} 0$ $(n \to +\infty)$, then using the definition of the ε-subdifferential of a function we get

$$\phi(x, y) - \phi(a, 0) \geq \langle x_n^*, x - a\rangle + \langle y_n^*, y\rangle - \varepsilon_n, \forall (x, y) \in X \times Y, \forall n \in \mathbb{N}.$$

We obtain

$$\phi(x, 0) - \phi(a, 0) \geq \langle x_n^*, x - a\rangle - \varepsilon_n, \forall x \in X, \forall n \in \mathbb{N}.$$

Passing to the limit as $n \to +\infty$, we get $\phi(x, 0) - \phi(a, 0) \geq 0$, $\forall x \in X$, so a is a minimizer of $\phi(\cdot, 0)$ on X. □

Combining this result with the Brøndsted-Rockafellar Theorem (Theorem 2.1) we get a necessary and sufficient optimality condition by means of the classical convex subdifferential.

Theorem 3.3 *Let $\phi : X \times Y \to \overline{\mathbb{R}}$ be a proper convex and lower semicontinuous function such that $\inf_{x \in X} \phi(x, 0) < +\infty$. The following statements are equivalent:*

(a) $a \in \operatorname{dom}(\phi(\cdot, 0))$ is a minimizer of $\phi(\cdot, 0)$ on X;

(b) there exist sequences $(x_n, y_n) \in \operatorname{dom}(\phi)$, $(x_n^*, y_n^*) \in \partial\phi(x_n, y_n)$ such that

$$x_n^* \xrightarrow{\|\cdot\|_*} 0, x_n \to a, y_n \to 0 \ (n \to +\infty) \text{ and}$$

$$\phi(x_n, y_n) - \langle y_n^*, y_n\rangle - \phi(a, 0) \to 0 \ (n \to +\infty);$$

(c) there exist sequences $(x_n, y_n) \in \operatorname{dom}(\phi)$, $(x_n^*, y_n^*) \in \partial\phi(x_n, y_n)$ such that

$$x_n^* \xrightarrow{\omega^*} 0, x_n \to a, y_n \to 0 \ (n \to +\infty) \text{ and}$$

$$\phi(x_n, y_n) - \langle y_n^*, y_n\rangle - \phi(a, 0) \to 0 \ (n \to +\infty).$$

Proof. As $(b) \Rightarrow (c)$ is always true, we prove just the implications $(a) \Rightarrow (b)$ and $(c) \Rightarrow (a)$.

$(a) \Rightarrow (b)$ Suppose that $a \in \operatorname{dom}(\phi(\cdot, 0))$ is a minimizer of $\phi(\cdot, 0)$ on X. By Theorem 3.2 there exist $\{\varepsilon_n\} \downarrow 0$ and $(\overline{x_n^*}, \overline{y_n^*}) \in \partial_{\varepsilon_n}\phi(a, 0)$ such that $\overline{x_n^*} \xrightarrow{\|\cdot\|_*} 0$ $(n \to +\infty)$. Applying Theorem 2.1 we get that $\forall n \in \mathbb{N}$ there exist $(x_n, y_n) \in X \times Y$ and $(x_n^*, y_n^*) \in \partial\phi(x_n, y_n)$ such that

$$\|(x_n, y_n) - (a, 0)\| \leq \sqrt{\varepsilon_n}, \|(x_n^*, y_n^*) - (\overline{x_n^*}, \overline{y_n^*})\|_* \leq \sqrt{\varepsilon_n}$$

and

$$|\phi(x_n, y_n) - \langle (x_n^*, y_n^*), (x_n, y_n) - (a, 0)\rangle - \phi(a, 0)| \leq 2\varepsilon_n,$$

from which we obtain $x_n^* \xrightarrow{\|\cdot\|_*} 0, x_n \to a, y_n \to 0$ $(n \to +\infty)$ and $\phi(x_n, y_n) - \langle x_n^*, x_n - a\rangle - \langle y_n^*, y_n\rangle - \phi(a, 0) \to 0$ $(n \to +\infty)$. Since $\langle x_n^*, x_n - a\rangle \to 0$ $(n \to +\infty)$, the desired result follows.

$(c) \Rightarrow (a)$ Assume that there exist sequences $(x_n, y_n) \in \operatorname{dom}(\phi)$, $(x_n^*, y_n^*) \in \partial\phi(x_n, y_n)$ such that $x_n^* \xrightarrow{\omega^*} 0, x_n \to a, y_n \to 0$ $(n \to +\infty)$ and $\phi(x_n, y_n) - \langle y_n^*, y_n\rangle - \phi(a, 0) \to 0$ $(n \to +\infty)$. Since $(x_n^*, y_n^*) \in \partial\phi(x_n, y_n)$, we have $\phi(x, y) \geq \phi(x_n, y_n) + \langle (x_n^*, y_n^*), (x - x_n, y - y_n)\rangle, \forall (x, y) \in X \times Y, \forall n \in \mathbb{N}$. Then, for every $x \in X$ the following inequality is true

$$\phi(x, 0) - \phi(a, 0) \geq \phi(x_n, y_n) - \langle y_n^*, y_n\rangle - \phi(a, 0) + \langle x_n^*, x - x_n\rangle, \forall n \in \mathbb{N}.$$

Passing to the limit as $n \to +\infty$, we get $\phi(x, 0) - \phi(a, 0) \geq 0$, $\forall x \in X$, so a is a minimizer of $\phi(\cdot, 0)$ on X. □

Remark 3.4 Using the convention mentioned in the beginning of the section, the above results can be reformulated as follows.

The following assertions are equivalent:

(a) $a \in \operatorname{dom}(\phi(\cdot, 0))$ is a minimizer of $\phi(\cdot, 0)$ on X;

(b) there exist sequences $\{\varepsilon_n\} \downarrow 0$ and $(x_n^*, y_n^*) \in \partial_{\varepsilon_n}\phi(a, 0)$ such that $x_n^* \to 0$ $(n \to +\infty)$;

(c) there exist sequences $(x_n, y_n) \in \operatorname{dom}(\phi)$, $(x_n^*, y_n^*) \in \partial\phi(x_n, y_n)$ such that

$$x_n^* \to 0, x_n \to a, y_n \to 0 \ (n \to +\infty) \text{ and}$$

$$\phi(x_n, y_n) - \langle y_n^*, y_n\rangle - \phi(a, 0) \to 0 \ (n \to +\infty).$$

4 Particular cases

In this section we show that some results from the literature concerning sequential optimality conditions are obtained as particular cases of the main results presented in the previous section. In the first part we give sequential optimality conditions for the optimization problem with the objective function being the sum of a proper convex and lower semicontinuous function with the composition of another proper convex and lower semicontinuous function with a continuous linear operator, obtaining as particular cases the results given by Jeyakumar and Wu in [10] for the problem (P_0). Further, taking a different perturbation function, we improve a sequential Lagrange multiplier condition given by Thibault in [15] for the problem (P_K).

4.1 Sequential generalizations of the Pshenichnyi-Rockafellar Lemma

Let us consider $A : X \to Y$ a linear continuous mapping and $A^* : Y^* \to X^*$ its *adjoint operator*, defined in the usual way $\langle A^*y^*, x\rangle = \langle y^*, Ax\rangle, \forall (y^*, x) \in Y^* \times X$. Let $f : X \to \overline{\mathbb{R}}$, $g : Y \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous functions such that $A(\mathrm{dom}(f)) \cap \mathrm{dom}(g) \neq \emptyset$. In the following we give sequential characterizations of optimal solutions regarding the convex optimization problem

$$(P_A) \quad \inf_{x\in X}\{f(x) + (g \circ A)(x)\}.$$

To this end we define the function $\phi : X \times Y \to \overline{\mathbb{R}}$ by $\phi(x, y) = f(x) + g(Ax + y)$, for $(x, y) \in X \times Y$. A simple computation shows that $\phi^*(x^*, y^*) = f^*(x^* - A^*y^*) + g^*(y^*)$, for $(x^*, y^*) \in X^* \times Y^*$. Let us prove first the following lemma.

Lemma 4.1 *Let $(x^*, y^*) \in X^* \times Y^*$, $a \in \mathrm{dom}(f) \cap A^{-1}(\mathrm{dom}(g))$ and $\varepsilon \geq 0$ be fixed. The following statements are true*

(a) if $(x^*, y^*) \in \partial_\varepsilon \phi(a, 0)$, then $x^* - A^*y^* \in \partial_\varepsilon f(a)$ and $y^* \in \partial_\varepsilon g(Aa)$;

(b) if $x^* - A^*y^* \in \partial_\varepsilon f(a)$ and $y^* \in \partial_\varepsilon g(Aa)$, then $(x^*, y^*) \in \partial_{2\varepsilon}\phi(a, 0)$.

Proof. The pair (x^*, y^*) belongs to $\partial_\varepsilon\phi(a, 0)$ if and only if $\phi(a, 0) + \phi^*(x^*, y^*) \leq \langle x^*, a\rangle + \varepsilon \Leftrightarrow f(a) + g(Aa) + f^*(x^* - A^*y^*) + g^*(y^*) \leq \langle x^*, a\rangle + \varepsilon$.

(a) If $(x^*, y^*) \in \partial_\varepsilon\phi(a, 0)$, then $f(a)+g(Aa)+f^*(x^*-A^*y^*)+g^*(y^*) \leq \langle x^*, a\rangle + \varepsilon$. Let us suppose that $x^* - A^*y^* \notin \partial_\varepsilon f(a)$. Then $f(a) + f^*(x^* - A^*y^*) > \langle x^* - A^*y^*, a\rangle + \varepsilon$. By the Young-Fenchel inequality we have $g(Aa) + g^*(y^*) \geq \langle y^*, Aa\rangle$. Adding the last two inequalities we obtain $f(a)+g(Aa)+f^*(x^*-A^*y^*)+g^*(y^*) > \langle x^*, a\rangle + \varepsilon$, which is a contradiction. Hence $x^* - A^*y^* \in \partial_\varepsilon f(a)$ and similarly we get $y^* \in \partial_\varepsilon g(Aa)$.

(b) As $x^*-A^*y^* \in \partial_\varepsilon f(a)$ and $y^* \in \partial_\varepsilon g(Aa)$, we obtain $f(a)+f^*(x^*-A^*y^*) \leq \langle x^* - A^*y^*, a\rangle + \varepsilon$ and $g(Aa) + g^*(y^*) \leq \langle y^*, Aa\rangle + \varepsilon$. The conclusion follows by adding the last two inequalities. □

Theorem 4.2 *Let $A : X \to Y$ be a linear continuous mapping, $f : X \to \overline{\mathbb{R}}$, $g : Y \to \overline{\mathbb{R}}$ be proper, convex and lower semicontinuous functions such that $A(\mathrm{dom}(f)) \cap \mathrm{dom}(g) \neq \emptyset$. Then $a \in \mathrm{dom}(f) \cap A^{-1}(\mathrm{dom}(g))$ is a minimizer of $f + g \circ A$ on X if and only if*

$$\exists\{\varepsilon_n\} \downarrow 0, \exists x_n^* \in \partial_{\varepsilon_n} f(a), \exists y_n^* \in \partial_{\varepsilon_n} g(Aa) \text{ such that } x_n^* + A^*y_n^* \to 0 \ (n \to +\infty). \tag{2}$$

Proof. The element $a \in \mathrm{dom}(f) \cap A^{-1}(\mathrm{dom}(g))$ is a minimizer of $f + g \circ A$ on X if and only if a is a minimizer of $\phi(\cdot, 0)$ on X, which is equivalent to (see

Theorem 3.2)

$$\exists\{\varepsilon_n\}\downarrow 0, \exists(x_n^*, y_n^*)\in\partial_{\varepsilon_n}\phi(a,0) \text{ such that } x_n^*\to 0\ (n\to+\infty). \tag{3}$$

We prove that the conditions (2) and (3) are equivalent.

"(3)⇒(2)" There exist $\{\overline{\varepsilon_n}\}\downarrow 0$ and $(\overline{x_n^*}, \overline{y_n^*})\in\partial_{\overline{\varepsilon_n}}\phi(a,0)$ such that $\overline{x_n^*}\to 0$ $(n\to+\infty)$. According to Lemma 4.1(a), $\overline{x_n^*}-A^*\overline{y_n^*}\in\partial_{\overline{\varepsilon_n}}f(a)$ and $\overline{y_n^*}\in\partial_{\overline{\varepsilon_n}}g(Aa)$. If we take $\varepsilon_n:=\overline{\varepsilon_n}$, $x_n^*:=\overline{x_n^*}-A^*\overline{y_n^*}$ and $y_n^*:=\overline{y_n^*}$, then (2) is fulfilled.

"(2) ⇒(3)" There exist $\{\overline{\varepsilon_n}\}\downarrow 0$, $\overline{x_n^*}\in\partial_{\overline{\varepsilon_n}}f(a)$ and $\overline{y_n^*}\in\partial_{\overline{\varepsilon_n}}g(Aa)$ such that $\overline{x_n^*}+A^*\overline{y_n^*}\to 0$ $(n\to+\infty)$. Take $\varepsilon_n:=2\overline{\varepsilon_n}$, $x_n^*:=\overline{x_n^*}+A^*\overline{y_n^*}$ and $y_n^*:=\overline{y_n^*}$. Then $x_n^*-A^*y_n^*=\overline{x_n^*}\in\partial_{\overline{\varepsilon_n}}f(a)$ and $y_n^*=\overline{y_n^*}\in\partial_{\overline{\varepsilon_n}}g(Aa)$, hence by Lemma 4.1(b) we have $(x_n^*, y_n^*)\in\partial_{\varepsilon_n}\phi(a,0)$. Moreover, $x_n^*=\overline{x_n^*}+A^*\overline{y_n^*}\to 0$ $(n\to+\infty)$, so (3) is fulfilled. □

We derive from Theorem 3.3 a sequential optimality condition for (P_A) involving only the convex subdifferentials of the functions f and g.

Theorem 4.3 *Let $A: X\to Y$ be a linear continuous mapping, $f: X\to\overline{\mathbb{R}}$, $g: Y\to\overline{\mathbb{R}}$ be proper, convex and lower semicontinuous functions such that $A(\mathrm{dom}(f))\cap\mathrm{dom}(g)\neq\emptyset$. Then $a\in\mathrm{dom}(f)\cap A^{-1}(\mathrm{dom}(g))$ is a minimizer of $f+g\circ A$ on X if and only if*

$$\begin{cases} \exists(x_n, y_n)\in\mathrm{dom}(f)\times\mathrm{dom}(g), \exists x_n^*\in\partial f(x_n), \exists y_n^*\in\partial g(y_n) \text{ such that} \\ x_n^*+A^*y_n^*\to 0, x_n\to a, y_n\to Aa\ (n\to+\infty), \\ f(x_n)-\langle x_n^*, x_n-a\rangle - f(a)\to 0,\ (n\to+\infty) \text{ and} \\ g(y_n)-\langle y_n^*, y_n-Aa\rangle - g(Aa)\to 0\ (n\to+\infty). \end{cases} \tag{4}$$

Proof. Applying Theorem 3.3, we get that a is a minimizer of $f+g\circ A$ on X if and only if $\exists(x_n, y_n)\in X\times Y$, $x_n\in\mathrm{dom}(f)$, $Ax_n+y_n\in\mathrm{dom}(g)$, $\exists(x_n^*, y_n^*)\in\partial\phi(x_n, y_n)$ such that $x_n^*\to 0, x_n\to a, y_n\to 0$ and $\phi(x_n, y_n)-\langle y_n^*, y_n\rangle-\phi(a,0)\to 0$ $(n\to+\infty)$. The last condition is equivalent to

$$f(x_n)+g(Ax_n+y_n)-\langle y_n^*, y_n\rangle - f(a)-g(Aa)\to 0\ (n\to+\infty).$$

We have $(x_n^*, y_n^*)\in\partial\phi(x_n, y_n)$ if and only if $\phi(x_n, y_n)+\phi^*(x_n^*, y_n^*)=\langle x_n^*, x_n\rangle+\langle y_n^*, y_n\rangle \Leftrightarrow f(x_n)+g(Ax_n+y_n)+f^*(x_n^*-A^*y_n^*)+g^*(y_n^*)=\langle x_n^*, x_n\rangle+\langle y_n^*, y_n\rangle$. Using the Young-Fenchel inequality we obtain

$$f(x_n)+f^*(x_n^*-A^*y_n^*)+g(Ax_n+y_n)+g^*(y_n^*)\geq\langle x_n^*-A^*y_n^*, x_n\rangle+\langle y_n^*, Ax_n+y_n\rangle$$

$$=\langle x_n^*, x_n\rangle+\langle y_n^*, y_n\rangle,$$

hence $(x_n^*, y_n^*)\in\partial\phi(x_n, y_n)$ if and only if $f(x_n)+f^*(x_n^*-A^*y_n^*)=\langle x_n^*-A^*y_n^*, x_n\rangle$ and $g(Ax_n+y_n)+g^*(y_n^*)=\langle y_n^*, Ax_n+y_n\rangle \Leftrightarrow x_n^*-A^*y_n^*\in\partial f(x_n)$ and $y_n^*\in$

$\partial g(Ax_n + y_n)$. We proved that $a \in \mathrm{dom}(f) \cap A^{-1}(\mathrm{dom}(g))$ is a minimizer of $f + g \circ A$ on X if and only if

$$\begin{cases} \exists (x_n, y_n) \in X \times Y, x_n \in \mathrm{dom}(f), Ax_n + y_n \in \mathrm{dom}(g), \\ \exists (x_n^*, y_n^*) \in X^* \times Y^*, x_n^* - A^* y_n^* \in \partial f(x_n), y_n^* \in \partial g(Ax_n + y_n) \text{ such that} \\ x_n^* \to 0, x_n \to a, y_n \to 0 \ (n \to +\infty) \text{ and} \\ f(x_n) + g(Ax_n + y_n) - \langle y_n^*, y_n \rangle - f(a) - g(Aa) \to 0 \ (n \to +\infty). \end{cases} \tag{5}$$

Next we show that the conditions (4) and (5) are equivalent.

"(5)⇒(4)" Suppose that

$$\begin{cases} \exists (\overline{x_n}, \overline{y_n}) \in X \times Y, \overline{x_n} \in \mathrm{dom}(f), A\overline{x_n} + \overline{y_n} \in \mathrm{dom}(g), \\ \exists (\overline{x_n^*}, \overline{y_n^*}) \in X^* \times Y^*, \overline{x_n^*} - A^* \overline{y_n^*} \in \partial f(\overline{x_n}), \overline{y_n^*} \in \partial g(A\overline{x_n} + \overline{y_n}) \text{ such that} \\ \overline{x_n^*} \to 0, \overline{x_n} \to a, \overline{y_n} \to 0 \ (n \to +\infty) \text{ and} \\ f(\overline{x_n}) + g(A\overline{x_n} + \overline{y_n}) - \langle \overline{y_n^*}, \overline{y_n} \rangle - f(a) - g(Aa) \to 0 \ (n \to +\infty). \end{cases}$$

Take $x_n := \overline{x_n}, y_n := A\overline{x_n} + \overline{y_n}, x_n^* := \overline{x_n^*} - A^* \overline{y_n^*}$ and $y_n^* := \overline{y_n^*}$. Then $x_n \in \mathrm{dom}(f), y_n \in \mathrm{dom}(g), x_n^* \in \partial f(x_n), y_n^* \in \partial g(y_n), x_n^* + A^* y_n^* \to 0, x_n \to a$ and $y_n \to Aa$ $(n \to +\infty)$. Moreover,

$$f(x_n) - \langle x_n^*, x_n - a \rangle - f(a) = f(\overline{x_n}) - \langle \overline{x_n^*} - A^* \overline{y_n^*}, \overline{x_n} - a \rangle - f(a) =$$

$$f(\overline{x_n}) + g(A\overline{x_n} + \overline{y_n}) - \langle \overline{x_n^*}, \overline{x_n} - a \rangle - \langle \overline{y_n^*}, \overline{y_n} \rangle - f(a) - g(Aa) - g(A\overline{x_n} + \overline{y_n})$$

$$+ \langle A^* \overline{y_n^*}, \overline{x_n} - a \rangle + \langle \overline{y_n^*}, \overline{y_n} \rangle + g(Aa) = f(\overline{x_n}) + g(A\overline{x_n} + \overline{y_n}) - \langle \overline{x_n^*}, \overline{x_n} - a \rangle - \langle \overline{y_n^*}, \overline{y_n} \rangle$$

$$- f(a) - g(Aa) - g(y_n) + g(Aa) + \langle y_n^*, y_n - Aa \rangle.$$

With the notations $a_n := f(x_n) - \langle x_n^*, x_n - a \rangle - f(a)$ and $b_n := g(Aa) - g(y_n) - \langle y_n^*, Aa - y_n \rangle$, we have $a_n - b_n = f(\overline{x_n}) + g(A\overline{x_n} + \overline{y_n}) - \langle \overline{y_n^*}, \overline{y_n} \rangle - f(a) - g(Aa) - \langle \overline{x_n^*}, \overline{x_n} - a \rangle \to 0$ $(n \to +\infty)$. Since $x_n^* \in \partial f(x_n)$ we have $f(x) - f(x_n) \geq \langle x_n^*, x - x_n \rangle, \forall x \in X$. For $x := a$ in the previous inequality we get $a_n = f(x_n) - \langle x_n^*, x_n - a \rangle - f(a) \leq 0$. Similarly, from $y_n^* \in \partial g(y_n)$ we have $b_n = g(Aa) - g(y_n) - \langle y_n^*, Aa - y_n \rangle \geq 0$. Thus $a_n \leq 0 \leq b_n$ and $a_n - b_n \to 0$ $(n \to +\infty)$. As in this case one must have that $a_n \to 0$ and $b_n \to 0$ $(n \to \infty)$, (4) is fulfilled.

"(4)⇒(5)" Assume now that (4) holds, namely

$$\begin{cases} \exists (\overline{x_n}, \overline{y_n}) \in \mathrm{dom}(f) \times \mathrm{dom}(g), \exists \overline{x_n^*} \in \partial f(\overline{x_n}), \exists \overline{y_n^*} \in \partial g(\overline{y_n}) \text{ such that} \\ \overline{x_n^*} + A^* \overline{y_n^*} \to 0, \overline{x_n} \to a, \overline{y_n} \to Aa \ (n \to +\infty), \\ f(\overline{x_n}) - \langle \overline{x_n^*}, \overline{x_n} - a \rangle - f(a) \to 0 \ (n \to +\infty) \text{ and} \\ g(\overline{y_n}) - \langle \overline{y_n^*}, \overline{y_n} - Aa \rangle - g(Aa) \to 0 \ (n \to +\infty). \end{cases}$$

Take $x_n := \overline{x_n}, y_n := \overline{y_n} - A\overline{x_n}, y_n^* := \overline{y_n^*}$ and $x_n^* := \overline{x_n^*} + A^* \overline{y_n^*}$. Then $x_n \in \mathrm{dom}(f), Ax_n + y_n \in \mathrm{dom}(g), x_n^* - A^* y_n^* \in \partial f(x_n), y_n^* \in \partial g(Ax_n + y_n), x_n^* \to 0, x_n \to a$ and $y_n \to 0$ $(n \to +\infty)$. Moreover,

$$f(x_n) + g(Ax_n + y_n) - \langle y_n^*, y_n \rangle - f(a) - g(Aa) = f(\overline{x_n}) + g(\overline{y_n}) - \langle \overline{y_n^*}, \overline{y_n} - A\overline{x_n} \rangle$$

$$-f(a)-g(Aa)=f(\overline{x_n})-\langle\overline{x_n^*},\overline{x_n}-a\rangle-f(a)+g(\overline{y_n})-\langle\overline{y_n^*},\overline{y_n}-Aa\rangle-g(Aa)$$
$$+\langle\overline{x_n^*},\overline{x_n}-a\rangle+\langle\overline{y_n^*},-Aa+A\overline{x_n}\rangle=f(\overline{x_n})-\langle\overline{x_n^*},\overline{x_n}-a\rangle-f(a)$$
$$+g(\overline{y_n})-\langle\overline{y_n^*},\overline{y_n}-Aa\rangle-g(Aa)+\langle\overline{x_n^*}+A^*\overline{y_n^*},\overline{x_n}-a\rangle\to 0\ (n\to+\infty),$$

hence (5) is fulfilled. □

If we take $X=Y$ and $A=\mathrm{id}_X$ in the above theorems we obtain the following sequential optimality conditions concerning the convex optimization problem

$$(P)\quad \inf_{x\in X}\{f(x)+g(x)\}.$$

They are presented in the following as two corollaries.

Corollary 4.4 *Let $f,g:X\to\overline{\mathbb{R}}$ be proper, convex and lower semicontinuous functions such that $\mathrm{dom}(f)\cap\mathrm{dom}(g)\neq\emptyset$. Then $a\in\mathrm{dom}(f)\cap\mathrm{dom}(g)$ is a minimizer of $f+g$ on X if and only if*

$$\exists\{\varepsilon_n\}\downarrow 0, \exists x_n^*\in\partial_{\varepsilon_n}f(a), \exists y_n^*\in\partial_{\varepsilon_n}g(a) \text{ such that } x_n^*+y_n^*\to 0\ (n\to+\infty).$$

Corollary 4.5 *Let $f,g:X\to\overline{\mathbb{R}}$ be proper, convex and lower semicontinuous functions such that $\mathrm{dom}(f)\cap\mathrm{dom}(g)\neq\emptyset$. Then $a\in\mathrm{dom}(f)\cap\mathrm{dom}(g)$ is a minimizer of $f+g$ on X if and only if*

$$\begin{cases}\exists(x_n,y_n)\in\mathrm{dom}(f)\times\mathrm{dom}(g), \exists x_n^*\in\partial f(x_n), \exists y_n^*\in\partial g(y_n) \text{ such that}\\ x_n^*+y_n^*\to 0, x_n\to a, y_n\to a\ (n\to+\infty),\\ f(x_n)-\langle x_n^*,x_n-a\rangle-f(a)\to 0\ (n\to+\infty) \text{ and}\\ g(y_n)-\langle y_n^*,y_n-a\rangle-g(a)\to 0\ (n\to+\infty).\end{cases}$$

Taking $g:=\delta_C$ in the previous corollaries, where $C\subseteq X$ is a closed convex set, we obtain the following sequential optimality conditions regarding the convex optimization problem

$$(P_0)\quad \inf_{x\in C} f(x).$$

Corollary 4.6 *Let $f:X\to\overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function and $C\subseteq X$ a closed convex set such that $C\cap\mathrm{dom}(f)\neq\emptyset$. Then $a\in C\cap\mathrm{dom}(f)$ is a minimizer of f on C if and only if*

$$\exists\{\varepsilon_n\}\downarrow 0, \exists x_n^*\in\partial_{\varepsilon_n}f(a), \exists y_n^*\in N_C^{\varepsilon_n}(a) \text{ such that } x_n^*+y_n^*\to 0\ (n\to+\infty).$$

Corollary 4.7 *Let $f : X \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function and $C \subseteq X$ a closed convex set such that $C \cap \operatorname{dom}(f) \neq \emptyset$. Then $a \in C \cap \operatorname{dom}(f)$ is a minimizer of f on C if and only if*

$$\begin{cases} \exists (x_n, y_n) \in \operatorname{dom}(f) \times C, \exists x_n^* \in \partial f(x_n), \exists y_n^* \in N_C(y_n) \text{ such that} \\ x_n^* + y_n^* \to 0, x_n \to a, y_n \to a \ (n \to +\infty), \\ f(x_n) - \langle x_n^*, x_n - a\rangle - f(a) \to 0 \ (n \to +\infty) \text{ and} \\ \langle y_n^*, y_n - a\rangle \to 0 \ (n \to +\infty). \end{cases}$$

Remark 4.8 Corollary 4.6 and Corollary 4.7 give the sequential generalizations of the well-known Pshenichnyi-Rockafellar Lemma, improving the results of Jeyakumar and Wu (see Theorem 3.3 and Corollary 3.5 in [10]). One can notice that in our case the convergence in X^* can be considered both in the weak* and strong topology, since in [10] just the weak* topology is considered. More than that, as shown in this section, the results given in [10] are obtained as particular cases of the main results of our paper, Theorem 3.2 and Theorem 3.3.

4.2 Sequential Lagrange multiplier conditions

In the following we consider the convex optimization problem with cone inequality constraints

$$(P_K) \quad \inf_{\substack{x \in C \\ g(x) \in -K}} f(x),$$

where $C \cap g^{-1}(-K) \cap \operatorname{dom}(f) \neq \emptyset$, C is a closed convex subset of a Banach space X, K is a closed convex cone of another Banach space Y, $f : X \to \overline{\mathbb{R}}$ is a proper, convex and lower semicontinuous function and $g : X \to Y$ is continuous and K-convex, that is $g((1-t)x + tx') - (1-t)g(x) - tg(x') \in -K$, $\forall t \in [0,1], \forall x, x' \in X$. Consider also $K^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0, \forall y \in K\}$ the dual cone of K. We derive a sequential form of the Lagrange multiplier condition for (P_K) by applying Theorem 3.3 to the following perturbation function

$$\phi : X \times X \times Y \to \overline{\mathbb{R}}, \ \phi(x, p, q) = \begin{cases} f(x), & \text{if } x + p \in C \text{ and } g(x) - q \in -K, \\ +\infty, & \text{otherwise.} \end{cases}$$

The conjugate of ϕ is $\phi^* : X^* \times X^* \times Y^* \to \overline{\mathbb{R}}$,

$$\phi^*(x^*, p^*, q^*) = \sup_{\substack{(x,p,q) \in X \times X \times Y \\ x+p \in C \\ g(x) - q \in -K}} \{\langle x^*, x\rangle + \langle p^*, p\rangle + \langle q^*, q\rangle - f(x)\}.$$

In order to compute ϕ^* we introduce new variables z and s by $z := x + p$ and $q - g(x) := s$. It follows

$$\phi^*(x^*, p^*, q^*) = \sup_{(x,z,s) \in X \times C \times K} \{\langle x^*, x\rangle + \langle p^*, z - x\rangle + \langle q^*, s + g(x)\rangle - f(x)\},$$

and, as the three variables are separated, we get $\phi^*(x^*, p^*, q^*) = \sup_{z\in C}\langle p^*, z\rangle + \sup_{x\in X}\{\langle x^* - p^*, x\rangle + \langle q^*, g(x)\rangle - f(x)\} + \sup_{s\in K}\langle q^*, s\rangle$. We obtain the following formula

$$\phi^*(x^*, p^*, q^*) = \begin{cases} \delta_C^*(p^*) + \sup_{x\in X}\{\langle x^* - p^*, x\rangle + \langle q^*, g(x)\rangle - f(x)\}, & \text{if } q^* \in -K^*, \\ +\infty, & \text{otherwise.} \end{cases}$$

For $q^* \in Y^*$ we define the function $q^*g : X \to \mathbb{R}$ by $(q^*g)(x) = \langle q^*, g(x)\rangle, \forall x \in X$. We obtain the following result.

Theorem 4.9 *The element $a \in C \cap g^{-1}(-K) \cap \text{dom}(f)$ is an optimal solution of the problem (P_K) if and only if*

$$\begin{cases} \exists(x_n, \omega_n, t_n) \in \text{dom}(f) \times C \times (-K), \exists(u_n^*, v_n^*, \omega_n^*, q_n^*) \in X^* \times X^* \times X^* \times K^*, \\ u_n^* \in \partial f(x_n), v_n^* \in \partial(q_n^*g)(x_n), \omega_n^* \in N_C(\omega_n), \langle q_n^*, t_n\rangle = 0, \forall n \in \mathbb{N}, \\ u_n^* + v_n^* + \omega_n^* \to 0, \omega_n \to a, x_n \to a, t_n \to g(a)\ (n \to +\infty) \textit{ and} \\ f(x_n) - f(a) + \langle q_n^*, g(x_n)\rangle - \langle \omega_n^*, \omega_n - x_n\rangle \to 0\ (n \to +\infty). \end{cases} \tag{6}$$

Proof. According to Theorem 3.3, the element $a \in C \cap g^{-1}(-K) \cap \text{dom}(f)$ solves the problem (P_K) if and only if there exist sequences $(x_n, p_n, q_n) \in \text{dom}(\phi)$, $(x_n^*, p_n^*, q_n^*) \in \partial\phi(x_n, p_n, q_n)$ such that

$$x_n^* \to 0, x_n \to a, (p_n, q_n) \to (0, 0)\ (n \to +\infty) \text{ and}$$

$$\phi(x_n, p_n, q_n) - \langle (p_n^*, q_n^*), (p_n, q_n)\rangle - \phi(a, 0, 0) \to 0\ (n \to +\infty).$$

Since $(x_n, p_n, q_n) \in \text{dom}(\phi)$ we get $x_n \in \text{dom}(f), x_n + p_n \in C$ and $g(x_n) - q_n \in -K$. We have $(x_n^*, p_n^*, q_n^*) \in \partial\phi(x_n, p_n, q_n)$ if and only if

$$\phi(x_n, p_n, q_n) + \phi^*(x_n^*, p_n^*, q_n^*) = \langle x_n^*, x_n\rangle + \langle p_n^*, p_n\rangle + \langle q_n^*, q_n\rangle$$

$$\Leftrightarrow f(x_n) + \delta_C^*(p_n^*) + (f + q_n^*g)^*(x_n^* - p_n^*) = \langle x_n^*, x_n\rangle + \langle p_n^*, p_n\rangle + \langle -q_n^*, q_n\rangle,$$

where $q_n^* \in -K^*$ was replaced by $-q_n^*$ with $q_n^* \in K^*$, $\forall n \in \mathbb{N}$. The previous relation holds if and only if

$$(f + q_n^*g)^*(x_n^* - p_n^*) + (f + q_n^*g)(x_n) - \langle x_n^* - p_n^*, x_n\rangle$$

$$+\langle q_n^*, q_n - g(x_n)\rangle + \delta_C^*(p_n^*) - \langle p_n^*, x_n + p_n\rangle = 0, \forall n \in \mathbb{N}.$$

As $q_n - g(x_n) \in K$ and $q_n^* \in K^*$, we have $\langle q_n^*, q_n - g(x_n)\rangle \geq 0, \forall n \in \mathbb{N}$. Also the Young-Fenchel inequality yields

$$(f + q_n^*g)^*(x_n^* - p_n^*) + (f + q_n^*g)(x_n) - \langle x_n^* - p_n^*, x_n\rangle \geq 0$$

and

$$\delta_C^*(p_n^*) - \langle p_n^*, x_n + p_n \rangle \geq 0,$$

hence $(x_n^*, p_n^*, q_n^*) \in \partial\phi(x_n, p_n, q_n)$ if and only if $x_n^* - p_n^* \in \partial(f + q_n^* g)(x_n), p_n^* \in \partial\delta_C(x_n + p_n) = N_C(x_n + p_n)$ and $\langle q_n^*, q_n - g(x_n)\rangle = 0, \forall n \in \mathbb{N}$. The relation $\phi(x_n, p_n, q_n) - \langle (p_n^*, -q_n^*), (p_n, q_n)\rangle - \phi(a, 0, 0) \to 0$ $(n \to +\infty)$ (remember that we replaced q_n^* by $-q_n^*$) is equivalent to $f(x_n) - \langle p_n^*, p_n\rangle + \langle q_n^*, q_n\rangle - f(a) \to 0$ $(n \to +\infty)$. Hence the point $a \in C \cap g^{-1}(-K) \cap \operatorname{dom}(f)$ solves the problem (P_K) if and only if

$$\begin{cases} \exists (x_n, p_n, q_n) \in \operatorname{dom}(f) \times X \times Y, x_n + p_n \in C, g(x_n) - q_n \in -K, \\ \exists (x_n^*, p_n^*, q_n^*) \in X^* \times X^* \times K^* \text{ such that} \\ x_n^* - p_n^* \in \partial(f + q_n^* g)(x_n), p_n^* \in N_C(x_n + p_n), \langle q_n^*, q_n - g(x_n)\rangle = 0, \forall n \in \mathbb{N}, \\ x_n^* \to 0, x_n \to a, p_n \to 0, q_n \to 0 \ (n \to +\infty) \text{ and} \\ f(x_n) - f(a) + \langle q_n^*, q_n\rangle - \langle p_n^*, p_n\rangle \to 0 \ (n \to +\infty). \end{cases} \tag{7}$$

Introducing the new variables $t_n, \omega_n, \overline{u_n^*}$ and ω_n^* instead of q_n, p_n, x_n^* and p_n^*, by $t_n := g(x_n) - q_n, \omega_n := p_n + x_n, \overline{u_n^*} := x_n^* - p_n^*$ and $\omega_n^* := p_n^*$ $\forall n \in \mathbb{N}$, respectively, the condition (7) can be reformulated as follows

$$\begin{cases} \exists (x_n, \omega_n, t_n) \in \operatorname{dom}(f) \times C \times (-K), \exists (\overline{u_n^*}, \omega_n^*, q_n^*) \in X^* \times X^* \times K^*, \\ \overline{u_n^*} \in \partial(f + q_n^* g)(x_n), \omega_n^* \in N_C(\omega_n), \langle q_n^*, t_n\rangle = 0, \forall n \in \mathbb{N}, \\ \overline{u_n^*} + \omega_n^* \to 0, \omega_n \to a, x_n \to a, t_n \to g(a) \ (n \to +\infty) \text{ and} \\ f(x_n) - f(a) + \langle q_n^*, g(x_n)\rangle - \langle \omega_n^*, \omega_n - x_n\rangle \to 0 \ (n \to +\infty). \end{cases} \tag{8}$$

The function g being continuous, we obtain that the following subdifferential sum formula holds

$$\partial(f + q_n^* g)(x_n) = \partial f(x_n) + \partial(q_n^* g)(x_n)$$

(see Theorem 2.8.7 in [16]). Thus $\overline{u_n^*} \in \partial(f + q_n^* g)(x_n)$ if and only if there exist $u_n^* \in \partial f(x_n)$ and $v_n^* \in \partial(q_n^* g)(x_n)$ such that $\overline{u_n^*} = u_n^* + v_n^*$ $\forall n \in \mathbb{N}$, so the desired conclusion follows. □

Let us introduce now the following real sequences: $l_n := f(x_n) - f(a) + \langle q_n^*, g(x_n)\rangle - \langle \omega_n^*, \omega_n - x_n\rangle$ (see Theorem 4.9), $l_n^1 := \langle q_n^*, t_n - g(a)\rangle + \langle \omega_n^*, \omega_n - a\rangle$ and $l_n^2 := f(x_n) - f(a) + \langle q_n^*, g(x_n) - g(a)\rangle + \langle \omega_n^*, x_n - a\rangle, \forall n \in \mathbb{N}$. We prove that if the condition

$$\begin{cases} \exists (x_n, \omega_n, t_n) \in \operatorname{dom}(f) \times C \times (-K), \exists (u_n^*, v_n^*, \omega_n^*, q_n^*) \in X^* \times X^* \times X^* \times K^*, \\ u_n^* \in \partial f(x_n), v_n^* \in \partial(q_n^* g)(x_n), \omega_n^* \in N_C(\omega_n), \langle q_n^*, t_n\rangle = 0, \forall n \in \mathbb{N} \text{ and} \\ u_n^* + v_n^* + \omega_n^* \to 0, x_n \to a \ (n \to +\infty), \end{cases} \tag{9}$$

is satisfied, then we have

$$l_n \to 0 \ (n \to +\infty) \text{ if and only if } l_n^1 \to 0 \text{ and } l_n^2 \to 0 \ (n \to +\infty). \tag{10}$$

Indeed, if (9) is fulfilled, then

$$l_n = l_n^2 - l_n^1 \ , \tag{11}$$

hence the sufficiency of relation (10) is trivial (in fact for this implication we need only the fulfillment of $\langle q_n^*, t_n \rangle = 0, \forall n \in \mathbb{N}$).

Assume now that $l_n \to 0$ $(n \to +\infty)$. Since $\omega_n^* \in N_C(\omega_n)$, we have $\langle \omega_n^*, a - \omega_n \rangle \leq 0$ and, as $q_n^* \in K^*$, we get

$$l_n^1 \geq 0, \forall n \in \mathbb{N}. \tag{12}$$

From $v_n^* \in \partial(q_n^* g)(x_n)$ we obtain the inequality $(q_n^* g)(a) - (q_n^* g)(x_n) \geq \langle v_n^*, a - x_n \rangle$, that is $\langle q_n^*, g(x_n) - g(a) \rangle \leq \langle v_n^*, x_n - a \rangle$. This inequality leads to $l_n^2 \leq f(x_n) - f(a) + \langle v_n^* + \omega_n^*, x_n - a \rangle, \forall n \in \mathbb{N}$. Since $u_n^* \in \partial f(x_n)$ we have $f(a) - f(x_n) \geq \langle u_n^*, a - x_n \rangle$, $\forall n \in \mathbb{N}$. Combining the last two inequalities we obtain $l_n^2 \leq \langle u_n^* + v_n^* + \omega_n^*, x_n - a \rangle$, $\forall n \in \mathbb{N}$. This implies, using relation (11) and inequality (12), that

$$0 \leq l_n^1 = l_n^2 - l_n \leq \langle u_n^* + v_n^* + \omega_n^*, x_n - a \rangle - l_n \ \forall n \in \mathbb{N},$$

and so $l_n^1 \to 0$ $(n \to +\infty)$. From (11) we obtain that $l_n^2 \to 0$ $(n \to +\infty)$.

Thus we can state the following result.

Theorem 4.10 *The point $a \in C \cap g^{-1}(-K) \cap \mathrm{dom}(f)$ is an optimal solution of the problem (P_K) if and only if*

$$\begin{cases} \exists (x_n, \omega_n, t_n) \in \mathrm{dom}(f) \times C \times (-K), \exists (u_n^*, v_n^*, \omega_n^*, q_n^*) \in X^* \times X^* \times X^* \times K^*, \\ u_n^* \in \partial f(x_n), v_n^* \in \partial(q_n^* g)(x_n), \omega_n^* \in N_C(\omega_n), \langle q_n^*, t_n \rangle = 0, \forall n \in \mathbb{N}, \\ u_n^* + v_n^* + \omega_n^* \to 0, \omega_n \to a, x_n \to a, t_n \to g(a) \ (n \to +\infty), \\ \langle q_n^*, t_n - g(a) \rangle + \langle \omega_n^*, \omega_n - a \rangle \to 0 \ (n \to +\infty) \ \textit{and} \\ f(x_n) - f(a) + \langle q_n^*, g(x_n) - g(a) \rangle + \langle \omega_n^*, x_n - a \rangle \to 0 \ (n \to +\infty). \end{cases} \tag{13}$$

In case f is continuous, we obtain the following corollary.

Corollary 4.11 *The point $a \in C \cap g^{-1}(-K) \cap \mathrm{dom}(f)$ is an optimal solution of the problem (P_K) if and only if*

$$\begin{cases} \exists (x_n, \omega_n, t_n) \in \mathrm{dom}(f) \times C \times (-K), \exists (u_n^*, v_n^*, \omega_n^*, q_n^*) \in X^* \times X^* \times X^* \times K^*, \\ u_n^* \in \partial f(x_n), v_n^* \in \partial(q_n^* g)(x_n), \omega_n^* \in N_C(\omega_n), \langle q_n^*, t_n \rangle = 0, \forall n \in \mathbb{N}, \\ u_n^* + v_n^* + \omega_n^* \to 0, \omega_n \to a, x_n \to a, t_n \to g(a) \ (n \to +\infty), \\ \langle q_n^*, t_n - g(a) \rangle + \langle \omega_n^*, \omega_n - a \rangle \to 0 \ (n \to +\infty) \ \textit{and} \\ \langle q_n^*, g(x_n) - g(a) \rangle + \langle \omega_n^*, x_n - a \rangle \to 0 \ (n \to +\infty). \end{cases} \tag{14}$$

Remark 4.12 Corollary 4.11 above is exactly the result given by Thibault (see Theorem 4.1 in [15]), in case X and Y are reflexive Banach spaces. Although in [15] it is not mentioned, the pair $(\omega_n, t_n), n \in \mathbb{N}$, must belong to the set $C \times (-K)$ and if one looks carefully at the proof given by Thibault, one can see that this must be assumed also in Theorem 4.1 in [15]. Moreover, we have established this result under weaker assumptions than in [15], since for the sequential Lagrange multiplier condition the cone K need not to be normal.

References

[1] Ben-Israel, A., Ben-Tal, A., Zlobec, S. (1979): *Optimality conditions in convex programming*, in: A. Prékopa, ed., Survey of mathematical programming, Proceedings of the 9th International Mathematical Programming Symposium (The Hungarian Academy of Sciences, Budapest and North-Holland, Amsterdam), 137–156.

[2] Ben-Israel, A., Ben-Tal, A., Zlobec, S. (1976): *Characterization of optimality in convex programming without a constraint qualification*, Journal of Optimization Theory and Applications 20(4), 417–437.

[3] Borwein, J.M., Wolkowicz, H. (1982): *Characterizations of optimality without constraint qualification for the abstract convex program*, Mathematical Programming Study 19, 77–100.

[4] Borwein, J.M., Wolkowicz, H. (1981): *Characterizations of optimality for the abstract convex program with finite dimensional range*, Journal of the Australian Mathematical Society Series A 30(4), 390–411.

[5] Brøndsted, A., Rockafellar, R.T. (1965): *On the subdifferential of convex functions*, Proceedings of the American Mathematical Society 16, 605–611.

[6] Craven, B.D., Zlobec, S. (1980): *Complete characterization of optimality of convex programming in Banach spaces*, Applicable Analysis 11(1), 61–78.

[7] Ekeland, I., Temam, R. (1976): *Convex analysis and variational problems*, North-Holland Publishing Company, Amsterdam.

[8] Jeyakumar, V. (1997): *Asymptotic dual conditions characterizing optimality for infinite convex programs*, Journal of Optimization Theory and Applications 93(1), 153–165.

[9] Jeyakumar, V., Lee, G.M., Dinh, N. (2003): *New sequential Lagrange multiplier conditions characterizing optimality without constraint qualifications for convex programs*, SIAM Journal on Optimization 14(2), 534–547.

[10] Jeyakumar, V., Wu, Z.Y. (2006): *A qualification free sequential Pshenichnyi-Rockafellar Lemma and convex semidefinite programming*, Journal of Convex Analysis 13 (3-4).

[11] Jeyakumar, V., Wu, Z.Y., Lee, G.M., Dinh, N. (2006): *Liberating the subgradient optimality conditions from constraint qualifications*, Journal of Global Optimization 36(1), 127–137.

[12] Jeyakumar, V., Zaffaroni, A. (1996): *Asymptotic conditions for weak and proper optimality in infinite dimensional convex vector optimization*, Numerical Functional Analysis and Optimization 17(3-4), 323–343.

[13] Pshenichnyi, B.N. (1971): *Necessary conditions for an extremum*, Marcel Dekker, New York.

[14] Rockafellar, R.T. (1970): *Convex analysis*, Princeton University Press, Princeton.

[15] Thibault, L. (1997): *Sequential convex subdifferential calculus and sequential Lagrange multipliers*, SIAM Journal on Control and Optimization 35(4), 1434–1444.

[16] Zălinescu, C. (2002): *Convex analysis in general vector spaces*, World Scientific, Singapore.