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Weaker constraint qualifications in maximal monotonicity

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Abstract. We give a sufficient condition, weaker than the others known so far, that guarantees that the sum of two maximal monotone operators on a reflexive Banach space is maximal monotone. Then we give a weak constraint qualification assuring the Brézis-Haraux-type approximation of the range of the sum of the subdifferentials of two proper convex lower-semicontinuous functions in non-reflexive Banach spaces, extending and correcting an earlier result due to Riahi.

Keywords. Maximal monotone operator, Fitzpatrick function, subdifferential, Brézis-Haraux-type approximation

1 Introduction

Finding a weaker sufficient condition under which the sum of two maximal monotone operators on a reflexive Banach space is maximal monotone has been a challenge for many mathematicians during the last four decades. From Browder ([6]) and Rockafellar ([19]) in the 60's to the very recent papers of Borwein ([3]), Simons and Zălinescu ([22]) or Zălinescu ([24]), the conditions imposed on two maximal monotone operators in order to assure the maximal monotonicity of their sum became weaker and weaker. We mention here also Simons' book [20] where many sufficient conditions for the mentioned problem are recalled, compared and unified, as many of them turned out to be actually equivalent. This book and the lecture notes [16] due to Phelps are excellent references for anyone interested in maximal monotone operators.

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Within this paper we give a new constraint qualification, that guarantees the maximal monotonicity of the sum of two maximal monotone operators, satisfied also by some maximal monotone operators that violate the other sufficient conditions known to us. This condition uses the so-called Fitzpatrick functions and has been developed from the one introduced by two of the present authors in [4] for Fenchel duality. The proof we give is inspired by the one due to Borwein ([3]), whose condition is weakened. An example showing that our constraint qualification is weaker than the other ones in the mentioned literature is provided.

Another result in maximal monotonicity improved within this paper is the one due to Riahi (cf. [17]) concerning the Brézis-Haraux-type approximation (cf. [20]) of the range of the sum of the subdifferentials of two lower-semicontinuous functions defined on a non-reflexive Banach space by the sum of the ranges of the two subdifferentials. We show by a counter-example that there was an error in his statement and we give a constraint qualification, weaker than the one considered there, under which the corrected assertion holds.

2 Preliminaries

The following notions and results are necessary in order to make the paper as self-contained as possible. Although the main results in the paper are given in (reflexive) Banach spaces, some of the preliminaries are valid also for more general spaces, thus we begin by considering a non-trivial locally convex topological space X and its continuous dual space X^*, endowed with the weak* topology $w(X^*, X)$. By $\langle x^*, x\rangle$ we denote the value of the linear continuous functional $x^* \in X^*$ at $x \in X$. For a subset C of X we denote by $\operatorname{int}(C)$ and $\operatorname{cl}(C)$ its *interior*, respectively its *closure* in the corresponding topology and we have the *indicator* function $\delta_C : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, defined by

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise.} \end{cases}$$

We consider also the *first projection*, i.e. the function $\mathrm{pr}_1 : X \times Y \to X$, for Y some non-trivial locally convex space, defined as follows $\mathrm{pr}_1(x, y) = x$ for any $(x, y) \in X \times Y$.

Having a function $f : X \to \overline{\mathbb{R}}$, we denote its *domain* by $\operatorname{dom}(f) = \{x \in X : f(x) < +\infty\}$ and its *epigraph* by $\operatorname{epi}(f) = \{(x, r) \in X \times \mathbb{R} : f(x) \le r\}$. For $x \in X$ such that $f(x) \in \mathbb{R}$ we define the *subdifferential* of f at x by $\partial f(x) = \{x^* \in X^* : f(y) - f(x) \ge \langle x^*, y - x\rangle\}$. We call f *proper* if $f(x) > -\infty \ \forall x \in X$ and $\operatorname{dom}(f) \ne \emptyset$. The *conjugate* of the function f is $f^* : X^* \to \overline{\mathbb{R}}$ introduced by

$$f^*(y) = \sup\big\{\langle y, x\rangle - f(x) : x \in X\big\}.$$

Between a function and its conjugate there is *Young's inequality*

$$f^*(y) + f(x) \ge \langle y, x\rangle \ \forall x \in X \ y \in X^*.$$

Consider also the *identity* function on X defined as follows, $\mathrm{id}_X : X \to X$, $\mathrm{id}_X(x) = x \ \forall x \in X$. When $f : X \to \overline{\mathbb{R}}$ and $g : Y \to \overline{\mathbb{R}}$, we define the function $f \times g : X \times Y \to \overline{\mathbb{R}} \times \overline{\mathbb{R}}$ through $f \times g(x, y) = (f(x), g(y))$, $(x, y) \in X \times Y$. When $f, g : X \to \overline{\mathbb{R}}$ are proper functions, we have the *infimal convolution* of f and g defined by

$$f \Box g : X \to \overline{\mathbb{R}},\ f \Box g(a) = \inf\{f(x) + g(a - x) : x \in X\}.$$

Given a linear continuous mapping $A : X \to Y$, we have its *image-set* $\mathrm{Im}(A) = \{Ax : x \in X\} \subseteq Y$ and its *adjoint* $A^* : Y^* \to X^*$ given by $\langle A^* y^*, x\rangle = \langle y^*, Ax\rangle$ for any $(x, y^*) \in X \times Y^*$. For the proper function $f : X \to \overline{\mathbb{R}}$ we define also the *marginal* function of f through A as $Af : Y \to \overline{\mathbb{R}}$, $Af(y) = \inf\big\{f(x) : x \in X, Ax = y\big\}$, $y \in Y$. All along the present paper when an infimum or a supremum is attained we write min, respectively max instead of inf and sup.

Lemma 1. *Let X and Y be non-trivial locally convex spaces, $A : X \to Y$ a linear continuous mapping and $f : Y \to \overline{\mathbb{R}}$ a proper, convex and lower-semicontinuous function such that $f \circ A$ is proper on X. Then one has*

$$\mathrm{epi}((f \circ A)^*) = \mathrm{cl}(\mathrm{epi}(A^* f^*)) = \mathrm{cl}(A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(f^*))), \tag{1}$$

where the closure is considered in the product topology of $(X^*, \tau) \times \mathbb{R}$, with τ any locally convex topology on X^* giving X as dual.

Proof. Use Theorem 2.7 in [9] and Theorem 2.4 in [4]. $\Box$

Definition 1. A set $M \subseteq X$ is said to be *closed regarding the subspace $Z \subseteq X$* if $M \cap Z = \mathrm{cl}(M) \cap Z$.

Proposition 1. *Let X, Y and U be non-trivial locally convex spaces, $A : X \to Y$ a linear continuous mapping and $f : Y \to \overline{\mathbb{R}}$ a proper, convex and lower-semicontinuous function such that $f \circ A$ is proper on X. Consider moreover the linear continuous mapping $M : U \to X^*$. Let τ be any locally convex topology on X^* giving X as dual. The following statements are equivalent:*

(a) $A^* \times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(f^*))$ *is closed regarding the subspace* $\mathrm{Im}(M) \times \mathbb{R}$ *in the product topology of* $(X^*, \tau) \times \mathbb{R}$,

(b) $(f \circ A)^*(Mu) = \min\big\{f^*(y^*) : A^* y^* = Mu\big\}$ *for all* $u \in U$.

Proof. Because f is proper, convex and lower-semicontinuous, A linear and continuous and $f \circ A$ proper it follows that $(f \circ A)^*$ is proper, convex and lower-semicontinuous.

"$(a) \Rightarrow (b)$" Let $u \in U$. For any $y^* \in Y^*$ fulfilling $A^* y^* = Mu$ we have because of Young's inequality

$$f^*(y^*) \geq \langle y^*, Ax\rangle - f(Ax) = \langle A^* y^*, x\rangle - (f \circ A)(x) = \langle Mu, x\rangle - (f \circ A)(x)\ \forall x \in X,$$

and when taking the supremum subject to $x \in X$ in the right-hand side we get $f^*(y^*) \geq \sup_{x \in X}\{\langle Mu, x\rangle - (f \circ A)(x)\} = (f \circ A)^*(Mu)$. This holds for any $y^* \in Y^*$ satisfying $A^*y^* = Mu$, so we conclude

$$\inf\left\{f^*(y^*) : A^*y^* = Mu\right\} \geq (f \circ A)^*(Mu). \tag{2}$$

Let us prove now the reverse inequality. If $(f \circ A)^*(Mu) = +\infty$ then (2) yields $f^*(y^*) = +\infty = (f \circ A)^*(Mu)$ for any $y^* \in Y^*$ such that $A^*y^* = Mu$. Consider further $(f \circ A)^*(Mu) \in \mathbb{R}$. It follows $(Mu, (f \circ A)^*(Mu)) \in \operatorname{epi}((f \circ A)^*)$ and it is clear that it belongs also to $\operatorname{Im}(M) \times \mathbb{R}$. By (1), (a) gives

$$\begin{aligned}(A^* \times \mathrm{id}_{\mathbb{R}}(\operatorname{epi}(f^*))) \cap (\operatorname{Im}(M) \times \mathbb{R}) &= \operatorname{cl}(A^* \times \mathrm{id}_{\mathbb{R}}(\operatorname{epi}(f^*))) \cap (\operatorname{Im}(M) \times \mathbb{R}) \\ &= \operatorname{epi}((f \circ A)^*) \cap (\operatorname{Im}(M) \times \mathbb{R}),\end{aligned}$$

so $(Mu, (f \circ A)^*(Mu))$ belongs to the set in the left-hand side, too. This means that there is some $\bar{y}^* \in Y^*$ such that $A^*\bar{y}^* = Mu$ and $(\bar{y}^*, (f \circ A)^*(Mu)) \in \operatorname{epi}(f^*)$. The latter relation can be rewritten as $f^*(\bar{y}^*) \leq (f \circ A)^*(Mu)$ and we get

$$\inf\left\{f^*(y^*) : A^*y^* = Mu\right\} \leq f^*(\bar{y}^*) \leq (f \circ A)^*(Mu). \tag{3}$$

Having (2) and (3) we are allowed to write

$$\inf\left\{f^*(y^*) : A^*y^* = Mu\right\} = (f \circ A)^*(Mu), \tag{4}$$

and the relations above regarding $\bar{y}^*$ show that the infimum in (4) is attained at $\bar{y}^*$, so (b) is true as $u \in U$ has been taken arbitrarily.

"$(b) \Rightarrow (a)$" From (1) one gets $\operatorname{epi}((f \circ A)^*) \supseteq A^* \times \mathrm{id}_{\mathbb{R}}(\operatorname{epi}(f^*))$, followed by

$$\operatorname{epi}((f \circ A)^*) \cap (\operatorname{Im}(M) \times \mathbb{R}) \supseteq (A^* \times \mathrm{id}_{\mathbb{R}}(\operatorname{epi}(f^*))) \cap (\operatorname{Im}(M) \times \mathbb{R}).$$

For any pair $(x^*, r) \in \operatorname{epi}((f \circ A)^*) \cap (\operatorname{Im}(M) \times \mathbb{R})$ there is some $u \in U$ such that $x^* = Mu$ and we have $(f \circ A)^*(x^*) = (f \circ A)^*(Mu) \leq r$. The hypothesis (b) grants the existence of an $\bar{y}^* \in Y^*$ satisfying both $A^*\bar{y}^* = Mu = x^*$ and $f^*(\bar{y}^*) = (f \circ A)^*(Mu) \leq r$, i.e. $(\bar{y}^*, r) \in \operatorname{epi}(f^*)$. Thus $(x^*, r) = (A^*\bar{y}^*, r) \in A^* \times \mathrm{id}_{\mathbb{R}}(\operatorname{epi}(f^*))$, and as it is in $\operatorname{Im}(M) \times \mathbb{R}$, too, and this pair has been arbitrarily chosen it follows

$$\operatorname{epi}((f \circ A)^*) \cap (\operatorname{Im}(M) \times \mathbb{R}) \subseteq (A^* \times \mathrm{id}_{\mathbb{R}}(\operatorname{epi}(f^*))) \cap (\operatorname{Im}(M) \times \mathbb{R}).$$

As the opposite inclusion stands, too, we get

$$\operatorname{epi}((f \circ A)^*) \cap (\operatorname{Im}(M) \times \mathbb{R}) = (A^* \times \mathrm{id}_{\mathbb{R}}(\operatorname{epi}(f^*))) \cap (\operatorname{Im}(M) \times \mathbb{R}),$$

which yields (a) by (1) and Definition 1. □

Corollary 1. ([4]) *Let X be a non-trivial locally convex space and $f, g : X \to \overline{\mathbb{R}}$ proper, convex and lower-semicontinuous functions whose domains have at least a point in common. Let τ be any locally convex topology on X^* giving X as dual. Then* *(i)* $\mathrm{epi}(f^*) + \mathrm{epi}(g^*)$ *is closed in the product topology of* $(X^*, \tau) \times \mathbb{R}$ *if and only if for all* $x^* \in X^*$ *one has*

$$(f+g)^*(x^*) = \min\{f^*(x^* - y^*) + g^*(y^*) : y^* \in X^*\}.$$

(ii) If $\mathrm{epi}(f^*) + \mathrm{epi}(g^*)$ *is closed in the product topology of* $(X^*, \tau) \times \mathbb{R}$*, then for all* $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ *one has* $\partial(f+g)(x) = \partial f(x) + \partial g(x)$.

Proof. Proposition 1 yields (i), while for (ii) we refer to Theorem 3.2 in [4] (see also [11]). □

The second part of this section in devoted to monotone operators and some of their properties. Consider further X a Banach space equipped with the norm $\|\cdot\|$, while the norm on X^* is $\|\cdot\|_*$.

Definition 2. ([19]) A mapping (generally multivalued) $T : X \rightarrow 2^{X^*}$ is called *monotone operator* provided that for any $x, y \in X$ one has

$$\langle y^* - x^*, y - x\rangle \geq 0 \text{ whenever } x^* \in T(x) \text{ and } y^* \in T(y).$$

Definition 3. ([19]) For any monotone operator $T : X \rightarrow 2^{X^*}$ we have

· its *effective domain* $D(T) = \{x \in X : T(x) \neq \emptyset\}$,

· its *range* $R(T) = \cup\{T(x) : x \in X\}$,

· its *graph* $G(T) = \{(x, x^*) : x \in X, x^* \in T(x)\}$.

Definition 4. ([19]) A monotone operator $T : X \rightarrow 2^{X^*}$ is called *maximal* when its graph is not properly included in the graph of any other monotone operator on the same space.

The subdifferential of a proper convex lower-semicontinuous function on X is a typical example of a maximal monotone operator (cf. [18]). As we shall see in Section 4, it belongs to many other classes of operators, too. We introduce also the *duality map* $J : X \rightarrow 2^{X^*}$ defined as follows

$$J(x) = \partial \frac{1}{2}\|x\|^2 = \Big\{x^* \in X^* : \|x\|^2 = \|x^*\|^2 = \langle x^*, x\rangle\Big\} \ \forall x \in X,$$

because it gives the following criterion for the maximal monotonicity of a monotone operator $T : X \rightarrow 2^{X^*}$.

Proposition 2. ([3], [20]) *A monotone operator* T *on a reflexive Banach space* X *is maximal if and only if the mapping* $T(x+\cdot) + J(\cdot)$ *is surjective for all* $x \in X$.

As underlined by many authors (cf. [3], [12], [15], [20], [22], [24]), there are strong connections between the maximal monotone operators and convex analysis. They are best noticeable by the *Fitzpatrick functions* associated to the monotone operators (cf. [8]). Rediscovered after some years, they proved to be crucial in treating the problem of maximal monotonicity of the sum of maximal monotone operators within the latest papers on the subject ([3], [22], [24], [25]). To a monotone operator $T : X \to 2^{X^*}$ Fitzpatrick attached the function

$$\varphi_T : X \times X^* \to \overline{\mathbb{R}},\ \varphi_T(x, x^*) = \sup\big\{\langle y^*, x\rangle + \langle x^*, y\rangle - \langle y^*, y\rangle : y^* \in T(y)\big\}.$$

For any monotone operator T it is quite clear that φ_T is a convex lower - semicontinuous function as an affine supremum. An important result regarding the Fitzpatrick functions and their conjugates in reflexive Banach spaces follows.

Proposition 3. ([22]) *Let T be a maximal monotone operator on a reflexive Banach space X. Then for any pair $(x, x^*) \in X \times X^*$ we have*

$$\varphi_T^*(x^*, x) \geq \varphi_T(x, x^*) \geq \langle x^*, x\rangle.$$

Moreover, $\varphi_T^*(x^*, x) = \varphi_T(x, x^*) = \langle x^*, x\rangle$ if and only if $(x, x^*) \in G(T)$.

3 Maximal monotonicity for the sum of two maximal monotone operators

Within this section X is a reflexive Banach space and let $S : X \to 2^{X^*}$ and $T : X \to 2^{X^*}$ be two maximal monotone operators such that $\mathrm{pr}_1(\mathrm{dom}(\varphi_S)) \cap \mathrm{pr}_1(\mathrm{dom}(\varphi_T)) \neq \emptyset$. It is known (cf. [19]) that the sum of two monotone operators is monotone, while the maximal monotonicity of the sum of two maximal monotone operators can fail (see [20] for examples). Many various conditions were imposed on two maximal monotone operators in order to assure that their sum is also maximal monotone. Some of them arose from convex analysis and optimization and we give another one following this line. First let us remind some of the conditions taken into consideration so far

(i) ([19]) $D(S) \cap \mathrm{int}(D(T)) \neq \emptyset$,

(ii) ([19]) there exists an $x \in \mathrm{cl}(D(S)) \cap \mathrm{cl}(D(T))$ where T is locally bounded,

(iii) ([15]) $\bigcup\limits_{\lambda \geq 0} \lambda\big[\,\mathrm{co}(D(S)) - \mathrm{co}(D(T))\big] = X$,

(iv) ([3]) $0 \in \mathrm{core}\,\big[\,\mathrm{co}(D(S)) - \mathrm{co}(D(T))\big]$,

(v) ([2], [24], [25]) $\bigcup\limits_{\lambda > 0} \lambda\big[D(S) - D(T)\big]$ is a closed linear subspace of X,

(vi) ([13]) $0 \in \mathrm{ri}(D(S) - D(T))$,

(vii) ([7]) $0 \in \mathrm{ri}(\mathrm{co}(D(S)) - \mathrm{co}(D(T)))$,

(viii) ([22]) $\bigcup_{\lambda>0} \lambda\big[\mathrm{pr}_1(\mathrm{dom}(\varphi_S)) - \mathrm{pr}_1(\mathrm{dom}(\varphi_T))\big]$ is a closed linear subspace of X.

Here co denotes the *convex hull*, ri the *relative interior* and core the *core*. As these notions are known and they are not used anywhere in our paper we do not define them here, referring the interested reader to [25], for instance. Let us mention that in [24] and [25] there are more sufficient conditions that assure the maximal monotonicity of $S+T$, but we mentioned here only one as they are equivalent to $(viii)$ or stronger than it. Each of the eight regularity conditions given above implies that $S+T$ is a maximal monotone operator, as well as some others, like the ones given in [20] or [23], for instance. It is also known that (i) is equivalent to (ii) (cf. [19]), each of them implying (iii) and (iv), which are equivalent. Then, $(v) - (vii)$ are also equivalent, being valid whenever one of $(i) - (iv)$ holds. The remaining condition, $(viii)$ is implied by any of $(i) - (vii)$. For the results concerning the comparisons between them we refer to the papers where the conditions are taken from.

We prove, using an idea due to Borwein ([3]), that $S+T$ is maximal monotone provided that the following constraint qualification is fulfilled,

(CQ) $\{(x^* + y^*, x, y, r) : \varphi_S^*(x^*, x) + \varphi_T^*(y^*, y) \leq r\}$ is closed regarding the subspace $X^* \times \Delta_X \times \mathbb{R}$,

where $\Delta_X = \{(x, x) : x \in X\}$. Later we prove that (CQ) is weaker than the conditions $(i) - (viii)$ mentioned above.

Theorem 1. *If (CQ) is fulfilled then $S+T$ is a maximal monotone operator.*

Proof. Fix first some $z \in X$ and $z^* \in X^*$. We prove that there is always an $\bar{x} \in X$ such that $z^* \in (S+T)(\bar{x}+z) + J(\bar{x})$. Consider the functions f, $g : X \times X^* \to \overline{\mathbb{R}}$, defined by

$$f(x, x^*) = \inf_{y^* \in X^*} \{\varphi_S(x+z, x^* + z^* - y^*) + \varphi_T(x+z, y^*)\} - \langle x^* + z^*, z\rangle$$

and $$g(x, x^*) = \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|_*^2 - \langle z^*, x\rangle,\ (x, x^*) \in X \times X^*.$$

Let us calculate the conjugates of f and g. For any $(w^*, w) \in X^* \times X$ we have

$f^*(w^*,w) = \sup_{\substack{x\in X,\\ x^*\in X^*}} \Big\{\langle w^*,x\rangle + \langle x^*,w\rangle - \inf_{y^*\in X^*}\{\varphi_S(x+z,x^*+z^*-y^*) + \varphi_T(x+z,y^*)\} + \langle x^*+z^*,z\rangle\Big\} = \sup_{\substack{x\in X,\\ x^*,y^*\in X^*}} \{\langle w^*,x\rangle + \langle x^*,w\rangle + \langle x^*+z^*,z\rangle - \varphi_S(x+z,x^*+z^*-y^*) - \varphi_T(x+z,y^*)\} = \sup_{\substack{u\in X,\\ u^*,y^*\in X^*}} \{\langle w^*,u-z\rangle + \langle u^*+y^*-z^*,w\rangle + \langle u^*+y^*,z\rangle - \varphi_S(u,u^*) - \varphi_T(u,y^*)\} = \sup_{\substack{u\in X,\\ u^*,y^*\in X^*}} \{\langle w^*,u\rangle + \langle u^*+y^*,w+z\rangle - \varphi_S(u,u^*) - \varphi_T(u,y^*)\} - \langle w^*,z\rangle - \langle z^*,w\rangle$.

Considering the functions $F : X\times X\times X^*\times X^* \to \overline{\mathbb{R}}$, $F(a,b,a^*,b^*) = \varphi_S(a,a^*) + \varphi_T(b,b^*)$, $A : X\times X^*\times X^* \to X\times X\times X^*\times X^*$, $A(a,a^*,b^*) = (a,a,a^*,b^*)$ and $M : X^*\times X \to X^*\times X\times X$, $M(a^*,a) = (a^*,a,a)$, we have that

$$f^*(w^*,w) = (F\circ A)^*(M(w^*,w+z)) - \langle w^*,z\rangle - \langle z^*,w\rangle \ \forall (w^*,w)\in X^*\times X.$$

Because $F^* : X^*\times X^*\times X\times X \to \overline{\mathbb{R}}$, $F^*(a^*,b^*,a,b) = \varphi_S^*(a^*,a) + \varphi_T^*(b^*,b)$ and $A^* : X^*\times X^*\times X\times X \to X^*\times X\times X$, $A^*(a^*,b^*,a,b) = (a^*+b^*,a,b)$, one has

$$A^*\times \mathrm{id}_{\mathbb{R}}(\mathrm{epi}(F^*)) = \{(a^*+b^*,a,b,r) : \varphi_S^*(a^*,a) + \varphi_T^*(b^*,b) \le r\}.$$

Knowing that $\mathrm{Im}(M)\times\mathbb{R} = X^*\times\Delta_X\times\mathbb{R}$, the constraint qualification (CQ) is nothing else than the fact $A^*\times\mathrm{id}_{\mathbb{R}}(\mathrm{epi}(F^*))$ is closed regarding the subspace $\mathrm{Im}(M)\times\mathbb{R}$. So, by Proposition 1, we have that for any $(w^*,w)\in X^*\times X$

$$(F\circ A)^*(M(w^*,w+z)) = \min\{F^*(a^*,b^*,a,b) : (a^*+b^*,a,b) = (w^*,w+z,w+z)\}.$$

Back to f^*, one gets immediately that for any $(w^*,w)\in X^*\times X$

$$f^*(w^*,w) = \min_{a^*+b^*=w^*} \big\{\varphi_S^*(a^*,w+z) + \varphi_T^*(b^*,w+z)\big\} - \langle w^*,z\rangle - \langle z^*,w\rangle.$$

Regarding g^*, the conjugate of g, for any $(w^*,w)\in X^*\times X$ one has

$$\begin{aligned} g^*(w^*,w) &= \sup_{\substack{x\in X,\\ x^*\in X^*}} \Big\{\langle w^*,x\rangle + \langle x^*,w\rangle - \frac{1}{2}\|x\|^2 - \frac{1}{2}\|x^*\|_*^2 + \langle z^*,x\rangle\Big\} \\ &= \sup_{x\in X}\Big\{\langle w^*+z^*,x\rangle - \frac{1}{2}\|x\|^2\Big\} + \sup_{x^*\in X^*}\Big\{\langle x^*,w\rangle - \frac{1}{2}\|x^*\|_*^2\Big\} \\ &= \frac{1}{2}\|w^*+z^*\|_*^2 + \frac{1}{2}\|w\|^2. \end{aligned}$$

For any $(x,x^*)\in X\times X^*$ and $y^*\in X^*$, by Proposition 3, we have

$$\varphi_S(x+z,x^*+z^*-y^*) + \varphi_T(x+z,y^*) - \langle x^*+z^*,z\rangle + g(x,x^*) \ge$$

$$\langle x^* + z^* - y^*, x + z\rangle + \langle y^*, x + z\rangle - \langle x^* + z^*, z\rangle +$$
$$\frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|_*^2 - \langle z^*, x\rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|_*^2 + \langle x^*, x\rangle \geq 0.$$

Taking in the left-hand side the infimum subject to all $y^* \in X^*$, we get $f(x, x^*) + g(x, x^*) \geq 0$. Thus $\inf_{(x,x^*)\in X\times X^*}\{f(x, x^*) + g(x, x^*)\} \geq 0$.

Because of the convexity of f and g and as the latter is continuous Fenchel's duality theorem (cf. [25]) guarantees the existence of some pair $(\bar{x}^*, \bar{x}) \in X^* \times X$ such that

$$\begin{aligned}
\inf_{(x,x^*)\in X\times X^*}\{f(x, x^*) + g(x, x^*)\} &= \max_{(x^*,x)\in X^*\times X}\{-f^*(x^*, x) - g^*(-x^*, -x)\} \\
&= -f^*(\bar{x}^*, \bar{x}) - g^*(-\bar{x}^*, -\bar{x}).
\end{aligned}$$

Using the result above, one gets $f^*(\bar{x}^*, \bar{x}) + g^*(-\bar{x}^*, -\bar{x}) \leq 0$. So there are some $\bar{a}^*$ and $\bar{b}^*$ in X^* such that $\bar{a}^* + \bar{b}^* = \bar{x}^*$ and

$$\varphi_S^*(\bar{a}^*, \bar{x} + z) + \varphi_T^*(\bar{b}^*, \bar{x} + z) - \langle \bar{x}^*, z\rangle - \langle z^*, \bar{x}\rangle + \frac{1}{2}\| - \bar{x}^* + z^*\|_*^2 + \frac{1}{2}\| - \bar{x}\|^2 \leq 0.$$

Taking into account that $\bar{a}^* + \bar{b}^* = \bar{x}^*$, after some minor calculations we get

$$\begin{aligned}
0 &\geq \big(\varphi_S^*(\bar{a}^*, \bar{x} + z) - \langle \bar{a}^*, \bar{x} + z\rangle\big) + \big(\varphi_T^*(\bar{b}^*, \bar{x} + z) - \langle \bar{b}^*, \bar{x} + z\rangle\big) \\
&+ \left(\langle \bar{x}^* - z^*, \bar{x}\rangle + \frac{1}{2}\|\bar{x}^* - z^*\|_*^2 + \frac{1}{2}\|\bar{x}\|^2\right) \geq 0,
\end{aligned}$$

where the last inequality comes from Proposition 3. Thus the inequalities above must be fulfilled as equalities, so

$$\varphi_S^*(\bar{a}^*, \bar{x} + z) = \langle \bar{a}^*, \bar{x} + z\rangle, \quad \varphi_T^*(\bar{b}^*, \bar{x} + z) = \langle \bar{b}^*, \bar{x} + z\rangle,$$

and

$$\langle \bar{a}^* + \bar{b}^* - z^*, \bar{x}\rangle + \frac{1}{2}\|\bar{a}^* + \bar{b}^* - z^*\|_*^2 + \frac{1}{2}\|\bar{x}\|^2 = 0.$$

These three equalities are equivalent, using Proposition 3, to $\bar{a}^* \in S(\bar{x} + z)$, $\bar{b}^* \in T(\bar{x} + z)$ and, respectively, $z^* - \bar{a}^* - \bar{b}^* \in \partial\frac{1}{2}\|\cdot\|^2(\bar{x}) = J(\bar{x})$. Summing these three relations one gets

$$z^* - \bar{a}^* - \bar{b}^* + \bar{a}^* + \bar{b}^* \in (S + T)(\bar{x} + z) + J(\bar{x}).$$

As z and z^* have been arbitrarily chosen, Proposition 2 yields the conclusion. □

Remark 1. We prove that the constraint qualification (CQ) is weaker than some generalized interior-point regularity conditions given in the literature in order to assure the maximality of the sum of two maximal monotone operators. We have recalled in the beginning of the section eight of the regularity conditions given for this purpose and the weakest of them is $(viii)$ (see Lemma 5.3 in [22]).

Consider the functions $s,t : X \times X^* \times X^* \to \overline{\mathbb{R}}$, defined by $s(x,x^*,y^*) = \varphi_S(x,x^*)$ and $t(x,x^*,y^*) = \varphi_T(x,y^*)$, respectively. As

$$\underset{\lambda>0}{\cup} \lambda\big[\operatorname{pr}_1(\operatorname{dom}(\varphi_S)) - \operatorname{pr}_1(\operatorname{dom}(\varphi_T))\big] \times X^* \times X^* = \underset{\lambda>0}{\cup} \lambda\big[\operatorname{dom}(s) - \operatorname{dom}(t)\big],$$

assuming $(viii)$ fulfilled, it follows that $\underset{\lambda>0}{\cup} \lambda\big[\operatorname{dom}(s) - \operatorname{dom}(t)\big]$ is a closed linear subspace in $X \times X^* \times X^*$. By Theorem 2.6 in [1] (see also [25]) it follows that for any $(x^*,x,y) \in X^* \times X \times X$, $(s+t)^*(x^*,x,y) = \inf\{s^*(x_1^*,x_1,y_1) + t^*(x_2^*,x_2,y_2) : x_1^* + x_2^* = x^*, x_1 + x_2 = x, y_1 + y_2 = y\}$.

On the other hand, by Corollary 1 (i), this last relation is true if and only if $\operatorname{epi}(s^*) + \operatorname{epi}(t^*)$ is closed in $X^* \times X \times X \times \mathbb{R}$. As $\operatorname{epi}(s^*) = \{(x^*,x,0,r) : \varphi_S^*(x^*,x) \le r\}$ and $\operatorname{epi}(t^*) = \{(y^*,0,y,r) : \varphi_T^*(y^*,y) \le r\}$, we have, in conclusion, that if $(viii)$ is fulfilled, then $\{(x^*+y^*,x,y,r) : \varphi_S^*(x^*,x) + \varphi_T^*(y^*,y) \le r\}$ is a closed set. Thus it is clear that $(viii)$ implies (CQ).

Remark 2. The maximal monotonicity of $S+T$ is valid also when imposing the constraint qualification

$$(\widetilde{CQ}) \qquad \{(x^*+y^*,x,y,r) : \varphi_S^*(x^*,x) + \varphi_T^*(y^*,y) \le r\} \text{ is closed.}$$

In the following we show that (CQ) and $(\widetilde{CQ})$ are indeed weaker than $(viii)$.

Example 1. Let $X = \mathbb{R}$. Then $X^* = \mathbb{R}$. Consider the operators $S,T : \mathbb{R} \to 2^{\mathbb{R}}$ defined by

$$S(x) = \begin{cases} \{0\}, & \text{if } x > 0, \\ (-\infty,0], & \text{if } x = 0, \\ \emptyset, & \text{otherwise,} \end{cases} \quad \text{and } T(x) = \begin{cases} \mathbb{R}, & \text{if } x = 0, \\ \emptyset, & \text{otherwise,} \end{cases} \quad \forall x \in \mathbb{R}.$$

One notices easily that, considering the functions $f,g : \mathbb{R} \to \overline{\mathbb{R}}$, $f = \delta_{[0,+\infty)}$ and $g = \delta_{\{0\}}$, which are proper, convex and lower-semicontinuous, we have $S = \partial f$ and $T = \partial g$.

As $\partial f + \partial g = \partial g$, it follows that $S+T$ is maximal monotone, being the subdifferential of a proper, convex and lower-semicontinuous function.

Let us calculate the conjugates of φ_S and φ_T to see if (CQ) is fulfilled. We have for all $x,x^* \in \mathbb{R}$

$$\varphi_S(x,x^*) = \begin{cases} 0, & \text{if } x \ge 0, x^* \le 0, \\ +\infty, & \text{otherwise,} \end{cases} \quad \text{and } \varphi_T(x,x^*) = \begin{cases} 0, & \text{if } x = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

so

$$\varphi_S^*(x^*,x) = \begin{cases} 0, & \text{if } x^* \le 0, x \ge 0, \\ +\infty, & \text{otherwise,} \end{cases} \quad \text{and } \varphi_T^*(x^*,x) = \begin{cases} 0, & \text{if } x = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

The set involved in (CQ) is in this case $\mathbb{R} \times [0,+\infty) \times \{0\} \times [0,+\infty)$, which is closed, i.e. $(\widetilde{CQ})$ is valid. Thus it is closed regarding the subspace $\mathbb{R} \times \Delta_{\mathbb{R}} \times \mathbb{R}$, too, i.e. (CQ) is satisfied in this case.

We show now that $\cup_{\lambda>0}\lambda\big[\operatorname{pr}_1(\operatorname{dom}(\varphi_S)) - \operatorname{pr}_1(\operatorname{dom}(\varphi_T))\big]$ is not a closed subspace of X. We have $\operatorname{pr}_1(\operatorname{dom}(\varphi_S)) = [0,+\infty)$ and $\operatorname{pr}_1(\operatorname{dom}(\varphi_T)) = \{0\}$, so

$$\cup_{\lambda>0}\lambda\big[\operatorname{pr}_1(\operatorname{dom}(\varphi_S)) - \operatorname{pr}_1(\operatorname{dom}(\varphi_T))\big] = [0,+\infty),$$

which is not a subspace, thus $(viii)$ fails in this case. Therefore, even if $(viii)$ implies (CQ) and $(\widetilde{CQ})$, the reverse implication does not always hold, i.e. our conditions are indeed weaker than $(viii)$, the most general known so far to us in the literature.

4 Brézis - Haraux - type approximation of the range of the sum of two subdifferentials

Within this part X is considered a non-reflexive Banach space. In the following we rectify a partially false statement due to Riahi ([17]) concerning the so-called Brézis-Haraux-type approximation of the range of the sum of the subdifferentials of two proper convex lower-semicontinuous functions by the subdifferential of their sum. Moreover, we prove that the correct result is valid under a weaker condition than the one considered in the original paper. Let us mention that Pennanen ([14]) has obtained some related results, but in reflexive spaces, while for more on the Brézis-Haraux approximation of the sum of two monotone operators we refer to [20].

Some new notions and results are necessary before giving the main statement in this part of the paper. We introduce and use the so-called monotone operators of type (D), originally introduced by Gossez in [10] as operators of dense type and known in the literature also as densely maximal, of type 3^*, also known as star monotone and of the type (BH), and operators of the type (NI). We stress once again that we work in non-reflexive Banach spaces.

Before this we need to introduce the operator $\overline{T} : X^{**} \to 2^{X^*}$ as follows

$$G(\overline{T}) = \{(x^{**}, x^*) \in X^{**} \times X^* : \langle x^{**} - \hat{y}, x^* - y^* \rangle \geq 0 \ \forall (y, y^*) \in G(T),$$

where $\hat{y}$ denotes the canonical image of y in X^{**}. The elements in $G(\overline{T})$ are called in the literature *monotonically related* to T.

Definition 5. ([16]) A monotone operator $T : X \to 2^{X^*}$ is called of *type* (D) provided that for any $(y^{**}, y^*) \in G(\overline{T})$ there is a net $(y_\alpha, y^*_\alpha)_\alpha \subseteq G(T)$ such that

$y_\alpha \to y^{**}$ in the $\sigma(X^{**}, X^*)$-topology (cf. [10], [16], [17]), $(y_\alpha)_\alpha$ is bounded and $y^*_\alpha \to y^*$ in the norm-topology.

Definition 6. ([14], [19]) A monotone operator $T : X \to 2^{X^*}$ is called 3^*-*monotone* if for all $x^* \in R(T)$ and $x \in D(T)$ there is some $\beta(x^*, x) \in \mathbb{R}$ such that $\inf_{y^* \in T(y)} \langle x^* - y^*, x - y\rangle \geq \beta(x^*, x)$.

Definition 7. ([14], [19]) An operator $T : X \to 2^{X^*}$ is called of *type* (NI) if for all $(x^{**}, x^*) \in X^{**} \times X^*$ one has $\inf_{y^* \in T(y)} \langle \hat{y} - x^{**}, y^* - x^*\rangle \leq 0$.

The following statement rectifies and weakens Corollary 2 in [17].

Theorem 2. *Let f and g be two proper convex lower-semicontinuous functions on the Banach space X with extended real values, such that* $\mathrm{dom}(f) \cap \mathrm{dom}(g) \neq \emptyset$. *Assume the satisfaction of the condition*

(C) $\mathrm{epi}(f^*) + \mathrm{epi}(g^*)$ *is closed in the product topology of* $(X^*, w(X^*, X)) \times \mathbb{R}$.

Then one has

(i) $\mathrm{cl}(R(\partial f) + R(\partial g)) = \mathrm{cl}(R(\partial(f + g)))$,

(ii) $\mathrm{int}(R(\partial(f + g))) \subseteq \mathrm{int}(R(\partial f) + R(\partial g)) \subseteq \mathrm{int}(D(\partial(f^* \Box g^*)))$.

Proof. Corollary $1(ii)$ states that (C) suffices in order to assure that $\partial(f + g) = \partial f + \partial g$ on $\mathrm{dom}(f) \cap \mathrm{dom}(g)$. According to Theorem B in [18] (see also [14], [17]) ∂f and ∂g are 3^*-monotone operators. Theorem 1 in [17] yields

$$\mathrm{cl}(R(\partial f + \partial g)) = \mathrm{cl}(R(\partial f) + R(\partial g)),$$

which delivers (i) by Corollary $1(ii)$ because (C) is valid and

$$\mathrm{int}(R(\partial f + \partial g)) \subseteq \mathrm{int}(R(\partial f) + R(\partial g)) \subseteq \mathrm{int}(R(\overline{\partial f + \partial g})). \tag{5}$$

By Lemma 35.2 in [20] one has

$$R(\overline{\partial f + \partial g}) = D(\partial(f^* \Box g^*)),$$

so, applying again Corollary $1(ii)$, (5) turns into (ii). $\Box$

A similar result has been obtained by Riahi in Corollary 2 in [17]. There he said that under the constraint qualification

(C_R) $\quad \bigcup_{\lambda > 0} \lambda(\mathrm{dom}(f) - \mathrm{dom}(g))$ is a closed linear subspace of X, one gets $\mathrm{cl}(R(\partial f)+R(\partial g)) = \mathrm{cl}(R(\partial(f+g)))$ and $\mathrm{int}(R(\partial f)+R(\partial g)) = \mathrm{int}(D(\partial(f^*\square g^*)))$.

We prove that the latter is not always true under (C_R). For a proper, convex and lower-semicontinuous function $g : X \to \overline{\mathbb{R}}$ Riahi's relation would become $\mathrm{int}(R(\partial g)) = \mathrm{int}(D(\partial g^*))$, which is equivalent, by Lemma 35.2 in [20] to

$$\mathrm{int}(R(\partial g)) = \mathrm{int}(R(\overline{\partial g})). \tag{6}$$

From Théoréme 3.1 in [10] we have that ∂g is a monotone operator of type (D) and it is also known that it is maximal monotone, too. According to Simons ([21]) ∂g is also of type (NI) so, by Theorem 20 in the same paper, $\mathrm{int}(R(\overline{\partial g}))$ is convex. Thus (6) yields $\mathrm{int}(R(\partial g))$ convex. Unfortunately this is not always true, as Example 2.21 in [16], originally given by Fitzpatrick, shows. Take $X = c_0$, which is a Banach space with the usual supremum norm, and $g(x) = \|x\| + \|x-(1,0,0,...)\|$, a proper, convex and continuous function on c_0. Skipping the calculatory details, it follows that $\mathrm{int}(R(\partial g))$ is not convex, unlike $\mathrm{int}\, R(\overline{\partial g})$. Thus (6) is false and the same happens to Riahi's allegation.

Remark 3. As proven in Proposition 3.1 in [5] (see also [4]), (C_R) implies (C), but the converse is not true, as shown by Example 3.1 in the same paper. Therefore our Theorem 2 improves, by weakening the regularity condition, and corrects Corollary 2 in [17].

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