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Fenchel's duality theorem for nearly convex functions

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Abstract. We present an extension of Fenchel's duality theorem to nearly convexity, giving weaker conditions under which it takes place. Instead of minimizing the difference between a convex and a concave function, we minimize the subtraction of a nearly concave function from a nearly convex one. The assertion in the special case of Fenchel's duality theorem that consists in minimizing the difference between a convex function and a concave function pre-composed with a linear transformation is also proven to remain valid when one considers nearly convexity. We deliver an example where the Fenchel's classical duality theorem is not applicable, unlike the extension we have introduced, and an application related to games theory.

Keywords. Fenchel duality, conjugate functions, nearly convex functions

AMS subject classification (2000). 26A51, 42A50, 49N15

1 Introduction

A cornerstone in Optimization, Fenchel's duality theorem ([21]) is one of the most applied results in Convex Analysis. It asserts that under a certain condition, namely the existence of a common element in the relative interiors of the effective domains of a convex and a concave function, respectively, both defined over the n-dimensional real space, the infimal value of the difference between the convex function and the concave one is actually equal to the maximal value of the function obtained by subtracting the conjugate of the convex function from the conjugate of the concave function. The minimization problem is usually called

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primal problem, while the maximization problem involving conjugate functions is known as the Fenchel dual problem attached to the primal. The areas of applicability for this basic result cover actually more than optimization and its offsprings and relatives. The interested reader is referred to Rockafellar's book [21] for a first and consistent contact with this theorem and some of its essential applications.

As Fenchel's duality theorem is given for convex optimization problems, there were many attempts to extend it in various directions, some of them proving to be successful. We cite here three of them. Kanniappan has given in [16] a Fenchel-type duality theorem for non-convex and non-differentiable maximization problems, Beoni ([2]) extended Fenchel's statement to fractional programming, while Penot and Volle ([20]) considered it for quasiconvex problems. Our paper is meant to present another extension of Fenchel's duality theorem, this time for a primal problem having as objective the difference between a nearly convex function and a nearly concave one. The nearly convex functions were introduced quite recently by Aleman ([1]) (see, also, [5] and [17]) who called them p-convex, while for the sets the concept of nearly convexity due to Green and Gustin ([11]) is already older than half a century. Among the papers that have dealt with nearly convex sets and functions let us mention [3], [4], [9], [15] and [18], the first and the third presenting also some results concerning their applicability in optimization, especially in duality. Within the last twenty years there has been published a series of papers dealing with and extending this concept. Among these generalizations let us recall nearly S-convexlikeness and nearly S-subconvexlikeness (see [6], [14] and [18]). Jeyakumar and Gwinner ([15]) have used this notions and their properties in order to derive solvability theorems for general convexlike inequality systems. Further applications in Lagrange duality, theorems of the alternative and optimality conditions for optimization problems involving nearly convex functions and sets (and their already mentioned extensions) were given in [8] and [13].

Our aim is to reveal some sufficient conditions, weaker than those in [21], that guarantee the equality between the infimal value of the objective function of the primal problem as given in Theorem 31.1 in the mentioned book and the supreme objective value of the Fenchel dual problem attached to it, which is actually attained at some point in $\mathbb{R}^n$. Because some authors cite Corollary 31.2.1 in [21] as Fenchel's duality theorem, we have taken it into consideration, too. We proved that its assertion holds under weaker conditions than the ones considered in [21], namely, alongside the regularity condition, instead of f convex and closed function and g concave and closed function we need just f nearly convex, g nearly concave and the relative interiors of their epigraphs non-empty. When f is convex and g concave the statement remains valid without assuming that these functions are closed.

Let us now present some things regarding the way this paper is organized. The next section is dedicated to some necessary preliminaries. The two theorems we generalize, some notions we will use throughout the paper, notations and some statements used later are presented. The third part of the paper contains the main result and some remarks concerning the way it generalizes Fenchel's results. An example on which this extension is applicable is also delivered alongside an application in games theory. We give in the fourth section an extension to the case of Fenchel's duality theorem where the concave function to be subtracted is post-composed with a linear transformation, too. A short conclusive section followed by the list of references close the paper.

2 Preliminaries

Because the notions of nearly convex sets and nearly convex/concave functions are not so widely-known, we dedicate this section to familiarize the reader with them and the most important results related to them that we will need further within this paper. Other notions, notations and assertions concerning convex sets and convex/concave functions required or mentioned later are also contained inside this section, as well as the theorems to whose generalization this paper is dedicated to.

As usual, $\mathbb{R}^n$ denotes the n-dimensional real space for $n \in \mathbb{N}$ and $\mathbb{Q}$ is the set of all rational real numbers. Throughout this paper all the vectors are considered as column vectors belonging to $\mathbb{R}^n$, unless otherwise specified. An upper index ^{T} transposes a column vector into a row one and viceversa. The inner product of two vectors $x = \big(x_1, ..., x_k\big)^T$ and $y = \big(y_1, ..., y_k\big)^T$ in the k-dimensional real space is denoted by $x^T y = \sum_{i=1}^k x_i y_i$. The closure of a certain set is distinguished from the set itself by a preceding cl, the prefix aff denotes the affine hull of the corresponding set, while to write the relative interior of a set we use the prefix ri. If $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then by $A^* : \mathbb{R}^m \to \mathbb{R}^n$ we denote its adjoint defined by $(Ax)^T y = x^T (A^* y)$ $\forall x \in \mathbb{R}^n$ $\forall y \in \mathbb{R}^m$. For some set $X \subseteq \mathbb{R}^n$ we use the well-known indicator function $\delta_X : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\delta_X(x) = \begin{cases} 0, & \text{if } x \in X, \\ +\infty, & \text{if } x \notin X. \end{cases}$$

For a convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ we consider the following notions and definitions

- effective domain: $\text{dom}(f) = \big\{x \in \mathbb{R}^n : f(x) < +\infty\big\}$,
- epigraph: $\text{epi}(f) = \big\{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\big\}$,
- f is proper: $\text{dom}(f) \neq \emptyset$ and $f(x) > -\infty$ $\forall x \in \mathbb{R}^n$,
- lower semi-continuous envelope: $\overline{f} : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that $\text{epi}(\overline{f}) = \text{cl}(\text{epi}(f))$,

· conjugate function: $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, $f^*(p) = \sup\limits_{x \in \mathbb{R}^n} \big\{ p^T x - f(x) \big\}$.

Similar notions are defined also for a concave function $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ as follows (cf. [21])

· effective domain: $\mathrm{dom}(g) = \big\{ x \in \mathbb{R}^n : g(x) > -\infty \big\}$,

· epigraph: $\mathrm{epi}(g) = \big\{ (x, r) \in \mathbb{R}^n \times \mathbb{R} : g(x) \geq r \big\}$,

· g is proper: $\mathrm{dom}(g) \neq \emptyset$ and $g(x) < +\infty \ \forall x \in \mathbb{R}^n$,

· upper semi-continuous envelope: $\overline{g} : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that $\mathrm{epi}(\overline{g}) = \mathrm{cl}(\mathrm{epi}(g))$,

· conjugate function: $g^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, $g^*(p) = \inf\limits_{x \in \mathbb{R}^n} \big[p^T x - g(x) \big]$.

To avoid more intricate notations we denote all these notions in the same way for both convex and concave functions, because the meaning arises always clearly from the context. When necessary, we shall refer to them as "convex" when considered as for convex functions and "concave" otherwise. Let us specify that for the nearly convex functions these notions will be considered in the convex sense, while for the nearly concave ones they are taken in the same way as for concave functions.

Regarding the conjugate functions there is the famous Young-Fenchel inequality formulated below for a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and its convex conjugate, respectively for a function $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ and its concave conjugate (cf. [21]),

$$f^*(u) + f(x) \geq u^T x \geq g^*(u) + g(x) \ \forall u, x \in \mathbb{R}^n.$$

The two important results due to Fenchel to whose generalization this paper is dedicated to follow. The first of them is referred to as Fenchel's duality theorem throughout the present paper.

Theorem 2.1. (Theorem 31.1 in [21]) *Let f be a proper convex function on $\mathbb{R}^n$ and let g be a proper concave function on $\mathbb{R}^n$. One has*

$$\inf_{x \in \mathbb{R}^n} \big[f(x) - g(x) \big] = \sup_{u \in \mathbb{R}^n} \big\{ g^*(u) - f^*(u) \big\}$$

if either of the following conditions is satisfied:

(a) $\mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{ri}(\mathrm{dom}(g)) \neq \emptyset$*;*

(b) f and g are closed, and $\mathrm{ri}(\mathrm{dom}(f^*)) \cap \mathrm{ri}(\mathrm{dom}(g^*)) \neq \emptyset$*.*

Under (a) the supremum is attained at some u, while under (b) the infimum is attained at some x; if (a) and (b) both hold, the infimum and supremum are necessarily finite.

Theorem 2.2. (Corollary 31.2.1 in [21]) *Let f be a closed proper convex function on $\mathbb{R}^n$, let g be a closed proper concave function on $\mathbb{R}^m$, and let A be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. One has*

$$\inf_{x\in\mathbb{R}^n}\big[f(x)-g(Ax)\big]=\sup_{v\in\mathbb{R}^m}\big\{g^*(v)-f^*(A^*v)\big\}$$

if either of the following conditions is satisfied:

(a) there exists an $x'\in \mathrm{ri}(\mathrm{dom}(f))$ such that $Ax'\in \mathrm{ri}(\mathrm{dom}(g))$;

(b) there exists a $u\in \mathrm{ri}(\mathrm{dom}(g^*))$ such that $A^*u\in \mathrm{ri}(\mathrm{dom}(f^*))$.

Under (a) the supremum is attained at some u, while under (b) the infimum is attained at some x.

Let us recall now some other notions (cf. [3]) that play an important role within this paper.

Definition 2.1. A set $S\subseteq\mathbb{R}^n$ is called nearly convex if there is a constant $\alpha\in(0,1)$ such that for each x and y belonging to S it follows that $\alpha x+(1-\alpha)y\in S$.

The name "nearly convex" has been used in the literature also for other concepts, but we followed the terminology used in some relevant optimization papers ([4], [9], [13], [15], [18], [19], [22]).

Obviously every convex set is nearly convex, while $\mathbb{Q}\subseteq\mathbb{R}$, for instance, is nearly convex (with $\alpha=1/2$), but not a convex set.

Definition 2.2. A function $f:\mathbb{R}^n\to\overline{\mathbb{R}}$ is said to be nearly convex when there is an $\alpha\in(0,1)$ such that for all x and y in $\mathrm{dom}(f)=\big\{x\in\mathbb{R}^n: f(x)<+\infty\big\}$ we have

$$f\big(\alpha x+(1-\alpha)y\big)\le \alpha f(x)+(1-\alpha)f(y).$$

Definition 2.3. A function $g:\mathbb{R}^n\to\overline{\mathbb{R}}$ is said to be nearly concave when there is an $\alpha\in(0,1)$ such that for all x and y in $\mathrm{dom}(g)=\big\{x\in\mathbb{R}^n: g(x)>-\infty\big\}$ we have

$$g\big(\alpha x+(1-\alpha)y\big)\ge \alpha g(x)+(1-\alpha)g(y).$$

Obviously, if the function f is nearly convex, then $-f$ is nearly concave and viceversa.

Remark 2.1. The nearly convex/concave functions have nearly convex effective domains.

Any convex function is also nearly convex, but there are nearly convex functions that are not convex as is to be seen in the following.

Example 2.1. Let $F : \mathbb{R} \to \mathbb{R}$ be any discontinuous solution of Cauchy's functional equation $F(x + y) = F(x) + F(y)\ \forall x, y \in \mathbb{R}$. For each of these functions, whose existence is guaranteed in [12], one has

$$F\Big(\frac{x+y}{2}\Big) = \frac{F(x)+F(y)}{2}\ \forall x, y \in \mathbb{R},$$

i.e. these functions are nearly convex. None of these functions is convex because of the absence of continuity.

We need to introduce also some statements which are to be used later during the proof of the main theorem and afterwards. For the reader's convenience some of them are given with proofs. This is not the case for the first of them, whose proof is elementary, following minutely the one in the convex case.

Lemma 2.1. *Consider the functions $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$.*

(a) f is nearly convex if and only if its convex epigraph $\operatorname{epi}(f) = \big\{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \le r\big\}$ *is nearly convex.*

(b) g is nearly concave if and only if its concave epigraph $\operatorname{epi}(g) = \big\{(x, r) \in \mathbb{R}^n \times \mathbb{R} : g(x) \ge r\big\}$ *is nearly convex.*

Lemma 2.2. ([8]) *For a convex set $C \subseteq \mathbb{R}^n$ and any set $S \subseteq \mathbb{R}^n$ satisfying $S \subseteq C$ we have* $\operatorname{ri}(C) \subseteq S$ *if and only if* $\operatorname{ri}(C) = \operatorname{ri}(S)$.

Proof. Assuming first $\operatorname{ri}(C) \subseteq S$ to be true, we distinguish two cases. If C is empty, then $\operatorname{ri}(C) = \emptyset$ and $S = \emptyset$ (being included into C). Therefore $\operatorname{ri}(S) = \emptyset = \operatorname{ri}(C)$. When C is not empty, then by Theorem 6.2 in [21] we have $\operatorname{ri}(C) \neq \emptyset$ and $\operatorname{aff}(C) = \operatorname{aff}(\operatorname{ri}(C)) \subseteq \operatorname{aff}(S)$.

Since $S \subseteq C$ we have $\operatorname{aff}(S) \subseteq \operatorname{aff}(C)$, hence $\operatorname{aff}(S) = \operatorname{aff}(C) = \operatorname{aff}(\operatorname{ri}(C))$, so $\operatorname{ri}(S) \subseteq \operatorname{ri}(C)$ and $\operatorname{ri}(\operatorname{ri}(C)) \subseteq \operatorname{ri}(S)$. On the other hand, from [21] we have $\operatorname{ri}(\operatorname{ri}(C)) = \operatorname{ri}(C)$, followed by $\operatorname{ri}(C) \subseteq \operatorname{ri}(S)$. Therefore $\operatorname{ri}(C) = \operatorname{ri}(S)$.

Reversely, suppose $\operatorname{ri}(C) = \operatorname{ri}(S)$. As $\operatorname{ri}(S) \subseteq S$ (by definition), it follows $\operatorname{ri}(C) \subseteq S$. □

Lemma 2.3. ([1]) *For every nearly convex set $S \subseteq \mathbb{R}^n$ the following properties are valid* *(i)* $\mathrm{ri}(S)$ *is convex (may be empty),*

(ii) $\mathrm{cl}(S)$ *is convex,*

(iii) for every $x \in \mathrm{cl}(S)$ *and* $y \in \mathrm{ri}(S)$ *we have* $tx + (1-t)y \in \mathrm{ri}(S)$ *for each* $0 \leq t < 1$.

Remark 2.2 As every nearly convex which is also closed is actually convex, it is obvious that every lower semi-continuous nearly convex function is a convex (see Lemma 2.1).

Lemma 2.4. ([3]) *Let* $S \subseteq \mathbb{R}^n$ *be a nearly convex set. Then* $\mathrm{ri}(S) \neq \emptyset$ *if and only if* $\mathrm{ri}(\mathrm{cl}(S)) \subseteq S$.

Proof. Let us suppose first that $\mathrm{ri}(\mathrm{cl}(S)) \subseteq S$. As $S \subseteq \mathrm{cl}(S)$ it follows that $\mathrm{aff}(\mathrm{ri}(\mathrm{cl}(S))) \subseteq \mathrm{aff}(S) \subseteq \mathrm{aff}(\mathrm{cl}(S))$. From Theorem 6.2 in [21] we know, since $\mathrm{cl}(S)$ is convex, that $\mathrm{aff}(\mathrm{ri}(\mathrm{cl}(S))) = \mathrm{aff}(\mathrm{cl}(S))$, so we deduce that $\mathrm{aff}(S) = \mathrm{aff}(\mathrm{cl}(S))$. By Lemma 2.3(*ii*) $\mathrm{cl}(S)$ is convex, so $\mathrm{ri}(\mathrm{cl}(S))$ is non-empty and convex (cf. [21]). Therefore by Theorem 6.2 in [21] follows $\mathrm{aff}(\mathrm{cl}(S)) = \mathrm{aff}(\mathrm{ri}(\mathrm{cl}(S)))$, hence $\mathrm{aff}(S) = \mathrm{aff}(\mathrm{ri}(\mathrm{cl}(S)))$. Using the initial assumption in this proof it follows $\mathrm{ri}(\mathrm{ri}(\mathrm{cl}(S))) \subseteq \mathrm{ri}(S)$, which leads to $\emptyset \neq \mathrm{ri}(\mathrm{cl}(S)) \subseteq \mathrm{ri}(S)$. Thus $\mathrm{ri}(S) \neq \emptyset$.

Now let us prove the reverse statement. Take an arbitrary element $x \in \mathrm{ri}(\mathrm{cl}(S))$. By definition, there is an $\varepsilon > 0$ such that $(x + B(0,\varepsilon)) \cap \mathrm{aff}(\mathrm{cl}(S)) \subseteq \mathrm{cl}(S)$, where $B(0,\varepsilon)$ is the closed ball centered in 0 and with radius ε in $\mathbb{R}^n$. Choosing an arbitrary element $x' \in \mathrm{ri}(S)$ and a $t \in (0,1)$ such that

$$\frac{t}{1-t}\|x - x'\| < \varepsilon,$$

it follows that $z := (1/(1-t))x + (-t/(1-t))x'$ belongs to $\mathrm{aff}(\mathrm{cl}(S))$ and

$$\|z - x\| = \left\| \frac{-t}{1-t}x' + \frac{t}{1-t}x \right\| = \frac{t}{1-t}\|x - x'\| < \varepsilon.$$

Therefore $z \in \mathrm{cl}(S)$. Since $x' \in \mathrm{ri}(S)$, Lemma 2.3(*iii*) leads to $x = tx' + (1-t)z \in \mathrm{ri}(S) \subseteq S$. □

Lemma 2.5. ([3]) *For a non-empty nearly convex set* $S \subseteq \mathbb{R}^n$, $\mathrm{ri}(S) \neq \emptyset$ *if and only if* $\mathrm{ri}(S) = \mathrm{ri}(\mathrm{cl}(S))$.

Proof. By Lemma 2.3(*ii*) $\mathrm{cl}(S)$ is convex, so Theorem 6.2 in [21] yields $\mathrm{ri}(\mathrm{cl}(S)) \neq \emptyset$. Assuming that $\mathrm{ri}(S) = \mathrm{ri}(\mathrm{cl}(S))$, it becomes obvious that $\mathrm{ri}(S) \neq \emptyset$.

Within the proof of the previous lemma we have shown the following equivalence

$$\text{ri}(S) \neq \emptyset \Leftrightarrow \text{ri}(\text{cl}(S)) \subseteq \text{ri}(S).$$

To prove the reverse inclusion in the right-hand side let us take an arbitrary element $x \in \text{ri}(S)$. By definition, there is an $\varepsilon > 0$ such that $(x + B(0, \varepsilon)) \cap \text{aff}(S) \subseteq S$. Moreover by [21] $\text{aff}(S) = \text{aff}(\text{cl}(S))$ and $S \subseteq \text{cl}(S)$, therefore

$$(x + B(0, \varepsilon)) \cap \text{aff}(\text{cl}(S)) \subseteq \text{cl}(S),$$

i.e. $x \in \text{ri}(\text{cl}(S))$. Hence $\text{ri}(S) \subseteq \text{ri}(\text{cl}(S))$ whenever $\text{ri}(S) \neq \emptyset$. The conclusion arises naturally. □

3 The extension of Fenchel's duality theorem

As said before, the importance of Fenchel's duality theorem in convex analysis and optimization is really huge. Let us briefly recall its content. For a proper convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and a proper concave one $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ the infimal value of the objective function of the convex optimization problem

$$(P) \qquad \inf_{x \in \mathbb{R}^n} \big[f(x) - g(x)\big]$$

and the supreme objective value of its Fenchel dual problem

$$\big(P^*\big) \qquad \sup_{u \in \mathbb{R}^n} \big\{g^*(u) - f^*(u)\big\}$$

are equal and the latter supremum is attained at some point $u \in \mathbb{R}^n$, provided that

$$\text{ri}(\text{dom}(f)) \cap \text{ri}(\text{dom}(g)) \neq \emptyset.$$

Remark 3.1. The Fenchel dual problem can be obtained also via the perturbation theory described in [7].

Our intention is to weaken the conditions imposed in the book of Rockafellar [21] without altering the conclusion. That is why we consider f nearly convex and g nearly concave, and not convex, respectively concave. We need also to assume two additional conditions to be fulfilled, which are satisfied when f is convex and g concave. But let us formulate and prove the main result in this paper.

Theorem 3.1 *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper nearly convex function and let $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper nearly concave function. If the following conditions are simultaneously satisfied*

(i) $\text{ri}(\text{dom}(f)) \cap \text{ri}(\text{dom}(g)) \neq \emptyset$*,*

(ii) $\mathrm{ri}(\mathrm{epi}(f)) \neq \emptyset$,

(iii) $\mathrm{ri}(\mathrm{epi}(g)) \neq \emptyset$,

then one has

$$\inf_{x \in \mathbb{R}^n} \big[f(x) - g(x)\big] = \sup_{u \in \mathbb{R}^n} \big\{g^*(u) - f^*(u)\big\}$$

and the supremum is attained at some $u \in \mathbb{R}^n$.

Proof. According to Lemma 2.1, the set $\mathrm{epi}(f)$ is nearly convex, the same property being valid for $\mathrm{epi}(g)$, too.

As $\mathrm{ri}(\mathrm{epi}(f)) \neq \emptyset$, by Lemma 2.4 it follows $\mathrm{ri}(\mathrm{cl}(\mathrm{epi}(f))) \subseteq \mathrm{epi}(f)$. On the other hand, by Lemma 2.3(*ii*), the set $\mathrm{epi}(\bar{f}) = \mathrm{cl}(\mathrm{epi}(f))$ is closed and also convex. Therefore $\bar{f}$ is a convex function and by Lemma 7.3 in [21] we have

$$\mathrm{ri}(\mathrm{epi}(\bar{f})) = \big\{(x, r) : \bar{f}(x) < r, x \in \mathrm{ri}(\mathrm{dom}(\bar{f}))\big\} = \mathrm{ri}(\mathrm{cl}(\mathrm{epi}(f))) \subseteq \mathrm{epi}(f).$$

Take an x from $\mathrm{ri}(\mathrm{dom}(\bar{f}))$. It follows that for all $\varepsilon > 0$ we have $(x, \bar{f}(x) + \varepsilon) \in \mathrm{ri}(\mathrm{epi}(\bar{f})) \subseteq \mathrm{epi}(f)$, so $f(x) \leq \bar{f}(x) + \varepsilon$. Letting ε tend to 0 it follows $f(x) \leq \bar{f}(x)$ $\forall x \in \mathrm{ri}(\mathrm{dom}(\bar{f}))$, which implies that $\bar{f}(x) = f(x)$ $\forall x \in \mathrm{ri}(\mathrm{dom}(\bar{f}))$.

Thus $\mathrm{ri}(\mathrm{dom}(\bar{f})) \subseteq \mathrm{dom}(f)$ and as $\mathrm{dom}(f) \subseteq \mathrm{dom}(\bar{f})$, by Lemma 2.2 it follows that $\mathrm{ri}(\mathrm{dom}(\bar{f})) = \mathrm{ri}(\mathrm{dom}(f))$.

As f is not identical $+\infty$ it follows that $\bar{f}$ is also not identical $+\infty$. Assume now that there exists an $x' \in \mathbb{R}^n$ such that $\bar{f}(x') = -\infty$. The function $\bar{f}$ being convex and lower semi-continuous we have that $\bar{f}(x) = -\infty$ $\forall x \in \mathrm{dom}(\bar{f})$. But, as $\bar{f}(x) = f(x)$ $\forall x \in \mathrm{ri}(\mathrm{dom}(\bar{f}))$ and $\mathrm{ri}(\mathrm{dom}(\bar{f})) \neq \emptyset$, this contradicts the properness of f. This proves that $\bar{f}$ is a proper function.

In a similar way, for $\bar{g}$ being the upper semi-continuous envelope of g we can prove that $\bar{g}$ is a proper, concave and upper semi-continuous function with the property that $\bar{g}(x) = g(x)$ $\forall x \in \mathrm{ri}(\mathrm{dom}(\bar{g}))$ and $\mathrm{ri}(\mathrm{dom}(\bar{g})) = \mathrm{ri}(\mathrm{dom}(g))$.

The assumption (*i*) implies that $\mathrm{ri}(\mathrm{dom}(\bar{f})) \cap \mathrm{ri}(\mathrm{dom}(\bar{g})) \neq \emptyset$. Let us denote now by $\alpha := \inf_{x \in \mathbb{R}^n} \big[\bar{f}(x) - \bar{g}(x)\big]$. It is obvious that $\alpha \in [-\infty, +\infty)$. Next we prove that $\alpha = \inf_{x \in \mathbb{R}^n} [f(x) - g(x)]$. It is obvious that $\alpha \leq \inf_{x \in \mathbb{R}^n} [f(x) - g(x)]$.

Assume for the beginning that $\alpha \in \mathbb{R}$. Given $\varepsilon > 0$, let $y \in \mathbb{R}^n$ be such that

$$\bar{f}(y) - \bar{g}(y) < \alpha + \varepsilon.$$

It is easy to see that $y \in \mathrm{dom}(\bar{f}) \cap \mathrm{dom}(\bar{g})$. Let now be $z \in \mathrm{ri}(\mathrm{dom}(\bar{f})) \cap \mathrm{ri}(\mathrm{dom}(\bar{g}))$. For all $\lambda \in (0, 1]$ we have then $(1-\lambda)y + \lambda z \in \mathrm{ri}(\mathrm{dom}(\bar{f})) \cap \mathrm{ri}(\mathrm{dom}(\bar{g}))$. As $f(x) - g(x) = \bar{f}(x) - \bar{g}(x)$ $\forall x \in \mathrm{ri}(\mathrm{dom}(\bar{f})) \cap \mathrm{ri}(\mathrm{dom}(\bar{g}))$, it follows that

$$f((1-\lambda)y + \lambda z) - g((1-\lambda)y + \lambda z) = \bar{f}((1-\lambda)y + \lambda z) - \bar{g}((1-\lambda)y + \lambda z) \leq$$

$$(1-\lambda)(\bar{f}(y) - \bar{g}(y)) + \lambda(\bar{f}(z) - \bar{g}(z)) =$$

$$\bar{f}(y) - \bar{g}(y) + \lambda(\bar{f}(z) - \bar{f}(y) + \bar{g}(y) - \bar{g}(z))\ \forall \lambda \in (0,1].$$

Choosing $\bar{\lambda} \in (0,1]$ such that $\bar{\lambda}(\bar{f}(z) - \bar{f}(y) + \bar{g}(y) - \bar{g}(z)) \leq \varepsilon$, we obtain an element $\bar{x} := (1-\bar{\lambda})y + \bar{\lambda}z \in \mathbb{R}^n$ such that $f(\bar{x}) - g(\bar{x}) \leq \bar{f}(y) - \bar{g}(y) + \varepsilon$ and therefore $\inf\limits_{x \in \mathbb{R}^n} [f(x) - g(x)] \leq \bar{f}(y) - \bar{g}(y) + \varepsilon < \alpha + 2\varepsilon$. Letting ε converge to 0 it follows that $\inf\limits_{x \in \mathbb{R}^n} [f(x) - g(x)] = \alpha$.

Assume now that $\alpha = -\infty$. For any $k \geq 1$ there exists $y_k \in \mathbb{R}^n$ such that

$$\bar{f}(y_k) - \bar{g}(y_k) < -k - 1.$$

Let be again $z \in \mathrm{ri}(\mathrm{dom}(\bar{f})) \cap \mathrm{ri}(\mathrm{dom}(\bar{g}))$. As $y_k \in \mathrm{dom}(\bar{f}) \cap \mathrm{dom}(\bar{g})$, we get $(1-\lambda)y_k + \lambda z \in \mathrm{ri}(\mathrm{dom}(\bar{f})) \cap \mathrm{ri}(\mathrm{dom}(\bar{g}))\ \forall k \geq 1$ and

$$f((1-\lambda)y_k + \lambda z) - g((1-\lambda)y_k + \lambda z) = \bar{f}((1-\lambda)y_k + \lambda z) - \bar{g}((1-\lambda)y_k + \lambda z) \leq$$

$$(1-\lambda)(\bar{f}(y_k) - \bar{g}(y_k)) + \lambda(\bar{f}(z) - \bar{g}(z)) =$$

$$\bar{f}(y_k) - \bar{g}(y_k) + \lambda(\bar{f}(z) - \bar{f}(y_k) + \bar{g}(y_k) - \bar{g}(z))\ \forall \lambda \in (0,1]\ \forall k \geq 1.$$

Choosing $\bar{\lambda}_k \in (0,1]$ such that $\bar{\lambda}_k(\bar{f}(z) - \bar{f}(y_k) + \bar{g}(y_k) - \bar{g}(z)) \leq 1$, we obtain an $x_k := (1-\bar{\lambda}_k)y_k + \bar{\lambda}_k z \in \mathbb{R}^n$ such that $f(x_k) - g(x_k) \leq \bar{f}(y_k) - \bar{g}(y_k) + 1 < -k\ \forall k \geq 1$. Therefore $\inf\limits_{x \in \mathbb{R}^n} [f(x) - g(x)] = -\infty = \alpha$.

Applying Fenchel's duality theorem (Theorem 31.1 in [21]) for the functions $\bar{f}$ and $\bar{g}$ we have that

$$\inf_{x \in \mathbb{R}^n} \left[\bar{f}(x) - \bar{g}(x)\right] = \sup_{u \in \mathbb{R}^n} \left\{(\bar{g})^*(u) - (\bar{f})^*(u)\right\}$$

and the supremum is attained at some $u \in \mathbb{R}^n$. As the equalities $f^* = (\bar{f})^*$ and $g^* = (\bar{g})^*$ are always fulfilled (cf. [21]) one has that

$$\inf_{x \in \mathbb{R}^n} \left[f(x) - g(x)\right] = \inf_{x \in \mathbb{R}^n} \left[\bar{f}(x) - \bar{g}(x)\right] = \sup_{u \in \mathbb{R}^n} \left\{g^*(u) - f^*(u)\right\}$$

and the supremum is attained at some $u \in \mathbb{R}^n$. □

Remark 3.2. Even if it seems to be very strong, the condition (i) does not always imply the other two conditions assumed in the hypothesis. Taking as f a discontinuous solutions of Cauchy's functional equation F (cf. Example 2.1) which is nearly convex as proved within the previous section, we have that $\mathrm{dom}(f) = \mathbb{R} = \mathrm{ri}(\mathrm{dom}(f))$, so (i) is fulfilled for any function g whose effective domain has non-empty relative interior. Let us consider also $g = -F$, which is naturally a nearly concave function and $\mathrm{ri}(\mathrm{dom}(g)) = \mathbb{R}$.

Assuming that $\mathrm{ri}(\mathrm{epi}(f)) \neq \emptyset$, this would imply that $\bar{f}(x) = f(x)\ \forall x \in \mathrm{ri}(\mathrm{dom}(\bar{f}))$. As $\mathbb{R} = \mathrm{dom}(f) \subseteq \mathrm{dom}(\bar{f}) \subseteq \mathbb{R}$ it follows $\mathrm{dom}(\bar{f}) = \mathbb{R} = \mathrm{ri}(\mathrm{dom}(\bar{f}))$, so $f = \bar{f}$, i.e. f is a lower semi-continuous function, hence its epigraph is closed.

Being a nearly convex set whose closure is convex, it follows that $\text{epi}(f)$ is convex, thus f is also convex. This reveals the continuity of f, which contradicts what has been proven in [12], thus the assumption on which this reasoning is based turns out to be false, i.e. $\text{ri}(\text{epi}(f)) = \emptyset$. Therefore F is an example of a function which satisfies (i), but does not meet (ii).

Let us assume now that (iii) is valid, i.e. $\text{ri}(\text{epi}(g)) \neq \emptyset$. We have

$$\text{epi}(g) = \big\{(x,r) : r \leq g(x)\big\} = \big\{(x,r) : F(x) \leq -r\big\}.$$

Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined as $L(x,r) = (x,-r)\ \forall (x,r) \in \mathbb{R}^2$. It is easy to notice that $L^{-1} = L$ and $L(\text{epi}(g)) = \text{epi}(F)$. From (iii) follows that (cf. Lemma 2.4) $\text{ri}(\text{cl}(\text{epi}(g))) \subseteq \text{epi}(g)$, so $L\big(\text{ri}(\text{cl}(\text{epi}(g)))\big) \subseteq L(\text{epi}(g))$. As $\text{cl}(\text{epi}(g))$ is convex (by Lemma 2.1 and Lemma 2.3(ii)), by Theorem 6.6 in [21] we get

$$\text{ri}\,\big(L(\text{cl}(\text{epi}(g)))\big) = L\big(\text{ri}(\text{cl}(\text{epi}(g)))\big) \subseteq L(\text{epi}(g)).$$

Because $L(\text{epi}(g)) \subseteq L(\text{cl}(\text{epi}(g)))$ and due to the fact that $L(\text{cl}(\text{epi}(g)))$ is convex, by Lemma 2.2 follows

$$\text{ri}(L(\text{epi}(g))) = \text{ri}(L(\text{cl}(\text{epi}(g)))).$$

The set in the left-hand side is actually $\text{ri}(\text{epi}(F))$ which has been proven before to be empty, while in the right-hand side there is the relative interior of a non-empty convex set, obviously non-empty. We have reached a contradiction, so for $g = -F$ (i) holds, but (iii) does not. Therefore conditions $(i) - (iii)$ are all required in order to formulate a hypothesis sufficient to imply the assertions of Theorem 3.1.

Remark 3.3. When the function f is proper convex (so also nearly convex) it follows that $\text{epi}(f)$ is convex and non-empty, so $\text{ri}(\text{epi}(f)) \neq \emptyset$. Therefore in this case the condition (ii) becomes redundant. The same applies when g is proper concave, i.e. (iii) is surely fulfilled. It is obvious that when both happen, i.e. f is convex and g is concave, Theorem 3.1 becomes actually Fenchel's duality theorem. This sustains our claim that we have extended Fenchel's statement to nearly convexity.

Remark 3.4. Fenchel's duality theorem contains a second part (Theorem 2.1(b)) where the condition (i) is replaced by the non-emptiness of the intersection of the relative interiors of f^* and g^* provided that f and g were closed functions. As (convex) closedness for functions means convexity plus lower semi-continuity, let us consider f nearly convex and lower semi-continuous. Then $\text{epi}(f)$ is closed, so also convex (cf. Lemma 2.3(ii)), thus f is convex and also closed. Analogously g becomes closed concave if it is supposed also upper semi-continuous additionally to being nearly convex. Therefore we can conclude that this part of Fenchel's theorem can be reformulated as follows, while its proof is actually the one in book [21] .

Theorem 3.2 *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a lower semi-continuous proper nearly convex function and let $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be an upper semi-continuous proper nearly concave function. One has*

$$\inf_{x\in\mathbb{R}^n} \big[f(x) - g(x)\big] = \sup_{u\in\mathbb{R}^n} \big\{g^*(u) - f^*(u)\big\},$$

provided that

$$\mathrm{ri}(\mathrm{dom}(f^*)) \cap \mathrm{ri}(\mathrm{dom}(g^*)) \neq \emptyset,$$

where the infimum is attained at some x.

The best way to prove the usefulness of a generalization is to give an example which cannot be properly treated by using the original statement, but is writable as a special case of the extended assertion.

Example 3.1. Consider the sets

$$\begin{aligned} \mathcal{F} &= \big\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\big\} \cup \big\{(x_1, 0) \in \mathbb{R}^2 : x_1 \in \mathbb{Q}, x_1 \geq 0\big\} \\ &\cup \big\{(0, x_2) \in \mathbb{R}^2 : x_2 \in \mathbb{Q}, x_2 \geq 0\big\} \end{aligned}$$

and

$$\mathcal{G} = \big\{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 < 3\big\} \cup \big\{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \in \mathbb{Q}, x_1 + x_2 = 3\big\}$$

and some real-valued functions defined on $\mathbb{R}^2$, f convex and g concave. Both $\mathcal{F}$ and $\mathcal{G}$ are nearly convex sets with $\alpha = 1/2$ playing the role of the constant required in the definition, but not convex. We are interested in treating by Fenchel duality the problem

$$(P_1) \qquad \inf_{x=(x_1,x_2)\in\mathcal{F}\cap\mathcal{G}} \big[f(x) - g(x)\big],$$

i.e. we would like to obtain the infimal objective value of (P_1) by using the conjugate functions of f and g. A Fenchel-type dual problem may be attached to this problem, but the conditions under which the primal and the dual have equal optimal objective values are not known to us as we cannot apply Fenchel's duality theorem because $\mathcal{F} \cap \mathcal{G}$ is not convex. Let us define now the functions

$$\tilde{f} : \mathbb{R}^2 \to \overline{\mathbb{R}}, \ \tilde{f}(x) = \begin{cases} f(x), & x \in \mathcal{F}, \\ +\infty, & \text{otherwise}, \end{cases}$$

and

$$\tilde{g} : \mathbb{R}^2 \to \overline{\mathbb{R}}, \ \tilde{g}(x) = \begin{cases} g(x), & x \in \mathcal{G}, \\ +\infty, & \text{otherwise}. \end{cases}$$

The function $\tilde{f}$ is clearly nearly convex (with constant 1/2), but not convex since $\mathrm{dom}(\tilde{f}) = \mathcal{F}$ is not convex. Analogously $\tilde{g}$ is nearly concave, but not concave. Therefore we are not yet in the situation to apply Fenchel's duality theorem for the problem

$$\inf_{x\in\mathbb{R}^2} \big[\tilde{f}(x) - \tilde{g}(x)\big],$$

which is actually equivalent to (P_1), but let us check whether the extension we have given in Theorem 3.1 is applicable. Condition (i) in Theorem 3.1 is satisfied in this case since

$$\begin{aligned}\mathrm{ri}(\mathrm{dom}(\tilde{f})) \cap \mathrm{ri}(\mathrm{dom}(\tilde{g})) &= \mathrm{ri}(\mathcal{F}) \cap \mathrm{ri}(\mathcal{G}) \\ &= (0,+\infty) \times (0,+\infty) \cap \big\{(x,y) \in \mathbb{R}^2 : x + y < 3\big\},\end{aligned}$$

which is non-empty since $(1,1)$, for instance, is contained in both sets.

Regarding the relative interiors of the epigraphs of $\tilde{f}$ and $\tilde{g}$, it is not difficult to check that $\big((1,1), f(1,1)+1\big) \in \mathrm{int}(\mathrm{epi}(\tilde{f})) = \mathrm{ri}(\mathrm{epi}(\tilde{f}))$ and $\big((1,1), g(1,1)-1\big) \in \mathrm{int}(\mathrm{epi}(\tilde{g})) = \mathrm{ri}(\mathrm{epi}(\tilde{g}))$.

Therefore the conditions (ii) and (iii) in the hypothesis of Theorem 3.1 are fulfilled for $\tilde{f}$ and $\tilde{g}$, respectively. So we can apply the statement and we get that

$$\inf_{x\in\mathcal{F}\cap\mathcal{G}} \big[f(x) - g(x)\big] = \inf_{x\in\mathbb{R}^2} \big[\tilde{f}(x) - \tilde{g}(x)\big]$$

$$= \sup_{u\in\mathbb{R}^2} \big\{\tilde{g}^*(u) - \tilde{f}^*(u)\big\} = \sup_{u\in\mathbb{R}^n} \big\{g^*_{\mathcal{G}}(u) - f^*_{\mathcal{F}}(u)\big\}$$

and the suprema are attained at some $u \in \mathbb{R}^n$. Here we have used the conjugate function of f regarding the set $\mathcal{F}$ defined as

$$f^*_{\mathcal{F}} : \mathbb{R}^2 \to \overline{\mathbb{R}}, \quad f^*_{\mathcal{F}}(u) = \sup_{x\in\mathcal{F}} \big\{u^T x - f(x)\big\}$$

and the one of g regarding the set $\mathcal{G}$,

$$g^*_{\mathcal{G}} : \mathbb{R}^2 \to \overline{\mathbb{R}}, \quad g^*_{\mathcal{G}}(u) = \inf_{x\in\mathcal{G}} \big\{u^T x - g(x)\big\}.$$

Knowing the connections between Lagrange duality and games theory, we give an application of our main result in the latter field opening the gate into the direction of Fenchel duality.

Example 3.2. Consider a two-person zero-sum game, where D and C are the sets of strategies for the players I and II, respectively, and $L : C \times D \to \mathbb{R}$ is the so-called payoff-function. By $\beta^L := \sup_{d\in D} \inf_{c\in C} L(c,d)$ and $\alpha^L := \inf_{c\in C} \sup_{d\in D} L(c,d)$ we denote the lower, respectively the upper values of the game (C, D, L). As the minmax inequality

$$\beta^L = \sup_{d\in D} \inf_{c\in C} L(c,d) \le \inf_{c\in C} \sup_{d\in D} L(c,d) = \alpha^L, \tag{1}$$

is always fulfilled, the challenge was to find weak conditions which guarantee equality in the relation above. Nearly convexity and its generalizations played an important role within, as [10], [18] or [19] show. Having an optimization problem with geometrical and inequality constraints,

$$(P_c) \quad \inf_{\substack{x \in X, \\ w(x) \leqq 0}} v(x),$$

where $X \subseteq \mathbb{R}^n$, $w : \mathbb{R}^n \to \mathbb{R}^k$ and $v : \mathbb{R}^n \to \overline{\mathbb{R}}$, the Lagrangian attached to it is $L : X \times \mathbb{R}^k_+ \to \overline{\mathbb{R}}$, $L(x, \lambda) = v(x) + \lambda^T w(x)$. The theorems mentioned above give sufficient conditions under which the strong duality occurs between (P_c) and its Lagrange dual

$$(D_c) \quad \sup_{\lambda \geqq 0} \inf_{x \in X} L(x, \lambda) = \sup_{\lambda \geqq 0} \inf_{x \in X} [v(x) + \lambda^T w(x)],$$

which is nothing else than the equality in (1).

Coming to the problem treated in Theorem 3.1,

$$\inf_{x \in \mathbb{R}^n} [f(x) - g(x)],$$

one can define the Lagrangian attached to it by (cf. [7])

$$L : \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}, \ L(x, u) = u^T x - g(x) - f^*(u).$$

As

$$\sup_{u \in \mathbb{R}^n} \inf_{x \in \mathbb{R}^n} L(x, u) = \sup_{u \in \mathbb{R}^n} [g^*(u) - f^*(u)]$$

and

$$\inf_{x \in \mathbb{R}^n} \sup_{u \in \mathbb{R}^n} L(x, u) = \inf_{x \in \mathbb{R}^n} [-g(x) + f^{**}(x)] \leq \inf_{x \in \mathbb{R}^n} [f(x) - g(x)],$$

under the hypotheses of Theorem 3.1 one has

$$\max_{u \in \mathbb{R}^n} \inf_{x \in \mathbb{R}^n} L(x, u) = \max_{u \in \mathbb{R}^n} [g^*(u) - f^*(u)] = \inf_{x \in \mathbb{R}^n} \sup_{u \in \mathbb{R}^n} L(x, u) = \inf_{x \in \mathbb{R}^n} [f(x) - g(x)].$$

The solution to the dual problem can be seen as an optimal strategy for the game having this Lagrangian as payoff-function.

4 The case of post-composition with a linear transformation

There is also another statement sometimes called Fenchel's duality theorem. It is given in [21] as Corollary 31.2.1, being also presented in our paper as Theorem

2.2. We have extended this result for nearly convexity, too, as follows.

Theorem 4.1. *Let f be a proper nearly convex function on $\mathbb{R}^n$, let g be a proper nearly concave function on $\mathbb{R}^m$, and let A be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. If the following conditions are satisfied*

(i) $\exists x' \in \operatorname{ri}(\operatorname{dom}(f))$ *such that* $Ax' \in \operatorname{ri}(\operatorname{dom}(g))$,

(ii) $\operatorname{ri}(\operatorname{epi}(f)) \neq \emptyset$,

(iii) $\operatorname{ri}(\operatorname{epi}(g)) \neq \emptyset$,

then one has

$$\inf_{x\in\mathbb{R}^n} \big[f(x) - g(Ax)\big] = \sup_{v\in\mathbb{R}^m} \big\{g^*(v) - f^*(A^*v)\big\}$$

and the supremum is attained at some $v \in \mathbb{R}^m$.

Proof. We apply the Theorem 3.1 for the functions

$$F, G : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}},\ F(x,y) = f(x) + \delta_{\{x\in\mathbb{R}^n : Ax=y\}}(x),\ G(x,y) = g(y),$$

which are easily verifiable as nearly convex, respectively nearly concave. We have $\operatorname{dom}(F) = \operatorname{dom}(f) \times A(\operatorname{dom} f)$ and $\operatorname{dom}(G) = \mathbb{R}^n \times \operatorname{dom}(g)$. Now let us see if F and G verify the conditions $(i) - (iii)$ in Theorem 3.1. We have

$$\operatorname{epi}(F) = \big\{(x,y,r) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} : Ax = y \text{ and } f(x) \leq r\big\}.$$

We have by (i) that $\operatorname{ri}(\operatorname{dom}(f)) \neq \emptyset$, thus, by Lemma 2.4, $\operatorname{ri}(\operatorname{cl}(\operatorname{dom}(f))) \subseteq \operatorname{dom}(f)$. By Theorem 6.6 in [21] one has

$$\operatorname{ri}(A(\operatorname{cl}(\operatorname{dom}(f)))) = A(\operatorname{ri}(\operatorname{cl}(\operatorname{dom}(f)))) \subseteq A(\operatorname{dom}(f)),$$

while it is also true that $A(\operatorname{dom}(f)) \subseteq A(\operatorname{cl}(\operatorname{dom}(f)))$, so Lemma 2.2 yields $\operatorname{ri}(A(\operatorname{dom} f)) = \operatorname{ri}(A(\operatorname{cl}(\operatorname{dom}(f))))$. Applying again Theorem 6.6 in [21] and then Lemma 2.5 we obtain

$$\operatorname{ri}(A(\operatorname{dom} f)) = \operatorname{ri}(A(\operatorname{cl}(\operatorname{dom}(f)))) = A(\operatorname{ri}(\operatorname{cl}(\operatorname{dom}(f)))) = A(\operatorname{ri}(\operatorname{dom}(f))),$$

so (i) implies that

$$\begin{aligned}(x', Ax') &\in \big(\operatorname{ri}(\operatorname{dom}(f)) \times \operatorname{ri}(A(\operatorname{dom}(f)))\big) \cap \big(\mathbb{R}^n \times \operatorname{ri}(\operatorname{dom}(g))\big)\\ &= \operatorname{ri}\big(\operatorname{dom}(f) \times A(\operatorname{dom}(f))\big) \cap \operatorname{ri}\big(\mathbb{R}^n \times \operatorname{dom}(g)\big)\\ &= \operatorname{ri}(\operatorname{dom}(F)) \cap \operatorname{ri}(\operatorname{dom}(G)),\end{aligned}$$

i.e. (i) in Theorem 3.1 is satisfied by F and G. Consider now the linear transformation $M : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ defined by $M(x,r) = (x, Ax, r)$. For any pair

$(x,r) \in \text{epi}(f)$ we have $M(x,r) = (x, Ax, r) \in \text{epi}(F)$, thus $M(\text{epi}(f)) \subseteq \text{epi}(F)$. On the other hand, for each triplet $(x,y,r) \in \text{epi}(F)$ we know that $(x,r) \in \text{epi}(f)$ and $Ax = y$, so $(x,y,r) = (x,Ax,r) = M(x,r) \in M(\text{epi}(f))$. Therefore, $M(\text{epi}(f)) = \text{epi}(F)$. By (ii) and Lemma 2.4 follows $\text{ri}(\text{cl}(\text{epi}(f))) \subseteq \text{epi}(f)$. Applying the transformation M to both sides, one gets,

$$M(\text{ri}(\text{cl}(\text{epi}(f)))) \subseteq M(\text{epi}(f)).$$

Because Lemma 2.3(ii) assures the convexity of $\text{cl}(\text{epi}(f))$, by Theorem 6.6 in [21] follows $M(\text{ri}(\text{cl}(\text{epi}(f)))) = \text{ri}(M(\text{cl}(\text{epi}(f))))$. Consequently,

$$\text{ri}(M(\text{cl}(\text{epi}(f)))) \subseteq M(\text{epi}(f)).$$

Applying M to both sides of the obvious inclusion $\text{epi}(f) \subseteq \text{cl}(\text{epi}(f))$ we get

$$M(\text{epi}(f)) \subseteq M(\text{cl}(\text{epi}(f))),$$

so we can apply Lemma 2.2 for the convex set $M(\text{cl}(\text{epi}(f)))$ and its subset $M(\text{epi}(f))$, obtaining

$$\text{ri}(M(\text{cl}(\text{epi}(f)))) = \text{ri}(M(\text{epi}(f))).$$

The left-hand side term is non-empty (by Theorem 6.2 in [21]), while the set in the right hand side is actually equal to $\text{ri}(\text{epi}(F))$, thus F satisfies condition (ii) in Theorem 3.1.

Let us see what happens with the relative interior of $\text{epi}(G)$. We have

$$\begin{aligned} \text{epi}(G) &= \big\{(x,y,r) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} : r \le G(x,y)\big\} \\ &= \big\{(x,y,r) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} : r \le g(y)\big\} = \mathbb{R}^n \times \text{epi}(g). \end{aligned}$$

As $\text{ri}(\text{epi}(g)) \neq \emptyset$, it follows that $\text{ri}(\text{epi}(G)) \neq \emptyset$. We are allowed to apply Theorem 3.1 for F and G. Consequently,

$$\inf_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m} \big[F(x,y) - G(x,y)\big] = \sup_{(u,v)\in\mathbb{R}^n\times\mathbb{R}^m} \big\{G^*(u,v) - F^*(u,v)\big\}$$

and the supremum in the right-hand side is attained at some $(u,v) \in \mathbb{R}^n \times \mathbb{R}^m$. Let us see what does this mean for f and g. Regarding the left-hand side of the relation above we have

$$\begin{aligned} \inf_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m} \big[F(x,y) - G(x,y)\big] &= \inf_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m} \big[f(x) + \delta_{\{x\in\mathbb{R}^n : Ax=y\}}(x) - g(y)\big] \\ &= \inf_{x\in\mathbb{R}^n} \big[f(x) - g(Ax)\big]. \end{aligned}$$

We calculate now the conjugate functions of F and G, respectively. So we get

$$\begin{aligned} F^*(u,v) &= \sup_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m} \big\{u^T x + v^T y - f(x) - \delta_{\{x\in\mathbb{R}^n : Ax=y\}}(x)\big\} \\ &= \sup_{x\in\mathbb{R}^n} \big\{u^T x + v^T (Ax) - f(x)\big\} \\ &= \sup_{x\in\mathbb{R}^n} \big\{(u + A^* v)^T x - f(x)\big\} = f^*(u + A^* v), \end{aligned}$$

$$G^*(u,v) = \inf_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m} \left[u^T x + v^T y - g(y)\right] = \begin{cases} g^*(v), & u = 0, \\ -\infty, & u \neq 0. \end{cases}$$

Therefore,

$$\begin{aligned} \sup_{(u,v)\in\mathbb{R}^n\times\mathbb{R}^m} \left\{G^*(u,v) - F^*(u,v)\right\} &= \sup_{\substack{(u,v)\in\mathbb{R}^n\times\mathbb{R}^m,\\ u=0}} \left\{g^*(v) - f^*(u + A^*v)\right\} \\ &= \sup_{v\in\mathbb{R}^m} \left\{g^*(v) - f^*(A^*v)\right\}, \end{aligned}$$

so the assertion follows. □

Remark 4.1. When f is convex and proper and g is concave and proper it follows that $\mathrm{ri}(\mathrm{epi}(f)) \neq \emptyset$ and $\mathrm{ri}(\mathrm{epi}(g)) \neq \emptyset$ (cf. Remark 3.3). Hence it becomes obvious that the assertion of Corollary 31.2.1 in [21] is valid under the condition (a) ((i) here) without any closedness assumption concerning f or g . Let us remind that in the mentioned book f and g are considered to be closed.

Remark 4.2. Since a nearly convex function that is also lower semi-continuous is convex and closed we are allowed to say that the variant of Theorem 2.2 when (b) is fulfilled may also be generalized by considering as hypothesis alongside the mentioned condition (b) that f is a proper nearly convex function that is also lower semi-continuous and g proper nearly concave and upper semi-continuous. The assertion remains valid and the proof does not differ from the one suggested in [21] as f turns out immediately to be closed convex and g closed concave.

Remark 4.3. One may notice that the assumption of nearly convexity applied to f and of nearly concavity concerning g simultaneously does not require the same nearly convexity constant to be attached to both of these functions.

5 Conclusions

Fenchel's duality theorem is very famous and widely-used, stating that $\inf_{x\in\mathbb{R}^n} \left[f(x) - g(x)\right] = \sup_{u\in\mathbb{R}^n} \left\{g^*(u) - f^*(u)\right\}$ and that there is an u for which the supremum in the right-hand side is attained when f is proper convex, g proper concave and $\mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{ri}(\mathrm{dom}(g)) \neq \emptyset$. It was given for convex optimization problems, but there may occur problems that are not convex, as the one presented in the third section. This paper is dedicated to the generalization of Fenchel's duality for so-called nearly convex and nearly concave functions (see [1], [3], [4], [11], [15], [18]). We have proved that Fenchel's statement remains valid under weaker conditions, i.e. when f is just proper nearly convex, g proper nearly concave, $\mathrm{ri}(\mathrm{epi}(f)) \neq \emptyset$ and $\mathrm{ri}(\mathrm{epi}(g)) \neq \emptyset$, together with $\mathrm{ri}(\mathrm{dom}(f)) \cap \mathrm{ri}(\mathrm{dom}(g)) \neq \emptyset$. When f is proper and convex $\mathrm{ri}(\mathrm{epi}(f))$ is non-empty, as is $\mathrm{ri}(\mathrm{epi}(g))$ when g is proper and concave, thus Fenchel's duality theorem proves to be a special case of our statement. Some authors call also the result given in the Corollary 31.2.1 in [21] Fenchel's duality theorem. We have proved that it is true under similar weaker conditions, too. Moreover, we noticed that when f is convex and g concave as in [21], the assertion stands under the assumption of the existence of an $x' \in \text{ri}(\text{dom}(f))$ such that $Ax' \in \text{ri}(\text{dom}(g))$. But the closedness assumptions regarding the functions f and g as supposed in [21] is not necessary as we have realized during our investigations. As the question concerning the applicability of these new results arises naturally, we have provided a problem which is not convex. So Fenchel's duality theorem is not applicable, while the generalization of Fenchel's duality given in this paper is capable of pointing out the strong duality. Relations between Fenchel duality for nearly convex functions and two-person zero-sum games with generalized convex payoff-functions have also been brought into attention.

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