On Tikhonov regularization for the inverse problem of option pricing in the price-dependent case

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Abstract

This paper deals with analytic studies for solving the inverse problem of identifying purely price-dependent volatilities from given option price data. Using the classical theory of parabolic differential equations we formulate and analyze the forward operator as a mapping between the Hilbert spaces $H^1(\mathbb{R})$ and $L^2(\mathbb{R})$. We investigate continuity and Fréchet differentiability of this operator and prove the discontinuity of the inverse operator. We use Tikhonov regularization and present assertions to the stable solvability of this problem.

Keywords: inverse problem of option pricing, identification of local volatilities, Black-Scholes model, parabolic equations, fundamental solutions, ill-posed problem, regularization, convergence rates

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1 Introduction

A European call option gives the holder the right to buy the underlying asset at the expiration date (or maturity) T for the strike price (or exercise price) K – independent of the actual price X of the asset at time T. We assume that the price X of the underlying asset follows a stochastic process of the form

$$\frac{dX}{X} = \mu(X)\, dt + \sigma(X)\, dW_t \tag{1}$$

with parameters drift μ and local volatility σ. Here W_t denotes a standard Wiener process. We suppose that the volatility σ is a deterministic function which depends on the asset price X. The model (1) represents a generalization of the model of geometric Brownian

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motion which forms the fundamentals for calculating option prices via the well-known Black-Scholes formula (see e.g. [4]).

We denote by $c(X, t, K, T)$ the (fair) market price of a call option as a function of the variables asset price X, time $t \geq 0$, strike price K and expiration date $T \geq t$. Let the strike price K and expiration date T be fixed. Using the Black-Scholes analysis we can show that the option price function c satisfies the (generalized) Black-Scholes equation

$$\frac{\partial c}{\partial t} = \frac{1}{2} X^2 \sigma^2(X) \frac{\partial^2 c}{\partial X^2} + r\, X \frac{\partial c}{\partial X} - r\, c, \qquad (X, t) \in (0, \infty) \times [0, T), \tag{2}$$

(see e.g. [4] or [16]). Furthermore the final condition

$$c(X, T, K, T) = \max(X - K, 0), \qquad X \in (0, \infty), \tag{3}$$

holds. The additional parameter r represents the interest rate of a risk-less investment and is assumed to be known. On the other hand we can fix the asset price X and the time t. Then the price function c fulfills the Dupire equation

$$\frac{\partial c}{\partial T} = \frac{1}{2} K^2 \sigma^2(K) \frac{\partial^2 c}{\partial K^2} - r\, K \frac{\partial c}{\partial K}, \qquad (K, T) \in (0, \infty) \times (t, \infty), \tag{4}$$

(see e.g. [8]) together with the initial condition

$$c(X, t, K, t) = \max(X - K, 0), \qquad K \in (0, \infty). \tag{5}$$

We see that the volatility σ plays an important role in option pricing. The Cauchy problems (2)-(3) respectively (4)-(5) suggest to introduce a mapping

$$\sigma \mapsto c(X, t, K, T). \tag{6}$$

Calculating option prices by a given volatility σ is called the forward problem of option pricing. Otherwise the inverse problem seems to be of high interest. The volatility function σ is a market parameter which is not directly observable. On the other hand, options are traded on the stock market for a given asset price X and time t but different strike prices K and maturities T. Therefore we formulate following question: Is it possible to identify the corresponding volatility function σ from given option price data?

This problem is known in the literature as the inverse problem of option pricing or model calibrating problem. It was first mentioned in [8] for the more general case that the volatility σ is a function as well of the asset price X as of the time t. Knowing the prices of European call options for for all strike prices $K > 0$ and all maturities $T > t$ we can determine the corresponding volatility uniquely via the Dupire formula (see [8],[2]). For practical determination of local volatilities the Dupire formula does not play an important role. One reason is due to the nature of the given data: Normally option prices are given only on a discrete set of strike prices and maturities. The second point is more crucial. Calculating local volatilities via the Dupire formula demands the differentiation of the given data. As it is well-known this leads to instability phenonemons. The obtained results do not depend continuously on the given data.

In [2],[3] and [18] different approaches are suggested to solve the inverse option pricing problem in a stable way. In this context the development of numerical methods plays the principle part without studying the analytic background. First [7] gives an in-depth analytic study of this problem. The direct problem (6) is formulated as mapping between the Hilbert space H^1 and Banach spaces L^p with $p \in (2,3)$. In particular the case $p = 2$ is excluded. On the other hand mappings between Hilbert spaces are of particular interest in the regularization theory.

In [5] and [6] another idea is suggested to study the inverse option pricing problem. To simplify the analysis the authors assume initially that the volatility is purely price-dependent. Later on these results are generalized to the case the volatility is a function which is piecewise constant in respect of time t. The inverse problem is analyzed in spaces of continuous functions. There is given a sufficient condition for uniqueness by using option prices for different strike prices but only one maturity. Some numerical methods for solving the inverse problem are introduced. Under strong conditions on the given data even stability was proved. In [19] a nonlinear Tikhonov regularization approach is used for stabilizing the inverse problem. Methods of optimal control are applied to derive necessary and sufficient optimality conditions for the corresponding optimization problem. But neither [6] nor [19] includes a deep analytic study of the direct mapping (6) in concrete function spaces. A first step in this direction is done in [10]. Specific Hilbert spaces are constructed to apply the well-known convergence analysis for nonlinear Tikhononv regularization (see e.g. [11]). The case that the volatility is purely time-dependent is analyzed in [13].

Our aim is to introduce a formulation of the mapping (6) as (nonlinear) operator between the Hilbert spaces H^1 and L^2 for purely price-dependent volatilities. A detailed analysis of the forward operator shall show that we can apply again the theory of nonlinear Tikhonov regularization to the inverse option pricing problem. Consequently all known stability and convergence results and assertions for convergence rates can be formulated.

The paper is organized as follows: In section 2 we introduce appropriate variable transformations which allow us to apply the known (classical) solvability theory of parabolic equations. Additionally a close relationship between option prices and the fundamental solutions of the Black-Scholes equation and Dupire equation is derived. This fact plays a crucial role in the further investigations. In section 3 we deal with formulating the forward operator as a mapping from $H^1(\mathbb{R})$ into $L^2(\mathbb{R})$. Continuity of the forward operator is proved and the specific inverse problem is formulated. Section 4 is dedicated to the Fréchet differentiability of the forward operator. Afterwards we show that the inverse problem is ill-posed. We can prove that the well-known theory of nonlinear Tikhonov regularization (see e.g. [11]) is applicable to obtain stability and convergence of regularized solutions of the inverse problem. Finally we deduce the adjoint of the Fréchet derivative to formulate necessary conditions to the given data to prove convergence rate results.

2 Option prices and fundamental solutions

The (classical) solvability theory of parabolic differential equation (see e.g. [9, Chapter 1] or [17, Chapter IV.5]) is not directly applicable to the Cauchy problems (2)-(3) and (4)-(5). It's due to the coefficient of the second derivative, which in booth equations is unbounded and tends to zero for $X \to 0$ respectively $K \to 0$. Therefore we introduce the following well-known transformations:

$$x := \ln X,\ y := \ln K,\ \tau := T - t,\ u(x,\tau,y) := c(X,t,K,T) \ \text{ and } \ a(x,t) := \frac{1}{2}\sigma^2(X,t).$$

In [19] a slightly modified transformation is used. Let $T > 0$ be fixed. Then we obtain from (2) respectively (4) the equations

$$\frac{\partial u}{\partial \tau} = a(x)\frac{\partial^2 u}{\partial x^2} + (r - a(x))\frac{\partial u}{\partial x} - r\,u, \qquad (x,\tau) \in \mathbb{R} \times (0,T], \tag{7}$$

respectively

$$\frac{\partial u}{\partial \tau} = a(y)\frac{\partial^2 u}{\partial y^2} - (r + a(y))\frac{\partial u}{\partial y}, \qquad (y,\tau) \in \mathbb{R} \times (0,T]. \tag{8}$$

Additionally the initial condition

$$u(x,0,y) = \max(e^x - e^y, 0) \qquad x, y \in \mathbb{R}, \tag{9}$$

holds for booth equations.

Now conditions to the parameter a can be specified to obtain unique solvability of the Cauchy problems (7),(9) and (8),(9). In this context we define

$$\mathcal{D}_c^\lambda := \{a \in C^\lambda(\mathbb{R}) \ : \ \underline{c} \le a(y) \le \overline{c},\ y \in \mathbb{R}\}$$

for two constants $0 < \underline{c} < \overline{c} < \infty$ and a Hölder index $0 < \lambda \le 1$. Thereby $C^\lambda(\mathbb{R})$ denotes as usual the Banach space of bounded and Hölder continuous functions (with Hölder index λ) with norm

$$\|a\|_{C^\lambda(\mathbb{R})} := \sup_{y\in\mathbb{R}} |a(y)| + \sup_{y_1 \ne y_2} \frac{|a(y_1) - a(y_2)|}{|y_1 - y_2|^\lambda} < \infty.$$

$\mathcal{D}_c^\lambda$ is a convex set as well in $C^\lambda(\mathbb{R})$ as in $C(\mathbb{R})$. In $C^\lambda(\mathbb{R})$ it is even closed.

Proposition 2.1 *Let $T > 0$, $r \in \mathbb{R}$ and an element $a \in \mathcal{D}_c^\lambda$ be fixed.*

(i) For every $y \in \mathbb{R}$ the Cauchy problem

$$\begin{cases} u_\tau(x,\tau) &= a(x)\,u_{xx}(x,\tau) + (r - a(x))\,u_x(x,\tau) - r\,u(x,\tau), \quad (x,\tau) \in \mathbb{R} \times (0,T], \\ u(x,0) &= \max(e^x - e^y, 0), \quad x \in \mathbb{R}, \end{cases}$$

possesses an unique (classical) continuous solution u on $\mathbb{R} \times [0,T]$ which satisfies a growth condition of the form

$$|u(x,\tau)| \le C_1 \exp\left(C_2\,|x|^2\right)$$

for two positive constants C_1 and C_2.

(ii) For every $x \in \mathbb{R}$ the Cauchy problem

$$\begin{cases} u_\tau(y,\tau) &= a(y)\,u_{yy}(y,\tau) - (r + a(y))\,u_y(y,\tau), \quad (y,\tau) \in \mathbb{R} \times (0,T], \\ u(y,0) &= \max(e^x - e^y, 0), \quad y \in \mathbb{R}, \end{cases}$$

possesses an unique (classical) bounded continuous solution u on $\mathbb{R} \times [0,T]$.

In booth cases the existence of a classical solution follows directly [9, Theorem 1.12]. The uniqueness we conclude from [9, Theorem 1.16]). The boundedness of the solution of the second Cauchy problem follows out of the boundedness of the initial condition (see [17, Theorem I.2.5]).

In the next step we will derive a correlation between the fundamental solutions of the equations (7) and (8) and the transformed option price function $u(x, \tau, y)$. For further readings on fundamental solutions we refer to [9, pp.4]. We prove the following lemma.

Lemma 2.1 *(i) For every $a \in \mathcal{D}_c^\lambda$ the parabolic equation (7) possesses a fundamental solution $\Gamma(x, \tau, \xi, \eta)$ $(x, \xi \in \mathbb{R},\ \tau > \eta \geq 0)$ and*

$$\Gamma(x, T, y, t) = e^{-y} \left(u_{yy}(x, T-t, y) - u_y(x, T-t, y) \right).$$

(ii) For every $a \in \mathcal{D}_c^\lambda$ the parabolic equation (8) possesses a fundamental solution $\hat{\Gamma}(y, \tau, \xi, \eta)$ $(y, \xi \in \mathbb{R},\ \tau > \eta \geq 0)$ and

$$\hat{\Gamma}(y, T, x, t) = e^{-x} \left(u_{xx}(x, T-t, y) - u_x(x, T-t, y) \right).$$

Proof. For the solution u of (7),(9) we derive from [9, Theorem 1.12] that

$$u(x, \tau, y) := \int_{-\infty}^{\infty} \max(e^\xi - e^y, 0)\, \Gamma(x, \tau, \xi, 0)\, d\xi = \int_{y}^{\infty} (e^\xi - e^y)\, \Gamma(x, \tau, \xi, 0)\, d\xi.$$

We differentiate twice by y to achieve

$$u_y(x, \tau, y) = -\int_{y}^{\infty} e^y\, \Gamma(x, \tau, \xi, 0)\, d\xi$$

respectively

$$u_{yy}(x, \tau, y) = e^y \Gamma(x, \tau, y, 0) - \int_{y}^{\infty} e^y\, \Gamma(x, \tau, \xi, 0)\, d\xi.$$

We take the difference and obtain

$$u_{yy}(x, \tau, y) - u_y(x, \tau, y) = e^y \Gamma(x, \tau, y, 0).$$

The first statement of the lemma we conclude now by using the relation

$$\Gamma(x, \tau, y, 0) = \Gamma(x, \tau + t, y, 0 + t) = \Gamma(x, T, y, t),$$

which is valid for parabolic equations with time-independent coefficients. The proof of the second part occurs analogous. ■

These results we use later to obtain L^2-estimates for solutions of appropriate Cauchy problems.

3 Formulation of the inverse problem

For evaluating option prices we consider the transformed Dupire equation

$$u_t = L(a)\,u \quad \text{on} \quad \mathbb{R} \times (0,T] \tag{10}$$

with the elliptic differential operator

$$L(a)\,u(x,t) := a(x)\,u_{xx}(x,t) - (r + a(x))\,u_x(x,t),$$

which depends on the parameter a.

As consequence of Proposition 2.1(ii) we can introduce the following notation: Let the maturity $T > 0$, the interest rate $r \in \mathbb{R}$ and the logarithmized asset price $x_0 \in \mathbb{R}$ be fixed. We set $Q_T := \mathbb{R} \times (0,T)$. For given $a \in \mathcal{D}_c^\lambda$ we define $u(a) \in C(\overline{Q_T}) \cap C^{2,1}(Q_T)$ as bounded classical solution of the Cauchy problem

$$\left\{ \begin{array}{rll} u_t(a) &= L(a)\,u(a) & \text{on} \ \ \mathbb{R} \times (0,T], \\ \left[u(a)\right](x,0) &= \max(e^{x_0} - e^x, 0), & x \in \mathbb{R}. \end{array} \right. \tag{11}$$

Now we investigate how variations in the volatility function a influence the option price function $u(a)$. Let $a_1, a_2 \in \mathcal{D}_c^\lambda$ and $u(a_1)$, $u(a_2)$ are the corresponding solutions of (11). We set $v := u(a_1) - u(a_2)$. Then

$$\begin{aligned} v_t &= u_t(a_1) - u_t(a_2) \\ &= L(a_1)\,u(a_1) - L(a_2)\,u(a_2) \\ &= a_1\,u_{xx}(a_1) - (r + a_1)u_x(a_1) - [a_2\,u_{xx}(a_2) - (r + a_2)u_x(a_2)] \\ &= a_2\,[u_{xx}(a_1) - u_{xx}(a_2)] - (r + a_2)\,[u_x(a_1) - u_x(a_2)] + (a_1 - a_2)\,[u_{xx}(a_1) - u_x(a_1)] \\ &= L(a_2)\,v + (a_1 - a_2)\,[u_{xx}(a_1) - u_x(a_1)] \\ &= L(a_1)\,v + (a_1 - a_2)\,[u_{xx}(a_2) - u_x(a_2)]\,. \end{aligned}$$

Additionally we get the initial condition

$$v(x,0) = 0, \qquad x \in \mathbb{R}.$$

This gives the motivation to establish the following notation: For a given function $a_0 \in \mathcal{D}_c^\lambda$ we set

$$\mathcal{D}(a_0) := \left\{ a \in C^\lambda(\mathbb{R}) \,:\, a + a_0 \in \mathcal{D}_c^\lambda \right\} \subset C^\lambda(\mathbb{R}). \tag{12}$$

Let a function $a \in \mathcal{D}(a_0)$ be given. We define $v(a) \in C(\overline{Q_T}) \cap C^{2,1}(Q_T)$ as solution of the Cauchy problem

$$\left\{ \begin{array}{rll} v_t(a) &= L(a_0)\,v(a) + a\,[u_{xx}(a_0 + a) - u_x(a_0 + a)] & \text{on} \ \ \mathbb{R} \times (0,T], \\ \left[v(a)\right](x,0) &= 0, & x \in \mathbb{R}. \end{array} \right. \tag{13}$$

Remark 3.1 *As seen in the derivation of (13) we can define $v(a)$ analogous via the Cauchy problem*

$$\left\{ \begin{array}{rll} v_t(a) &= L(a_0 + a)\,v(a) + a\,[u_{xx}(a_0) - u_x(a_0)] & \text{on} \ \ \mathbb{R} \times (0,T], \\ \left[v(a)\right](x,0) &= 0, & x \in \mathbb{R}. \end{array} \right. \tag{14}$$

In the further investigations we will use booth versions to derive continuity and differentiability properties of the option pricing problem.

As consequence of Lemma 2.1 we prove the following estimate.

Lemma 3.1 *For given function $a \in \mathcal{D}_c^\lambda$ and $u(a)$ as the classical solution of (11) the property $u_{xx}(a) - u_x(a) \in L^2(Q_T)$ holds.*

PROOF. Using Lemma 2.1 we know

$$[u_{xx}(a)](x,T) - [u_x(a)](x,T) = e^x\, \Gamma(x_0,t,x,0).$$

whereby $\Gamma(x_0,t,x,0)$ is the fundamental solution of (7). From the estimate

$$0 < \Gamma(x_0,T,x,t) \le \frac{c_1}{\sqrt{T-t}} \exp\left(-c_2\, \frac{(x_0-x)^2}{T-t}\right) \tag{15}$$

for two positive constants c_1 and c_2 (which depends only on $\mathcal{D}_c^\lambda$, see e.g. [9, Theorem 1.11]), we derive

$$\begin{aligned}
\|u_{xx}(a) - u_x(a)\|^2_{L^2(Q_T)} &\le c_1^2 \iint\limits_{Q_T} \frac{1}{t} \exp\left(-2\,c_2\, \frac{(x_0-x)^2}{t} + 2\,x\right)\, dy\, dt \\
&\le c_1^2 \sqrt{\frac{\pi}{2\,c_2}} \exp(2\,x_0) \int\limits_0^T \frac{1}{\sqrt{t}} \exp\left(\frac{t}{2\,c_2}\right)\, dt \\
&\le c_1^2 \sqrt{\frac{\pi}{2\,c_2}} \exp\left(2\,x_0 + \frac{T}{2\,c_2}\right) \int\limits_0^T \frac{1}{\sqrt{t}}\, dt \\
&= c_1^2 \sqrt{\frac{2\,\pi\,T}{c_2}} \exp\left(2\,x_0 + \frac{T}{2\,c_2}\right) = C^2(x_0,T) < \infty. \qquad \blacksquare
\end{aligned}$$

Now we prove a first important statement.

Theorem 3.1 *For given $a_0 \in \mathcal{D}_c^\lambda$ let $a \in \mathcal{D}(a_0)$ be arbitrarily and $v(a)$ is the solution of (13). Then $v(a) \in L^2(0,T;H^2(\mathbb{R})) \cap C([0,T];H^1(\mathbb{R}))$. In particular the estimate*

$$\|v(a)\|_{L^2(0,T;H^2(\mathbb{R})))} + \|[v(a)](\cdot,T)\|_{L^2(\mathbb{R})} \le C\|a\|_{L^\infty(\mathbb{R})} \tag{16}$$

holds for a constant $C > 0$ independent of a.

PROOF. Since the differential operator $L(a_0)$ is given in non-divergence form we divide by a_0. Then from (13) we receive the Cauchy problem

$$\left\{\begin{array}{lll} a_0^{-1}\, v_t(a) &= v_{xx}(a) - (a_0^{-1}\, r + 1)\, v_x(a) + a_0^{-1}\, a\, [u_{xx}(a_0) - u_x(a_0)] & \text{on } \mathbb{R}\times(0,T], \\ {[v(a)]}\,(x,0) &= 0, & x \in \mathbb{R}. \end{array}\right. \tag{17}$$

We examine the source term $a_0^{-1}\, a\, [u_{xx}(a_0) - u_x(a_0)]$ and conclude from Lemma 3.1

$$\begin{aligned}
\|a_0^{-1}\, a\, [u_{xx}(a_0) - u_x(a_0)]\,\|_{L^2(Q_T)} &\le \|a_0^{-1}\|_{L^\infty(Q_T)} \|a\|_{L^\infty(\mathbb{R})} \|u_{xx}(a_0) - u_x(a_0)\|_{L^2(Q_T)} \\
&\le \underline{c}^{-1} C(x_0,T)\, \|a\|_{L^\infty(\mathbb{R})},
\end{aligned}$$

in particular $a_0^{-1}\, a\, [u_{xx}(a_0) - u_x(a_0)] \in L^2(Q_T)$. Therefore from [14, Theorem 2.1+Theorem 2.4] we conclude that the (classical) solution $v(a)$ of (13) is also a weak solution in $L^2(0,T;H^2(\mathbb{R})) \cap C([0,T];H^1(\mathbb{R}))$ of (17) and the estimate

$$\begin{aligned} \|v(a)\|_{L^2(0,T;H^2(\mathbb{R})))} + \|[v(a)](\cdot,T)\|_{L^2(\mathbb{R})} &\leq \|v(a)\|_{L^2(0,T;H^2(\mathbb{R})))} + \sup_{t\in[0,T]} \|[v(a)](\cdot,t)\|_{L^2(\mathbb{R})} \\ &\leq C\, \|a_0^{-1}\, a\, [u_{xx}(a_0) - u_x(a_0)]\, \|_{L^2(Q_T)} \\ &\leq \tilde{C}\, \|a\|_{L^\infty(Q_T)} \end{aligned}$$

holds. The proof is complete. ■

Remark 3.2 *In [14] similar results are established for the more general case that the volatility a depends on the price x and time t (see [14, Corollary 4.2] respectively [14, Theorem 4.1]).*

Using the fact that $H^1(\mathbb{R})$ is continuously embedded in $C(\mathbb{R})$ (and therefore in $L^\infty(\mathbb{R})$) we can define the following operator.

Definition 3.1 *For given $a_0 \in \mathcal{D}_c^\lambda$, $x_0, r \in \mathbb{R}$ and $T > 0$ we define the nonlinear operator $F : \mathcal{D}(a_0) \subset H^1(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ via*

$$[F(a)](x) := [v(a)](x,T) = [u(a_0 + a)](x,T) - [u(a_0)](x,T), \qquad x \in \mathbb{R}. \tag{18}$$

The following result is an immediate consequence of Theorem 3.1.

Corollary 3.1 *The operator F is Lipschitz continuous for every element $a \in \mathcal{D}(a_0)$.*

PROOF. For given $a \in \mathcal{D}(a_0)$ let $\tilde{a} \in \mathcal{D}(a_0)$ be arbitrarily. We set $h := \tilde{a} - a$ and consider $\tilde{v} := v(\tilde{a}) - v(a)$. Then

$$\begin{aligned} \tilde{v} &= v(a+h) - v(a) \\ &= v(a+h) - v(0) - [v(a) - v(0)] \\ &= u(a_0 + a + h) - u(a_0) - [u(a_0 + a) - u(a_0)] \\ &= u(a_0 + a + h) - u(a_0 + a). \end{aligned}$$

We can apply Theorem 3.1 with $a_0 + a$ instead of a_0 and h instead of a to obtain

$$\|F(\tilde{a}) - F(a)\|_{L^2(\mathbb{R})} = \|\tilde{v}\|_{L^2(\mathbb{R})} \leq C\, \|h\|_{L^\infty(\mathbb{R})} \leq \tilde{C}\, \|h\|_{H^1(\mathbb{R})} = \tilde{C}\, \|\tilde{a} - a\|_{H^1(\mathbb{R})},$$

whereby the constant $\tilde{C}$ does not depend on the element $\tilde{a}$. ■

Now Definition 3.1 of the operator F allows us to formulate the following inverse problem.

Definition 3.2 (Inverse Problem-(IP)) *Let $x_0 \in \mathbb{R}$ be the actual logarithmized asset price at time $t := 0$. Furthermore let $u_d(x)$ be a function of option prices for a fixed maturity $T > 0$ and all logarithmized exercise prices $x \in \mathbb{R}$. For given interest rate r and given a priori guess $a_0 \in \mathcal{D}_c^\lambda$ of the unknown local volatility we try to find a function $a \in \mathcal{D}(a_0)$, which satisfies the equation*

$$F(a) = u_d - [u(a_0)](\cdot,T) =: \tilde{u}. \tag{19}$$

4 Differentiability of the forward operator

We will examine the Fréchet differentiability of the operator F. To do this let the Hölder coefficient λ in the definition of $\mathcal{D}_c^\lambda$ be in the interval $(0, \frac{1}{2})$. Then $\mathcal{D}_c^\lambda \subset H^1(\mathbb{R})$, the interior of $\mathcal{D}(a_0)$ in $H^1(\mathbb{R})$ is not empty and

$$a \in \text{int } \mathcal{D}(a_0) \iff \underline{c} < \gamma_1 \leq a + a_0 \leq \gamma_2 < \overline{c} \ \text{ on } \ \mathbb{R}$$

for two constants γ_1 and γ_2. Then we prove that F is Fréchet differentiable for all interior points a of $\mathcal{D}(a_0)$.

In a first step we show the existence of directional derivatives.

Theorem 4.1 *Let $a \in \mathcal{D}(a_0)$. Then for every $h \neq 0$ with $a + h \in \mathcal{D}(a_0)$ there exists the directional derivative $F'(a)\,h$. Let $w(a,h)$ be the solution of the Cauchy problem*

$$\left\{\begin{aligned} w_t(a,h) &= L(a_0+a)\,w(a,h) + h\,[u_{xx}(a_0+a) - u_x(a_0+a)] && \text{on } \ \mathbb{R}\times(0,T], \\ [w(a,h)]\,(x,0) &= 0, && x \in \mathbb{R}, \end{aligned}\right. \tag{20}$$

then

$$[F'(a)\,h](x) = [w(a,h)](x,T), \qquad x \in \mathbb{R}.$$

Proof. Let $h \neq 0$ with $a + h \in \mathcal{D}(a_0)$ be arbitrarily. Then $a + \varepsilon\, h \in \mathcal{D}(a_0)$ for every $0 \leq \varepsilon \leq 1$ since $\mathcal{D}(a_0)$ is a convex set in $H^1(\mathbb{R})$. We set

$$w^\varepsilon := \frac{1}{\varepsilon}\,(F(a+\varepsilon\,h) - F(a)) = \frac{1}{\varepsilon}\,(v(a+\varepsilon\,h) - v(a)) = \frac{1}{\varepsilon}\,(u(a_0+a+\varepsilon\,h) - u(a_0+a))$$

and show $w^\varepsilon \to w(a,h)$ for $\varepsilon \to 0$. Since

$$\begin{aligned} w_t^\varepsilon &= \frac{1}{\varepsilon}\,(L(a_0+a+\varepsilon\,h)\,v(a+\varepsilon\,h) - L(a_0+a)\,v(a)) + \ldots \\ &\quad \ldots + \frac{1}{\varepsilon}\,[(a+\varepsilon\,h)(u_{xx}(a_0) - u_x(a_0)) - a\,(u_{xx}(a_0) - u_x(a_0))] \\ &= L(a_0+a)\,w^\varepsilon + h\,(v_{xx}(a+\varepsilon\,h) - v_x(a+\varepsilon,h) + u_{xx}(a_0) - u_x(a_0)) \\ &= L(a_0+a)\,w^\varepsilon + h\,(u_{xx}(a_0+a+\varepsilon\,h) - u_x(a_0+a+\varepsilon,h))\,, \end{aligned}$$

because

$$v(a+\varepsilon\,h) + u(a_0) = u(a_0+a+\varepsilon\,h) - u(a_0) + u(a_0) = u(a_0+a+\varepsilon\,h),$$

we can w^ε represent as solution of the Cauchy problem

$$\left\{\begin{aligned} w_t^\varepsilon &= L(a_0+a)\,w^\varepsilon + h\,[u_{xx}(a_0+a+\varepsilon\,h) - u_x(a_0+a+\varepsilon\,h)] && \text{on } \ \mathbb{R}\times(0,T], \\ w^\varepsilon(x,0) &= 0, && x \in \mathbb{R}, \end{aligned}\right.$$

We consider the limit $\varepsilon \to 0$. Let $w(a,h)$ be the solution of (20) and $\tilde{w} := w^\varepsilon - w(a,h)$. Then $\tilde{w}$ satisfies

$$\left\{\begin{aligned} \tilde{w}_t &= L(a_0+a)\,\tilde{w} + h\,[(u_{xx}(a_0+a+\varepsilon\,h) - u_x(a_0+a+\varepsilon\,h)) - \ldots \\ &\qquad \ldots - (u_{xx}(a_0+a) - u_x(a_0+a))] && \text{on } \mathbb{R}\times(0,T], \\ \tilde{w}(x,0) &= 0, && x \in \mathbb{R}. \end{aligned}\right.$$

Obviously $v := u(a_0 + a + \varepsilon\, h) - u(a_0 + a)$ is a solution of the Cauchy problem (13) with $a_0 + a$ instead of a_0 and $\varepsilon\, h$ instead of a. Then – from Theorem 3.1 – $v \in L^2(0, T; H^2(\mathbb{R}))$ and

$$\|v_{xx} - v_x\|_{L^2(Q_T)} \leq \|v\|_{L^2(0,T;H^2(\mathbb{R}))} \leq C\, \varepsilon\, \|h\|_{L^\infty(Q_T)} \to 0 \quad \text{for} \quad \varepsilon \to 0.$$

Consequently $\tilde{w} \to 0$ for $\varepsilon \to 0$ and therefore $w^\varepsilon \to w$. The proof is complete. ■

Let now $a \in \mathcal{D}(a_0)$ be fixed. Theorem 4.1 indicates the definition of a linear operator $F'(a) : H^1(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ via

$$[F'(a)\, h](x) := [w(a, h)](x, T), \qquad x \in \mathbb{R}, \tag{21}$$

whereby now $h \in H^1(\mathbb{R})$ is arbitrarily and $w(a, h)$ is the corresponding solution of (20). Since

$$\begin{aligned}
\|F'(a)\, h\|_{L^2(\mathbb{R})} &= \|[w(a, h)](\cdot, T)\|_{L^2(\mathbb{R})} \\
&\leq C\, \|h\|_{L^\infty(\mathbb{R})} \|u_{xx}(a_0 + a) - u_x(a_0 + a)\|_{L^2(Q_T)} \\
&\leq \tilde{C}\, \|h\|_{H^1(\mathbb{R})}
\end{aligned}$$

the linear operator $F'(a)$ is bounded. Therefore F is Gâteaux differentiable for each $a \in \operatorname{int} \mathcal{D}(a_0)$ and $F'(a)$ denotes the Gâteaux derivative of F at the point a. Now we can prove the Fréchet differentiability. Therefore we need the following lemma.

Lemma 4.1 *For arbitrary elements $a, a + h \in \mathcal{D}(a_0)$ the estimate*

$$\|F'(a + h) - F'(a)\| \leq C\, \|h\|_{H^1(\mathbb{R})}$$

holds for a constant $C > 0$ which does not depend on h.

PROOF. Let $\hat{h} \in H^1(\mathbb{R})$ be arbitrarily. We set $\hat{w} := F'(a + h)\, \hat{h} - F'(a)\, \hat{h} = w(a + h, \hat{h}) - w(a, \hat{h})$ whereby $w(a, \hat{h})$, $w(a + h, \hat{h})$ are the corresponding solutions of (20). Then $\hat{w}$ satisfies

$$\begin{aligned}
\hat{w}_t &= L(a_0 + a + h)\, w(a + h, \hat{h}) + \hat{h}\, [u_{xx}(a_0 + a + h) - u_x(a_0 + a + h)] - \ldots \\
&\quad \ldots - L(a_0 + a)\, w(a, \hat{h}) - \hat{h}\, [u_{xx}(a_0 + a) - u_x(a_0 + a)] \\
&= L(a_0 + a)\, \hat{w} + h \left[w_{xx}(a + h, \hat{h}) - w_x(a + h, \hat{h}) \right] + \ldots \\
&\quad \ldots + \hat{h} \left[[u_{xx}(a_0 + a + h) - u_x(a_0 + a + h)] - [u_{xx}(a_0 + a) - u_x(a_0 + a)] \right],
\end{aligned}$$

together with the initial condition

$$\hat{w}(x, 0) = 0, \qquad x \in \mathbb{R}.$$

From Theorem 3.1 with $a_0 + a$ instead of a_0 and h instead of a we derive

$$\|[u_{xx}(a_0 + a + h) - u_x(a_0 + a + h)] - [u_{xx}(a_0 + a) - u_x(a_0 + a)]\|_{L^2(Q_T)} \leq C\, \|h\|_{L^\infty(\mathbb{R})}.$$

Analogous to Theorem 3.1 we can conclude that $w(a + h, \hat{h}) \in L^2(0, T; H^2(\mathbb{R}))$ and

$$\begin{aligned}
\left\| h \left[w_{xx}(a + h, \hat{h}) - w_x(a + h, \hat{h}) \right] \right\|_{L^2(Q_T)} &\leq \|h\|_{L^\infty(\mathbb{R})} \left\| w_{xx}(a + h, \hat{h}) - w_x(a + h, \hat{h}) \right\|_{L^2(Q_T)} \\
&\leq C\, \|h\|_{L^\infty(\mathbb{R})} \|\hat{h}\|_{L^\infty(\mathbb{R})}.
\end{aligned}$$

Together with [14, Theroem 2.1] and the continuous embedding of $H^1(\mathbb{R})$ in $L^\infty(\mathbb{R})$ we obtain

$$\|\hat{w}(\cdot,T)\|_{L^2(\mathbb{R})} = \|F'(a+h)\,\hat{h} - F'(a)\,\hat{h}\|_{L^2(\mathbb{R})} \le \tilde{C}\,\|h\|_{H^1(\mathbb{R})}\|\hat{h}\|_{H^1(\mathbb{R})}.$$

This proves the lemma. ■

The Fréchet differentiability of F follows now immediately out of this lemma (see e.g. [15, Proposition 1]).

Theorem 4.2 *For every $a \in int\,\mathcal{D}(a_0)$ the operator F is Fréchet differentiable. $F'(a)$ is defined via (21) and for every h with $a+h \in \mathcal{D}(a_0)$ the estimate*

$$\|F(a+h) - F(a) - F'(a)\,h\|_{L^2(\mathbb{R})} \le \frac{L}{2}\|h\|^2_{H^1(\mathbb{R})}$$

holds for a constant $L > 0$, which does not depend on h.

5 Tikhonov regularization of the inverse problem

First of all we show that the problem (IP) is ill-posed. Following [15, Definition 2] we prove the local ill-posedness of equation (19) for every function $a \in \mathcal{D}(a_0)$. Consequently solutions a of (19) does not depend continuously from the given data $\tilde{u}$ (respectively u_d).

Theorem 5.1 *For every element $a \in \mathcal{D}(a_0)$ and every ball $B_\varepsilon(a) := \{\hat{a} \in H^1(\mathbb{R}) : \|\hat{a} - a\|_{H^1(\mathbb{R})} < \varepsilon\}$ ($\varepsilon > 0$) there exists a sequence $\{a_n\} \subset \mathcal{D}(a_0) \cap B_\varepsilon(a)$ with $a_n \not\to a$ but $F(a_n) \to F(a)$, in particular equation (19) is locally ill-posed.*

Proof. Let $a \in \mathcal{D}(a_0)$ be arbitrarily. Then

$$\int\limits_{-\infty}^{\infty} \left[a_0(x) + a(x) - \underline{c}\right]^2\,dx = \infty \quad \text{or} \quad \int\limits_{-\infty}^{\infty} \left[\overline{c} - (a_0(x) + a(x))\right]^2\,dx = \infty.$$

We assume the first case. Then we define a sequence of functions $\{a_n\}$ $n = 1, 2, \ldots$, with the following properties

$$a_n(x) := \begin{cases} \underline{c} - a_0(x), & |x| < n, \\ a(x), & |x| > n+1. \end{cases}$$

On the intervals $[-n-1, -n]$ and $[n, n+1]$ we choose $a_n(x)$ in the way that they are elements of $\mathcal{D}(a_0) \subset H^1(\mathbb{R})$ and

$$\|a_n - a\|_{L^\infty(\mathbb{R})} = \sup_{x\in[-n,n]} |a_n(x) - a(x)|.$$

For given $0 < \varepsilon \le 1$ we introduce

$$\tilde{a}_n := a + \frac{\varepsilon}{2}\,\frac{a_n - a}{\max\{1, \|a_n - a\|_{H^1(\mathbb{R})}\}}.$$

Then $\{\tilde{a}_n\} \subset \mathcal{D}(a_0) \cap B_\varepsilon(a)$, $\|\tilde{a}_n - a\|_{H^1(\mathbb{R})} = \frac{\varepsilon}{2}$ for n large enough and $\|\tilde{a}_n - a\|_{L^\infty(\mathbb{R})} \to 0$ for $n \to \infty$ since $\|a_n - a\|_{L^2(\mathbb{R})} \to \infty$ for $n \to \infty$ (and therefore $\|a_n - a\|_{H^1(\mathbb{R})} \to \infty$). From Lemma 3.1 we obtain now

$$\|F(\tilde{a}_n) - F(a)\|_{L^2(\mathbb{R})} \leq C\,\|\tilde{a}_n - a\|_{L^\infty(\mathbb{R})} \to 0 \ \text{ for } \ n \to \infty$$

for a constant $C > 0$. Thus we have proved the instability of equation (19). ■

Theorem 5.2 *The operator F is weakly closed. Moreover for $a_n \in \mathcal{D}(a_0)$ and $a_n \rightharpoonup a$ it follows that $a \in \mathcal{D}(a_0)$ and $F(a_n) \to F(a)$.*

Proof. From the convexity and closedness of $\mathcal{D}(a_0)$ follows the weak closedness of $\mathcal{D}(a_0)$. Let $\{a_n\} \subset \mathcal{D}(a_0)$ be a sequence which converges weakly to an element $a \in H^1(\mathbb{R})$. Consequently $a \in \mathcal{D}(a_0)$ holds. We consider $F(a_n) - F(a)$. Again we have

$$F(a_n) - F(a) = v_n(\cdot, T),$$

where v_n is the solution of (13) with $a_0 + a$ instead of a_0 and $a_n - a$ instead of a. We show

$$\|\underbrace{(a_n - a)\,(u_{xx}(a_0 + a) - u_x(a_0 + a))}_{=:\, f}\|_{L^2(Q_T)} \to 0 \ \text{ for } \ n \to \infty.$$

Then – from [14, Theorem 2.1] – we can conclude $v_n \to 0$ and therefore $F(a_n) \to F(a)$ for $n \to \infty$. We consider

$$\begin{aligned}
\|f\|^2_{L^2(Q_T)} &= \int_0^T \int_{-\infty}^{\infty} f^2(x,t)\,dx\,dt \\
&= \underbrace{\int_0^T \int_{-\infty}^{y_1} f^2(x,t)\,dx\,dt}_{=:\, I_1(y_1)} + \underbrace{\int_0^T \int_{y_1}^{y_2} f^2(x,t)\,dx\,dt}_{=:\, I_2(y_1, y2)} + \underbrace{\int_0^T \int_{y_2}^{\infty} f^2(x,t)\,dx\,dt}_{=:\, I_3(y_2)}.
\end{aligned}$$

We prove: for every $\varepsilon > 0$ there exists $y_1, y_2 \in \mathbb{R}$ and $n \in \mathbb{N}$ so that

$$I_1(y_1) < \frac{\varepsilon}{3},\ I_2(y_1, y2) < \frac{\varepsilon}{3} \ \text{ and } \ I_3(y_2) < \frac{\varepsilon}{3}.$$

To show the first part we assume $y_1 < \min\{0, x_0\}$. Then

$$\begin{aligned}
I_1(y_1) &= \int_0^T \int_{-\infty}^{y_1} (a_n - a)^2\,(u_{xx}(a_0 + a) - u_x(a_0 + a))^2\ dx\,dt \\
&\leq (\overline{c} - \underline{c})^2 \int_0^T \int_{-\infty}^{y_1} (u_{xx}(a_0 + a) - u_x(a_0 + a))^2\ dx\,dt \\
&\leq (\overline{c} - \underline{c})^2\, c_1^2 \int_0^T \frac{1}{t} \int_{-\infty}^{y_1} \exp\left(-2\,c_2 \frac{(x - x_0)^2}{t} + 2\,x\right)\ dx\,dt.
\end{aligned}$$

We set

$$\xi := 2\sqrt{c_2}\frac{x-x_0}{\sqrt{t}} \quad \text{and} \quad \tilde{y}_1(t) := 2\sqrt{c_2}\frac{y_1-x_0}{\sqrt{t}}.$$

Then

$$\begin{aligned}
I_1(y_1) &\le (\overline{c}-\underline{c})^2\, c_1^2 \int_0^T \frac{1}{t} \int_{-\infty}^{\tilde{y}_1(t)} \exp\left(-\frac{(\xi)^2}{2}\right) \underbrace{\exp(2\,x)}_{<1} \frac{\sqrt{t}}{2\sqrt{c_2}}\, d\xi\, dt \\
&\le \frac{(\overline{c}-\underline{c})^2\, c_1^2}{2\sqrt{c_2}} \int_0^T \frac{1}{\sqrt{t}}\, \Phi(\tilde{y}_1(t))\, \sqrt{2\,\pi}\, dt \\
&\le \frac{(\overline{c}-\underline{c})^2\, c_1^2}{2\sqrt{c_2}} \sqrt{2\,\pi}\, \Phi(\tilde{y}_1(T)) \int_0^T \frac{1}{\sqrt{t}}\, dt \\
&\le \frac{(\overline{c}-\underline{c})^2\, c_1^2}{\sqrt{c_2}} \sqrt{2\,\pi\, T} \to 0 \quad \text{for} \quad y_1 \to -\infty.
\end{aligned}$$

Thereby $\Phi(z)$ denotes the cumulative density function of the standard normal distribution. Analogous we can show that $I_3(y_2) \to 0$ for $y_2 \to +\infty$.

Let now y_1, y_2 be fixed. We prove the last part. We have

$$\begin{aligned}
I_2(y_1,y_2) &= \int_0^T\!\!\int_{y_1}^{y_2} (a_n-a)^2\, \left(u_{xx}(a_0+a)-u_x(a_0+a)\right)^2\, dx\, dt \\
&\le \|a_n-a\|^2_{L^\infty(y_1,y_2)} \|u_{xx}(a_0+a)-u_x(a_0+a)\|^2_{L^2(0,T;L^2(y_1,y_2))} \\
&\le \|a_n-a\|^2_{L^\infty(y_1,y_2)} \|u_{xx}(a_0+a)-u_x(a_0+a)\|^2_{L^2(Q_T)}.
\end{aligned}$$

Since $a_n - a$ converges weakly to 0 in $H^1(\mathbb{R})$ we can conclude the strong convergence $a_n - a|_{(y_1,y_2)} \to 0$ in $L^\infty(y_1, y_2)$ by the Rellich-Kondrachov Theorem (see [1, Theorem 6.2]) and consequently $I_2(y_1, y_2) \to 0$ for $n \to \infty$. ■

Theorem 5.2 allows us to apply the convergence and stability results of the well-known theory of nonlinear Tikhonov regularization (see [11]). To to this we consider instead of equation (19) the minimization problem

$$\|F(a)-\tilde{u}^\delta\|^2_{L^2(\mathbb{R})} + \alpha\,\|a-a_0\|^2_{H^1(\mathbb{R})} \longrightarrow \min_{a\in\mathcal{D}(a_0)} \tag{22}$$

for given noisy data $\tilde{u}^\delta$ (instead of the exact data $\tilde{u}$) with

$$\|\tilde{u}^\delta - \tilde{u}\| \le \delta, \qquad \delta > 0.$$

Then (22) admits for every $\alpha > 0$ a (not necessarily unique) solution $a_\alpha^\delta \in \mathcal{D}(a_0)$ which depends continuously of the data $\tilde{u}^\delta$ (see [11, Theorem 2.1]). We formulate the following lemma.

Lemma 5.1 *Let $\alpha > 0$, $\{u_k\}$ and $\{a_k\}$ are sequences with $u_k \to \tilde{u}^\delta$ and a_k is a solution of (22) with u_k instead of $\tilde{u}^\delta$. Then there exists as convergent subsequence of $\{a_k\}$ and the limit of each convergent subsequence is a solution of (22).*

Under the additional assumption that equation (19) admits a solution a_{ex} for exact data $\tilde{u}$ we can prove the convergence $a_\alpha^\delta \to a_{ex}$ for $\tilde{u}^\delta \to \tilde{u}$ and an appropriate parameter choice $\alpha = \alpha(\delta)$ (see [11, Theorem 2.3]).

Now we turn to convergence rates. We analyze the usual sufficient condition for obtaining a convergence rate

$$\|a_\alpha^\delta - a_{ex}\|_{H^1(\mathbb{R})} = \mathrm{O}\left(\sqrt{\delta}\right). \tag{23}$$

As consequence of Theorem 4.2 we can apply [11, Theorem 2.4] to present the following statement.

Proposition 5.1 *Assume there exists a solution $a_{ex} \in \mathcal{D}(a_0)$ of (19) for exact data $\tilde{u}$. The we obtain for the parameter choise $\alpha \sim \delta$ a convergence rate (23) of the Tikhonov regularization if there exists an element $h \in L^2(\mathbb{R})$ such that*

(i) $a_{ex} - a_0 = F'(a_{ex})^* h$, whereby $F'(a_{ex})^*$ denotes the adjoint operator of $F'(a_{ex})$ and

(ii) $L\,\|h\|_{L^2(\mathbb{R})} < 1$, whereby L is the constant in Theorem 4.2.

6 The adjoint of the Fréchet derivative

As seen in Proposition 5.1 the adjoint operator $F'(a)^*$ of the Fréchet derivative plays an important role the convergence theory of regularization methods. Therefore we will now establish the structure of $F'(a)^*$. For that purpose we use the integral description of solution of parabolic equations by fundamental solutions. Let $\Gamma(x,t,\xi,\tau)$, $x,\xi \in \mathbb{R}$, $t > \tau \geq 0$, again denote the fundamental solution of equation (8). For simplicity we set $f := u_{xx}(a_0+a) - u_x(a_0+a)$. Then $F'(a)$ can be written as

$$[F'(a)\,h](x) := \iint\limits_{Q_T} \Gamma(x,T,\xi,\tau)\,f(\xi,\tau)\,h(\xi)\,d\xi\,d\tau, \quad x \in \mathbb{R}.$$

This expression allows us to give a decomposition of the operator $F'(a)$. Therefore we introduce $A_1 : H^1(\mathbb{R}) \longrightarrow L^2(Q_T)$ as

$$[A_1 h](x,t) := f(x,t)\,h(x), \qquad (x,t) \in Q_T,$$

the operator $A_2 : L^2(Q_T) \longrightarrow H^1(Q_T)$ as

$$[A_2 v](x,t) := \iint\limits_{Q_t} \Gamma(x,t,\xi,\tau)\,v(\xi,\tau)\,d\xi\,d\tau, \quad (x,t) \in Q_T,$$

and finally the trace operator $A_3 : H^1(Q_T) \longrightarrow L^2(\mathbb{R})$ via

$$[A_3 w](x) := w(x,T) = \int_0^T \delta(t-T)\,w(x,t)\,dt, \qquad x \in \mathbb{R}.$$

With these notations we can write

$$F'(a) = A_3 \circ A_2 \circ A_1.$$

In the following we denote with $(\cdot,\cdot)_{0,\Omega}$ the L^2-scalar product over the domain $\Omega \subseteq \mathbb{R}^n$. Furthermore $\langle\cdot,\cdot\rangle$ means the duality product $\mathcal{X} \times \mathcal{X}^* \longrightarrow \mathbb{R}$ for a Hilbert space $\mathcal{X}^*$.

Lemma 6.1 *$A_1^* : L^2(Q_T) \longrightarrow H^{-1}(\mathbb{R})$ is defined via*

$$(A_1^* v)(x) = \int_0^T v(x,t)\, f(x,t)\, dT, \qquad x \in \mathbb{R},$$

where $\langle h, A_1^* v\rangle = (h, A_1^* v)_{0,\mathbb{R}}$ for all $h \in H^1(\mathbb{R})$ and all $v \in L^2(Q_T)$.

PROOF. Let $v \in L^2(Q_T)$ and $h \in H^1(\mathbb{R})$. Then

$$\begin{aligned}
\langle A_1 h, v\rangle = (A_1 h, v)_{0,Q_T} &= \iint_{Q_T} h(x)\, f(x,t)\, v(x,t)\, dx\, dt \\
&= \int_{-\infty}^{\infty} \underbrace{\left[\int_0^T f(x,t)\, v(x,t)\, dt\right]}_{= A_1^* v} h(x)\, dx \\
&= (h, A_1^* v)_{0,\mathbb{R}} = \langle h, A_1^* v\rangle. \qquad \blacksquare
\end{aligned}$$

Lemma 6.2 *$A_2^* : L^2(Q_T) \subset H^{-1}(Q_T) \longrightarrow L^2(Q_T)$ is defined via*

$$(A_2^* w)(\xi,\tau) = \int_\xi^\tau \int_{-\infty}^{\infty} \Gamma(x,t,\xi,\tau)\, w(x,t)\, dx\, dt, \qquad (\xi,\tau) \in Q_T,$$

where $\langle v, A_2^* w\rangle = (v, A_2^* w)_{0,Q_T}$ for all $v \in L^2(\mathbb{R})$ and all $w \in L^2(Q_T)$.

PROOF. Let $w \in L^2(Q_T) \subset H^{-1}(Q_T)$ and $v \in L^2(Q_T)$. Then

$$\begin{aligned}
\langle A_2 v, w\rangle = (A_2 v, w)_{0,Q_T} &= \iint_{Q_T} \left[\iint_{Q_t} \Gamma(x,t,\xi,\tau)\, v(\xi,\tau)\, d\xi\, d\tau\right] w(x,t)\, dx\, dt \\
&= \iint_{Q_T} \underbrace{\left[\int_\xi^\tau \int_{-\infty}^{\infty} \Gamma(x,t,\xi,\tau)\, w(x,t)\, dx\, dt\right]}_{= A_2^* w} v(\xi,\tau)\, d\xi\, d\tau \\
&= (v, A_2^* w)_{0,Q_T} = \langle v, A_2^* w\rangle. \qquad \blacksquare
\end{aligned}$$

Lemma 6.3 *$A_3^* : L^2(\mathbb{R}) \longrightarrow H^{-1}(Q_T)$ is defined via*

$$(A_3^* u)(x,t) = \delta(t-T)\, u(x), \qquad (x,t) \in Q_T,$$

where $\langle w, A_3^* u\rangle = (w, A_3^* u)_{0,Q_T}$ for all $u \in L^2(\mathbb{R})$ and all $w \in H^1(Q_T)$.

PROOF. Let $u \in L^2(\mathbb{R})$ and $w \in H^1(Q_T)$. Then

$$\begin{aligned}
\langle A_3 w, u\rangle = (A_3 w, u)_{0,\mathbb{R}} &= \int_{-\infty}^{\infty} w(x,T)\, u(x)\, dx \\
&= \int_{-\infty}^{\infty} \left(\int_0^T \delta(t-T)\, w(x,t)\, dt \right) u(x)\, dx \\
&= \iint_{Q_T} \underbrace{\delta(t-T)\, u(x)}_{=A_3^* u} w(x,t)\, dx\, dt \\
&= (w, A_3^* u)_{0,Q_T} = \langle w, A_3^* u\rangle. \qquad \blacksquare
\end{aligned}$$

Corollary 6.1 *$F'(a)^* : L^2(\mathbb{R}) \longrightarrow H^{-1}(\mathbb{R})$ is defined via*

$$[F'(a)^* u](\xi) = \iint_{Q_T} \Gamma(x,T,\xi,\tau)\, f(\xi,\tau)\, u(x)\, dx\, dt, \qquad \xi \in \mathbb{R}.$$

where $\langle v, F'(a)^* u\rangle = (v, F'(a)^* u)_{0,\mathbb{R}}$ for all $u \in L^2(\mathbb{R})$ and all $v \in H^1(\mathbb{R})$.

PROOF. We know

$$F'(a)^* = (A_3 \circ A_2 \circ A_1)^* = A_1^* \circ A_2^* \circ A_3^*.$$

Consequently

$$\begin{aligned}
[F'(a)^* u](\xi) &= \int_0^T \underbrace{\left(\int_\tau^T \int_{-\infty}^{\infty} \delta(t-T)\, u(x)\, \Gamma(x,t,\xi,\tau)\, dx\, dt \right)}_{= \int_{-\infty}^{\infty} u(x)\, \Gamma(x,T,\xi,\tau)\, dx} f(\xi,\tau)\, d\tau \\
&= \iint_{Q_T} \Gamma(x,T,\xi,\tau)\, f(\xi,\tau)\, u(x)\, dx\, d\tau. \qquad \blacksquare
\end{aligned}$$

How can we represent $F'(a)^*$ in the differential equation frame work? Let us consider $A := A_2^* \circ A_3^*$. Then we can A express as

$$[A\, h](x,t) := \int_{-\infty}^{\infty} h(\xi)\, \Gamma(\xi,t,x,0)\, d\xi, \qquad (x,t) \in \mathbb{R} \times [0,T]. \tag{24}$$

We consider the adjoint equation of (8) which is defined via

$$u_t + L^*(a)\, u = u_t + (a\, u)_{xx} + ((r+a)\, u)_x = 0 \qquad \text{on } \mathbb{R} \times [0,T). \tag{25}$$

Let $z(a,h)$ denotes the solution of the Cauchy problem

$$\begin{cases} z_t(a,h) + L^*(a)\, z(a,h) = 0 & \text{on } \mathbb{R} \times [0,T), \\ [z(a,h)]\,(x,T) = h(x), & x \in \mathbb{R}. \end{cases} \tag{26}$$

Let $\Gamma^*(x,t,\xi,\tau)$ be the fundamental solution of (25). Then we can conclude

$$\Gamma^*(x,t,\xi,\tau) = \Gamma(\xi,\tau,x,t).$$

On the other hand we know that we can represent $z(a,h)$ as

$$\begin{aligned}
[z(a,h)](x,t) &= \int_{-\infty}^{\infty} \Gamma^*(x,t,\xi,T)\, h(\xi)\, d\xi \\
&= \int_{-\infty}^{\infty} \Gamma(\xi,T,x,t)\, h(\xi)\, d\xi \\
&= [A\, h](x,t)
\end{aligned}$$

(see e.g. [9, pp.26]). We summarize all results in the following theorem.

Theorem 6.1 *For given $a_0 \in \mathcal{D}_c^\lambda$, $T > 0$ and $a \in \mathcal{D}(a_0)$ let $F'(a)$ be defined via (21). Then we can represent $F'(a)^* : L^2(\mathbb{R}) \longrightarrow H^{-1}(\mathbb{R})$ as*

$$\langle x, F'(a)^*\, h\rangle = \int_{-\infty}^{\infty} x(\xi)\, [g(h)](\xi)\, d\xi, \qquad \forall\, x \in H^1(\mathbb{R}), \tag{27}$$

where $g(h)$ is defined by

$$[g(h)](x) = \int_0^T [z(a,h)](x,t)\, \Big([u_{xx}(a_0+a)](x,t) - [u_x(a_0+a)](x,t)\Big)\, dt, \qquad x \in \mathbb{R}. \tag{28}$$

Thereby $z(a,h)$ denotes the solution of the Cauchy problem (26).

7 L^2-representation of the adjoint operator

The aim of this section is to show that $g(h)$, defined by (28), is an element of $L^2(\mathbb{R})$. Consequently the description (27) motivates a L^2-representation of $F'(a)^*h \in H^{-1}(\mathbb{R})$ by $g(h) \in L^2(\mathbb{R})$. For a general characterization of elements of H^{-1} we refer to [12, Theorem 5.9.1].

In a first step we derive an estimation of the solution $z(a,h)$ of (26).

Lemma 7.1 *For the solution $z(a,h)$ of (26) for all $0 \le t < T$ the estimate*

$$\|[z(a,h)](\cdot,t)\|_{L^\infty(\mathbb{R})} \le C\, (T-t)^{-\frac{1}{4}}\, \|h\|_{L^2(\mathbb{R})}$$

holds for a constant $C > 0$ which depends only on $\mathcal{D}_c^\lambda$.

Proof. We use the integral expression (24) of $z(a,h)$. Using the estimate

$$0 < \Gamma(x,t,\xi,\tau) \le \frac{c_1}{\sqrt{t-\tau}} \exp\left(-c_2\, \frac{(x-\xi)^2}{t-\tau}\right)$$

for two constants $c_1, c_2 > 0$, we derive for every $(x,t) \in \overline{Q_T}$ and every $h \in L^2(\mathbb{R})$

$$\begin{aligned}
|[z(a,h)](x,t)| &= \left| \int_{-\infty}^{\infty} h(\xi)\,\Gamma(\xi,T,x,t)\,d\xi \right| \\
&\le \frac{c_1}{\sqrt{T-t}} \|h\|_{L^2(\mathbb{R})} \left(\int_{-\infty}^{\infty} \exp\left(-2\,c_2 \frac{(x-\xi)^2}{T-t} \right) d\xi \right)^{\frac{1}{2}} \\
&= c_1\,(T-t)^{-\frac{1}{4}} \left(\frac{\pi}{2\,c_2} \right)^{\frac{1}{4}} \|h\|_{L^2(\mathbb{R})}. \qquad \blacksquare
\end{aligned}$$

Now we can prove the following result.

Lemma 7.2 *For an arbitrary $h \in L^2(\mathbb{R})$ let $g(h)$ be given by (28). Then $g(h) \in L^2(\mathbb{R})$ and the estimate*

$$\|g(h)\|_{L^2(\mathbb{R})} \le C_1 \|h\|_{L^2(\mathbb{R})}$$

holds for a constant $C_1 > 0$ which does not depend on h.

Proof. Using Lemma 7.1 we get

$$\begin{aligned}
\|g(h)\|^2_{L^2(\mathbb{R})} &= \int_{-\infty}^{\infty} \left[\int_0^T [z(a,h)](x,t) \Big([u_{xx}(a_0+a)](x,t) - [u_x(a_0+a)](x,t) \Big)\,dt \right]^2 dx \\
&\le C^2\,\|h\|^2_{L^2(\mathbb{R})} \int_{-\infty}^{\infty} \left[\int_0^T (T-t)^{-\frac{1}{4}} \Big([u_{xx}(a_0+a)](x,t) - [u_x(a_0+a)](x,t) \Big)\,dt \right]^2 dx \\
&\le C^2\,\|h\|^2_{L^2(\mathbb{R})} \int_{-\infty}^{\infty} \left[\int_0^T (T-t)^{-\frac{1}{4}} \frac{c_1}{\sqrt{t}} \exp\left(-c_2 \frac{(x-x_0)^2}{t} + x \right) dt \right]^2 dx \\
&= C^2\,\|h\|^2_{L^2(\mathbb{R})} c_1^2 \int_{-\infty}^{\infty} \exp\left(-2\,c_2 \frac{(x-x_0)^2}{T} + 2\,x \right) \left[\int_0^T (T-t)^{-\frac{1}{4}} \frac{1}{\sqrt{t}}\,dt \right]^2 dx \\
&= C^2\,\|h\|^2_{L^2(\mathbb{R})} c_1^2 \frac{8\,\sqrt{T}\,\Gamma\left(\frac{3}{4}\right)^4}{\pi} \int_{-\infty}^{\infty} \exp\left(-2\,c_2 \frac{(x-x_0)^2}{T} + 2\,x \right) dx \\
&= C^2\,\|h\|^2_{L^2(\mathbb{R})} c_1^2 \frac{8\,\sqrt{T}\,\Gamma\left(\frac{3}{4}\right)^4}{\pi} \exp\left(\frac{T}{2\,c_2} \right) \exp(2\,x_0) \sqrt{\frac{\pi\,T}{2\,c_2}}.
\end{aligned}$$

Thereby $\Gamma(\cdot)$ denotes the Eulerian Gamma function and $\Gamma(\frac{3}{4}) \approx 1.2254\ldots$. $\blacksquare$

8 H^1-representation of the adjoint operator

Theorem 6.1 gives an identification of the $F'(a)^*h$ with an element $g(h) \in L^2(\mathbb{R})$. However, to examine the conditions (i) and (ii) of Proposition 5.1 we need the identification of

$H^{-1}(\mathbb{R})$ with $H^1(\mathbb{R})$. In particular we are looking for a function $v(h) \in H^1(\mathbb{R})$ with

$$\langle x, F'(a)^* h\rangle = \int\limits_{-\infty}^{\infty} x(\xi)\,[g(h)](\xi)\,d\xi = \int\limits_{-\infty}^{\infty} x(\xi)\,[v(h)](\xi) + x'(\xi)\,[v'(h)](\xi)\,d\xi, \quad \forall\, x \in H^1(\mathbb{R}).$$

As derived in Appendix A we can $v(h)$ represent as

$$[v(h)](x) = k_1\, e^x + k_2\, e^{-x} - \frac{1}{2} e^x \int e^{-x}[g(h)](x)\,dx + \frac{1}{2} e^{-x} \int e^x [g(h)](x)\,dx, \quad x \in \mathbb{R}, \tag{29}$$

where the constants k_1 and k_2 still have to be determined. We formulate the following first lemma.

Lemma 8.1 *For an arbitrary $h \in L^2(\mathbb{R})$ let $g(h)$ be given by (28). Then the (pointwise) estimate*

$$|[g(h)](x)| \le C_2\, \|h\|_{L^2(\mathbb{R})} \exp\left(-c_2 \frac{(x-x_0)^2}{T} + x\right)$$

holds for two constants $C_2, c_2 > 0$ and all $x \in \mathbb{R}$, independent of h.

Proof. From the estimate of Lemma 7.1 we get

$$\begin{aligned}
|[g(h)](x)| &\le \int\limits_0^T \Big| [z(a,h)](x,t)\,\Big([u_{xx}(a_0+a)](x,t) - [u_x(a_0+a)](x,t)\Big)\Big|\, dt \\
&\le C\, \|h\|_{L^2(\mathbb{R})} \int\limits_0^T (T-t)^{-\frac{1}{4}} \frac{1}{\sqrt{t}} \exp\left(-c_2 \frac{(x-x_0)^2}{t} + x\right) dt \\
&\le C\, \|h\|_{L^2(\mathbb{R})} \exp\left(-c_2 \frac{(x-x_0)^2}{T} + x\right) \underbrace{\int\limits_0^T (T-t)^{-\frac{1}{4}} \frac{1}{\sqrt{t}}\, dt}_{= \frac{2\sqrt{2}\,\Gamma\left(\frac{3}{4}\right)^2 T^{\frac{1}{4}}}{\sqrt{\pi}} < \infty} \\
&= C_2\, \|h\|_{L^2(\mathbb{R})} \exp\left(-c_2 \frac{(x-x_0)^2}{T} + x\right).
\end{aligned}$$

Thereby $\Gamma(\cdot)$ denotes again the Eulerian Gamma function. ■

Now we can examine the function $v(h)$, defined by (29). At first we proof the following result.

Lemma 8.2 *For an arbitraily $h \in L^2(\mathbb{R})$ let $g(h)$ be given by (28). Then we have*

$$\left| \int\limits_{-\infty}^{\infty} e^x [g(h)](x)\,dx \right| < \infty \qquad \textit{and} \qquad \left| \int\limits_{-\infty}^{\infty} e^{-x} [g(h)](x)\,dx \right| < \infty.$$

PROOF. From the estimate of Lemma 8.1 we obtain

$$\left|\int_{-\infty}^{\infty} e^{-x}[g(h)](x)\,dx\right| \leq C_2\,\|h\|_{L^2(\mathbb{R})}\int_{-\infty}^{\infty}\exp\left(-c_2\frac{(x-x_0)^2}{T}\right)\,dx = C_2\,\|h\|_{L^2(\mathbb{R})}\sqrt{\frac{\pi\,T}{c_2}} < \infty.$$

Furthermore we get

$$\begin{aligned}\left|\int_{-\infty}^{\infty} e^{x}[g(h)](x)\,dx\right| &\leq C_2\,\|h\|_{L^2(\mathbb{R})}\int_{-\infty}^{\infty}\exp\left(-c_2\frac{(x-x_0)^2}{T}+2\,x\right)\,dx\\ &= C_2\,\|h\|_{L^2(\mathbb{R})}\sqrt{\frac{\pi\,T}{c_2}}\exp\left(2x_0+\frac{T}{c_2}\right) < \infty. \qquad \blacksquare\end{aligned}$$

Now we can eliminate the constants k_1 and k_2 in the definition (29) of $v(h)$. Therefore we set

$$k_1 := \frac{1}{2}\int_{-\infty}^{\infty} e^{-x}[g(h)](x)\,dx \quad\text{and}\quad k_2 := 0$$

to derive

$$[v(h)](x) = \frac{1}{2}\left(e^{x}\int_{x}^{\infty} e^{-\xi}[g(h)](\xi)\,d\xi + e^{-x}\int_{-\infty}^{x} e^{\xi}[g(h)](\xi)\,d\xi\right), \qquad x\in\mathbb{R}. \tag{30}$$

We can prove the following pointwise estimate of $v(h)$.

Lemma 8.3 *For an arbitrary $h\in L^2(\mathbb{R})$ let $v(h)$ be given by (30). Then there exists a constant $K>0$ which does not depend on h such that the estimate*

$$|[v(h)](x)| \leq K\,\|h\|_{L^2(\mathbb{R})}e^{-|x|}$$

holds for all $x\in\mathbb{R}$.

PROOF. We consider the first integral. Let $x\leq 0$. Then we conclude from Lemma 8.1

$$\begin{aligned}\left|e^{x}\int_{x}^{\infty} e^{-\xi}[g(h)](\xi)\,d\xi\right| &\leq e^{x}C_2\,\|h\|_{L^2(\mathbb{R})}\int_{x}^{\infty}\exp(-\xi)\exp\left(-c_2\frac{(\xi-x_0)^2}{T}+\xi\right)\,d\xi\\ &= C_2\,\|h\|_{L^2(\mathbb{R})}e^{-|x|}\int_{x}^{\infty}\exp\left(-c_2\frac{(\xi-x_0)^2}{T}\right)\,d\xi\\ &\leq C_2\,\|h\|_{L^2(\mathbb{R})}e^{-|x|}\int_{-\infty}^{\infty}\exp\left(-c_2\frac{(\xi-x_0)^2}{T}\right)\,d\xi\\ &= C_2\,\|h\|_{L^2(\mathbb{R})}e^{-|x|}\sqrt{\frac{\pi\,T}{c_2}}.\end{aligned}$$

For $x > 0$ we derive

$$\begin{aligned}
\left| e^x \int\limits_x^\infty e^{-\xi}[g(h)](\xi)\, d\xi \right| &\leq e^x C_2 \, \|h\|_{L^2(\mathbb{R})} \int\limits_x^\infty \exp\left(-c_2 \frac{(\xi - x_0)^2}{T} \right) \, d\xi \\
&\leq e^x C_2 \, \|h\|_{L^2(\mathbb{R})} \int\limits_x^\infty \exp\left(-c_2 \frac{(\xi - x_0)^2}{T} + 2\,\xi - 2\,\xi \right) \, d\xi \\
&\leq e^x C_2 \, \|h\|_{L^2(\mathbb{R})} \int\limits_x^\infty \exp\left(-c_2 \frac{(\xi - x_0)^2}{T} + 2\,\xi - 2\,x \right) \, d\xi \\
&\leq C_2 \, \|h\|_{L^2(\mathbb{R})} e^{-|x|} \int\limits_{-\infty}^\infty \exp\left(-c_2 \frac{(\xi - x_0)^2}{T} + 2\,\xi \right) \, d\xi \\
&= C_2 \, \|h\|_{L^2(\mathbb{R})} e^{-|x|} \exp\left(2\,x_0 + \frac{T}{c_2} \right) \sqrt{\frac{\pi\, T}{c_2}}.
\end{aligned}$$

Analogous we treat the second integral. For $x > 0$ we derive

$$\begin{aligned}
\left| e^{-x} \int\limits_{-\infty}^x e^{\xi}[g(h)](\xi)\, d\xi \right| &\leq C_2 \, \|h\|_{L^2(\mathbb{R})} e^{-|x|} \int -\infty^x \exp\left(-c_2 \frac{(\xi - x_0)^2}{T} + 2\,\xi \right) \, d\xi \\
&\leq C_2 \, \|h\|_{L^2(\mathbb{R})} e^{-|x|} \int\limits_{-\infty}^\infty \exp\left(-c_2 \frac{(\xi - x_0)^2}{T} + 2\,\xi \right) \, d\xi \\
&= C_2 \, \|h\|_{L^2(\mathbb{R})} e^{-|x|} \exp\left(2\,x_0 + \frac{T}{c_2} \right) \sqrt{\frac{\pi\, T}{c_2}}.
\end{aligned}$$

For $x \leq 0$ we get

$$\begin{aligned}
\left| e^{-x} \int\limits_{-\infty}^x e^{\xi}[g(h)](\xi)\, d\xi \right| &\leq e^{-x} C_2 \, \|h\|_{L^2(\mathbb{R})} \int\limits_{-\infty}^x \exp\left(-c_2 \frac{(\xi - x_0)^2}{T} + 2\,\xi \right) \, d\xi \\
&\leq C_2 \, \|h\|_{L^2(\mathbb{R})} e^{x} \int\limits_{-\infty}^x \exp\left(-c_2 \frac{(\xi - x_0)^2}{T} \right) \, d\xi \\
&\leq C_2 \, \|h\|_{L^2(\mathbb{R})} e^{-|x|} \int\limits_{-\infty}^\infty \exp\left(-c_2 \frac{(\xi - x_0)^2}{T} \right) \, d\xi \\
&= C_2 \, \|h\|_{L^2(\mathbb{R})} e^{-|x|} \sqrt{\frac{\pi\, T}{c_2}}. \qquad \blacksquare
\end{aligned}$$

Theorem 8.1 *For an arbitrary* $h \in L^2(\mathbb{R})$ *let* $v(h)$ *be given by (30). Then we have* $v(h) \in H^1(\mathbb{R})$.

PROOF. From the lemma above we conclude

$$\|v(h)\|_{L^2(\mathbb{R})} \leq K \left(\int_{-\infty}^{\infty} e^{-2|x|}\, dx \right)^{\frac{1}{2}} = K.$$

Furthermore by differentiation we obtain

$$\begin{aligned}
[v'(h)](x) &= \frac{1}{2} \Bigg(-e^{-x} \int_{-\infty}^{x} e^{\xi} [g(h)](\xi)\, d\xi + e^{-x} e^{x} [g(h)](x) + \ldots \\
&\qquad \ldots + e^{x} \int_{x}^{\infty} e^{-\xi} [g(h)](\xi)\, d\xi - e^{x} e^{-x} [g(h)](x) \Bigg) \\
&= \frac{1}{2} \left(e^{x} \int_{x}^{\infty} e^{-\xi} [g(h)](\xi)\, d\xi - e^{-x} \int_{-\infty}^{x} e^{\xi} [g(h)](\xi)\, d\xi \right).
\end{aligned}$$

Consequently $v'(h) \in L^2(\mathbb{R})$ follows again from the proof of the lemma above. ■

Since $v''(h) = v(h) - g(h) \in L^2(\mathbb{R})$ we can conclude even higher regularity.

Corollary 8.1 *For an arbitrary $h \in L^2(\mathbb{R})$ let $v(h)$ be given by (30). Then we have $v(h) \in H^2(\mathbb{R}) \cap H^1(\mathbb{R})$.*

Using Lemma 8.3 and Corollary 8.1 we can formulate consequences out of the source condition (i) and the smallness condition (ii) of Proposition 5.1. From the condition (ii) we conclude $L\,\|h\|_{L^2(\mathbb{R})} < 1$ and therefore $\|h\|_{L^2(\mathbb{R})} < L^{-1}$. Combining this with the results of Lemma 8.3 and Corollary 8.1 we can conclude the following interpretation of the conditions of Proposition 5.1.

Corollary 8.2 *Assume there exists a solution $a_{ex} \in H^1(\mathbb{R})$ of equation (19) for exact data $\tilde{u}$. Then the conditions of Proposition 5.1 say that $a_{ex} - a_0 \in H^2(\mathbb{R}) \cap H^1(\mathbb{R})$ and*

$$|a_{ex}(x) - a_0(x)| \leq \frac{K}{L} \exp(-|x|), \qquad \forall\, x \in \mathbb{R}.$$

Thereby L denotes the constant in Theorem 4.2 and K is the constant in Lemma 8.3.

A The adjoint operator as element of $H^1(\mathbb{R})$

Let $f \in \left(H^1(\mathbb{R})\right)^* = H^{-1}(\mathbb{R})$ be given by

$$\langle x, f \rangle := \int_{-\infty}^{\infty} u(\xi)\, x(\xi)\, d\xi, \qquad x \in H^1(\mathbb{R}),$$

for a function u (u sufficiently smooth). On the other hand we can $H^{-1}(\mathbb{R})$ with $H^1(\mathbb{R})$ by the Riesz Theorem. That means there exists an element $v \in H^1(\mathbb{R})$ with

$$\begin{aligned} \langle x, f \rangle = \int_{-\infty}^{\infty} u(\xi)\, x(\xi)\, d\xi &= (x, v)_{H^1(\mathbb{R})} \\ &= (x, v)_{0,\mathbb{R}} + (x', v')_{0,\mathbb{R}} \\ &= \int_{-\infty}^{\infty} x(\xi)\, v(\xi) + x'(\xi)\, v'(\xi)\, d\xi \qquad \forall\, x \in H^1(\mathbb{R}). \end{aligned}$$

We now derive a relation between the represention elements u and v of f. Therefore we differentiate by parts to obtain

$$\int_{-\infty}^{\infty} x'(\xi)\, v'(\xi)\, d\xi = x\, v' \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} x(\xi)\, v''(\xi)\, d\xi = - \int_{-\infty}^{\infty} x(\xi)\, v''(\xi)\, d\xi$$

since $x(\xi) \to 0$ for $|\xi| \to \infty$ as element of $H^1(\mathbb{R})$. Consequently

$$\int_{-\infty}^{\infty} u(\xi)\, x(\xi)\, d\xi = \int_{-\infty}^{\infty} (v(\xi) - v''(\xi))\, x(\xi)\, d\xi \iff u = v - v'' \;\text{ a.e. on }\; \mathbb{R}.$$

That's why we consider the ordinary differential equation

$$u(x) = v(x) - v''(x), \qquad x \in \mathbb{R}. \tag{31}$$

The general solution v_h of the homogenous equation $v - v'' = 0$ is given by

$$v_h(x) = c_1\, e^{x} + c_2\, e^{-x}, \qquad x \in \mathbb{R}.$$

A particular solution v_s is given by

$$v_s(x) := -\frac{1}{2}\, e^{x} \int e^{-x}\, u(x)\, dx + \frac{1}{2}\, e^{-x} \int e^{x} u(x)\, dx, \qquad x \in \mathbb{R}.$$

Now we can represent a solution v of (31) as

$$v(x) = v_h(x) + v_s(x) = c_1\, e^{x} + c_2\, e^{-x} - \frac{1}{2}\, e^{x} \int e^{-x} u(x)\, dx + \frac{1}{2}\, e^{-x} \int e^{x} u(x)\, dx, \quad x \in \mathbb{R}.$$

Since we are looking for a solution $v \in H^1(\mathbb{R})$ of (31) we determine the constants c_1 and c_2 out of the boundary conditions $v(x) \to 0$ for $|x| \to \infty$.

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