L^2-estimates of weak solutions of parabolic Cauchy problems and their application to the option pricing theory

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Abstract

This paper deals with L^2-estimates for weak solutions of one-dimensional parabolic Cauchy problems. These results were applied to the option pricing problem that arises in the Black-Scholes framework. In particular the continuous dependence of option prices form the underlying volatility are shown and the Fréchet differentiability is examined in specific spaces.

Keywords: parabolic equation, Cauchy problem, weak solution, L^2-estimates, Black-Scholes equation, option pricing.

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1 Introduction

We consider two concrete one-dimensional parabolic equations which occur in financial mathematics. They are well-known as Black-Scholes equation as well as Dupire equation and are fundamental in the option pricing theory.

An European call option gives the holder the right to buy the underlying asset at the expiration date T for the strike price K. We assume that the price X of the underlying asset follows a stochastic process of the form

$$\frac{dX}{X} = \mu(X,t)\,dt + \sigma(X,t)\,dW_t \qquad (1)$$

with parameters drift μ and volatility σ. W_t denotes a standard Wiener process. We suppose that the volatility σ is a deterministic function which depends on the asset price

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X and/or the time t. The model (1) represents one possible generalization of the model of geometric Brownian motion which forms the basis for calculating option prices via the well-known Black-Scholes formula (see e.g. [1]).

We denote with $c(X, t, K, T)$ the (fair) market price of a call option as function of the variables asset price X, time $t \geq 0$, strike price K and expiration date $T \geq t$. Let the strike price K and expiration date T be fixed. Using the Black-Scholes analysis we can show that the option price function c satisfies the (generalized) Black-Scholes equation

$$\frac{\partial c}{\partial t} + \frac{1}{2} X^2 \sigma^2(X,t) \frac{\partial^2 c}{\partial X^2} + r\, X \frac{\partial c}{\partial X} - r\, c = 0, \qquad (X,t) \in (0,\infty) \times [0,T). \tag{2}$$

Furthermore the final condition

$$c(X, T, K, T) = \max(X - K, 0), \qquad X \in (0, \infty), \tag{3}$$

holds. The additional parameter r represents the interest rate of a riskless investment. Fixing the asset price X and time t the option price function c fulfills the Dupire equation (see [2])

$$\frac{\partial c}{\partial T} = \frac{1}{2} K^2 \sigma^2(K,T) \frac{\partial^2 c}{\partial K^2} - r\, K \frac{\partial c}{\partial K}, \qquad (K,T) \in (0,\infty) \times (t,\infty), \tag{4}$$

together with the initial condition

$$c(X, t, K, t) = \max(X - K, 0), \qquad K \in (0, \infty). \tag{5}$$

Known solvability theory of parabolic equations is not directly applicable to the Cauchy problems (2)-(3) and (4)-(5). It's due to the coefficient of the second derivative, which in booth equations is unbounded and tends to zero for $X \to 0$ respectively $K \to 0$. Therefore we introduce the following well-known transformations:

$$x := \ln X, \;\; y := \ln K, \;\; u(x,t,y,T) := c(X,t,K,T) \;\text{ and }\; a(x,t) := \frac{1}{2}\sigma^2(X,t).$$

Consequently we obtain from (2)-(3) respectively (4)-(5) the Cauchy problems

$$\begin{cases} -\dfrac{\partial u}{\partial t} &= a(x,t) \dfrac{\partial^2 u}{\partial x^2} + (r - a(x,t)) \dfrac{\partial u}{\partial x} - r\, u, \quad (x,t) \in \mathbb{R} \times [0,T), \\ u(x,T,y,T) &= \max\left(e^x - e^y, 0\right), \qquad x \in \mathbb{R}, \end{cases} \tag{6}$$

respectively

$$\begin{cases} \dfrac{\partial u}{\partial T} &= a(y,T) \dfrac{\partial^2 u}{\partial y^2} - (r + a(y,T)) \dfrac{\partial u}{\partial y}, \quad (y,T) \in \mathbb{R} \times (t,\infty), \\ u(x,t,y,t) &= \max\left(e^x - e^y, 0\right), \qquad y \in \mathbb{R}. \end{cases} \tag{7}$$

Now conditions to the parameter a can be specified to obtain the unique solvability of (6) respectively (7) (see e.g. [4],[6]).

We see that the (transformed) volatility function a plays an important role in option pricing. Therefore it seems to be crucial to investigate the influence of the volatility a on the option price u. In particular it is of high interest to study the mapping

$$a \mapsto u(x,t,y,T)$$

which can be defined via the Cauchy problems (6) respectively (7).

This practical example gives the motivation to investigate the solvability of general Cauchy problems in parabolic equations. In particular we will use the concept of weak solutions to derive L^2-estimates of solutions of parabolic equations. In the standard literature the study of weak solutions of Cauchy problems does not seem to play an important role. Often the investigations are restricted only on boundary value problems on bounded domains (see e.g. [3, Section 7.1])or the generalization of these results to Cauchy problems are just brief mentioned (see e.g. [6, Section 3.5]). Therefore in a first step we will derive L^2-estimates for weak solutions of a wide class of one-dimensional parabolic Cauchy problems. This is done by using and generalizing well-known techniques for derivations of estimates for solutions of differential equations on bounded domains. Later on we will use these results to analyze the introducing option pricing problem.

2 L^2-estimates of weak solutions

The derivation of L^2-estimates for solution of parabolic equations is basically established by application of two different inequalities. The first one is the Cauchy inequality with ε (see e.g. [3, p.622]). It says in our context that for $\varepsilon > 0$ and two functions $f, g \in L^2(\mathbb{R})$ the estimate

$$\int_{-\infty}^{\infty} f(x)\, g(x)\; dx \leq \varepsilon\, \|f\|^2_{L^2(\mathbb{R})} + \frac{1}{4\,\varepsilon}\, \|g\|^2_{L^2(\mathbb{R})}$$

holds. The other mathematical tool is Gronwall's inequality (see e.g. [3, pp. 625]). We will formulate the statement in the following proposition.

Proposition 2.1 *Let y be a nonnegative absolutely continuous function on $[0, T]$, which satisfies for a.e. t the differential equation*

$$y'(t) \leq \phi(t)\, y(t) + \psi(t),$$

where ϕ and ψ are nonnegative, summable functions on $[0, T]$. Then

$$y(t) \leq \exp\left(\int_0^t \phi(\tau)\, d\tau\right)\left[y(0) + \int_0^t \psi(\tau)\, d\tau\right]$$

for all $0 \leq t \leq T$.

We will use the solvability concept for a widest possible class of parabolic equations. Therefore we consider for given $T > 0$ the abstract Cauchy problem

$$\left\{\begin{aligned} a_0(x,t)\, u_t(x,t) - L\, u(x,t) \; &= f(x,t), \qquad && (x,t) \in \mathbb{R} \times (0,T] \\ u(x,0) \; &= g(x), && x \in \mathbb{R}, \end{aligned}\right. \tag{8}$$

with the time-dependent elliptic differential operator

$$L\, u(x,t) := \left(a(x,t)\, u_x(x,t)\right)_x + b(x,t)\, u_x(x,t) + c(x,t)\, u(x,t)$$

in differgence form. We set $Q_T := \mathbb{R} \times (0,T)$. In a first step we make the following assumptions to coefficients. Later on we have to formulate further restrictions to receive higher regularity results of weak solutions of (8).

Assumption 2.1 *We assume*

(i) $a \in L^\infty(Q_T)$ *and* $\alpha_1 \le a \le \alpha_2$ *a.e. on* Q_T,

(ii) $b \in L^\infty(Q_T)$ *and* $\beta := \|b\|_{L^\infty(Q_T)}$,

(iii) $c \in L^\infty(Q_T)$ *and* $\gamma := \|c\|_{L^\infty(Q_T)}$,

(iv) $a_0 \in C([0,T]; L^\infty(\mathbb{R}))$, $\nu_1 \le a_0 \le \nu_2$ *a.e. on* Q_T, $\|a_{0_t}(\cdot,t)\|_{L^\infty(\mathbb{R})} \le \mu_1(t)$ *and*

$$\int_0^T \mu_1(t)\, dt =: \nu < \infty,$$

(v) $f \in L^2(Q_T)$ *and* $g \in L^2(\mathbb{R})$.

We develop a concept of a weak solution of (8). Let $v \in L^2(0,T;H^1(\mathbb{R}))$ be arbitrarily. Multiplying with $L\,u$ and integrating over Q_{t^*} for $t^* > 0$ we derive

$$\begin{aligned}
\iint_{Q_{t^*}} L\,u(x,t)\,v(x,t)\;dt\,dx &= \iint_{Q_{t^*}} (a\,u_x)_x\,v\;dt\,dx + \iint_{Q_{t^*}} b\,u_x\,v\;dt\,dx + \iint_{Q_{t^*}} c\,u\,v\;dt\,dx \\
&= -\iint_{Q_{t^*}} a\,u_x\,v_x\;dt\,dx + \iint_{Q_{t^*}} b\,u_x\,v\;dt\,dx + \iint_{Q_{t^*}} c\,u\,v\;dt\,dx \\
&=: -B(u,v;t^*)
\end{aligned}$$

with a time-dependent bilinear form

$$B : L^2(0,T;H^1(\mathbb{R})) \times L^2(0,T;H^1(\mathbb{R})) \times [0,T] \longrightarrow \mathbb{R}.$$

Definition 2.1 *We call a function* $u \in L^2(0,T;H^1(\mathbb{R}))$ *with* $u_t \in L^2(0,T;H^{-1}(\mathbb{R}))$ *a weak solution of (8) if*

$$\iint_{Q_{t^*}} a_0\,u_t\,v\;dx\,dt + B(u,v;t^*) = \iint_{Q_{t^*}} f\,v\;dx\,dt \tag{9}$$

for all $v \in L^2(0,T;H^1(\mathbb{R}))$ *and a.e.* $t^* \in (0,T]$ *and*

$$u(\cdot,0) = g. \tag{10}$$

Remark 2.1 *The expression (10) makes sense since the condition* $u_t \in L^2(0,T;H^{-1}(\mathbb{R}))$ *guarantees* $u \in C([0,T];L^2(\mathbb{R}))$ *(see e.g. [3, Theorem 5.9.3]).*

Now we will prove a first statement. We suppose the the existence of a weak solution of (8) and prove an estimate for this solution. Afterwards we can derive uniqueness and existence results out of this estimate.

Theorem 2.1 *Assume there exists a weak solution $u \in L^2(0,T;H^1(\mathbb{R}))$ of (8). Then $u \in L^\infty(0,T;L^2(\mathbb{R}))$ and the estimate*

$$\sup_{0\le t\le T} \|u(\cdot,t)\|^2_{L^2(\mathbb{R})} + \|u\|^2_{L^2(0,T;H^1(\mathbb{R}))} \le C_1\|g\|^2_{L^2(\mathbb{R})} + C_2\|f\|^2_{L^2(Q_T)} \tag{11}$$

holds for positive constants C_1, $C_2 > 0$.

PROOF. We consider for $0 < t^* \le T$

$$\begin{aligned}
\iint\limits_{Q_{t^*}} a_0\, u_t\, u\; dx\, dt &= \frac{1}{2}\left(\iint\limits_{Q_{t^*}} \frac{\partial}{\partial t}\left(a_0\, u^2\right) - a_{0t}\, u^2\, dx\, dt\right) \\
&= \frac{1}{2}\left(\left\|\sqrt{a_0(\cdot,t^*)}\, u(\cdot,t^*)\right\|^2_{L^2(\mathbb{R})} - \left\|\sqrt{a_0(\cdot,0)}\, u(\cdot,0)\right\|^2_{L^2(\mathbb{R})} - \dots\right. \\
&\quad \left. \dots - \iint\limits_{Q_{t^*}} a_{0_t}\, u^2\, dx\, dt\right).
\end{aligned}$$

Furthermore

$$\begin{aligned}
\alpha_1 \iint\limits_{Q_{t^*}} u_x^2\; dx\, dt &\le \iint\limits_{Q_{t^*}} a\, u_x^2\; dx\, dt \\
&= B(u,v;t^*) + \iint\limits_{Q_{t^*}} b\, u_x\, u\; dx\, dt + \iint\limits_{Q_{t^*}} c\, u^2\; dx\, dt \\
&\le B(u,v;t^*) + \beta\left(\varepsilon \iint\limits_{Q_{t^*}} u_x^2\; dx\, dt + \frac{1}{4\,\varepsilon}\iint\limits_{Q_{t^*}} u^2\; dx\, dt\right) + \gamma \iint\limits_{Q_{t^*}} u^2\; dx\, dt.
\end{aligned}$$

Choosing $\varepsilon := \frac{\alpha_1}{2\,\beta}$ for $\beta > 0$ (the case $\beta = 0$ is trivial) we get

$$\frac{\alpha_1}{2}\|u\|^2_{L^2(0,t^*;H^1(\mathbb{R}))} \le B(u,v;t^*) + \tilde{C}\|u\|^2_{L^2(Q_{t^*})}$$

with $\tilde{C} := \gamma + \frac{\alpha_1}{2} + \frac{\beta^2}{2\,\alpha_1}$. Combining these two results we get

$$\begin{aligned}
\frac{\nu_1}{2}\|u(\cdot,t^*)\|^2_{L^2(\mathbb{R})} + \frac{\alpha_1}{2}\|u\|^2_{L^2(0,t^*;H^1(\mathbb{R}))} &\le \underbrace{\iint\limits_{Q_{t^*}} a_0\, u_t\, u\; dx\, dt + B(u,v;t^*)}_{=\iint\limits_{Q_{t^*}} f\, u\; dx\, dt} + \tilde{C}\|u\|^2_{L^2(Q_{t^*})} + \dots \\
&\quad \dots + \frac{\nu_2}{2}\|u(\cdot,0)\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\iint\limits_{Q_{t^*}} a_{0_t}\, u^2\; dx\, dt \\
&= \iint\limits_{Q_{t^*}} f\, u\; dx\, dt + \tilde{C}\|u\|^2_{L^2(Q_{t^*})} + \dots
\end{aligned}$$

$$
\begin{aligned}
&\quad \ldots + \frac{\nu_2}{2}\|g\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\iint\limits_{Q_{t^*}} a_{0_t}\, u^2\, dx\, dt \\
&\leq \frac{\nu_2}{2}\|f\|^2_{L^2(Q_{t^*})} + \left(\tilde{C} + \frac{1}{2\,\nu_2}\right)\|u\|^2_{L^2(Q_{t^*})} + \ldots \\
&\quad \ldots + \frac{\nu_2}{2}\|g\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\int\limits_0^{t^*} \mu_1(t)\,\|u(\cdot,t\|^2_{L^2(\mathbb{R})}\, dt \\
&\leq \frac{\nu_2}{2}\|f\|^2_{L^2(Q_T)} + \frac{\nu_2}{2}\|g\|^2_{L^2(\mathbb{R})} + \ldots \\
&\quad \ldots + \frac{1}{2}\int\limits_0^{t^*} \underbrace{\left(2\,\tilde{C} + \frac{1}{\nu_2} + \mu_1(t)\right)}_{=:\,\tilde{\mu}(t)}\|u(\cdot,t\|^2_{L^2(\mathbb{R})}\, dt
\end{aligned}
$$

since u is a weak solution of (8). We multiply by 2 and set

$$
y(t^*) := \nu_1 \int\limits_0^{t^*} \tilde{\mu}(t)\,\|u(\cdot,t)\|^2_{L^2(\mathbb{R})}, \quad y'(t^*) = \nu_1\,\tilde{\mu}(t^*)\,\|u(\cdot,t)\|^2_{L^2(\mathbb{R})}.
$$

From the upper inequality we find that

$$
y'(t^*) \leq \tilde{\mu}(t^*)\left(\nu_2\left(\|f\|^2_{L^2(Q_T)} + \|g\|^2_{L^2(\mathbb{R})}\right) + \frac{1}{\nu_1}y(t^*)\right).
$$

We use Gronwall's inequality with

$$
\phi(t^*) := \frac{\tilde{\mu}(t^*)}{\nu_1} \quad \text{and} \quad \psi(t^*) := \tilde{\mu}(t^*)\,\nu_2\left(\|f\|^2_{L^2(Q_T)} + \|g\|^2_{L^2(\mathbb{R})}\right)
$$

and obtain

$$
\begin{aligned}
y(t^*) &\leq \exp\left(\frac{\tilde{\mu}(t^*)}{\nu_1}\right)\left(0 + \int\limits_0^{t^*} \tilde{\mu}(t)\,\nu_2\left(\|f\|^2_{L^2(Q_T)} + \|g\|^2_{L^2(\mathbb{R})}\right)\, dt\right) \\
&\leq \exp\left(\frac{2\,\tilde{C} + \nu_2^{-1} + \nu}{\nu_1}\right)\nu\,\nu_2\left(\|f\|^2_{L^2(Q_T)} + \|g\|^2_{L^2(\mathbb{R})}\right).
\end{aligned}
$$

Combining these two results we derive

$$
\begin{aligned}
&\nu_1\|u(\cdot,t^*)\|^2_{L^2(\mathbb{R})} + \alpha_1\|u\|^2_{L^2(0,t^*;H^1(\mathbb{R}))} \\
&\qquad \leq \left(\nu_2 + \frac{\nu\,\nu_2}{\nu_1}\right)\exp\left(\frac{2\,\tilde{C} + \nu_2^{-1} + \nu}{\nu_1}\right)\left(\|f\|^2_{L^2(Q_T)} + \|g\|^2_{L^2(\mathbb{R})}\right).
\end{aligned}
$$

The setting

$$
C_1 = C_2 := \frac{\left(\nu_2 + \frac{\nu\,\nu_2}{\nu_1}\right)\exp\left(\frac{2\,\tilde{C}+\nu_2^{-1}+\nu}{\nu_1}\right)}{\min(\nu_1,\alpha_1)}
$$

proves the theorem. ■

Remark 2.2 *In [3, chapter 7.1] an estimate of the form*

$$\sup_{0\le t\le T} \|u(\cdot,t)\|_{L^2(\mathbb{R})} + \|u\|_{L^2(0,T;H^1(\mathbb{R}))} \le C\left(\|g\|_{L^2(\mathbb{R})} + \|f\|_{L^2(Q_T)}\right)$$

is shown.

The uniqueness of a weak solution of (8) follows directly from these estimates.

Corollary 2.1 *A weak solution of (8) is unique.*

Proof. Assume there exist two weak solutions u_1, $u_2 \in L^2(0,T;H^1(\mathbb{R}))$. We set $u := u_1 - u_2$. Then u is obviously the weak solution of (8) with $g \equiv 0$ and $f \equiv 0$. From the estimate (11) we conclude $u \equiv 0$. ■

It can be shown that from (11) we can conclude the existence of a weak solution of (8). We will give here only the idea of the proof. For a more detailed derivation we refer to [3, Theorems 7.1.1-7.1.3].

Theorem 2.2 *Under the conditions of assumption 2.1 there exists a weak solution of (8).*

Idea of the proof. The theorem is proved by Galerkin approximation of the solution u of (8). Let $\{e_k\}$ be a sequence of smooth functions which form an orthogonal basis in $H^1(\mathbb{R})$ and an orthonormal basis in $L^2(\mathbb{R})$. For fixed n we try to find a function u_n of the form

$$u_n(x,t) := \sum_{k=1}^{n} d_k^n(t)\, e_k(x),$$

whereby

$$d_k^n(0) = \int\limits_{-\infty}^{\infty} g(x)\, e_k(x)\, dx, \quad 1 \le k \le n$$

and

$$\int\limits_{-\infty}^{\infty} u_m(x,t)\, e_k(x)\, dx + B(u_m, e_k; t) = \int\limits_{-\infty}^{\infty} f(x,t)\, e_k(x)\, dx, \quad 1 \le k \le n,\ 0 \le t \le T.$$

Using the orthogonal property of the e_k this problem can be transformed to a linear system of ordinary differential equations, which is uniquely solvable. The solutions u_n satisfy an estimate of the form (11), whereby the constants not depend on n. Consequently the sequence $\{u_n\}$ is bounded and hence there exists a weakly convergent subsequence $u_{nk} \rightharpoonup u \in L^2(0,T;H^1(\mathbb{R}))$. Finally we can show that u is a solution of (8). ■

Under additional assumptions to the parameters of (8) we can prove higher regularity.

Assumption 2.2 *Additionally to Assumption 2.1 we suppose for the parameters of (8)*

(vi) $\|a_t(\cdot,t)\|_{L^\infty(\mathbb{R})} \le \mu_2(t)$ *for a.e.* $t \in (0,T)$ *and*

$$\int\limits_{0}^{T} \mu_2(t)\, dt = \mu < \infty$$

(vii) $g \in H^1(\mathbb{R})$.

Theorem 2.3 *Under the conditions of Assumption 2.2 for the weak solution u of (8) holds $u_t \in L^2(0,T;L^2(\mathbb{R}))$ and $u_x \in L^\infty(0,T;L^2(\mathbb{R}))$ and the estimate*

$$\|u_t\|^2_{L^2(\Omega_T)} + \sup_{0\le t\le T} \|u_x(\cdot,t)\|^2_{L^2(\mathbb{R})} \le C_3\|g\|^2_{H^1(\mathbb{R})} + C_4\|f\|^2_{L^2(Q_T)} \tag{12}$$

is valid for positive constants C_3, $C_4 > 0$.

Proof. First we assume that $u_t \in L^2(0,T;L^2(\mathbb{R}))$, $u_x \in L^\infty(0,T;L^2(\mathbb{R}))$ and prove the estimate (12). Hence we conclude these properties analogous to Theorem 2.2.

Assume $u_t \in L^2(0,T;L^2(\mathbb{R}))$ and $u_x \in L^\infty(0,T;L^2(\mathbb{R}))$. Then we consider for $t^* \in [0,T]$

$$\iint\limits_{Q_{t^*}} a\, u_x\, u_{xt}\; dx\, dt = \int\limits_{-\infty}^{\infty} a(x,t^*)\, u_x^2(x,t^*)\; dx - \int\limits_{-\infty}^{\infty} a(x,0)\, u_x^2(x,0)\; dx - \iint\limits_{Q_{t^*}} (a\, u_x)_t\; u_x\; dx\, dt$$

and consequently

$$\iint\limits_{Q_{t^*}} a\, u_x\, u_{xt}\; dx\, dt = \frac{1}{2}\left(\int\limits_{-\infty}^{\infty} a(x,t^*)\, u_x^2(x,t^*)\; dx - \int\limits_{-\infty}^{\infty} a(x,0)\, u_x^2(x,0)\; dx - \iint\limits_{Q_{t^*}} a_t\, u_x^2\; dx\, dt\right).$$

From this we derive

$$\begin{aligned}
\nu_1\|u_t\|^2_{L^2(\Omega_{t^*})} + \frac{\alpha_1}{2}\|u_x(\cdot,t^*)\|^2_{L^2(\mathbb{R})} \;&\le\; \iint\limits_{Q_{t^*}} a_0\, u_t^2\; dx\, dt + B(u,u_t;t^*) + \ldots \\
&\quad \ldots + \iint\limits_{Q_{t^*}} b\, u_x\, u_t\; dx\, dt + \iint\limits_{Q_{t^*}} c\, u\, u_t\; dx\, dt + \ldots \\
&\quad \ldots + \frac{1}{2}\int\limits_{-\infty}^{\infty} a\, u_x^2(\cdot,0)\; dx + \iint\limits_{Q_{t^*}} a_t\, u_x^2\; dx\, dt \\
&= \iint\limits_{Q_{t^*}} f\, u_t\; dx\, dt + \frac{1}{2}\int\limits_{-\infty}^{\infty} a\, g_x^2\; dx + \ldots \\
&\quad \ldots + \iint\limits_{Q_{t^*}} b\, u_x\, u_t\; dx\, dt + \iint\limits_{Q_{t^*}} c\, u\, u_t\; dx\, dt + \ldots \\
&\quad \ldots + \iint\limits_{Q_{t^*}} a_t\, u_x^2\; dx\, dt \\
&\le \varepsilon_1\|f\|^2_{L^2(\Omega_{t^*})} + \frac{1}{4\,\varepsilon_1}\|u_t\|^2_{L^2(\Omega_{t^*})} + \frac{\alpha_1}{2}\|g_x\|^2_{L^2(\Omega_{t^*})} + \ldots \\
&\quad \ldots + \beta\left(\varepsilon_2\|u_x\|^2_{L^2(\Omega_{t^*})} + \frac{1}{4\,\varepsilon_2}\|u_t\|^2_{L^2(\Omega_{t^*})}\right) + \ldots
\end{aligned}$$

$$\ldots + \gamma\left(\varepsilon_3 \|u\|^2_{L^2(\Omega_{t^*})} + \frac{1}{4\,\varepsilon_3}\|u_t\|^2_{L^2(\Omega_{t^*})}\right) + \ldots$$
$$\ldots + \int_0^{t^*} \mu_2(t)\,\|u_x(\cdot,t)\|^2_{L^2(\mathbb{R})}\,dt.$$

Assume $\gamma > 0$ and $\beta > 0$ (it's becomes simpler for $\gamma = 0$ or $\beta = 0$). Then we choose $\varepsilon_1 := \frac{1}{\nu_1}$ and $\varepsilon_3 := \varepsilon_2 \frac{\beta}{\gamma}$ to obtain

$$\begin{aligned}
\nu_1 \|u_t\|^2_{L^2(\Omega_{t^*})} + \frac{\alpha_1}{2}\|u_x(\cdot,t^*)\|^2_{L^2(\mathbb{R})} \;\le\;& \frac{1}{\nu_1}\|f\|^2_{L^2(\Omega_{t^*})} + \frac{\alpha_1}{2}\|g_x\|^2_{L^2(\Omega_{t^*})} + \ldots\\
&\ldots + \beta\,\varepsilon_2\,\|u\|^2_{L^2(0,t^*;H^1(\mathbb{R}))} + \ldots\\
&\ldots + \left(\frac{\nu_1}{4} + \frac{\beta}{4\,\varepsilon_2} + \frac{\gamma^2}{4\,\varepsilon_2\,\beta}\right)\|u_t\|^2_{L^2(\Omega_{t^*})} + \ldots\\
&\ldots + \int_0^{t^*} \mu_2(t)\,\|u_x(\cdot,t)\|^2_{L^2(\mathbb{R})}\,dt.
\end{aligned}$$

Choosing

$$\frac{\nu}{4} = \frac{\beta}{4\,\varepsilon_2} + \frac{\gamma^2}{4\,\varepsilon_2\,\beta} \;\Leftrightarrow\; \varepsilon_2 := \frac{1}{\nu_1\,\beta}\left(\beta^2 + \gamma^2\right)$$

we get

$$\begin{aligned}
\frac{\nu_1}{2}\|u_t\|^2_{L^2(\Omega_{t^*})} + \frac{\alpha_1}{2}\|u_x(\cdot,t^*)\|^2_{L^2(\mathbb{R})} \;\le\;& \frac{1}{\nu_1}\|f\|_{L^2(\Omega_{t^*})} + \frac{\beta^2+\gamma^2}{\nu_1}\|u\|^2_{L^2(0,t^*;H^1(\mathbb{R}))} + \ldots\\
&\ldots + \frac{\alpha_1}{2}\|g_x\|^2_{L^2(\mathbb{R})} + \int_0^{t^*} \mu_2(t)\,\|u_x(\cdot,t)\|^2_{L^2(\mathbb{R})}\,dt\\
\le\;& \frac{1}{\nu_1}\|f\|_{L^2(\Omega_T)} + \frac{\beta^2+\gamma^2}{\nu_1}\|u\|^2_{L^2(0,T;H^1(\mathbb{R}))} + \ldots\\
&\ldots + \frac{\alpha_1}{2}\|g_x\|^2_{L^2(\mathbb{R})} + \int_0^{t^*} \mu_2(t)\,\|u_x(\cdot,t)\|^2_{L^2(\mathbb{R})}\,dt.
\end{aligned}$$

We multiply by 2 and set

$$y(t^*) := \alpha_1 \int_0^{t^*} \mu_2(t)\,\|u_x(\cdot,t)\|^2_{L^2(\mathbb{R})}\,dt, \quad y'(t^*) = \alpha_1\,\mu_2(t^*)\|u_x(\cdot,t^*)\|^2_{L^2(\mathbb{R})}.$$

From the inequality above we obtain

$$\begin{aligned}
y'(t^*) \;\le\;& 2\,\mu_2(t^*)\left(\frac{1}{\nu_1}\|f\|_{L^2(\Omega_T)} + \frac{\beta^2+\gamma^2}{\nu_1}\|u\|^2_{L^2(0,T;H^1(\mathbb{R}))} + \frac{\alpha_1}{2}\|g_x\|^2_{L^2(\mathbb{R})}\right) + \ldots\\
&\ldots + \frac{2\,\mu_2(t^*)}{\alpha_1}\,y(t^*).
\end{aligned}$$

From Gronwall's inequality we receive

$$
\begin{aligned}
y(t^*) \;&\le\; \exp\left(\frac{2}{\alpha_1}\int\limits_0^{t^*}\mu_2(t)\,dt\right)\,2\,\left(\int\limits_0^{t^*}\mu_2(t)\,dt\right)\left(\frac{1}{\nu_1}\|f\|_{L^2(\Omega_T)}+\dots\right.\\
&\qquad\left.\dots+\frac{\beta^2+\gamma^2}{\nu_1}\|u\|^2_{L^2(0,T;H^1(\mathbb{R}))}+\frac{\alpha_1}{2}\|g_x\|^2_{L^2(\mathbb{R})}\right)\\
&\le\; 2\,\mu\,\exp\left(\frac{2\,\mu}{\alpha_1}\right)\left(\frac{1}{\nu_1}\|f\|_{L^2(\Omega_T)}+\frac{\beta^2+\gamma^2}{\nu_1}\|u\|^2_{L^2(0,T;H^1(\mathbb{R}))}+\frac{\alpha_1}{2}\|g_x\|^2_{L^2(\mathbb{R})}\right)
\end{aligned}
$$

and finally

$$
\begin{aligned}
\nu_1\|u_t\|^2_{L^2(\Omega_{t^*})}+\alpha_1\|u_x(\cdot,t^*)\|^2_{L^2(\mathbb{R})} \;&\le\; \left(1+2\,\mu\,\exp\left(\frac{2\,\mu}{\alpha_1}\right)\right)\left(\frac{1}{\nu_1}\|f\|_{L^2(\Omega_T)}+\dots\right.\\
&\qquad\left.\dots+\frac{\beta^2+\gamma^2}{\nu_1}\|u\|^2_{L^2(0,T;H^1(\mathbb{R}))}+\frac{\alpha_1}{2}\|g_x\|^2_{L^2(\mathbb{R})}\right).
\end{aligned}
$$

The proof is completed by using the estimate (11) from Theorem 2.1. ■

Assumption 2.3 *Additionally to Assumption 2.2 we suppose for the parameters of (8)*

(viii) $a_x \in L^\infty(Q_T)$ *and* $\|a_x\|_{L^\infty(Q_T)} = \mu_3 < \infty$

Theorem 2.4 *Under the conditions of Assumption 2.3 for the weak solution* u *of (8) holds* $u \in L^2(0,T;H^2(\mathbb{R}))$ *and the estimate*

$$
\|u_{xx}\|^2_{L^2(Q_T)} \le C_5\|g\|^2_{H^1(\mathbb{R})} + C_6\|f\|^2_{L^2(Q_T)}. \tag{13}
$$

is valid for two positive constants $C_5,\, C_6 > 0$.

Proof. Since $u \in L^2(0,T;H^2(\mathbb{R}))$, $u_t \in L^2(0,T;L^2(\mathbb{R}))$ we conclude

$$
a_0\,u_t - L\,u = f \;\text{ a.e. on }\; Q_T.
$$

We multiply by u_{xx} and integrate over Q_T to receive

$$
\begin{aligned}
\iint\limits_{Q_T} f\,u_{xx}\,dx\,dt \;&=\; \iint\limits_{Q_T} a_0\,u_t\,u_{xx}\,dx\,dt - \iint\limits_{Q_T}(a\,u_x)_x\,u_{xx}\,dx\,dt - \dots\\
&\qquad \dots - \iint\limits_{Q_T} b\,u_x\,u_{xx}\,dx\,dt - \iint\limits_{Q_T} c\,u\,u_{xx}\,dx\,dt\\
&=\; \iint\limits_{Q_T} a_0\,u_t\,u_{xx}\,dx\,dt - \iint\limits_{Q_T} a\,u_{xx}^2\,dx\,dt - \dots\\
&\qquad \dots - \iint\limits_{Q_T}(b+a_x)\,u_x\,u_{xx}\,dx\,dt - \iint\limits_{Q_T} c\,u\,u_{xx}\,dx\,dt.
\end{aligned}
$$

Consequently

$$\begin{aligned}
\alpha_1 \|u_{xx}\|^2_{L(Q_T)} &\leq \iint\limits_{Q_T} a\, u_{xx}^2 \, dx\, dt \\
&= \iint\limits_{Q_T} a_0\, u_t\, u_{xx} \, dx\, dt - \iint\limits_{Q_T} f\, u_{xx} \, dx\, dt - \dots \\
&\quad \dots - \iint\limits_{Q_T} (b + a_x)\, u_x\, u_{xx} \, dx\, dt - \iint\limits_{Q_T} c\, u\, u_{xx} \, dx\, dt \\
&\leq \nu_2 \left(\varepsilon_1 \|u_t\|^2_{L^2(Q_T)} + \frac{1}{4\,\varepsilon_1} \|u_{xx}\|^2_{L^2(Q_T)} \right) + \dots \\
&\quad \dots + \varepsilon_2 \|f\|^2_{L^2(Q_T)} + \frac{1}{4\,\varepsilon_2} \|u_{xx}\|^2_{L^2(Q_T)} + \dots \\
&\quad \dots + (\beta + \mu_3) \left(\varepsilon_3 \|u_x\|^2_{L^2(Q_T)} + \frac{1}{4\,\varepsilon_3} \|u_{xx}\|^2_{L^2(Q_T)} \right) + \dots \\
&\quad \dots + \gamma \left(\varepsilon_4 \|u\|^2_{L^2(Q_T)} + \frac{1}{4\,\varepsilon_4} \|u_{xx}\|^2_{L^2(Q_T)} \right).
\end{aligned}$$

Choosing

$$\varepsilon_1 := \frac{2\,\nu_2}{\alpha_1},\ \varepsilon_2 := \frac{2}{\alpha_1},\ \varepsilon_3 := \frac{2\ (\beta + \mu_3)}{\alpha_1} \ \text{ and } \ \varepsilon_4 := \frac{2\,\gamma}{\alpha_1}$$

we obtain

$$\frac{\alpha_1}{2} \|u_{xx}\|^2_{L(Q_T)} \leq \frac{2}{\alpha_1} \|f\|^2_{L^2(Q_T)} + \frac{2\,\gamma^2}{\alpha_1} \|u\|^2_{L^2(Q_T)} + \frac{2\ (\beta + \mu_3)^2}{\alpha_1} \|u_x\|^2_{L^2(Q_T)} + \frac{2\,\nu_2^2}{\alpha_1} \|u_t\|^2_{L^2(Q_T)}.$$

Together with the estimates (11) and (12) this proves the theorem. ■

3 Parabolic equations in non-differgence form

As seen in the introducing examples (6) and (7) often parabolic equations arise in a non-differgence form. Therefore let L now have the form

$$L\, u(x,t) := a(x,t)\, u_{xx}(x,t) + b(x,t)\, u_x(x,t) + c(x,t)\, u(x,t).$$

We analyze again the solvability of the Cauchy problem (8). Therefore we will suggest two possible concepts to generalize the results of section 2 to this case. In booth cases we have demand further restrictions on the coefficient a to apply theorem 2.1.

Let a satisfy the condition $(viii)$, in particular

$$a_x \in L^\infty(Q_T) \ \text{ and } \ \|a_x\|_{L^\infty(Q_T)} = \mu_3 < \infty.$$

Then we L write in differgence form as

$$L\, u(x,T) = (a(x,t)\, u_x(x,t))_x + (b(x,t) - a_x(x,t))\ u_x(x,t) + c(x,t)\, u(x,t)$$

and Theorem 2.1 is applicable of the Cauchy problem (8).

The second idea uses the fact that the function a is strictly positive a.e. on Q_T. We divide (8) by a and obtain the Cauchy problem

$$\begin{cases} \tilde{a}_0(x,t)\, u_t(x,t) - \tilde{L}\, u(x,t) &= \tilde{f}(x,t), \qquad (x,t) \in \mathbb{R} \times (0,T], \\ u(x,0) &= \tilde{g}(x), \qquad x \in \mathbb{R}, \end{cases} \tag{14}$$

with

$$\tilde{a}_0 = \frac{a_0}{a},\ \tilde{f} = \frac{f}{a},\ \tilde{g} = \frac{g}{a(\cdot,0)}$$

and

$$\tilde{L}\, u(x,t) = u_{xx}(x,t) + \frac{b(x,t)}{a(x,t)}\, u_x(x,t) + \frac{c(x,t)}{a(x,t)}\, u(x,t).$$

Let $a \in C([0,T]; L^\infty(\mathbb{R}))$ and satisfy condition (vi), in particular

$$\|a_t(\cdot,t)\|_{L^\infty(\mathbb{R})} \le \mu_2(t) \ \text{ for a.e. } \ t \in (0,T) \ \text{ and } \int_0^T \mu_2(t)\, dt = \mu < \infty.$$

Obviously $\tilde{a}_0$ satisfies condition (iv) of Assumption 2.1 since

$$\tilde{a}_{0_t} = \frac{a_{0_t}}{a} - \frac{a_0\, a_t}{a^2}.$$

Then all requirements of Theorem 2.1 are fulfilled and we obtain the appropriate solvability result of the Cauchy problem (14).

Remark 3.1 *It can be shown that under conditions (vi) and $(viii)$ on the parameter a booth approaches of the definition of weak formulation of Cauchy problems in non-differgence form are equivalent.*

4 Application to the option pricing problem

For evaluating option prices we consider the transformed Dupire equation

$$u_t = L(a)\, u \ \text{ on } \ Q_T := \mathbb{R} \times (0,T) \tag{15}$$

with the elliptic differential operator

$$L(a)\, u := a\, u_{xx} - (r+a)\, u_x,$$

which depends on the parameter a. In order to use the classical solvability theory of parabolic equation we demand

$$a \in \mathcal{D}_c^\lambda := \left\{ a \in C^{\lambda,\frac{\lambda}{2}}(Q_T) \ : \ \underline{c} \le a \le \overline{c} \ \text{ on } \ Q_T \right\}$$

for constants $0 < \lambda \le 1$ and $0 < \underline{c} < \overline{c} < \infty$.

Definition 4.1 *Let be $T > 0$, $x_0 \in \mathbb{R}$ and $r \geq 0$ be fixed. For given $a \in \mathcal{D}_c^\lambda$ we define $u(a) \in C^{2,1}(Q_T)$ as classical solution of the Cauchy problem*

$$\left\{ \begin{array}{rll} u_t(x,t) & = L(a)\, u(x,t), & (x,t) \in Q_T, \\ u(x,0) & = \max(e^{x_0} - e^x, 0), & x \in \mathbb{R}. \end{array} \right. \tag{16}$$

We can show that this definition is correct since under these conditions there exists a unique solution of (16). We will formulate an appropriate statement.

Lemma 4.1 *For every $a \in \mathcal{D}_c^\lambda$ there exists a unique bounded (classical) solution $u(a)$ of (16).*

Proof. This is a well-known result from the theory of parabolic equation (see e.g. [4, Theorem 12, Chapter 1 + Theorem 10, Chapter 2]). ∎

In the next step we derive a correlation between the fundamental solution of the (transformed) Black-Scholes equation (3) and the solution $u(a)$ of (16). For further reading on fundamental solutions we refer to [4, pp.22].

Lemma 4.2 *For every $a \in \mathcal{D}_c^\lambda$ and every $t > 0$ the parabolic equation*

$$\frac{\partial v}{\partial \tau} = a\,\frac{\partial^2 v}{\partial x^2} + (r - a)\frac{\partial v}{\partial x} + r\,v, \qquad (x,\tau) \in \mathbb{R} \times (0,t],$$

admits a fundamental solution $\Gamma(x,\tau,\xi,\eta)$ ($x, \xi \in \mathbb{R}$, $t \geq \tau > \eta \geq 0$) and

$$\Gamma(x_0, t, x, 0) = e^{-x}\left(u_{xx}(x,t;a) - u_x(x,t;a)\right), \tag{17}$$

whereby – for given x_0 – $u(a)$ is the solution of (16).

Proof. Let $u(a)$ be a solution of (16). Then $u(a)$ describes an option price function for fixed asset price x_0 and fixed time $t_0 = 0$ but variable strike price x and expiration date t. Let now x and t be fixed. Then we consider $v(x_0, t_0)$ as option price function of the variables asset price and time. As in the introduction mentioned v satisfies the transformed Black-Scholes equation

$$-\frac{\partial v}{\partial t_0} = a(x_0,t_0)\frac{\partial^2 v}{\partial x_0^2} + (r - a(x_0,t_0))\frac{\partial v}{\partial x_0} - r\,v, \qquad (x_0,t_0) \in \mathbb{R} \times [0,t).$$

We use the transformations

$$\tau := t - t_0,\ \tilde{v}(x_0,\tau) := v(x_0,t) \ \text{ and } \ \tilde{a}(x_0,\tau) := a(x_0,t_0).$$

Then we get the equation

$$\frac{\partial \tilde{v}}{\partial \tau} = \tilde{a}(x_0,\tau)\frac{\partial^2 \tilde{v}}{\partial x_0^2} + (r - \tilde{a}(x_0,\tau))\frac{\partial \tilde{v}}{\partial x_0} - r\,\tilde{v}, \qquad (x_0,\tau) \in \mathbb{R} \times (0,t]. \tag{18}$$

Additionally the initial condition

$$\tilde{v}(x_0, 0) = \max\left(e^{x_0} - e^x, 0\right), \qquad x_0 \in \mathbb{R}, \tag{19}$$

holds. Then equation (18) admits a fundamental solution $\Gamma(x_0, \tau, \xi, \eta)$ for $\tau > \eta \geq 0$. Furthermore the Cauchy problem (18)-(19) possesses an unique classical solution $\tilde{v}$ which can be represented as

$$\tilde{v}(x_0, \tau) := \int_{-\infty}^{\infty} \max\left(e^{\xi} - e^{x}, 0\right) \Gamma(x_0, \tau, \xi, 0)\, d\xi = \int_{x}^{\infty} \left(e^{\xi} - e^{x}\right) \Gamma(x_0, \tau, \xi, 0)\, d\xi.$$

We set $t_0 = 0$. Then $\tau = t$ and

$$\tilde{v}(x_0, t) = \int_{x}^{\infty} \left(e^{\xi} - e^{x}\right) \Gamma(x_0, t, \xi, 0)\, d\xi = u(x, t; a).$$

We differentiate this integral twice by x to receive

$$u_x(x, t; a) = -\int_{x}^{\infty} e^{x}\, \Gamma(x_0, t, \xi, 0)\, d\xi$$

and

$$u_{xx}(x, t; a) = e^{x}\, \Gamma(x_0, t, x, 0) - \int_{x}^{\infty} e^{x}\, \Gamma(x_0, t, \xi, 0)\, d\xi.$$

Finally we take the difference and multiply by e^{-x} to obtain (17). ■

Important is the following consequence.

Corollary 4.1 *Let $u(a)$ be the solution of (16). Then $u_{xx}(a) - u_x(a) \in L^p(0, T; L^q(\mathbb{R}))$ for $1 \leq p \leq \infty$ if $q = 1$, $p < \frac{2\,q}{q-1}$ if $1 < q < \infty$ and $p < 2$ if $q = \infty$.*

PROOF. First we consider the case $1 \leq q < \infty$. Since

$$0 < \Gamma(x_0, t, x, 0) < \frac{c_1}{\sqrt{t}} \exp\left(-c_2 \frac{(x - x_0)^2}{t}\right)$$

for two positive constants c_1, c_2, which do not depend on a (see e.g. [6, Section 4.13]), we get

$$\begin{aligned}
\|u_{xx}(a) - u_x(a)\|_{L^p(0,T;L^q(\mathbb{R}))} &:= \left[\int_0^T \left[\int_{-\infty}^{\infty} |u_{xx}(x, t; a) - u_x(x, t; a)|^q\, dx\right]^{\frac{p}{q}} dt\right]^{\frac{1}{p}} \\
&\leq \left[\int_0^T \left[\int_{-\infty}^{\infty} \frac{c_1^q}{t^{\frac{q}{2}}} \exp\left(-q\, c_2 \frac{(x - x_0)^2}{t} + q\, x\right) dx\right]^{\frac{p}{q}} dt\right]^{\frac{1}{p}} \\
&= c_1 \left(\sqrt{\frac{\pi}{q\, c_2}}\right)^{\frac{1}{q}} \exp(x_0) \left[\int_0^T \left(t^{\frac{-q+1}{2}}\right)^{\frac{p}{q}} \exp\left(\frac{p}{4\, c_2} t\right) dt\right]^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&\le c_1 \left(\frac{\pi}{q\,c_2}\right)^{\frac{1}{2q}} \exp\left(x_0 + \frac{1}{4\,c_2}T\right) \left[\int_0^T t^{-\frac{p}{2}+\frac{p}{2q}}\,dt\right]^{\frac{1}{p}} \\
&< \infty \Leftrightarrow -\frac{p}{2} + \frac{p}{2\,q} > -1 \Leftrightarrow -p + \frac{p}{q} > -2
\end{aligned}
$$

For $q = 1$ this equation is fulfilled for all $1 \le p \le \infty$. In the case $1 < q < \infty$ we deduce

$$
p < \frac{2\,q}{q-1}.
$$

Let us now treat the case $q = \infty$. Then

$$
\begin{aligned}
\|u_{xx}(a) - u_x(a)\|_{L^p(0,T;L^\infty(\mathbb{R}))} &:= \left[\int_0^T \left[\sup_{x\in\mathbb{R}} |u_{xx}(x,t;a) - u_x(x,t;a)|\right]^p dt\right]^{\frac{1}{p}} \\
&\le c_1 \left[\int_0^T \frac{1}{t^{\frac{p}{2}}} \sup_{x\in\mathbb{R}} \left|\exp\left(-c_2 \frac{(x-x_0)^2}{t} + x\right)\right| dt\right]^{\frac{1}{p}} \\
&= c_1 \left[\int_0^T \frac{1}{t^{\frac{p}{2}}} \exp\left(x_0 + \frac{t}{4\,c_2}\right) dt\right]^{\frac{1}{p}} \\
&\le c_1 \exp\left(x_0 + \frac{T}{4\,c_2}\right) \left[\int_0^T \frac{1}{t^{\frac{p}{2}}}\,dt\right]^{\frac{1}{p}} \\
&< \infty \Leftrightarrow p < 2.
\end{aligned}
$$

The proof is complete. ■

By setting $p = q$ we immediately achieve the following result.

Corollary 4.2 $u_{xx}(a) - u_x(a) \in L^p(Q_T)$ *for* $1 \le p < 3$.

We investigate how variations in the volatility function a influence the option price function $u(a)$. Let be $a_1,\, a_2 \in \mathcal{D}_c^\lambda$ and $u(a_1)$, $u(a_2)$ are the corresponding solutions of (16). We set $v := u(a_1) - u(a_2)$. Then

$$
\begin{aligned}
v_t &= u_t(a_1) - u_t(a_2) \\
&= L(a_1)\,u(a_1) - L(a_2)\,u(a_2) \\
&= a_1\,u_{xx}(a_1) - (r + a_1)u_x(a_1) - [a_2\,u_{xx}(a_2) - (r + a_2)u_x(a_2)] \\
&= a_2\,[u_{xx}(a_1) - u_{xx}(a_2)] - (r + a_2)\,[u_x(a_1) - u_x(a_2)] + (a_1 - a_2)\,[u_{xx}(a_1) - u_x(a_1)] \\
&= L(a_2)\,v + (a_1 - a_2)\,[u_{xx}(a_1) - u_x(a_1)] \\
&= L(a_1)\,v + (a_1 - a_2)\,[u_{xx}(a_2) - u_x(a_2)]\,.
\end{aligned}
$$

Additionally we get the initial condition

$$
v(x,0) = 0, \qquad x \in \mathbb{R}.
$$

This motivates the following definition.

Definition 4.2 *For given $a_0 \in \mathcal{D}_c^\lambda$ we set*

$$\mathcal{D}(a_0) := \left\{ a \,:\, a + a_0 \in \mathcal{D}_c^\lambda \right\}. \tag{20}$$

Let $a \in \mathcal{D}(a_0)$. We define $v(a) \in C^{2,1}(Q_T)$ as solution of the Cauchy problem

$$\left\{ \begin{aligned} v_t(a) &= L(a_0)\, v(a) + a\, [u_{xx}(a_0 + a) - u_x(a_0 + a)] \qquad && on \;\; Q_T, \\ v(x, 0; a) &= 0, && x \in \mathbb{R}. \end{aligned} \right. \tag{21}$$

Remark 4.1 *As seen in derivation of (21) we can define $v(a)$ analogous via the Cauchy problem*

$$\left\{ \begin{aligned} v_t(a) &= L(a_0 + a)\, v(a) + a\, [u_{xx}(a_0) - u_x(a_0)] \qquad && on \;\; Q_T, \\ v(x, 0; a) &= 0, && x \in \mathbb{R}. \end{aligned} \right. \tag{22}$$

In the further investigations we will use booth versions to derive continuity and differentiability properties of the option pricing problem.

We will use the results from section 2 to derive L^2-estimates for the solution $v(a)$ of (21). For this we specify some properties which must be fulfilled by the functions a_0 respectively a. We can formulate the following conditions.

Assumption 4.1 *We say that a function $a \in \mathcal{D}_c^\lambda$ satisfy a regularity condition if one of the two assumptions is fulfilled:*

(A1) $a_x \in L^\infty(Q_T)$ *or*

(A2) $\|a_t(\cdot, t)\|_{L^2(\mathbb{R})} = \mu_1(t)$ *and*

$$\int_0^T \mu_1(t)\, dt = \mu < \infty.$$

Now we can prove a first result.

Theorem 4.1 *Let $a_0 \in \mathcal{D}_c^\lambda$ satisfy a regularity condition. Then $v(a) \in L^2(0, T; H^2(\mathbb{R}))$ for every $a \in \mathcal{D}(a_0)$ and the estimate*

$$\|v(a)\|_{L^2(0,T;H^2(\mathbb{R}))} \leq C\, \|a\|_{L^\infty(Q_T)} \tag{23}$$

holds for a constant $C > 0$, which do not depend on the parameter a.

Proof. From Corollary 4.1 we get $u_{xx}(a_0 + a) - u_x(a_0 + a) \in L^2(Q_T)$. Since a is bounded we conclude

$$a\, [u_{xx}(a_0 + a) - u_x(a_0 + a)] \in L^2(Q_T).$$

Let now $(A1)$ be satisfied. Then we can write $L(a_0)$ as

$$L(a_0)\, u = a_0\, u_{xx} - (r + a_0) u_x = (a_o\, u_x)_x - (r + a_0 + a_{0x})\, u_x.$$

Under these conditions we can apply Theorem 2.4 to obtain the the estimate

$$\|v(a)\|_{L^2(0,T;H^2(\mathbb{R}))} \leq \tilde{C}\,\|a\,[u_{xx}(a_0+a)-u_x(a_0+a)]\|_{L^2(Q_T)} \leq C\,\|a\|_{L^\infty(Q_T)}. \tag{24}$$

On the other hand let $(A2)$ be satisfied. Then we divide by a_0 to get the Cauchy problem

$$\begin{cases} \dfrac{1}{a_0}v_t(a) &= v_{xx}(a) - \left(\dfrac{r}{a_0}+1\right)v_x - \dfrac{a}{a_0}\,[u_{xx}(a_0+a)-u_x(a_0+a)] \qquad \text{on } Q_T, \\ v(x,0;a) &= 0, \qquad x\in\mathbb{R}. \end{cases}$$

Since

$$\frac{1}{\bar{c}} \leq \frac{1}{a_0} \leq \frac{1}{\underline{c}} \quad \text{and} \quad \left(\frac{1}{a_0}\right)_t = -\frac{1}{a_0^2}\,a_{0t},$$

all conditions of Theorem 2.4 are satisfies and we derive again the estimate (24). ■

Definition 4.3 *Let $a_0 \in \mathcal{D}_c^\lambda$ satisfy a regularity condition. Then we define the nonlinear operator $F:\mathcal{D}(a_0)\subset L^\infty(Q_T)\longrightarrow L^2(Q_T)$ via*

$$F(a) := v(a) = u(a_0+a)-u(a_0).$$

The following result is an immediate consequence of Theorem 4.1.

Corollary 4.3 *Let a_0 and a_0+a satisfy a regularity condition. Then F is Lipschitz continuous in a.*

PROOF. For given $a\in\mathcal{D}(a_0)$ let $\tilde{a}\in\mathcal{D}(a_0)$ be arbitrary. We set $h:=\tilde{a}-a$ and consider $\tilde{v}:=v(\tilde{a})-v(a)$. Then

$$\begin{aligned} \tilde{v} &= v(a+h)-v(a) \\ &= v(a+h)-v(0)-[v(a)-v(0)] \\ &= u(a_0+a+h)-u(a_0)-[u(a_0+a)-u(a_0)] \\ &= u(a_0+a+h)-u(a_0+a). \end{aligned}$$

Since a_0+a satisfies a regularity condition, we can apply Theorem 24 with a_0+a instead of a_0 and h instead of a to obtain

$$\|F(\tilde{a})-F(a)\|_{L^2(Q_T)} = \|\tilde{v}\|_{L^2(Q_T)} \leq C\,\|h\|_{L^\infty(Q_T)} = C\,\|\tilde{a}-a\|_{L^\infty(Q_T)},$$

whereby the constant C does not depends on $\tilde{a}$. ■

We examine the differentiability. First we show the existence of the first and the second directional derivatives when a_0+a satisfies a regularity condition. Later on we verify that the operator F is even Fréchet differentiable when additionally a is an interior point of $\mathcal{D}(a_0)$.

Theorem 4.2 *Let $a\in\mathcal{D}(a_0)$ and a_0+a satisfy a regularity condition. Then for every $h\neq 0$ with $a+h\in\mathcal{D}(a_0)$ there exists the directional derivative $F'(a)\,h$. Let $w(a,h)$ be the solution of the Cauchy problem*

$$\begin{cases} w_t(a,h) &= L(a_0+a)\,w(a,h)+h\,[u_{xx}(a_0+a)-u_x(a_0+a)] \qquad \text{on } Q_T, \\ w(x,0;a,h) &= 0, \qquad x\in\mathbb{R}, \end{cases} \tag{25}$$

then

$$F'(a)\,h = w(a,h).$$

PROOF. Let $h \neq 0$ with $a + h \in \mathcal{D}(a_0)$ be arbitrarily. Then $a + \varepsilon\, h \in \mathcal{D}(a_0)$ for every $0 \le \varepsilon \le 1$ since $\mathcal{D}(a_0)$ is a convex set in $L^\infty(Q_T)$. We set

$$w^\varepsilon := \frac{1}{\varepsilon}\left(F(a+\varepsilon\, h) - F(a)\right) = \frac{1}{\varepsilon}\left(v(a+\varepsilon\, h) - v(a)\right) = \frac{1}{\varepsilon}\left(u(a_0+a+\varepsilon\, h) - u(a_0+a)\right)$$

and show $w^\varepsilon \to w(a,h)$ for $\varepsilon \to 0$. Since

$$\begin{aligned} w_t^\varepsilon &= \frac{1}{\varepsilon}\left(L(a_0+a+\varepsilon\, h)\, v(a+\varepsilon\, h) - L(a_0+a)\, v(a)\right) + \dots \\ &\quad \dots + \frac{1}{\varepsilon}\left[(a+\varepsilon\, h)(u_{xx}(a_0) - u_x(a_0)) - a\,(u_{xx}(a_0) - u_x(a_0))\right] \\ &= L(a_0+a)\, w^\varepsilon + h\,(v_{xx}(a+\varepsilon\, h) - v_x(a+\varepsilon, h) + u_{xx}(a_0) - u_x(a_0)) \\ &= L(a_0+a)\, w^\varepsilon + h\,(u_{xx}(a_0+a+\varepsilon\, h) - u_x(a_0+a+\varepsilon, h))\,, \end{aligned}$$

because

$$v(a+\varepsilon\, h) + u(a_0) = u(a_0+a+\varepsilon\, h) - u(a_0) + u(a_0) = u(a_0+a+\varepsilon\, h),$$

we can w^ε represent as solution of the Cauchy problem

$$\begin{cases} w_t^\varepsilon &= L(a_0+a)\, w^\varepsilon + h\,[u_{xx}(a_0+a+\varepsilon\, h) - u_x(a_0+a+\varepsilon\, h)] \qquad \text{on } Q_T, \\ w^\varepsilon(x,0) &= 0, \qquad x \in \mathbb{R}, \end{cases}$$

We consider the limit $\varepsilon \to 0$. Let $w(a,h)$ be the solution of (25) and $\tilde{w} := w^\varepsilon - w(a,h)$. Then $\tilde{w}$ satisfies

$$\begin{cases} \tilde{w}_t &= L(a_0+a)\,\tilde{w} + h\,[(u_{xx}(a_0+a+\varepsilon\, h) - u_x(a_0+a+\varepsilon\, h)) - \dots \\ &\qquad \dots - (u_{xx}(a_0+a) - u_x(a_0+a))] \qquad \text{on } Q_T, \\ \tilde{w}(x,0) &= 0, \qquad x \in \mathbb{R}, \end{cases}$$

Since $a_0 + a$ satisfies a regularity condition $v := u(a_0+a+\varepsilon\, h) - u(a_0+a)$ is a solution of the Cauchy problem (21) with $a_0 + a$ instead of a_0 and $\varepsilon\, h$ instead of a. Then $v \in L^2(0,T;H^2(\mathbb{R}))$ and

$$\|v_{xx} - v_x\|_{L^2(Q_T)} \le C\,\varepsilon\,\|h\|_{L^\infty(Q_T)} \to 0 \quad \text{for} \quad \varepsilon \to 0.$$

Consequently $\tilde{w} \to 0$ for $\varepsilon \to 0$ and therefore $w^\varepsilon \to w$. The proof is complete. ■

Before we examine higher differentiability we prove the following lemma.

Lemma 4.3 *Let a_1, $a_2 \in \mathcal{D}(a_0)$ and $a_0 + a_i$, $i = 1,2$, satisfy a regularity condition. Then*

$$\|F'(a_2) - F'(a_1)\| \le C\,\|a_2 - a_1\|_{L^\infty(Q_T)}.$$

PROOF. Let $h \in L^\infty(Q_T)$ be arbitrarily. We set $\hat{w} := F'(a_2)\,h - F'(a_1)\,h = w(a_2,h) - w(a_1,h)$ whereby $w(a_i,h)$, $i = 1,2$, are the corresponding solutions of (25). Then $\hat{w}$ satisfies

$$\begin{aligned} \hat{w}_t &= L(a_0+a_2)\, w(a_2,h) + h\,[u_{xx}(a_0+a_2) - u_x(a_0+a_2)] - \dots \\ &\quad \dots - L(a_0+a_1)\, w(a_1,h) - h\,[u_{xx}(a_0+a_1) - u_x(a_0+a_1)] \\ &= L(a_0+a_1)\,\hat{w} + (a_2-a_1)\,[w_{xx}(a_2,h) - w_x(a_2,h)] + \dots \\ &\quad \dots + h\,[[u_{xx}(a_0+a_2) - u_x(a_0+a_2)] - [u_{xx}(a_0+a_1) - u_x(a_0+a_1)]]\,, \end{aligned}$$

together with the initial condition

$$\hat{w}(x,0) = 0, \qquad x \in \mathbb{R}.$$

From Theorem 4.1 with $a_0 + a_1$ instead of a_0 and $a_2 - a_1$ instead of a we derive

$$\|[u_{xx}(a_0+a_2) - u_x(a_0+a_2)] - [u_{xx}(a_0+a_1) - u_x(a_0+a_1)]\|_{L^2(Q_T)} \leq C\|a_2 - a_1\|_{L^\infty(Q_T)}.$$

Furthermore from Theorem 4.2 follows $w(a_2,h) \in L^2(0,T;H^2(\mathbb{R}))$ and therefore

$$\|(a_2 - a_1)\,[w_{xx}(a_2,h) - w_x(a_2,h)]\|_{L^2(Q_T)} \leq C\,\|a_2 - a_1\|_{L^\infty(Q_T)}\|h\|_{L^\infty(Q_T)}.$$

Together with the results of section 2 we obtain

$$\|\hat{w}\|_{L^2(0,T;H^2(\mathbb{R}))} = \|F'(a_2)\,h - F'(a_1)\,h\|_{L^2(0,T;H^2(\mathbb{R}))} \leq C\,\|a_2 - a_1\|_{L^\infty(Q_T)}\|h\|_{L^\infty(Q_T)}.$$

This proves the lemma. ■

Under the same conditions as in Theorem 4.3 there exists also the second directional derivative.

Theorem 4.3 *Let $a \in \mathcal{D}(a_0)$ and $a_0 + a$ satisfy a regularity condition. Then for every $h_1\,h_2 \neq 0$ with $a + h_i \in \mathcal{D}(a_0)$, $i = 1,2$, there exists the second directional derivative $F''(a)(h_1,h_2)$. Let $z(a,h_1,h_2)$ be the solution of the Cauchy problem*

$$\left\{\begin{aligned} z_t(a,h_1,h_2) &= L(a_0+a)\,z(a,h_1,h_2) + h_1\,[w_{xx}(a,h_2) - w_x(a,h_2)] + \dots \\ &\quad \dots + h_2\,[w_{xx}(a,h_1) - w_x(a,h_1)] && \text{on } Q_T, \\ z(x,0;a,h_1,h_2) &= 0, && x \in \mathbb{R}, \end{aligned}\right. \tag{26}$$

then

$$F''(a)(h_1,h_2) = z(a,h_1,h_2).$$

PROOF. At first we assume that $a_0 + a + h_2$ satisfies a regularity condition. Then we set

$$z^\varepsilon := \frac{1}{\varepsilon}\,(F'(a+\varepsilon\,h_2)\,h_1 - F'(a)\,h_2) = \frac{1}{\varepsilon}\,(w(a+\varepsilon\,h_2,h_1) - w(a,h_1))\,.$$

Obviously z^ε satisfies the parabolic equation

$$\begin{aligned} z^\varepsilon &= L(a_0+a)\,z^\varepsilon + h_2\,[w_{xx}(a+\varepsilon\,h_2,h_1) - w_x(a+\varepsilon\,h_2,h_1)] + \dots \\ &\quad \dots + \frac{h_1}{\varepsilon}\,[(u_{xx}(a_0+a+\varepsilon\,h_2) - u_x(a_0+a+\varepsilon\,h_2)) - (u_{xx}(a_0+a) - u_x(a_0+a))] \\ &= L(a_0+a)\,z^\varepsilon + h_2\,[w_{xx}(a+\varepsilon\,h_2,h_1) - w_x(a+\varepsilon\,h_2,h_1)] + \dots \\ &\quad \dots + h_1\,\frac{1}{\varepsilon}\,[(v_{xx}(a+\varepsilon\,h_2) - v_x(a+\varepsilon\,h_2)) - (v_{xx}(a) - v_x(a))] \\ &= L(a_0+a)\,z^\varepsilon + h_2\,[w_{xx}(a+\varepsilon\,h_2,h_1) - w_x(a+\varepsilon\,h_2,h_1)] + \dots \\ &\qquad \dots + h_1\,[w^\varepsilon_{xx} - w^\varepsilon_x]\,, \end{aligned}$$

whereby w^ε is defined analogous to the proof of Theorem 4.2 with h_2 instead of h. Additionally we get the initial condition

$$z^\varepsilon(x,0) = 0, \qquad x \in \mathbb{R}.$$

We examine again the limit $\varepsilon \to 0$. From the proof of Lemma 4.3 we get

$$w_{xx}(a+\varepsilon\, h_2, h_1) - w_x(a+\varepsilon\, h_2, h_1) \;\to\; w_{xx}(a, h_1) - w_x(a, h_1)$$

and from the proof of Theorem 4.2 we obtain

$$w^{\varepsilon}_{xx} - w^{\varepsilon}_{x} \;\to\; w_{xx}(a, h_2) - w_x(a, h_2)$$

in $L^2(Q_T)$ for $\varepsilon \to 0$. Consequently $z^{\varepsilon} \to z(a, h_1, h_2)$, whereby $z(a, h_1, h_2)$ is defined via (26).

Now we consider the case that $a_0 + a + h_2$ does not satisfy a regularity condition. Then we can approximate h_2 by a sequence h_2^k which satisfy a regularity condition and $h_2^k \to h_2$ for $k \to \infty$ in $L^{\infty}(Q_T)$. Considering the second directional derivatives $z(a, h_1, h_2^k)$ and taking the limit we get $z(a, h_1, h_2^k) \to z(a, h_1, h_2)$ in $L^2(0, T; H^2(\mathbb{R}))$. So the statements of the theorem remain true also in this case. ■

We examine Fréchet differentiability. A function $a \in \mathcal{D}(a_0)$ is an interior point of $\mathcal{D}(a_0)$ ($a \in \operatorname{int}\mathcal{D}(a_0)$) if and only if

$$\underline{c} < \gamma_1 \le a_0 + a \le \gamma_2 < \overline{c} \quad \text{on} \quad Q_T$$

for two constants γ_1, γ_2. We will prove the following theorem.

Theorem 4.4 *Let $a \in int\mathcal{D}(a_0)$ satisfy a regularity condition. Then F is in a Fréchet differentiable and for all h with $a + h \in \mathcal{D}(a_0)$ an estimate of the form*

$$\|F(a+h) - F(a) - F'(a)\, h\|_{L^2(Q_T)} \le \frac{L}{2}\|h\|^2_{L^{\infty}(Q_T)}$$

holds for a constant $L > 0$ independent of h.

Proof. For given a and h we set

$$z := F(a+h) - F(a) - F'(a)\, h = v(a+h) - v(a) - w(a, h).$$

Then z satisfies

$$\begin{aligned}
z_t &= L(a_0+a+h)\, v(a+h) + (a+h)(u_{xx}(a_0) - u_x(a_0)) - \ldots \\
&\quad \ldots - L(a_0+a)\, v(a) + a\,(u_{xx}(a_0) - u_x(a_0)) - \ldots \\
&\quad \ldots - L(a_0+a)\, w(a,h) + h\,(u_{xx}(a_0+a) - u_x(a_0+a)) \\
&= L(a_0+a)\, z + h\,(v_{xx}(a+h) - v_x(a+h)) - \ldots \\
&\quad \ldots + h\,(u_{xx}(a_0) - u_x(a_0)) - h\,(u_{xx}(a_0+a) - u_x(a_0+a)) \\
&= L(a_0+a)\, z + h\,[(u_{xx}(a_0+a+h) - u_x(a_0+a+h)) - (u_{xx}(a_0+a) - u_x(a_0+a))]
\end{aligned}$$

as well as the initial condition

$$z(x, 0) = 0, \qquad x \in \mathbb{R}.$$

From Theorem 4.1 with $a_0 + a$ instead of a_0 and h instead of a we obtain

$$\|(u_{xx}(a_0+a+h) - u_x(a_0+a+h)) - (u_{xx}(a_0+a) - u_x(a_0+a))\|_{L^2(Q_T)} \le C\|h\|_{L^{\infty}(Q_T)}$$

and therefore

$\|h\,[(u_{xx}(a_0+a+h)-u_x(a_0+a+h))-(u_{xx}(a_0+a)-u_x(a_0+a))]\|_{L^2(Q_T)} \leq C\|h\|^2_{L^\infty(Q_T)}.$

From the estimates of section 2 we can conclude

$$\|z\|_{L^2(Q_T)} \leq \|z\|_{L^2(0,T;H^1(\mathbb{R}))} \leq \frac{L}{2}\|h\|^2_{L^\infty(Q_T)}$$

for a positive constant L. ■

A Constant volatilities and fundamental solutions

We have shown that

$$e^{-x}\left(u_{xx}(x,t;a)-u_x(x,t;a)\right)=\Gamma(x_0,t,x,0)<\frac{c_1}{\sqrt{t}}\exp\left(-c_2\frac{(x-x_0)^2}{t}\right)$$

for two positive constants c_1 and c_2. In the case $a\equiv const.$ we can verify those constants closer. $\Gamma(x_0,\tau,x,\eta)$ denotes the fundamental solution of the transformed Black-Scholes equation

$$\frac{\partial v}{\partial \tau}=a\frac{\partial^2 v}{\partial x_0^2}+(r-a)\frac{\partial v}{\partial x_0}+r\,v, \qquad (x_0,\tau)\in\mathbb{R}\times(0,T]. \tag{27}$$

To show this we use the transformed Black-Scholes formula. Since $\sigma=\sqrt{2\,a}=const.$ The solution of the Cauchy problem (16) can be calculated explicitly by

$$u(x,t;a)=e^{x_0}\Phi(d_1)-e^{x-r\,t}\Phi(d_2)$$

with

$$d_1:=\frac{x_0-x+(r+a)t}{\sqrt{2\,a\,t}},\ d_2:=d_1-\sqrt{2\,a\,t}\ \text{ and } \Phi(d):=\frac{1}{\sqrt{2\,\pi}}\int\limits_{-\infty}^{d}\exp\left(-\frac{\xi^2}{2}\right)\,d\xi.$$

We differentiate by x. Then we obtain

$$u_x(x,t;a)=e^{x_0}\Phi'(d_1)\frac{\partial d_1}{\partial x}-e^{x-r\,t}\Phi'(d_2)\frac{\partial d_2}{\partial x}-e^{x-r\,t}\Phi(d_2)=-e^{x-r\,t}\Phi(d_2)$$

since

$$e^{x_0}\Phi'(d_1)=e^{x-r\,t}\Phi'(d_2)\ \text{ and }\ \frac{\partial d_1}{\partial x}=\frac{\partial d_2}{\partial x}=-\frac{1}{\sqrt{2\,a\,t}}$$

(see e.g. [5, Exercise 11.17]). By differentiating by x again we receive

$$u_{xx}(x,t;a)=-e^{x-r\,t}\Phi(d_2)-e^{x-r\,t}\Phi'(d_2)\frac{\partial d_2}{\partial x}.$$

Consequently

$$\begin{aligned}\Gamma(x_0,t,x,0) &= e^{-x}\left(u_{xx}(x,t;a)-u_x(x,t;a)\right)\\ &= -e^{-r\,t}\Phi'(d_2)\frac{\partial d_2}{\partial x}\\ &= \frac{1}{2\sqrt{\pi\,a\,t}}\exp\left(-r\,t+\frac{(x_0-x+(r-a)\,t)^2}{4\,a\,t}\right).\end{aligned}$$

That means we have found an explicite expression for the fundamental solution of the transformed Black-Scholes equation (27) when the volatility function is constant. Now we can derive estimates of the constants c_1 and c_2. For x_1, $x_2 \in \mathbb{R}$ and $c > 0$ we estimate by the Cauchy inequality with $\varepsilon := \frac{1}{4}$

$$\begin{aligned}
\exp\left(-\frac{(x_1+x_2)^2}{c}\right) &= \exp\left(-\frac{x_2^2}{c}\right)\exp\left(-\frac{x_1^2}{c}+\frac{2\,x_1\,x_2}{c}\right) \\
&\le \exp\left(-\frac{x_2^2}{c}\right)\exp\left(-\frac{x_1^2}{c}+\frac{2\,|x_1|\,|x_2|}{c}\right) \\
&\le \exp\left(-\frac{x_2^2}{c}\right)\exp\left(-\frac{x_1^2}{c}+\frac{x_1^2}{2\,c}+\frac{2\,x_2^2}{c}\right) \\
&= \exp\left(\frac{x_2^2}{c}\right)\exp\left(-\frac{x_1^2}{2\,c}\right).
\end{aligned}$$

We set

$$c := 4\,a\,t, \quad x_1 := x_0 - x \quad \text{and} \quad x_2 := (r-a)\,t$$

and receive

$$\begin{aligned}
\Gamma(x_0,t,x,0) &\le \frac{1}{2\sqrt{\pi\,a\,t}}\exp\left(-r\,t+\frac{(r-a)^2\,t}{4\,a}\right)\exp\left(-\frac{(x_0-x)^2}{8\,a\,t}\right) \\
&= \frac{1}{2\sqrt{\pi\,a\,t}}\exp\left(\frac{r^2-6\,r\,a+a^2\,t}{4\,a}\right)\exp\left(-\frac{(x_0-x)^2}{8\,a\,t}\right) \\
&= \frac{1}{2\sqrt{\pi\,a\,t}}\exp\left(\frac{(r-3\,a)^2\,t}{4\,a}-2\,a\,t\right)\exp\left(-\frac{(x_0-x)^2}{8\,a\,t}\right) \\
&\le \frac{1}{2\sqrt{\pi\,a\,t}}\exp\left(\frac{(r-3\,a)^2\,T}{4\,a}\right)\exp\left(-\frac{(x_0-x)^2}{8\,a\,t}\right)
\end{aligned}$$

on $Q_T = \mathbb{R}\times(0,T)$. So we have with

$$c_1 := \frac{1}{2\sqrt{\pi\,a}}\exp\left(\frac{(r-3\,a)^2\,T}{4\,a}\right) \quad \text{and} \quad c_2 := \frac{1}{8\,a}$$

estimates of the two constants c_1 and c_2 in the proof of Corollary 4.1. We can go even further. It is possible to calculate the norm $\|u_{xx}(a)-u_x(a)\|_{L^p(0,T;L^q(\mathbb{R}))}$ exactly. We will do this here for the case $p=q=2$.

Lemma A.1 *If $a \equiv const.$ then*

$$\|u_{xx}(a)-u_x(a)\|_{L^2(Q_T)} = \exp(x_0)\left(\frac{T}{2\,\pi\,a}\right)^{\frac{1}{4}}.$$

PROOF. As just derived we have

$$\begin{aligned}
u_{xx}(x,t;a)-u_x(x,t;a) &= -e^{x-r\,t}\Phi'(d_2)\frac{\partial d_2}{\partial x} \\
&= -e^{x_0}\Phi'(d_1)\frac{-1}{\sqrt{2\,a\,t}} \\
&= \frac{1}{2\sqrt{\pi\,a\,t}}\exp\left(x_0-\frac{(x_0-x+(r+a)\,t)^2}{4\,a\,t}\right)
\end{aligned}$$

and therefore

$$
\begin{aligned}
\|u_{xx}(a) - u_x(a)\|_{L^2(Q_T)} &= \left[\int\limits_0^T \int\limits_{-\infty}^{\infty} \left(\frac{1}{2\sqrt{\pi\, a\, t}} \exp\left(x_0 - \frac{(x_0 - x + (r+a)\,t)^2}{4\,a\,t}\right)\right)^2 dx\; dt\right]^{\frac{1}{2}} \\
&= \frac{\exp(x_0)}{2\sqrt{\pi\, a}} \left[\int\limits_0^T \frac{1}{t} \int\limits_{-\infty}^{\infty} \exp\left(-\frac{(x_0 - x + (r+a)\,t)^2}{2\,a\,t}\right) dx\; dt\right]^{\frac{1}{2}} \\
&= \frac{\exp(x_0)}{2\sqrt{\pi\, a}} \left[\int\limits_0^T t^{-1}\sqrt{2\,a\,t\,\pi}dt\right]^{\frac{1}{2}} \\
&= \frac{\exp(x_0)}{\sqrt{2}\,(2\,\pi\, a)^{\frac{1}{4}}} \left(\int\limits_0^T t^{-\frac{1}{2}}dt\right)^{\frac{1}{2}} \\
&= \frac{\exp(x_0)}{\sqrt{2}\,(2\,\pi\, a)^{\frac{1}{4}}} \sqrt{2}\,T^{\frac{1}{4}} \\
&= \exp(x_0)\left(\frac{T}{2\,\pi\, a}\right)^{\frac{1}{4}}.
\end{aligned}
$$

The proof is complete. ■

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