TECHNISCHE UNIVERSITÄT CHEMNITZ

A United Approach to Illumination and Visibility Problems via Closure Operators

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Preprint 2004-1

Fakultät für Mathematik

A Unified Approach to Illumination and Visibility Problems via Closure Operators

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Abstract

The notions of illumination and visibility of convex bodies are well known in combinatorial and computational geometry. With the help of related closure operators we present a unified approach to both these notions, studying also the complete lattice structure of a more general framework of closure operators. Moreover, precisely the extreme elements of this complete lattice control illumination and visibility, respectively.

AMS (MOS) Subject Classification:
Primary: 52A20
Secondary: 52A37, 03G05, 06A15, 06B23

Keywords: convex bodies, illumination, visibility, closure operators, complete lattices, Boolean lattices, Hadwiger's covering problem

§ 1 Introduction

The notion of *illumination* of a convex body K in $\mathbb{R}^n$ can be considered as the starting point for various interesting problems and as a useful tool in *combinatorial geometry* (cf. Chapter VI and VII in [2] and the survey [8]). This notion was independently introduced by V. Boltyanski [1] and H. Hadwiger [5], motivated by the famous (and still unsettled) question how many light sources are needed to illuminate the whole boundary of K. The conjectured upper bound 2^n, attained when K is an n-dimensional parallelotope, is verified only for special types of convex bodies, e.g. for all centrally symmetric convex bodies in $\mathbb{R}^3$ (as shown by M. Lassak in [7]), or for convex bodies in $\mathbb{R}^n$ whose supporting cones at singular points are not too acute (proved by B. Weißbach in [15]). Variations of the Boltyanski-Hadwiger notion of illumination were considered in [9] and [16].

A modified type of illumination, called *visibility*, was introduced by F. A. Valentine [14], see also [4], [3], and again the survey [8]. Visibility problems play an essential role in *computational geometry*, e.g. in connection with art gallery questions and the watchman route problem, cf. [12] and [13]. Valentine's notion of visibility can be seen as a weakening of the above mentioned notion of illumination; so already $n + 1$ points are sufficient to *see* the whole boundary of a convex body K in $\mathbb{R}^n$.

In this paper we want to present a unified approach to both these notions in terms of *closure operators*, which themselves are tailored to convex sets. In case of visibility, the corresponding closure operator is even new.

In Section 2 (Theorem 2.4) we show already basic connections between the notions of illumination and visibility in view of closure operators, and in Section 3 the new closure operator of the visibility notion is introduced and discussed. In Section 4 we give a detailed study of the lattice structure of a large family of closure operators, containing those related to illumination and visibility as extreme elements. It should be noticed that this extension to a whole family of closure operators is not only of theoretical interest: One might apply this more general framework by considering configurations of light sources having different intensities (Remark 4.5). The final Section 5 presents analogous investigations for parallel illumination of convex bodies, surprisingly showing that in this case we cannot have an approach via closure operators (at least not in a canonical way).

Finally we shortly mention two of the main motivations for our investigations presented here. First it is our hope that an extended framework of tools and methods might help to attack more successfully certain longstanding open problems from the combinatorial geometry of convex bodies, such as the Boltyanski-Hadwiger illumination problem (also known as the Gohberg-Markus-Hadwiger covering problem asking for the minimum number of smaller homothets of K sufficient to cover that convex body). Second, our more general approach seems to be suitable for investigating also extended illumination and visibility questions which occur already in the literature. For example, one might no longer restrict to translation classes of convex bodies (with respect to fixed configurations of light sources), but even consider congruence classes. For the case of illumination, this more general point of view was proposed by B. Weißbach [16].

§ 2 Illumination described in terms of a closure operator

For two points $a, b \in \mathbb{R}^n$ with $a \neq b$, $n \geq 1$, let

(2.1) $\overline{ab} := \{a + \lambda \cdot (b - a) \mid 0 \leq \lambda \leq 1\}$

denote the closed *line segment* between a and b, while

(2.2) $s(a, b) := \{a + \lambda \cdot (b - a) \mid \lambda \geq 0\}$

is written for the *ray* with initial point a and passing through b.

Assume that K is a *convex body*, i.e., a compact, convex set with interior points in $\mathbb{R}^n$. Due to H. Hadwiger [5] we say that a boundary point x of K is *illuminated* by a point $z \in \mathbb{R}^n \setminus K$ if

(2.3) $(s(z, x) \setminus \overline{zx}) \cap \text{ int } K \neq \emptyset\,.$

(We note that the analogous notion for parallel illumination was introduced by V. Boltyanski [1], see also [2], Chapter VI, for a historical survey, and [8].) A point set $A \subseteq \mathbb{R}^n \setminus K$ is said to *illuminate* a subset B of the *boundary* ∂K of K if every $x \in B$ is illuminated by at least one element $a \in A$. If A illuminates the whole boundary ∂K of K, we say also that A illuminates the body K.

To describe illumination more generally, namely in terms of closure operators, we define for every subset M of $\mathbb{R}^n$ and its complement $E = \mathbb{R}^n \setminus M$ the operator $\sigma_M : \mathcal{P}(E) \to \mathcal{P}(E)$ by

$$(2.4) \quad \sigma_M(A) := A \cup \{b \in E \setminus A \,|\, \exists a \in A : \overline{ab} \cap M = \emptyset, s(a,b) \cap M \neq \emptyset\}.$$

Thus $\sigma_M(A) \setminus A$ consists of those points of $E \setminus A$ which lie in front of M relative to some point $a \in A$.

We have the following proposition which was already proved in [10], see Theorem 2.5 there.

Proposition 2.1: *Assume that $M \subseteq \mathbb{R}^n$ is convex. Then the operator $\sigma_M : \mathcal{P}(E) \to \mathcal{P}(E)$, with $E := \mathbb{R}^n \setminus M$, is a closure operator, i.e.,*

(H0) *σ is increasing:* $A \subseteq E$ *implies* $A \subseteq \sigma(A)$,

(H1) *σ is monotone:* $A_1 \subseteq A_2 \subseteq E$ *implies* $\sigma(A_1) \subseteq \sigma(A_2)$,

(H2) *σ is idempotent:* $A \subseteq E$ *implies* $\sigma(\sigma(A)) = \sigma(A)$.

Note that the convexity of M is only used to verify (H2); the axioms (H0) and (H1) hold for all subsets M of $\mathbb{R}^n$.

In the remaining part of this paper, assume again that K is a convex body in $\mathbb{R}^n$. Then the *interior* $K_0 =$ int K of K is also convex; thus, for $E := \mathbb{R}^n \setminus K$ and $E_0 := \mathbb{R}^n \setminus K_0$ we get closure operators

$$\sigma := \sigma_K : \mathcal{P}(E) \to \mathcal{P}(E), \quad \sigma_0 := \sigma_{K_0} : \mathcal{P}(E_0) \to \mathcal{P}(E_0).$$

The next result shows that σ_0 is precisely adapted to the illumination problem.

Proposition 2.2: *For a subset A of $\mathbb{R}^n \setminus K$ and a subset B of $\partial K = K \setminus K_0$ the following statements are equivalent:*

(i) *A illuminates B.*

(ii) $B \subseteq \sigma_0(A)$.

Proof: By definition, (i) is equivalent to the statement that for every $b \in B$ there exists some $a \in A$ with

$$(2.3\text{ a}) \quad \left(s(a,b) \setminus \overline{ab}\right) \cap K_0 \neq \emptyset.$$

We verify that this relation holds if and only if

$$(2.4\text{ a}) \quad \overline{ab} \cap K_0 = \emptyset \text{ and } s(a,b) \cap K_0 \neq \emptyset.$$

Clearly, (2.4 a) implies (2.3 a). Vice versa, if (2.3 a) holds, we get $\overline{ab} \cap K_0 = \emptyset$, because $b \notin K_0$ and K_0 is convex. Thus (2.3 a) and (2.4 a) are equivalent, whence, by (2.4), the equivalence of (i) and (ii) follows. $\square$

Still in this section we want to prove an extension of Proposition 2.2 in the case $B = \partial K$. To this end, we consider already now the concept of *visibility* as introduced by F. A. Valentine in [14], see also [4] and [3]. Namely, a point $z \in \mathbb{R}^n \setminus K$ *sees* the point $x \in \partial K$ if

$$(2.5) \quad \overline{zx} \cap K = \{x\}.$$

A point set $A \subseteq \mathbb{R}^n \setminus K$ *sees* a subset B of ∂K if every $b \in B$ is seen from at least one point $a \in A$. If A sees ∂K, we say also that A sees (the whole of) K or that K is visible from A.

Remark 2.3:

(i) For $y \in \text{int } K$ and $x_1 \in \partial K$ there does not exist some further boundary point $x_2 \in \partial K \setminus \{x_1\}$ with $x_2 \in \overline{x_1 y}$: If $x_2 \in \overline{x_1 y} \setminus \{x_1, y\}$ is arbitrary and $r > 0$ satisfies

$$(2.6) \quad B(y, r) := \{z \in \mathbb{R}^n : ||y - z|| < r\} \subseteq \text{ int } K,$$

where $||\cdot||$ means the Euclidean norm, then we get also

$$B\left(x_2, \frac{||x_2 - x_1||}{||y - x_1||} \cdot r\right) \subseteq \text{ int } K$$

by the convexity of K. In particular, this means

$$\overline{x_1 y} \setminus \{x_1, y\} \subseteq \text{ int } K.$$

(ii) From (i) we conclude at once: A subset A of $E = \mathbb{R}^n \setminus K$, which illuminates a subset B of ∂K, also sees B. $\square$

Now we can prove the following

Theorem 2.4: *For every subset S of E the following statements are equivalent:*

(i) *S illuminates ∂K.*

(ii) $\partial K \subseteq \sigma_0(S)$.

(iii) *For every $x \in \partial K$ there exists some $\delta > 0$ such that $B(x, \delta) \cap \partial K$ is illuminated by some $z \in S$.*

(iv) *For every $x \in \partial K$ there exists some $\delta > 0$ such that $B(x, \delta) \cap \partial K$ is seen by some $z \in S$.*

(v) *There exists some open subset U of $\mathbb{R}^n$ with*

$$K \subseteq U \subseteq K \cup \sigma(S).$$

(vi) *There exists some open subset U of $\mathbb{R}^n$ with*

$$K \subseteq U \subseteq K_0 \cup \sigma_0(S).$$

Proof:

(i) $\Leftrightarrow$ (ii) is Proposition 2.2 for $B = \partial K$.

(iii) $\Rightarrow$ (i) is trivial. (iii) $\Rightarrow$ (iv) is clear by Remark 2.3 (ii).

(vi) $\Rightarrow$ (v) follows from the relation $K_0 \cup \sigma_0(S) \subseteq K_0 \cup (\sigma(S) \cup \partial K) = K \cup \sigma(S)$.

(i) $\Rightarrow$ (iii): Assume $x \in \partial K$, and choose $z \in S$ and $y \in K_0$ with $x \in \overline{zy}$. Suppose $r > 0$ satisfies $B(y,r) \subseteq K_0$. Then for every $x' \in \partial K$ with $||x - x'|| < \frac{||z-x||}{||z-y||} \cdot r$ there exists some $y' \in B(y,r)$ with $x' \in s(z,y')$. Thus x' is illuminated by z.

(iv) $\Rightarrow$ (ii): Assume $x \in \partial K$, and choose $\delta > 0$ and $z \in S$ such that $B(x,\delta) \cap \partial K$ is seen from z. It suffices to prove that $s(z,x) \cap K_0 \neq \emptyset$. Otherwise we would have $s(z,x) \cap K = s(z,x) \cap \partial K = \{x\}$, because x is the unique point in $s(z,x) \cap \partial K$ which is seen from z. Now assume that $y \in K_0$ is arbitrary, and for $\eta > 0$ put $w_\eta := x + \eta \cdot (x - z)$, see Fig. 1. If η is small enough, we get $x' \in B(x,\delta) \cap \partial K$ for some $x' \in \overline{yw_\eta} \setminus \{y\}$. On the other hand, we have $\overline{xy} \cap \overline{zx'} \neq \emptyset$. That means, x' is not seen from z, a contradiction to $x' \in B(x,\delta) \cap \partial K$.

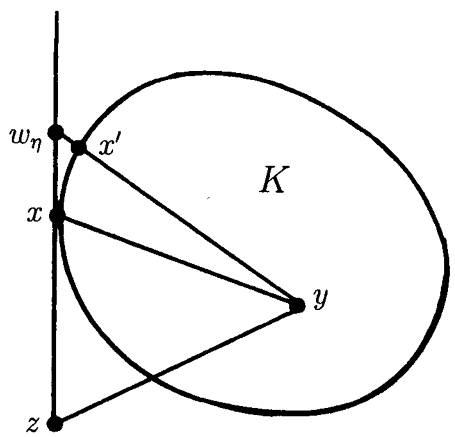

Figure 1

(i) $\Rightarrow$ (vi): For every $x \in \partial K$, choose some $y = y_x \in K_0$ and some $z = z_x \in S$ with $x \in \overline{zy}$. Furthermore, choose $r = r_x > 0$ with $B(y,r) \subseteq K_0$ as well as $w = w_x \in \overline{zx} \setminus \{z, x\}$ such that for

$$r' = r'_x := \frac{||z - w||}{||z - y||} \cdot r$$

we have $B(w,r') \subseteq E$, see Fig. 2.

Then we have $B(w,r') \subseteq \sigma_0(\{z\})$. Therefore, the set $U_x = \operatorname{conv}(B(w,r') \cup B(y,r))$ is an open subset of the convex set

$$(2.7) \quad K_0 \cup \sigma_0(\{z\}) = \operatorname{conv}(K_0 \cup \{z\}) \, .$$

(For a proof of the elementary relation (2.7) see [10], Proposition 2.6 i).) Now put

$$U := K \bigcup \bigcup_{x \in \partial K} U_x = K_0 \bigcup \bigcup_{x \in \partial K} U_x \, .$$

U is an open subset of $\mathbb{R}^n$ with $K \subseteq U \subseteq K_0 \cup \sigma_0(S)$.

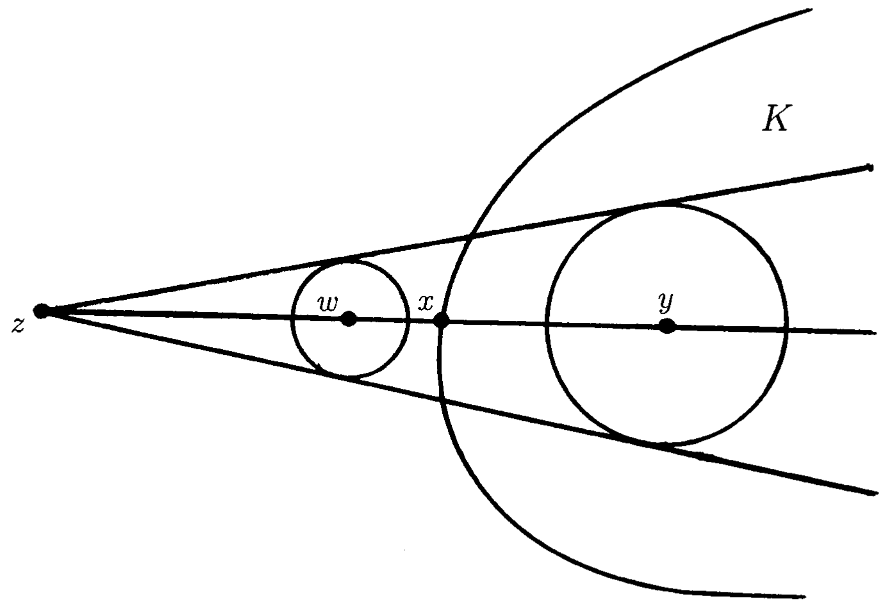

Figure 2

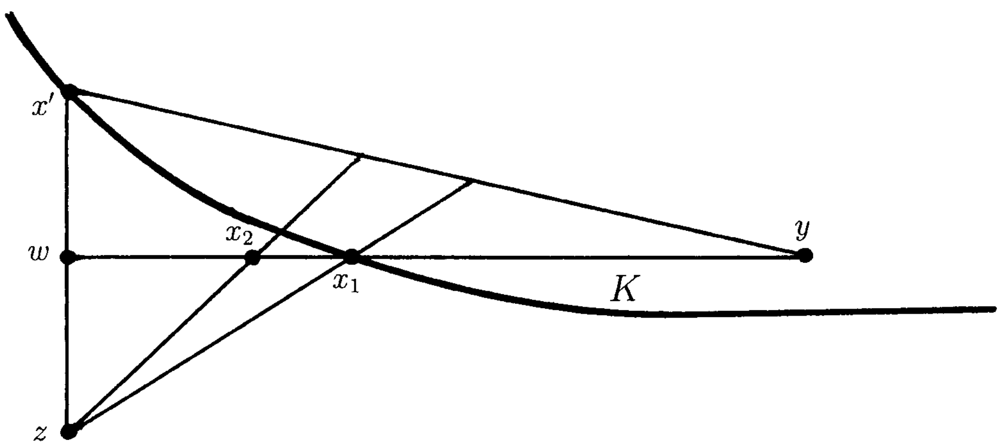

Figure 3

(vi) $\Rightarrow$ (ii): (vi) implies directly that $\partial K = K \setminus K_0 \subseteq \sigma_0(S)$.

(v) $\Rightarrow$ (vi): We prove that, if $U \subseteq \mathbb{R}^n$ is open and satisfies (v), then U satisfies (vi), too. Namely, assume that $x \in U \setminus K_0$. We must prove that $x \in \sigma_0(S)$. By (v) we have $x \in \partial K$ or $x \in \sigma(S)$. (In Fig. 3 we have $x_1 \in \partial K$ and $x_2 \in \sigma(S)$.) Assume that $y \in K_0$ is arbitrary. Then there exists some $w \in U \setminus (K \cup \{x\})$ with $x \in \overline{wy}$, because U is open. (v) implies $w \in \sigma(S)$; thus there exists some $x' \in \partial K$ with $w \in \overline{zx'}$ for some suitable $z \in S$. By Remark 2.3 (i) we have $\overline{yx'} \setminus \{x'\} \subseteq K_0$, and hence $x \in \sigma_0(\{z\}) \subseteq \sigma_0(S)$. □

§ 3 Visibility described in terms of a closure operator

In contrast to Proposition 2.2, the closure operator $\sigma = \sigma_K : \mathcal{P}(E) \to \mathcal{P}(E)$ is not precisely adapted to the visibility problem, because the boundary points of the convex body $K \subseteq \mathbb{R}^n$ do not belong to $E = \mathbb{R}^n \setminus K$. Thus they cannot lie in $\sigma_K(A)$ for a subset $A \subseteq E$. However, for any closed subset M of $\mathbb{R}^n$ we define an operator $\hat{\sigma}_M : \mathcal{P}(\mathbb{R}^n \setminus \text{int } M) \to \mathcal{P}(\mathbb{R}^n \setminus \text{ int } M)$ which in case $M = K$ precisely controls visibility: For closed $M \subseteq \mathbb{R}^n, A \subseteq \mathbb{R}^n \setminus M$ and $B \subseteq \partial M = M \setminus \text{ int } M$ put

$$(3.1) \quad \hat{\sigma}_M(A \cup B) := \sigma_M(A) \cup B \cup \{x \in \partial M \mid \overline{zx} \cap M = \{x\} \text{ for some } z \in A\}.$$

By the definitions it is clear that for all $A \subseteq F_0 := \mathbb{R}^n \setminus \text{int } M$ we have

$$(3.2) \quad \hat{\sigma}_M(A) = A \cup \{b \in F_0 \setminus A \mid \exists a \in A,\ \exists x \in \partial M : b \in \overline{ax},\ \overline{ax} \cap M = \{x\}\}.$$

By (2.5) we have the following trivial

Proposition 3.1: *For a subset A of $\mathbb{R}^n \setminus K$ and a subset B of $\partial K = K \setminus K_0$ the following statements are equivalent:*

(i) *A sees B.*

(ii) *$B \subseteq \hat{\sigma}_K(A)$.*

To prove that $\hat{\sigma}_M$ is a closure operator in case M equals the convex body K, we show first the following

Lemma 3.2: *For a boundary point $x \in \partial K$ of the convex body K and a subset A of $\mathbb{R}^n \setminus K$ the following statements are equivalent:*

(i) *There exists some $z \in A$ with $\overline{zx} \cap K = \{x\}$.*

(ii) *There exists some $z' \in \sigma_K(A)$ with $\overline{z'x} \cap K = \{x\}$.*

Proof:

(i) $\Rightarrow$ (ii): This implication is trivial in view of $A \subseteq \sigma_K(A)$.

(ii) $\Rightarrow$ (i): Choose some $z \in A$ with $z' \in \sigma_K(\{z\})$ and assume, without loss of generality, that $z' \neq z$. Then there exists some $x' \in \partial K$ with $z' \in \overline{zx'}$ and $\overline{zx'} \cap K = \{x'\}$. We prove that $\overline{zx} \cap K = \{x\}$. Otherwise there would exist some $y \in \overline{zx} \cap K$ with $y \neq x$. Since K is convex, we get even $\overline{yx'} \subseteq K$. On the other hand, we have $\overline{yx'} \cap \left(\overline{z'x} \setminus \{x\}\right) \neq \emptyset$ and thus also $K \cap \left(\overline{z'x} \setminus \{x\}\right) \neq \emptyset$, which contradicts (ii). □

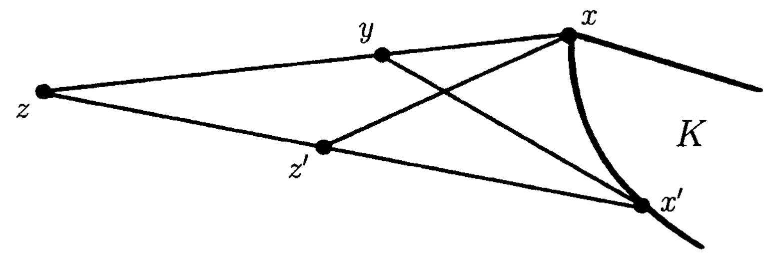

Figure 4

In Fig. 4 one sees that Lemma 3.2 is not a trivial corollary of Proposition 2.1. Now we can prove

Theorem 3.3: *The operator $\hat{\sigma}_M : \mathcal{P}(\mathbb{R}^n \setminus \operatorname{int} M) \to \mathcal{P}(\mathbb{R}^n \setminus \operatorname{int} M)$, where $M \subseteq \mathbb{R}^n$ is closed, satisfies*

(H0) $\hat{\sigma}_M$ *is increasing:* $A \subseteq \mathbb{R}^n \setminus \operatorname{int} M$ *implies* $A \subseteq \hat{\sigma}_M(A)$.

(H1) $\hat{\sigma}_M$ *is monotone:* $A_1 \subseteq A_2 \subseteq \mathbb{R}^n \setminus \operatorname{int} M$ *implies* $\hat{\sigma}_M(A_1) \subseteq \hat{\sigma}_M(A_2)$.

If M equals a convex body $K \subset \mathbb{R}^n$, then we have also

(H2) $\hat{\sigma}_M$ *is idempotent:* $A \subseteq \mathbb{R}^n \setminus \operatorname{int} M$ *implies* $\hat{\sigma}_M(\hat{\sigma}_M(A)) = \hat{\sigma}_M(A)$.

Thus $\hat{\sigma}_K$ is a closure operator for every convex body K in $\mathbb{R}^n$.

Proof: By (3.2), (H0) and (H1) are trivial for arbitrary closed $M \subseteq \mathbb{R}^n$. To prove (H2) for $M = K$, assume that $A \subseteq \mathbb{R}^n \setminus K$ and $B \subseteq \partial K$. Then we obtain by (3.1), Lemma 3.2, and the fact that σ_K is a closure operator:

$$\begin{aligned}\hat{\sigma}_K\left(\hat{\sigma}_K(A \cup B)\right) &= \hat{\sigma}_K\left(\sigma_K(A) \cup B \cup \{x \in \partial K \mid \overline{zx} \cap K = \{x\} \text{ for some } z \in A\}\right) \\ &= \sigma_K\left(\sigma_K(A)\right) \cup B \cup \{x \in \partial K \mid \overline{zx} \cap K = \{x\} \text{ for some } z \in \sigma_K(A)\} \\ &= \sigma_K(A) \cup B \cup \{x \in \partial K \mid \overline{zx} \cap K = \{x\} \text{ for some } z \in A\} \\ &= \hat{\sigma}_K(A \cup B)\,.\end{aligned}$$

□

We can now also prove some converse of Theorem 3.3. This, however, will not only be an analogue, but also a consequence of Theorem 2.9 in [10].

Proposition 3.4: *Assume that $M \subseteq \mathbb{R}^n$ is compact and that $\mathbb{R}^n \setminus M$ is connected. If the operator $\hat{\sigma}_M : \mathcal{P}(\mathbb{R}^n \setminus \operatorname{int} M) \to \mathcal{P}(\mathbb{R}^n \setminus \operatorname{int} M)$ is a closure operator, then M is convex.*

Proof: We verify that $\sigma_M : \mathcal{P}(\mathbb{R}^n \setminus M) \to \mathcal{P}(\mathbb{R}^n \setminus M)$ is a closure operator. Assume that $A \subseteq \mathbb{R}^n \setminus M$. Since $\hat{\sigma}_M$ satisfies (H2), we get

$$\sigma_M(\sigma_M(A)) \subseteq \hat{\sigma}_M(\hat{\sigma}_M(A)) = \hat{\sigma}_M(A) \subseteq \sigma_M(A) \cup \partial M .$$

However, $\sigma_M(\sigma_M(A))$ is contained in the open set $\mathbb{R}^n \setminus M$, which does not intersect ∂M. This means that $\sigma_M(\sigma_M(A)) = \sigma_M(A)$. Thus σ_M is a closure operator, whence M is convex by [10], Theorem 2.9. □

§ 4 Lattices of closure operators adapted to illumination and visibility

In this section we want to study families of closure operators $\sigma : \mathcal{P}(E_0) \to \mathcal{P}(E_0)$ including σ_0 and $\hat{\sigma}_K$ where, now as before, $E_0 = \mathbb{R}^n \setminus \text{ int } K$ for the given convex body K in $\mathbb{R}^n$. First we repeat the following definition from [10] (see Definition 2.2 there).

Definition 4.1: For an arbitrary set P, an operator $\sigma : \mathcal{P}(P) \to \mathcal{P}(P)$ is called a *visibility operator* if σ is increasing, monotone, and the following *visibility condition* holds:

(V) For every subset A of P and every $p \in \sigma(A)$ there exists some $a \in A$ with $p \in \sigma(\{a\})$.

Remark 4.2:

i) Note that a visibility operator is not necessarily a closure operator, because idempotence is not required.

ii) By definition, the operator $\sigma_M : \mathcal{P}(\mathbb{R}^n \setminus M) \to \mathcal{P}(\mathbb{R}^n \setminus M)$ as defined in (2.4) is a visibility operator for every subset M of $\mathbb{R}^n$. In particular, $\sigma_0 : \mathcal{P}(E_0) \to \mathcal{P}(E_0)$ is a visibility operator. Moreover, for closed $M \subseteq \mathbb{R}^n$ the operator $\hat{\sigma}_M : \mathcal{P}(\mathbb{R}^n \setminus \text{int } M) \to \mathcal{P}(\mathbb{R}^n \setminus \text{int } M)$ as defined in (3.1) is also a visibility operator.

(iii) If $\sigma : \mathcal{P}(P) \to \mathcal{P}(P)$ is an arbitrary visibility operator and $(A_i)_{i \in I}$ is a family of subsets of P, then monotony and the visibility condition imply at once

$$(4.1) \qquad \sigma\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} \sigma(A_i) .$$ □

Before we introduce new closure operators $\sigma : \mathcal{P}(E_0) \to \mathcal{P}(E_0)$, we want to prove two elementary relations between σ_0 and $\hat{\sigma}_K$.

Proposition 4.3: *For every subset A of E_0 one has*

$$(4.2) \qquad \sigma_0(A) \subseteq \hat{\sigma}_K(A) .$$

Proof: Assume that $b \in \sigma_0(A) \setminus A$. If $b \in \partial K$, then A illuminates b by Proposition 2.2. Thus, by Remark 2.3 (ii), A also sees b, whence $b \in \hat{\sigma}_K(A)$ by Proposition 3.1. Now assume that $b \notin \partial K$, that means, $b \notin K$. Choose some $a \in A$ with $\overline{ab} \cap K_0 = \emptyset$ and $s(a,b) \cap K_0 \neq \emptyset$. Thus there exists some $p \in s(a,b) \setminus \overline{ab}$ with $p \in K_0 \subseteq K$. Since K is convex and $b \notin K$, we must have $\overline{ab} \cap K = \emptyset$ and thus $b \in \sigma_K(A) \subseteq \hat{\sigma}_K(A)$. □

Lemma 4.4: *For arbitrary subsets A, Q of E_0 one has*

$$(4.3) \quad \sigma_0\left(\hat{\sigma}_K(A) \cap Q\right) \subseteq \sigma_0(A) \cup \left(\hat{\sigma}_K(A) \cap Q\right) .$$

Proof: Assume that $q \in \hat{\sigma}_K(A) \cap Q$. Since σ_0 is a visibility operator, it suffices to prove that

$$(4.4) \quad \sigma_0(\{q\}) \subseteq \sigma_0(A) \cup \left(\hat{\sigma}_K(A) \cap Q\right) .$$

Suppose that $p \in \sigma_0(\{q\})$. To prove that we have $p \in \sigma_0(A) \cup (\hat{\sigma}_K(A) \cap Q)$, we may assume that $q \notin A$ and that $p \neq q$; otherwise the assertion is trivial. $q \in \hat{\sigma}_K(A) \setminus A$ implies that there exists some $a \in A$ and some $x_1 \in \partial K$ with $\overline{ax_1} \cap K = \{x_1\}$ and $q \in \overline{ax_1}$. Since $p \in \sigma_0(\{q\})$, there exists some $x_2 \in K_0$ with $p \in \overline{qx_2}$. Thus we get also $p \in \text{ conv } \{q, x_2\} \subseteq \text{ conv } \{a, x_1, x_2\}$, whence there exists some $x_3 \in \overline{x_1x_2} \subseteq K$ with $p \in \overline{ax_3}$. If $x_3 = x_1$, then we have $p, q \in \overline{ax_1}$ and thus $p \in \sigma_0(\{a\})$ as claimed, because $\overline{ap} \cap K_0 \subseteq \overline{ax_1} \cap K_0 = \emptyset$ and $s(a,p) \cap K_0 = s(q,p) \cap K_0 \neq \emptyset$. If, on the other hand, $x_3 \neq x_1$, then Remark 2.3 (i) implies $x_3 \in s(a,p) \cap K_0$. Since $p \notin K_0$ and K_0 is convex, we have $\overline{ap} \cap K_0 = \emptyset$ and thus $p \in \sigma_0(\{a\})$. □

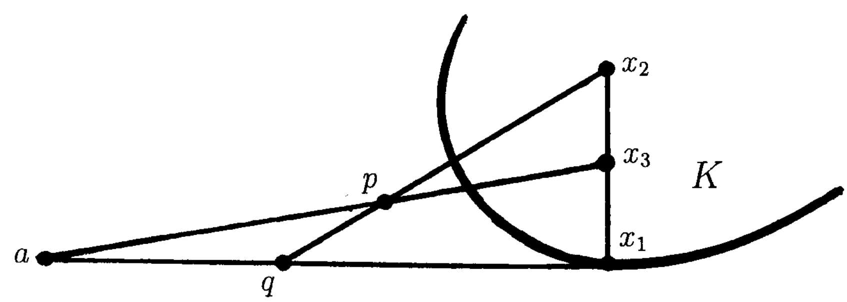

Figure 5

Now for every subset Q of E_0 we define the operator $\sigma_K^Q : \mathcal{P}(E_0) \to \mathcal{P}(E_0)$ by

$$(4.5) \quad \sigma_K^Q(A) := \sigma_0(A) \cup \left(\hat{\sigma}_K(A) \cap Q\right) = \hat{\sigma}_K(A) \cap \left(\sigma_0(A) \cup Q\right) .$$

Remark 4.5: The second equation in (4.5) holds by Proposition 4.3. Clearly, one has

$$(4.5\text{ a}) \quad \sigma_K^{\emptyset} = \sigma_0 ,$$

$$(4.5\text{ b}) \quad \sigma_K^{E_0} = \hat{\sigma}_K ,$$

and for all $A, Q \subseteq E_0$ we have

$$(4.6) \quad \sigma_0(A) \subseteq \sigma_K^Q(A) \subseteq \hat{\sigma}_K(A) .$$

It is trivial that all operators σ_K^Q for $Q \subseteq E_0$ are visibility operators. Before we prove that these are also idempotent and thus closure operators, we would like to mention that they are not only theoretically interesting because of encompassing the closure operators σ_0 and $\hat{\sigma}_K$, adapted to illumination and visibility, respectively. They may also be of practical interest for the following reason: Assume that $S \subseteq E_0$ consists of light sources of weak intensity. Those boundary points of K which lie in $Q = \sigma_0(S)$ should additionally be at least seen by some further light source $a \in A$, while the other elements of ∂K should, even harsher, be illuminated by some $a \in A$. If, on the other hand, S consists of light sources of very strong intensity, then, in view of blinding effects, it might be interesting to consider σ_K^Q for $Q := E_0 \setminus \sigma_0(S)$. □

Theorem 4.6: *For every subset Q of E_0, the operator σ_K^Q is a closure operator.*

Proof: It remains to prove that σ_K^Q is idempotent. By (4.5), by the facts that σ_K^Q is a visibility operator and that σ_0 and $\hat{\sigma}_K$ are closure operators as well as by Proposition 4.3 and Lemma 4.4 we get for $A \subseteq E_0$ that

$$\begin{aligned}
\sigma_K^Q\left(\sigma_K^Q(A)\right) &= \sigma_K^Q\left(\sigma_0(A) \cup (\hat{\sigma}_K(A) \cap Q)\right) \\
&= \sigma_K^Q\left(\sigma_0(A)\right) \cup \sigma_K^Q\left(\hat{\sigma}_K(A) \cap Q\right) \\
&= \sigma_0(\sigma_0(A)) \cup (\hat{\sigma}_K(\sigma_0(A)) \cap Q) \\
&\quad \cup \sigma_0\left(\hat{\sigma}_K(A) \cap Q\right) \cup (\hat{\sigma}_K(\hat{\sigma}_K(A) \cap Q) \cap Q) \\
&\subseteq \sigma_0(A) \cup (\hat{\sigma}_K(\hat{\sigma}_K(A)) \cap Q) \\
&\quad \cup (\sigma_0(A) \cup (\hat{\sigma}_K(A) \cap Q)) \cup (\hat{\sigma}_K(\hat{\sigma}_K(A)) \cap Q) \\
&= \sigma_0(A) \cup (\hat{\sigma}_K(A) \cap Q) \\
&= \sigma_K^Q(A)\,.
\end{aligned}$$

□

For arbitrary closure operators $\sigma, \sigma' : \mathcal{P}(E_0) \to \mathcal{P}(E_0)$ we write $\sigma \leq \sigma'$ if $\sigma(A) \subseteq \sigma'(A)$ holds for all $A \subseteq E_0$. Now we consider the complete lattice Φ_0 of all closure operators $\sigma : \mathcal{P}(E_0) \to \mathcal{P}(E_0)$ with $\sigma_0 \leq \sigma \leq \hat{\sigma}_K$. Clearly, $(\Phi_0, \leq)$ is an ordered set with σ_0 as its smallest and $\sigma_1 := \hat{\sigma}_K$ as its largest element. Moreover, $(\Phi_0, \leq)$ is indeed a complete lattice, as the following arguments will demonstrate.

For every family of closure operators $(\sigma_i)_{i \in I}$ in Φ_0 we have

$$\text{(4.7 a)} \qquad \inf_{\Phi_0} \{\sigma_i \mid i \in I\} = \sigma'$$

with

$$\text{(4.7 b)} \qquad \sigma'(A) := \bigcap_{i \in I} \sigma_i(A) \text{ for all } A \subseteq E_0$$

and

$$\text{(4.7 c)} \qquad \sup_{\Phi_0} \{\sigma_i \mid i \in I\} = \inf_{\Phi_0} \{\sigma \in \Phi_0 \mid \sigma_i \leq \sigma \text{ for all } i \in I\}\,.$$

(See also [11] as well as [6].)

Next we study complete sublattices $(\Phi, \leq)$ of $(\Phi_0, \leq)$. That means, one has

(4.8 a) $\{\sigma_0, \sigma_1\} \subseteq \Phi \subseteq \Phi_0$

as well as

(4.8 b) $\inf_{\Phi} \{\sigma_i \,|\, i \in I\} = \inf_{\Phi_0} \{\sigma_i \,|\, i \in I\}$,

(4.8 c) $\sup_{\Phi} \{\sigma_i \,|\, i \in I\} = \sup_{\Phi_0} \{\sigma_i \,|\, i \in I\}$

for every family of closure operators $(\sigma_i)_{i \in I}$ in Φ.

Put

(4.9) $\Phi_1 := \left\{\sigma_K^Q \,|\, Q \subseteq E_0\right\}$.

We have

(4.9 a) $\sigma_0 = \sigma_K^{\emptyset} \in \Phi_1$, $\sigma_1 = \hat{\sigma}_K = \sigma_K^{E_0} \in \Phi_1$.

Moreover, for any family $(Q_i)_{i \in I}$ of subsets of E_0 we get by (4.5):

(4.9 b) $\inf_{\Phi_1} \left\{\sigma_K^{Q_i} \,|\, i \in I\right\} = \inf_{\Phi_0} \left\{\sigma_K^{Q_i} \,|\, i \in I\right\} = \sigma_K^{\underline{Q}}$

with

(4.9 b′) $\underline{Q} := \bigcap_{i \in I} Q_i$

as well as

(4.9 c) $\sup_{\Phi_1} \left\{\sigma_K^{Q_i} \,|\, i \in I\right\} = \sup_{\Phi_0} \left\{\sigma_K^{Q_i} \,|\, i \in I\right\} = \sigma_K^{\overline{Q}}$

with

(4.9 c′) $\overline{Q} := \bigcup_{i \in I} Q_i$.

Now (4.6), (4.9 a), (4.9 b), and (4.9 c) imply at once the following

Proposition 4.7: $(\Phi_1, \leq)$ *is a Boolean lattice which, in addition, is a complete sublattice of* $(\Phi_0, \leq)$.

For a fixed subset P of E_0 put

(4.10 a) $\mathcal{V}(P) := \{\sigma_K^P(A) \mid A \subseteq E_0\}$,

(4.10 b) $\mathcal{V}'(P) := \{E_0 \setminus \sigma_K^P(A) \mid A \subseteq E_0\}$,

(4.10 c) $\Phi(P) := \left\{\sigma_K^Q \mid Q \in \mathcal{V}(P)\right\}$,

(4.10 d) $\Phi'(P) := \left\{\sigma_K^Q \mid Q \in \mathcal{V}'(P)\right\}$.

A motivation for these definitions can be taken from Remark 4.5, where the corresponding terms occured already in case $P = \emptyset$. Since σ_K^P is a closure operator as well as a visibility operator, $\mathcal{V}(P)$ and $\mathcal{V}'(P)$ are closed with respect to arbitrary intersections and unions. Moreover, we have

(4.11) $\{\emptyset, E_0\} = \{\sigma_K^P(\emptyset), \sigma_K^P(E_0)\} \subseteq \mathcal{V}(P) \cap \mathcal{V}'(P)$,

and therefore

(4.11 a) $\{\sigma_0, \sigma_1\} \subseteq \Phi(P) \cap \Phi'(P)$.

Thus we obtain the following

Theorem 4.8: *For every subset P of E_0, the ordered sets $(\Phi(P), \leq)$ and $(\Phi'(P), \leq)$ are complete sublattices of $(\Phi_1, \leq)$ and thus also of $(\Phi_0, \leq)$.*

It should be noticed that the closure operators σ_K^Q for $Q \subseteq E_0$ are not necessarily pairwise distinct. More precisely, we have the following

Proposition 4.9: *For $Q, Q' \subseteq E_0$ the following statements are equivalent:*

(i) $\sigma_K^Q = \sigma_K^{Q'}$.

(ii) *$Q \triangle Q' \subseteq \partial K$, and every $q \in Q \triangle Q'$ is contained in a (uniquely determined) supporting hyperplane H of K such that there exists some $\delta = \delta_q > 0$ with $B(q, \delta) \cap H \subseteq K$.*

Proof:

(i) $\Rightarrow$ (ii): First assume that $Q \triangle Q' \not\subseteq \partial K$, say $q \in Q \setminus Q'$ holds for some $q \in E = \mathbb{R}^n \setminus K$. Then there exists some $b \in \partial K$ with $\overline{qb} \cap K = \{b\}$ such that the line containing q and b does not intersect K_0. Now choose some $a \in E \setminus \{q\}$ with $q \in \overline{ab}$. Then we have also $\overline{ab} \cap K = \{b\}$ and $s(a, q) \cap K_0 = \emptyset$. This means that $q \in \hat{\sigma}_K(\{a\}) \setminus \sigma_0(\{a\})$ and thus $q \in \sigma_K^Q(\{a\})$ but $q \notin \sigma_K^{Q'}(\{a\})$, which contradicts (i). Now suppose that $q \in Q \setminus Q' \subseteq \partial K$, and choose any supporting hyperplane H to K with $q \in H$. Assume that q does not lie in the relative interior of $H \cap K$ in H. Then there exists some $a \in H \setminus \{q\}$ with $\overline{aq} \cap K = \{q\}$, and the line containing a and q does not intersect K_0. This means that $q \in \hat{\sigma}_K(\{a\}) \setminus \sigma_0(\{a\})$, whence again $q \in \sigma_K^Q(\{a\})$ but $q \notin \sigma_K^{Q'}(\{a\})$, contradicting (i).

(ii) $\Rightarrow$ (i): We must prove that for given $q \in Q \Delta Q'$ and all $A \subseteq E_0$ the relation $q \in \sigma_0(A) \cup (\hat{\sigma}_K(A) \cap Q)$ holds if and only if $q \in \sigma_0(A) \cup (\hat{\sigma}_K(A) \cap Q')$. This amounts to prove that for all $a \in E_0$ we have $q \in \sigma_0(\{a\})$ whenever $q \in \hat{\sigma}_K(\{a\})$. If $a \in E = \mathbb{R}^n \setminus K$, our assumption in (ii) implies that the ray $s(a,q)$ meets K_0 whenever $\overline{aq} \cap K = \{q\}$ as claimed, because then a cannot lie in the supporting hyperplane H of K with $q \in H$. If, however, $a \in \partial K$, we have $q \in \hat{\sigma}_K(\{a\})$ only for $a = q$, in which case we also have $q \in \sigma_0(\{a\})$. □

Remark 4.10: Proposition 4.9 implies the following observations:

(i) Two closure operators $\sigma_K^Q, \sigma_K^{Q'}$ are different whenever $Q \neq Q'$ and $Q \cup Q' \subseteq E = \mathbb{R}^n \setminus K$.

ii) If K is a ball, then all of the closure operators σ_K^Q, $Q \subseteq E_0$, are pairwise different.

iii) If K is a polytope, then not all of the closure operators σ_K^Q, $Q \subseteq E_0$, are pairwise different. □

At the end of this section we want to look at further lattices contained in $(\Phi_1, \leq)$. For fixed sets $P_1 \subseteq E_0$ and $P_2 \in \mathcal{V}(P_1)$ put

$$(4.12) \qquad \Phi(P_1, P_2) := \left\{ \sigma_K^Q \mid Q \in \mathcal{V}(P_1), Q \subseteq P_2 \right\} = \left\{ \sigma \in \Phi(P_1) \mid \sigma \leq \sigma_K^{P_2} \right\}.$$

Clearly, $\Phi(P_1, P_2)$ is a lattice. More precisely, for any family of closure operators in $\Phi(P_1, P_2)$, infimum and supremum mean the same as in Φ_0, Φ_1 and $\Phi(P_1)$. However, $\Phi(P_1, P_2)$ is in general not a sublattice of these lattices, because σ_1 need not lie in $\Phi(P_1, P_2)$.

With the intention to consider the Gohberg-Markus-Hadwiger covering conjecture (cf. Chapter VI of [2]) it might be interesting to consider $\Phi(\emptyset, S)$ for subsets $S \subseteq E = \mathbb{R}^n \setminus K$ with $\partial K \subseteq \sigma_0(S)$. If S illuminates K, Theorem 2.4 shows that for every $x \in \partial K$ there exists an open set $W \subseteq \mathbb{R}^n$ with $x \in W$ such that $W \cap \partial K$ is illuminated by some $a \in S$. Thus there exists a finite subset S_0 of S which illuminates the compact set ∂K. For the minimal subsets S_0 of S illuminating ∂K – with respect to inclusion or cardinality – the lattice

$$(4.12\text{ a}) \qquad \Phi(\emptyset, \sigma_0(S_0)) = \left\{ \sigma_K^Q \mid Q = \sigma_0(A) \subseteq \sigma_0(S_0) \text{ for some } A \subseteq E_0 \right\}$$

might be of importance. Note that this lattice is not isomorphic to the Boolean lattice $(\mathcal{P}(S_0), \subseteq)$, because for $s_1, s_2 \in S_0$ with $s_1 \neq s_2$ we have in general $\sigma_0(\{s_1\}) \cap \sigma_0(\{s_2\}) \neq \emptyset$.

§ 5 Illumination by directions

Throughout this section assume that we have $0 \in \text{int } K$ for a convex body K in $\mathbb{R}^n$. Due to [1], a point $x \in \partial K$ is said to be *illuminated by the direction* $u \in S^{n-1} := \{x \in \mathbb{R}^n : ||x|| = 1\}$

if the ray $s(x, x+u)$ with initial point x and direction u meets the interior of K. Therefore it seems natural to consider the operator $\tau_0 : \mathcal{P}(S^{n-1}) \to \mathcal{P}(S^{n-1})$ given by

$$(5.1) \quad \tau_0(A) := \left\{ b \in S^{n-1} \mid \exists u \in A, \exists x \in \partial K : \frac{1}{||x||} \cdot x = b, s(x, x+u) \cap \text{int } K \neq \emptyset \right\},$$

which is adapted to the problem of illumination by directions. Note that $0 \in \text{int } K$ implies that for every $b \in S^{n-1}$ there exists a unique element $x \in \partial K$ with $\frac{1}{||x||} \cdot x = b$. However, τ_0 is not increasing, since for $u \in S^{n-1}$ and $x \in \partial K$ with $\frac{1}{||x||} \cdot x = u$ we have $s(x, x+u) \cap K = \{x\}$ and thus $s(x, x+u) \cap \text{int } K = \emptyset$. This means $u \notin \tau_0(\{u\})$. Therefore let us look at the modified operator $\tau_1 : \mathcal{P}(S^{n-1}) \to \mathcal{P}(S^{n-1})$ given by

$$(5.2) \quad \tau_1(A) := \left\{ b \in S^{n-1} \mid \exists u \in A, \exists x \in \partial K : \frac{1}{||x||} \cdot x = b, s(x, x-u) \cap \text{int } K \neq \emptyset \right\}.$$

Clearly, we have $\tau_1(A) = \tau_0(-A)$ for $A \subseteq S^{n-1}$. Now $0 \in \text{int } K$ implies that τ_1 is increasing. Thus it is also evident that τ_1 is a visibility operator which is in some sense analogous to σ_0 by relating $u \in S^{n-1}$ to points $x \in \mathbb{R}^n$ for which $||x||$ is large and $\frac{1}{||x||} \cdot x = u$.

However, τ_1 is in general not a closure operator though σ_0 is. If, for instance, K is a ball with 0 as its center, then we have $\tau_1(\{u\}) = \{v \in S^{n-1} | \cos(\sphericalangle(u, v)) > 0\}$ for all $u \in S^{n-1}$. Thus we get $\tau_1(\tau_1(\{u\})) = S^{n-1} \setminus \{-u\}$ and $\tau_1(\tau_1(\tau_1(\{u\}))) = S^{n-1}$, whence τ_1 is not idempotent.

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