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On Planar Convex Bodies of Given Minkovskian Thickness and Least Possible Area

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Abstract

Let K be a convex body in a Minkowski plane, i.e., in a two-dimensional real Banach space. The Minkowskian thickness of K is the minimal possible Minkowskian distance between two points of K lying in different parallel supporting lines of that convex body. Let $\mathcal{X}$ be the class of planar convex bodies having a given Minkowskian thickness, say one, and least possible area. We prove that each body K from $\mathcal{X}$ is necessarily a triangle or a quadrilateral. Furthermore, under certain conditions involving the Minkowskian unit ball, the class $\mathcal{X}$ consists only of triangles. The result of Pál [BF74, §10], stating that in Euclidean case $\mathcal{X}$ is the class of equilateral triangles with altitudes of length one, is obtained as a simple consequence of our main theorem.

MSC (2000): 52A21, 52A10, 52A38

1 Convex bodies in the Euclidean space

By $\mathbb{E}^d$, $d \geq 2$, we denote the d-dimensional Euclidean space with origin o and norm $|\cdot|$. Length and volume in $\mathbb{E}^d$ are denoted by μ and V, respectively. The abbreviations cl, bd, int, and conv stand for closure, boundary, interior, and convex hull, respectively. A set $K \subseteq \mathbb{E}^d$ is said to be a *convex body* if it is convex, compact and has non-empty interior, cf. [BF74] and [Sch93]. A chord $[p_1, p_2]$, $p_1, p_2 \in \operatorname{bd} K$, of K is called an *affine diameter* if there exist two different parallel supporting hyperplanes H_1 and H_2 of K with $p_1 \in H_1$ and $p_2 \in H_2$. Let u be a variable in $\mathbb{E}^d \setminus \{o\}$. Then with K we associate the following functions of u and sets depending on u :

$$\begin{aligned}
r_K(u) &:= \max\{\alpha \in \mathbb{R} : \alpha u \in K\} && \textit{(radius function)},\\
h_K(u) &:= \max\{\langle x, u\rangle : x \in K\} && \textit{(support function)},\\
H(K,u) &:= \{x \in \mathbb{E}^d : \langle x, u\rangle = h_K(u)\} && \textit{(supporting hyperplane at direction } u\textit{)},\\
F(K,u) &:= H(K,u) \cap K && \textit{(face at direction } u\textit{)}.
\end{aligned}$$

For a point $x \in \operatorname{bd} K$ the *normal cone* of K at x is defined by

$$N(K,x) := \{u \in \mathbb{E}^d \setminus \{o\} : x \in H(K,u)\} \cup \{o\}.$$

If $o \in \operatorname{int} K$, then the convex body $K^* = \{u \in \mathbb{E}^d : h_K(u) \leq 1\}$ is called the *polar body of* K (or the *dual body of* K). Using the definition of the polar body one can obtain the well-known equalities $r_{K^*}(u)h_K(u) = 1$ ($u \in \mathbb{E}^d\setminus\{o\}$) and $K^{**} = K$. The following proposition follows from basic properties of the duality transformation, see [Sch93, Section 1.6]. The proof of its first part is more or less straightforward, and the second part follows directly from the first one.

Proposition 1. *Let K be a convex body with $o \in \operatorname{int} K$. We consider a direction $v \in \mathbb{E}^d\setminus\{o\}$ and the boundary point p of K with radius vector of direction v. Let u be an outward Euclidean normal of K at p, and N be the normal cone of K at p. Then*

(i) the vector v is the outward Euclidean normal of K^* at the boundary point q of K^* with radius vector of direction u,

(ii) the set $N \cap \operatorname{bd} K^*$ is the face of K^* at direction v.

□

The convex body $DK := K + (-K)$ is called the *difference body of K*. The *width function* $w_K(u) := h_{DK}(u) = h_K(u) + h_K(-u)$ and the *maximal chord-length function* $l_K(u) := r_{DK}(u)$ yield analytical representations of the so-called *one-dimensional cross-section measures* of K, cf. [Mar94] and [Gar95]. If u is a unit vector, then $w_K(u)$ is the distance between two different supporting hyperplanes of K with Euclidean normal u, and $l_K(u)$ is the length of the affine diameter of K having direction u.

2 Selected topics from Minkowski spaces

A finite dimensional real Banach space is called a *Minkowski space*, cf. the monograph [Tho96] and the surveys [MSW01] and [MS]. If $B \subseteq \mathbb{E}^d$ is a convex body centered at the origin, then by $\mathcal{M}^d(B)$ we denote the Minkowski space with unit ball B. The norm in $\mathcal{M}^d(B)$ is denoted by $\|\,.\,\|_B$. The body $\alpha B + p$, where $\alpha > 0$ and $p \in \mathcal{M}^d(B)$, is called the *Minkowskian ball of radius α centered at p*. Let μ_B denote the Minkowskian length in $\mathcal{M}^d(B)$. The *volume* in $\mathcal{M}^d(B)$ can be defined exactly as in $\mathbb{E}^d$, for the justification see [Tho96, Section 1.4].

Given a convex body K in $\mathcal{M}^d(B)$ and a vector u ranging over $\mathbb{E}^d\setminus\{o\}$, we introduce the *Minkowskian width function* $w_{K,B}(u) := w_K(u)/h_B(u)$ and the *Minkowskian maximal chord-length function* $l_{K,B}(u) := l_K(u)/r_B(u)$ of the body K. The *Minkowskian diameter* $\operatorname{diam}_B(K) := \max\{\|x-y\|_B : x, y \in K\}$ of K is equal to the maximum of both $w_{K,B}(u)$ and $l_{K,B}(u)$, cf. [Ave03, Theorem 2]. Analogously, the minima of $w_{K,B}(u)$ and $l_{K,B}(u)$ are equal and called *Minkowskian thickness* $\Delta_B(K)$ of K, cf. [Ave03, Theorem 3]. It is not hard to show that the Minkowskian thickness of K is equal to the Minkowskian inradius of DK, see [Ave03, Theorem 3], i.e.,

$$\Delta_B(K) = \max\{\alpha > 0 : \alpha B \subseteq DK\}. \tag{1}$$

Let B_1 and B_2 be two convex bodies centered at the origin. In view of (1), we easily see that the inclusion $B_1 \subseteq B_2$ implies the inequality $\Delta_{B_1}(K) \geq \Delta_{B_2}(K)$, where K is an arbitrary convex body.

A triangle T is said to be *equilateral in a Minkowski plane* $\mathcal{M}^2(B)$ if all its sides have the same length in $\mathcal{M}^2(B)$. It can be shown that for every direction u there exists an equilateral triangle in $\mathcal{M}^2(B)$ with a side parallel to u. Obviously, if T is an equilateral triangle in $\mathcal{M}^2(B)$ with sides of Minkowskian length one, then all vertices of the hexagon DT lie in $\operatorname{bd} B$ and all sides of DT have Minkowskian length one.

A d-dimensional convex body K is said to be of *constant width* $\lambda > 0$ *in* $\mathcal{M}^d(B)$ if for any direction $u \in \mathbb{E}^d \setminus \{o\}$ we have $w_{K,B}(u) = \lambda$, cf. [CG83], [HM93, Section 5] and [MS, Section 2]. Equivalently, the latter condition can be given as the equality $DK = \lambda \cdot B$. If $T \subseteq \mathcal{M}^2(B)$ is a Minkowskian equilateral triangle with sides of Minkowskian length $\lambda > 0$, then the intersection $W_B(T)$ of three Minkowskian balls of radius λ centered at the vertices of T is called a *Minkowskian Reuleaux triangle.* One can express the area of $W_B(T)$ by the areas of T and B as follows:

$$V(W_B(T)) = \frac{\lambda^2}{2} V(B) - 2V(T). \tag{2}$$

It turns out that a convex body K of constant Minkowskian width in $\mathcal{M}^2(B)$ is a Minkowskian Reuleaux triangle if there exist boundary points p_1, p_2, p_3 of K such that the chords $[p_1, p_2]$, $[p_2, p_3]$, and $[p_3, p_1]$ are affine diameters of K.

The Minkowskian analogue of the classical *Blaschke-Lebesgue Theorem* states that every planar convex body of given constant width in $\mathcal{M}^2(B)$ and least possible area is necessarily a Reuleaux triangle in $\mathcal{M}^2(B)$, for the proof see [Ohm52], [Cha66] and [Tho96, Theorem 4.2.8].

A d-dimensional convex body K is called *reduced in* $\mathcal{M}^d(B)$ if it does not properly contain a convex body of the same thickness in $\mathcal{M}^d(B)$, cf. the papers [Las90] and [LM] containing general results on reduced bodies in Euclidean and Minkowski spaces. Let B be a d-dimensional convex polytope with vertices $\pm b_k$, $k \in \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$, and K be a Minkowskian reduced body in $\mathcal{M}^2(B)$ having Minkowskian thickness one. Then

$$K = \operatorname{conv} \bigcup_{k=1}^{n} I'_k, \tag{3}$$

where I'_k is a suitable translate of $[o, b_k]$, cf. [LM, Corollary 1]. Suppose B is a parallelogram. Then, using the above statement, we can show that K is a reduced body in $\mathcal{M}^2(B)$ with $\Delta_B(K) = 1$ if and only if $K = \operatorname{conv}(I_1 \cup I_2)$, where I_1 and I_2 are intersecting translates of the diagonals of B. One can easily verify that the area of any such K is equal to the area of B.

The *isoperimetrix* $\tilde{B}$ in a Minkowski plane $\mathcal{M}^2(B)$ is the polar body of B rotated by the angle $\frac{\pi}{2}$ about the origin. The *Minkowskian height* of a triangle $T \subseteq \mathcal{M}^2(B)$ with respect to its side I is the value $w_{T,B}(u)$, where u is the Euclidean normal of I. The following theorem gives a characterization of Minkowskian reduced triangles, see [Ave, Theorem 7] and [CG85, Section 6].

Theorem 2. *Let $\mathcal{M}^2(B)$ be an arbitrary Minkowski plane. Then a triangle T is reduced in $\mathcal{M}^2(B)$ if and only if T has equal heights in $\mathcal{M}^2(B)$ or, equivalently, T is equilateral in $\mathcal{M}^2(\tilde{B})$.* □

If B and $\tilde{B}$ are homothetic convex bodies, then $\operatorname{bd} B$ is called a *Radon curve,* and the corresponding Minkowski plane $\mathcal{M}^2(B)$ is said to be a *Radon plane,* see

[Tho96, Section 4.7]. Clearly, the Euclidean plane is a special Radon plane. The difference body of a triangle is said to be an *affine regular hexagon.* The boundary of any affine regular hexagon is a Radon curve, implying that the polar body of any affine regular hexagon is also an affine regular hexagon. From Theorem 2 we can see that in any Radon plane the classes of Minkowskian equilateral triangles and Minkowskian reduced triangles coincide.

Let p, q be points in a Minkowski space $\mathcal{M}^d(B)$. Then the set

$$[p,q]_B := \{x \in \mathcal{M}^d(B) : \|p-q\|_B = \|p-x\|_B + \|x-q\|_B\}$$

is called the *d-segment connecting p with q*. A set $X \subseteq \mathcal{M}^d(B)$ is said to be *d-convex* if and only if for any $p, q \in X$ the d-segment $[p,q]_B$ lies in X. The smallest (with respect to inclusion) d-convex set containing a given set $X \subseteq \mathcal{M}^d(B)$ is called a *d-convex hull of X* and denoted by $\mathrm{conv}_B(X)$. The following theorem gives a characterization of those Minkowski planes where the unit Minkowskian ball is d-convex, cf. [BMS97, Theorem 11.4].

Theorem 3. *Let $\mathcal{M}^2(B)$ be an arbitrary Minkowski plane. Then we have $\mathrm{conv}_B(B) = B$ if and only if B for any line l supporting B at a smooth boundary point the line parallel to l and passing through the origin intersects $\mathrm{bd}\, B$ at extreme boundary points of B.* □

Using Proposition 1, we can obtain the following. If a line l supports B at a smooth point, then the line parallel to l and passing through the origin intersects $\mathrm{bd}\, \tilde{B}$ at extreme points. Consequently, in view of Theorem 3, for Radon planes $\mathcal{M}^2(B)$ we have $\mathrm{conv}_B(B) = B$.

3 The results

Let $\mathcal{M}^d(B)$ be an arbitrary Minkowski space. By $\mathcal{X}(B)$ we denote the class of convex bodies in $\mathcal{M}^d(B)$ having Minkowskian thickness one and least possible volume. Trivially, the homothetical copies of convex bodies from $\mathcal{X}(B)$ correspond to the equality case in the *geometric inequality* $V(K) \geq \alpha(B) \cdot \Delta_B(K)^d$, where $\alpha(B)$ is the volume of convex bodies from $\mathcal{X}(B)$. We cite the books [BF74, §10], [JB56, §6], [BZ80], and [Tho96, Sections 4.4 and 4.5], where various geometric inequalities in Minkowski and Euclidean spaces are discussed. If $\mathcal{M}^d(B)$ is the Euclidean plane (i.e., $d = 2$ and B is an ellipse), then $\mathcal{X}(B)$ is the class of equilateral triangles with altitudes of length one, which is proved by Pál in [Pál21] and also, in another way, below in this paper. For $\mathcal{M}^d(B)$ being the Euclidean space with $d \geq 3$, no elements from $\mathcal{X}(B)$ are known, but see [Hei78] for a related discussion. The following theorem gives a complete description of $\mathcal{X}(B)$ in the important case when $d = 2$ and B is a hexagon.

Theorem 4. *Let $B \subseteq \mathbb{E}^2$ be a convex hexagon with vertices $\pm b_j$, $j \in \{1,2,3\}$, and let $I_j := [o, b_j]$. Then the following statements hold.*

(i) For some $k \in \{1,2,3\}$ there exists a triangle T which contains a translate of I_k and whose two sides are translates of the remaining two segments I_j, $j \in \{1,2,3\} \setminus \{k\}$.

(ii) A convex body K belongs to $\mathcal{X}(B)$ if and only if for some $k \in \{1,2,3\}$ the body K is the convex hull of two intersecting translates of I_j, $j \in \{1,2,3\} \setminus \{k\}$, and some translate of I_k is contained in K.

Proof. Without loss of generality we assume that b_1, b_2 as well as b_2, b_3 are neighboring vertices of B. First we prove *(i)*. Let us introduce the affine regular hexagon $B' := \operatorname{conv}\{\pm b_1, \pm b_2, \pm b_3'\}$ with $b_3' := b_2 - b_1$. If $B \subseteq B'$, see Fig. 1, then the triangle $\operatorname{conv}\{o, b_1, b_2\}$ contains a translate of I_3. Otherwise, the line $\operatorname{aff}\{b_2, b_3\}$ or the line $\operatorname{aff}\{b_3, -b_1\}$ supports B', see Fig. 2. In the first case the triangle $\operatorname{conv}\{o, b_3, -b_1\}$ contains a translate of I_2, while in the latter case the triangle $\operatorname{conv}\{o, b_2, b_3\}$ contains a translate of I_1.

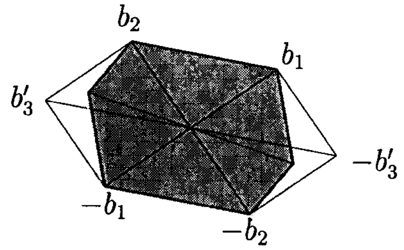

Figure 1

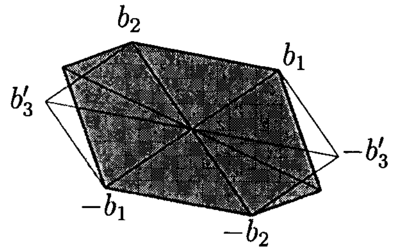

Figure 2

Next we prove *(ii)*. First we verify the sufficiency. Let I_j' be a translate of I_j, $j \in \{1,2,3\}$. Suppose that for some $k \in \{1,2,3\}$ the convex body $K := \operatorname{conv} \bigcup_{j \in \{1,2,3\} \setminus \{k\}} I_j'$ contains I_k' and the two segments I_j', $j \in \{1,2,3\} \setminus \{k\}$, intersect. It suffices to show that $V(K) \le V(R)$ for any reduced body R in $\mathcal{M}^2(B)$ with $\Delta_B(R) = 1$. By (3) we have $R = \operatorname{conv} \bigcup_{i \in \{1,2,3\}} I_j''$, where I_j'' is a translate of I_j, $j \in \{1,2,3\}$. But then we obtain $V(K) = V(\operatorname{conv} \bigcup_{j \in \{1,2,3\} \setminus \{k\}} I_j') \le V(\operatorname{conv} \bigcup_{j \in \{1,2,3\} \setminus \{k\}} I_j'') \le V(R)$.

For showing the necessity, let $k \in \{1,2,3\}$, a triangle T be as in *(i)*, and K be an arbitrary body from $\mathcal{X}(B)$. Of course, K is reduced in $\mathcal{M}^2(B)$ and therefore, in view of (3), it is represented by $K = \operatorname{conv} \bigcup_{i \in \{1,2,3\}} I_j'$, where I_j' is a translate of I_j, $j \in \{1,2,3\}$. Consequently,

$$V(K) \ge V(\operatorname{conv} \bigcup_{j \in \{1,2,3\} \setminus \{k\}} I_j') \ge V(T). \tag{4}$$

We notice that $\Delta_B(T) = 1$, which can be verified by (1). But since $K \in \mathcal{X}(B)$, the area of T cannot be strictly less than the area of K. Thus the three quantities involved in (4) are equal. But then the equality $V(K) = V(\operatorname{conv} \bigcup_{j \in \{1,2,3\} \setminus \{k\}} I_j')$ implies that $I_k' \subseteq K$, while the equality $V(\operatorname{conv} \bigcup_{j \in \{1,2,3\} \setminus \{k\}} I_j') = V(T)$ implies that the two segments I_j', $j \in \{1,2,3\} \setminus \{k\}$, intersect. □

The following lemma shows how to transform a Reuleaux triangle K in $\mathcal{M}^2(B)$ with $B := DK$ to a convex body K' such that $B \subseteq B' := DK'$ and K' is a Reuleaux triangle in $\mathcal{M}^2(B')$. The main part of this lemma was proven in [KH53, Lemma 2]. We extend this proof from [KH53] by adding a characterization of the equality case in (5).

Lemma 5. *Let T be an equilateral triangle in a Minkowski plane $\mathcal{M}^2(B)$ having Minkowskian side length one. Suppose $B \ne DT$, i.e., some side I of the affine regular* *hexagon $H := DT$ is not contained in* $\operatorname{bd} B$. *Let q_1, q_2 be the vertices of H not lying in $I \cup (-I)$, and $u \in \mathbb{E}^2 \setminus \{o\}$ be a direction of a line supporting B at q_1. We introduce the line $l := \operatorname{aff}\{q_1, q_2\}$ and the halfplane l^+ bounded by l and containing I. Let us choose an $\alpha \in \mathbb{R}\setminus\{0\}$ such that $I' := I + \alpha u$ intersects $\operatorname{bd} B \cap l^+$, and we introduce the triangle $T' := \operatorname{conv}(\{o\} \cup I')$ and the convex body $B' := \operatorname{conv}(B \cup DT')$, see **Fig. 3**. **Furthermore,** let S_k, $k \in \{0,1,2\}$, be three compact, convex sets with $S_0 \cup S_1 \cup S_2 = l^+ \cap (B \setminus \operatorname{int} H)$ and $q_k \in S_k$ for $k \in \{1,2\}$. Then for the convex bodies $K := W_B(T)$ and $K' := W_{B'}(T')$ we have $\Delta_B(K') = 1$ and*

$$V(K') \leq V(K), \tag{5}$$

*with equality if and only if S_0 is a triangle, and for $k \in \{1,2\}$ the set S_k is a segment or a triangle with one side parallel to u, see **Fig. 4**.*

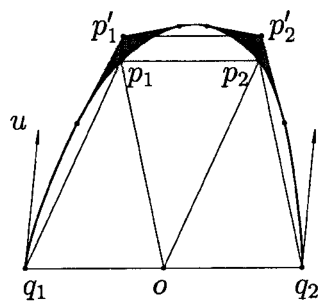

Figure 3

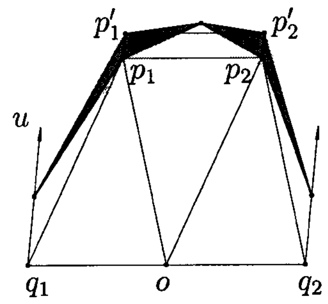

Figure 4

Proof. The equality $\Delta_B(K') = 1$ is a direct consequence of (1). Let us prove (5). The set $l^+ \cap \operatorname{cl}(B' \setminus B)$ is the union of two compact sets P_i, $i \in \{1,2\}$, with disjoint interiors and $P_i \cap S_i \neq \emptyset$. By p_i, $i \in \{1,2\}$, we denote the endpoint of I with $\{p_i\} = P_i \cap I$. Then $p_i' := p_i + \alpha u$ are the endpoints of I'. The segment $J_i := [p_i, p_i']$ splits P_i into two compact sets $P_{i,1}$ and $P_{i,2}$. We have

$$\frac{1}{2}(V(B') - V(B)) = \sum_{i,j=1,2}^{2} V(P_{i,j}) \leq \sum_{i,j=1}^{2} V(T_{i,j}), \tag{6}$$

where $T_{i,j} := \operatorname{conv} P_{i,j}$, $i,j \in \{1,2\}$. Clearly, $T_{i,j}$ is a triangle with side J_i. Let v denote the Euclidean unit vector orthogonal to u. Then we have

$$\sum_{i,j=1}^{2} w_{T_{i,j}}(v) \leq w_B(v) = 2w_P(v), \tag{7}$$

see Fig. 3. Since $V(T_{i,j}) = \frac{1}{2} w_{T_{i,j}}(v)\mu(J_i)$ and $V(P) = w_P(v)\mu(J_i)$, we can rewrite (7) as $\sum_{i,j=1,2}^{2} V(T_{i,j}) \leq V(P)$. The latter together with (6) yields that $\frac{1}{2}(V(B') - V(B)) \leq V(P)$. But the area of P can be expressed by the obvious formula $V(P) = 2(V(T') - V(T))$. Thus, $\frac{1}{2}(V(B') - V(B)) \leq 2(V(T') - V(T))$, which implies

$$V(K') \overset{(2)}{=} \frac{1}{2}V(B') - 2V(T') \leq \frac{1}{2}V(B) - 2V(T) \overset{(2)}{=} V(K)$$

and yields (5). Equality in (5) is attained if and only if it is attained in both (6) and (7). Obviously, the latter is equivalent to the conditions on the sets S_k given in the statement of the lemma, see also Fig. 4. □

The following theorem is the main result of our paper.

Theorem 6. *Let $\mathcal{M}^2(B)$ be an arbitrary Minkowski plane. Then*

(i) the class $\mathcal{X}(B)$ necessarily contains triangles;

(ii) the elements of $\mathcal{X}(B)$ distinct from triangles (if they exist) are necessarily quadrilaterals;

(iii) the class $\mathcal{X}(B)$ contains quadrilaterals if and only if for some boundary point x of B^*, which belongs to an open segment lying in $\operatorname{bd} B^*$, *the faces of* $\operatorname{bd} B^*$ *parallel to $[o,x]$ are strictly longer than $[o,x]$, see Fig. 8.*

Proof. I. First we prove that any convex body from $\mathcal{X}(B)$ is necessarily a triangle or a quadrilateral showing by this *(ii)*. For this we consider an arbitrary planar convex body K with $\Delta_B(K) = 1$ and single out the cases when we can find a convex body K' (which is different in each of these cases) with $\Delta_B(K') \geq 1$ and $V(K') < V(K)$. The latter shows that in such cases K does not belong to $\mathcal{X}(B)$. In the remaining cases K turns out to be a quadrilateral or a triangle.

If K is not a Reuleaux triangle in $\mathcal{M}^2(B_0)$ with $B_0 := DK$, then by the Minkowskian analogue of the Blaschke-Lebesgue Theorem K cannot have minimal area in the class of bodies having constant Minkowskian width one in $\mathcal{M}^2(B_0)$. Hence there exists some planar convex body K' of constant Minkowskian width one in $\mathcal{M}^2(B_0)$ with strictly smaller area than $V(K)$. Using the equality $DK' = DK$ and (1) we get that $\Delta_B(K') = \Delta_B(K) = 1$. From now on let K be a Reuleaux triangle in $\mathcal{M}^2(B_0)$, i.e., $K = W_{B_0}(T)$ for some triangle T equilateral in $\mathcal{M}^2(B_0)$. The set $K \setminus \operatorname{int} T$ is the union of three compact, convex sets S_k, $k \in \{1,2,3\}$.

If for some $k \in \{1,2,3\}$ the set S_k is neither a triangle nor a segment, then by Lemma 5 applied for the Reuleaux triangle K in the Minkowski plane $\mathcal{M}^2(B_0)$ there exists a planar convex body K' with $V(K') < V(K)$ and $\Delta_{B_0}(K') = \Delta_{B_0}(K) = 1$. For $\Delta_B(K')$ we have $\Delta_B(K') \geq \Delta_{B_0}(K') = 1$ (see the remark after Formula (1)).

Now suppose that for any $k \in \{1,2,3\}$ the set S_k is either a triangle or a segment. Then B_0 is a polygon with at least 4 and at most 12 vertices. Further on, we consider the following cases.

Case 1: All three sets S_k are segments. Then K is a triangle.

Case 2: Precisely two sets S_k are segments. Then K is a quadrilateral.

Case 3: Precisely one set S_k, say S_1, is a segment. If the intersection point p of S_2 and S_3 is not a vertex of K, then K is a quadrilateral. Otherwise, let us denote by u the direction of a line supporting K precisely at the point p. Applying Lemma 5 for K (with u in Lemma 5 chosen as above) we come to a convex body K' with $V(K') < V(K)$ and $\Delta_{B_0}(K') = \Delta_{B_0}(K)$. The estimate $\Delta_B(K') \geq 1$ is then obtained in the same way as before.

Case 4: None of the sets S_k is a segment. If some vertex p of DT is also a vertex of DK, then we can choose a direction u of a line supporting DK precisely at p. Then, applying Lemma 5 for K (with u in Lemma 5 chosen as here), we find a convex body K' with $V(K') < V(K)$ and $\Delta_B(K') \geq \Delta_B(K)$. Otherwise (i.e., no vertex of DT is a vertex of DK) the convex body DK is a hexagon or a parallelogram. If under the latter assumption K has more than four vertices, then by Theorem 4 applied for the Minkowski plane $\mathcal{M}^2(B_0)$ we can find a triangle K' with $V(K') < V(K)$ and $\Delta_{B_0}(K') = \Delta_{B_0}(K)$. The latter yields in the usual way that $\Delta_B(K) \geq 1$.

Summarizing we see that the area of a planar convex body K with $\Delta_B(K) = 1$ was not minimized only in the cases when K is a polygon with at most 4 vertices, which yields Part *(ii)* of our theorem.

II. Meanwhile we have shown the following. Let K be a convex body from $\mathcal{X}(B)$. Then K is a Reuleaux triangle in $\mathcal{M}^2(B_0)$ with $B_0 := DK$, i.e., $K = W_{B_0}(T)$ for some triangle T equilateral in $\mathcal{M}^2(B_0)$. Furthermore, for every triangle T as above at least two vertices of T are also vertices of K (to verify this, see Cases 1-4). We can even show that there exists an equilateral triangle T' in $\mathcal{M}^2(B_0)$ with $K = W_{B_0}(T')$ and $\operatorname{ext} T' \subseteq \operatorname{ext} K$. If K is a triangle then the above statement is trivial. Therefore, suppose that K is a quadrilateral and consider a triangle $T := \operatorname{conv}\{p_1, p_2, p_3\}$, $p_i \in \mathcal{M}^2(B_0)$ $(i = 1, 2)$, with $K = W_{B_0}(T)$, $\{p_2, p_3\} \subseteq \operatorname{ext} K$ and $p_1 \notin \operatorname{ext} K$. Let I be the side of K containing p_1. It turns out that I is parallel to $[p_2, p_3]$, since otherwise $[p_1, p_3]$ or $[p_2, p_3]$ would not be an affine diameter of K. But then for any point p from I both $[p, p_2]$ and $[p, p_3]$ are affine diameters of K. Consequently $T' := \operatorname{conv}\{p_1', p_2, p_3\}$, where p_1' is an endpoint of I, is a triangle whose existence we wanted to verify.

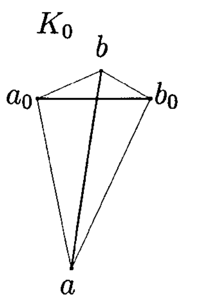

Figure 5

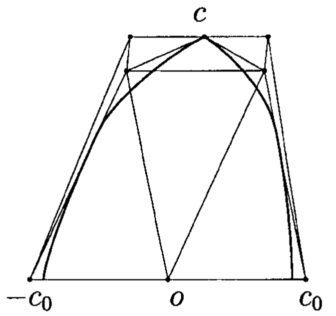

Figure 6

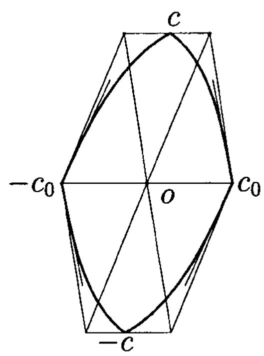

Figure 7

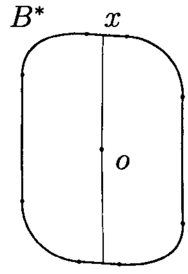

Figure 8

III. Now we will prove *(iii)*. Let us start with the necessity. Suppose $\mathcal{X}(B)$ contains quadrilaterals and K_0 be an arbitrary quadrilateral from $\mathcal{X}(B)$. Let $B_0 := DK_0$, and T be an equilateral triangle in $\mathcal{M}^2(B_0)$ with $W_{B_0}(T) = K_0$ and $\operatorname{ext} T \subseteq \operatorname{ext} K_0$. Let $I_0 := [a_0, b_0]$ and $I := [a, b]$ be diagonals of the quadrilateral K_0 such that $b \notin T$. It turns out that the boundary points $c_0 := b_0 - a_0$ and $c := b - a$ of B_0 belong to B. Arguing by contradiction, we assume first that $c \notin B$. Then some line parallel to I_0 and lying between I_0 and b can cut off a small piece from K_0 containing b such that we get some other convex body of the same Minkowskian thickness and strictly smaller area, a contradiction. Now let us prove that $c_0 \in B$. Suppose the contrary, i.e., $c_0 \notin B$. Let $u \in \mathbb{E}^2 \setminus \{o\}$ be the direction of a line supporting B_0 precisely at c_0. Then we fix an $\alpha \in \mathbb{R} \setminus \{0\}$ such that the segment $I_1 := I_0 + \alpha u$ passes through b. Trivially, $T_1 := \operatorname{conv}(I_1 \cup I)$ is a triangle whose area is equal to the area of K_0. Furthermore, the affine regular hexagon $B_1 := DT_1$ contains B, see Fig. 7. From our construction it is clear that $\operatorname{bd} B_1 \setminus \{\pm c\}$ and B are disjoint. Therefore, taking some segment $I_2 \subsetneq I_1$, whose length is sufficiently close to the length of I_1, we obtain a triangle $\mathrm{conv}(I_2 \cup I)$ with the same Minkowskian thickness as K_0 and strictly smaller area, a contradiction. Thus we see that the boundary points $\pm c_0$ and $\pm c$ of B_0 belong to B, see Fig. 6.

Since c_0 is a boundary point of both B and B_1, we have $r_B(c_0) = r_{B_1}(c_0)$ or, equivalently, $h_{B^*}(c_0) = h_{B_1^*}(c_0)$. The latter means that the supporting hyperplanes of B^* and B_1^* at direction c_0 coincide. Furthermore, taking into account the inclusion $B \subseteq B_1$ and the relation $c_0 \in \mathrm{bd}\, B_1 \cap \mathrm{bd}\, B$ we get the following strict inclusion of normal cones: $N(B_1, c_0) \subsetneq N(B, c_0)$, see Fig. 7. Therefore, applying Proposition 1*(ii)* we come to the strict inclusion $F(B_1^*, c_0) \subsetneq F(B^*, c_0)$. Clearly, $F(B_1^*, c_0)$ is a side of the hexagon B_1^*. Let v be the Euclidean outward normal of B_1 at c. It is not hard to see that $v \in \mathrm{int}\, N(B, c)$. Consequently, by Proposition 1 we have that $x := r_{B_1^*}(v)v$ is a vertex of B_1^* lying in the relative interior of $F(B^*, c)$. Since the hexagon B_1^* is affine regular, the segments $[o, x]$ and $F(B_1^*, c_0)$ have the same length and direction, which yields the necessity in *(iii)*.

IV. Now let us prove the sufficiency in *(iii)*. Suppose that a boundary point x of B^* belongs to an open segment J_1 lying in $\mathrm{bd}\, B^*$ and for the Euclidean normal $c_0 \in \mathrm{bd}\, B$ of the vector x the segment $F(B^*, c_0)$ is strictly longer than $[o, x]$. Let J_2 be a segment lying in the relative interior of $F(B^*, c_0)$ and having the same length as $[o, x]$ and $c \in \mathrm{bd}\, B$ be the Euclidean normal of J_1. We consider the affine regular hexagon B_0 being the dual body of $\mathrm{conv}(\{x, -x\} \cup J_2 \cup (-J_2))$. By assumptions we get that the points $\pm c_0$ are vertices of B_0, the vector x is a side normal of B_0, and the boundaries of B and B_0 intersect precisely at points $\pm c_0, \pm c$. Furthermore, using Proposition 1 it is not hard to derive the following relations:

$$N(B_0, c_0) \subseteq \mathrm{int}\, N(B, c_0) \cup \{o\}, \tag{8}$$

$$x \in \mathrm{int}\, N(B, c). \tag{9}$$

We consider a triangle T_0 with $DT_0 = B_0$. Let I_0 be the side of T_0 with the Euclidean normal x, and a be the vertex of T_0 not lying in I_1. Further on, let b be a point from I_1 such that the segment $I := [a, b]$ has the same direction and the same length as $[o, c]$. We also choose a direction u of a line supporting B_0 precisely at c_0. In view of (8) and (9) it is possible to find a segment the segment $I_1 := I_0 + \alpha u$, $\alpha \in \mathbb{R} \setminus \{o\}$, such that the difference body B_1 of the quadrilateral $K_1 := \mathrm{conv}(I_1 \cup I)$ contains B. It turns out that that the quadrilateral K_1 belongs to $\mathcal{X}(B)$, since its area is not larger than the area of any Minkowskian reduced body $K \in \mathcal{M}^2(B)$ having Minkowskian thickness one. Indeed, let I_1' and I' be the affine diameters of K parallel to I_1 and I, respectively. Of course, $\mu_B(I_0) \le \mu_B(I_0')$ and $\mu_B(J) \le \mu_B(J')$. But then $V(K_0) \le V(\mathrm{conv}(I_0' \cup J')) \le V(K')$, and the sufficiency is verified.

V. It turns out that *(i)* follows directly from the derivations given above. Indeed, *(ii)* states that all elements of $\mathcal{X}(B)$ distinct from triangles are necessarily quadrilaterals. But in the proof of *(iii)*, taking a quadrilateral K_0 from $\mathcal{X}(B)$, we construct a triangle K_1 which also belongs to $\mathcal{X}(B)$. Consequently, $\mathcal{X}(B)$ cannot consist only of quadrilaterals and has to contain some triangles. □

Further on, we wish to enumerate several simple properties of B which imply that $\mathcal{X}(B)$ consists only of triangles but which are not equivalent to the condition that all elements $\mathcal{X}(B)$ are triangles. Namely, using Theorems 6*(iii)* and 3 we get the following. Let a planar convex body B, which is centered at the origin, possess at least one of the following properties: *(i)* $\mathrm{conv}_{B^*}(B^*) = B^*$; *(ii)* B has at most two non-smooth boundary points; *(iii)* $\mathrm{bd}\, B$ is a Radon curve. Then $\mathcal{X}(B)$ consists only of triangles.

As a consequence of Theorem 2 and the previous statement we obtain that *if $\mathcal{M}^2(B)$ is the Euclidean plane (i.e., B is an ellipse) then $\mathcal{X}(B)$ is precisely the class of equilateral triangles in $\mathcal{M}^2(B)$ with heights of length one.* It is, however, an open question whether the converse implication holds.

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