CAUCHY SINGULAR INTEGRAL OPERATORS IN WEIGHTED SPACES OF CONTINUOUS FUNCTIONS

U. LUTHER

Dedicated to Professor P. Junghanns on the occasion of his 50th birthday

We study the Cauchy singular integral operator SwI on $(-1,1)$, where $|w|$ is a generalized Jacobi weight. This operator is considered in pairs of weighted spaces of continuous functions, where the weights u and v are generalized Jacobi weights with nonnegative exponents such that $|w| = u/v$. We introduce a certain polynomial approximation space which is well appropriated to serve as domain of definition of SwI. A description of this space in terms of smoothness properties shows that it can be viewed as a limit case of weighted Besov spaces of continuous functions. We use our results to characterize those of the operators $awI + SbwI$ and $\varrho^{-1}(aw\varrho I + bSw\varrho I)$, $\varrho^{-1} \in b^{-1}\Pi$, which act in certain pairs of Ditzian-Totik type Besov spaces.

1 Introduction

In many mathematical models the Cauchy singular integral operator S appears. It is well-known that S is bounded in Hölder-Zygmund spaces of functions defined on a closed (and sufficiently smooth) curve (see [15], Chapter 2, §6, and [17], Section 6.25). But in the present paper we treat the case of an open curve in which one has to be careful with the behaviour of Sf at the endpoints of the curve. For the sake of simplicity we consider the interval $(-1,1)$, i.e., Sf is defined by

$$(Sf)(x) = \int_{-1}^{1} \frac{f(t)}{t-x}\, dt := \lim_{\varepsilon \downarrow 0} \left(\int_{-1}^{x-\varepsilon} \frac{f(t)}{t-x}\, dt + \int_{x+\varepsilon}^{1} \frac{f(t)}{t-x}\, dt \right), \quad x \in (-1,1)\,.$$

If $f \in \mathbf{L}^p(-1,1)$ with some $p > 1$, then $(Sf)(x)$ exists for almost every $x \in (-1,1)$ ([15], Chapter 2, §2). But in practice one often wants to know exactly for which points x the value $(Sf)(x)$ is defined. This requires knowledge about the continuity and the singularities of f.

A good candidate of a function space which represents the continuity points as well as the singularities of its elements is the so-called weighted space of continuous functions

$$\mathbf{C}_u := \{f : \operatorname{supp} u \to \mathbb{C} \ \text{ such that } fu \in \mathbf{C} := \mathbf{C}[-1,1]\,\}\,,$$

where $u : [-1,1] \to \mathbb{R}$ is a given continuous function (the weight) with a support $\operatorname{supp} u := \{x \in [-1,1] : u(x) \neq 0\}$ which is dense in $[-1,1]$. By $fu \in \mathbf{C}$ we mean that fu possesses a continuous extension on $[-1,1]$ (which is also denoted by fu). This implies that the elements f of $\mathbf{C}_u$ are continuous on $\operatorname{supp} u$ and that f may have singularities in the zeros of u. It is clear that $\mathbf{C}_u$, endowed with the norm

$$\|f\|_u := \|fu\| \qquad (\,\|g\| = \max\{|g(x)| : x \in [-1,1]\}\,)\,,$$

is a Banach space which is isometrically isomorphic to $\mathbf{C}$.

From a certain point of view, the condition $fu \in \mathbf{C}$ in the definition of $\mathbf{C}_u$ is too restrictive if u possesses zeros inside $(-1,1)$: Why should we consider weighted spaces which do not only depend on the absolute value of the weight u, but also on sign changes of u? (For example, for $u(x) = x$ and $u(x) = |x|$, we get different spaces $\mathbf{C}_x$ and $\mathbf{C}_{|x|}$.) For this reason, we also introduce the following weighted space of piecewise continuous functions (by piecewise continuous we mean continuous with possible exception of finitely many jumps), which makes sense if u has only finitely many zeros:

$$\mathbf{PC}_u := \left\{ \begin{array}{ll} f : \operatorname{supp} u \to \mathbb{C} : & fu \text{ is piecewise continuous on } [-1,1] \\ & \text{with jumps only in the zeros of } u \end{array} \right\}, \quad \text{endowed with}$$

$$\|f\|_u := \sup\{\,|(fu)(x)| : x \in \operatorname{supp} u\,\}\,.$$

Obviously, this is a Banach space which does only depend on $|u|$.

Unfortunately, the operator S is not bounded in $\mathbf{C}_u$ or $\mathbf{PC}_u$ (however we choose u). For example, in case $u \equiv 1$, the image of $f \equiv 1$ is an unbounded function:

$$\int_{-1}^{1} \frac{1}{t-x}\,dt = \ln\frac{1-x}{1+x}\,. \tag{1.1}$$

So we have to restrict S onto a subspace of $\mathbf{C}_u$ to ensure that the images belong to $\mathbf{PC}_u$. For example, in case of a power weight $u \in \mathbf{C} \cap \{u : u(\pm 1) = 0\}$ (see below) with $u^{-1} \in \mathbf{L}^1$, S is an endomorphism of the space of all f for which fu is Hölder continuous and vanishes in all zeros of u ([9], Section 9.10). In the present paper we will give a much bigger subspace of $\mathbf{C}_u$ which may serve as domain of definition of S, in the sense that S is a bounded operator from this space into $\mathbf{PC}_u$ (or $\mathbf{PC}_{\widetilde{u}}$, $\widetilde{u}$ some modified weight, if $u(\pm 1) \neq 0$). The definition and the properties of this space are given in Section 2 and the corresponding mapping properties of S or, more general, SwI (w: some weight) are proved in Section 3. Of course, we are not able to deal with arbitrary weights w and u: We will consider so-called power weights, i.e., w (or only $|w|$) and u are weights of the form

$$u(x) = \prod_{i=1}^{N} |x - x_i|^{\alpha_i} \quad \text{with} \quad -1 \le x_1 < \ldots < x_N \le 1 \quad \text{and} \quad \alpha_i \neq 0\,.$$

For $N = 0$ this means $u \equiv 1$ in agreement with the conventions $\prod_{i\in\emptyset} . = 1$ and $\sum_{i\in\emptyset} . = 0$. (In this sense, the weight $u \equiv 1$ is also admitted if we speak, for example, about power weights with positive exponents.) If $x_1 = -1$ and $x_2 = 1$, then we also use the notation

$$v^{\alpha,\beta}(x) := (1-x)^{\alpha}(1+x)^{\beta}\,.$$

In case of a Jacobi weight, i.e., $w = v^{\alpha,\beta}$ with $\alpha, \beta > -1$, it is known that the operator SwI maps a certain subspace of $\mathbf{C}_u$ (namely, the space $\mathbf{C}_u^0$ which is defined in Section 2) into $\mathbf{C}_v$, if u and v are Jacobi weights with nonnegative exponents such that

$$w = \frac{u}{v}\,, \quad v^{-1} \in \mathbf{L}^1(-1,1)\,, \quad \text{and} \quad (uv)(-1) = (uv)(1) = 0\,.$$

This deep result is proved in [12]. In the present paper we will generalize this result to the case of power weights $|w| = u/v$ (and even generalized Jacobi weights). As consequence, we will obtain criteria which ensure that operators of the type $awI + SbwI$ and $\varrho^{-1}(aw\varrho I + bSw\varrho I)$, $\varrho^{-1} \in b^{-1}\Pi$, Π: set of all polynomials, act between certain weighted (Ditzian-Totik type) Besov spaces of continuous functions. These spaces can be defined in terms of polynomial best approximation errors of their elements. For this reason, the approximation-theoretical definition of the space $\mathbf{C}_u^0$ from the next section will be very useful in the second part of the paper. But first we have to consider SwI on $\mathbf{C}_u^0$. It turns out that this can be done with the help of a nice characterization of $\mathbf{C}_u^0$ in terms of smoothness properties of its elements.

2 The space $\mathbf{C}_u^0$

In all what follows, $u(x) = \prod_{i=1}^{N} |x - x_i|^{\alpha_i}$ is a fixed power weight with exponents

$$\alpha_i > 0 \quad \text{for all} \quad i = 1, \ldots, N\,.$$

Definition 2.1 *For $f \in \mathbf{C}_u$, we define the weighted polynomial best approximation errors*

$$E_n^u(f) := \inf_{P_n \in \Pi_n} \|f - P_n\|_u \quad (n = 0, 1, \ldots)\,, \quad \textit{where} \quad \Pi_n := \operatorname{span}\{x^k : k = 0, \ldots, n-1\}\,.$$

(Especially, $E_0^u(f) = \|f\|_u$.) The space $\mathbf{C}_u^0$ is given by

$$\mathbf{C}_u^0 := \left\{ f \in \mathbf{C}_u : \|f\|_{u,0} := \sum_{n=0}^{\infty} \frac{E_n^u(f)}{n+1} < \infty \right\}.$$

In case $u \equiv 1$ we write shortly $E_n(f)$ and $\mathbf{C}^0$ instead of $E_n^u(f)$ and $\mathbf{C}_u^0$.

$\mathbf{C}_u^0$ is a so-called approximation space, i.e., a space of the type $A(X, S; \{X_n\}) = \{f \in X : \{E(f; X_n)\} \in S\}$ ($E(f; X_n)$: best approximation errors w.r.t. to nested subspaces X_n of a Banach space X; here: $X = \mathbf{C}_u$, $X_n = \Pi_n$). For $\mathbf{C}_u^0$, the sequence space $S = \mathbf{l}^1(\{1/(n+1)\}) := \{\{a_n\}_{n=0}^{\infty} : \{a_n/(n+1)\} \in \mathbf{l}^1\}$ does not fit into the classical concept of approximation spaces, in which $S = \mathbf{l}^q(\{(n+1)^{s-(1/q)}\})$ with $s > 0$ and $0 < q \leq \infty$ ([16]). Especially (in case of a Ditzian-Totik weight $u \in J_\infty^*$; see [7]), $\mathbf{C}_u^0$ is different from the weighted Besov spaces $\mathbf{B}_{s,1}^{\infty}(\varphi, u) = A(\mathbf{C}_u, \mathbf{l}^1(\{(n+1)^{s-1}\}); \{\Pi_n\})$ (see [8]), since it appears as the "limit case" $s \to 0$ of these spaces. (This is the reason for using the index 0 in the notation $\mathbf{C}_u^0$.) However, from the general theory of approximation spaces $A(X, S; \{X_n\})$ (which is well-known today; see [2], [1], [3], [4], [10]) one can conclude a lot of nice properties of the space $\mathbf{C}_u^0$. For example, the following proposition holds true. (An easy proof can be found in [12], Lemma 4.1.)

Proposition 2.2 *$\mathbf{C}_u^0$ is a Banach space and the set $\Pi = \bigcup \Pi_n$ of all algebraic polynomials is dense in $\mathbf{C}_u^0$.*

Of course, the above definition of $\mathbf{C}_u^0$ is only of theoretical interest as long as we do not have a practical criterion to check whether a function f belongs to $\mathbf{C}_u^0$ or not. But it turns out that there exists a surprising and easy smoothness property which characterizes the elements of $\mathbf{C}_u^0$. The present section is devoted to this characterization.

Let us first introduce some notation: In the sequel we shall denote by c positive constants that may have different values at different places. By $c \neq c(n, f, \dots)$ we will indicate that c is independent of $n, f, \dots$. If A and B are two nonnegative quantities, then $A \sim B$ means that there exists some constant $c > 0$, independent of the variables under consideration, such that $c^{-1}A \leq B \leq cA$.

In the proof of the following lemma we need the Schur type inequality

$$\|P_n\| \leq c\, n^{\gamma} \|P_n\|_u\,, \quad P_n \in \Pi_n \quad (\gamma := \max_i 2^{[|x_i|]}\alpha_i\,,\ c \neq c(n, P_n)) \tag{2.1}$$

([14], estimate (7.33)), which is also of own interest. (By $[x]$ we denote the integer part of x.)

Lemma 2.3 *There are constants $c > 0$ and $k \in \mathbb{N}$ ($c \neq c(n, f)$, $k \neq k(n, f)$) such that, for $f \in \mathbf{C}_u$ and $n \in \mathbb{N}$,*

$$E_{n^k}(fu) \leq c \left[E_n^u(f) + \frac{\|f\|_u}{n} \right] \quad \textit{and}$$

$$E_{n^k}^u(f) \leq c \left[E_n(fu) + \frac{\|f\|_u}{n} \right] \quad \text{if } (fu)(x_i) = 0 \text{ for all } i\,.$$

Proof. The proof of the first assertion is left to the reader. (Use that the Hölder continuity of u yields $E_m(u) \leq c\, m^{-\mu}$ and that (2.1) implies $E_{n+n^l-1}(fu) \leq \|(f - f_n)u\| + c\, n^{\gamma} \|f_n\|_u \|u - u_n\|$ for all $f_n \in \Pi_n$ and all $u_n \in \Pi_{n^l}$.)

Now, let $N > 0$ (for $N = 0$ we have nothing to prove), let us fix some $\xi = x_j$ and set $\alpha = \alpha_j$. Then we may consider the power weight $v(x) := u(x)/|x-\xi|^\alpha$. The second assertion is proved if we have shown that, with some constant k,

$$E^u_{n^k}(f) \le c\left[E^v_n(fu/v) + \frac{\|f\|_u}{n}\right]. \tag{2.2}$$

Indeed, we may apply this estimate with n^l instead of n (l large enough) and for the term $E^v_{n^l}(fu/v)$ which now appears on the right hand side we use again the above estimate, but with v instead of u and another x_i. In this way it follows, with $w(x) = v(x)/|x - x_i|^{\alpha_i}$,

$$E^u_{n^{kl}}(f) \le c\left[E^w_n(fu/w) + \frac{\|f\|_u}{n}\right].$$

Repeating this procedure we finally get the assertion. Now we prove (2.2). Set $g(x) = f(x)|x - \xi|^\alpha$ and define

$$\widetilde{P}_n(x) = P_n(x) - P_n(\xi)\,, \quad \text{where} \quad P_n \in \Pi_n \ \text{ with } \ \|g - P_n\|_v = E^v_n(g)\,.$$

Then we have $|P_n(\xi)| = |P_n(\xi) - g(\xi)| = C|(P_n(\xi) - g(\xi))v(\xi)|$ ($C = 1/v(\xi)$) and, consequently, $\|P_n(\xi)\, v\| \le C\|v\|\, E^v_n(g) = c\, E^v_n(g)$. Hence,

$$\|g - \widetilde{P}_n\|_v \le c\, E^v_n(g) \quad \text{and} \quad \widetilde{P}_n(\xi) = 0\,.$$

Especially, $Q_n(x) := (x-\xi)^{-1}\widetilde{P}_n(x)$ is a polynomial of degree less than $n-1$ and, by (2.1),

$$\begin{aligned} \|Q_n\| &\le c\, n^{\max\{\gamma,2\}}\|(\,.-\xi)v\, Q_n\| = c\, n^{\max\{\gamma,2\}}\|\widetilde{P}_n\|_v \\ &\le c\, n^{\max\{\gamma,2\}}\|g\|_v = c\, n^{\max\{\gamma,2\}}\|f\|_u\,. \end{aligned}$$

Moreover, we can write

$$\widetilde{P}_n\, v = Q_n\, r\, u \quad \text{with} \quad r(x) = |x-\xi|^{1-\alpha}\operatorname{sign}(x-\xi)\,.$$

Let us suppose, for a moment, that $\alpha < 1$. Then r is Hölder continuous with exponent $\mu := 1-\alpha$. Hence, $E_n(r) \le c\, n^{-\mu}$ and this implies

$$E^u_n(r) \le c\, n^{-\mu}\,. \tag{2.3}$$

Now we choose some natural number l with $l \ge (\max\{2,\gamma\}+1)/\mu$ and some $R_n \in \Pi_{n^l}$ with $\|r - R_n\|_u = E^u_{n^l}(r)$. Then it follows

$$\begin{aligned} E^u_{n^{l+1}}(f) &\le E^u_{n^l+n-2}(f) \le \|(f - Q_nR_n)u\| \\ &\le \|(g - \widetilde{P}_n)v\| + \|Q_n(r - R_n)u\| \\ &\le c\, E^v_n(g) + c\,\frac{n^{\max\{2,\gamma\}}\|f\|_u}{n^{l\mu}} \le c\, E^v_n(g) + c\,\frac{\|f\|_u}{n} \end{aligned}$$

and (2.2) is proved in case $\alpha < 1$. This implies that the lemma is proved if $\alpha_j < 1$ for all j. Now we consider the case $\alpha_j < 2$ for all j. Then it turns out that the proof is the same with one exception: For those $\alpha = \alpha_j$, for which $\alpha \in [1, 2)$, the estimate (2.3) has to be proved in a different way. For this aim, we choose some $\eta < 1$ such that $\alpha \in [1, 1 + \eta)$. Then the exponent of the weight $\varrho(x) = |x - \xi|^{\alpha-\eta}$ lies in $(0, 1)$ and $r(x)\rho(x) = |x - \xi|^{1-\eta}\mathrm{sign}\,(x - \xi)$ is Hölder continuous with exponent $1 - \eta$ and vanishes in ξ. Thus, we can use what we have already proved:

$$E^{\rho}_{n^k}(r) \le c \left[E_n(r\rho) + \frac{\|r\rho\|}{n} \right] \le \frac{c}{n^{1-\eta}} \,.$$

This implies $E^u_n(r) \le c\, E^{\rho}_n(r) \le c\, n^{-\mu}$ with $\mu = (1-\eta)/k$. Similarly one can prove the lemma in case $\max \alpha_j < 3$, then in case $\max \alpha_j < 4$, and so on (induction).

■

Remark 2.4 *The exact value of the constant γ in the Schur type inequality (2.1) is not needed in the proof of Lemma 2.3. Therefore, it is worth to mention that, for bigger values of γ, (2.1) is neither surprising nor new. For example, the well-known estimate*

$$\|P_n\| \le c\, \|P_n\|_{\mathbf{C}[-\sqrt{1-n^{-2}}, \sqrt{1-n^{-2}}]} \le c\, n^{2\max\{\alpha,\beta\}} \|v^{\alpha,\beta} P_n\| \,, \quad P_n \in \Pi_{n+1} \,, \ n \in \mathbb{N}$$

($\alpha, \beta \ge 0$, $c \ne c(n, P_n)$; see, e.g., [6], inequality (2.2) of Chapter 8) can be transformed onto $[x_{i-1}, x_i]$ which yields (2.1) with $\gamma = \max_i 2\alpha_i$.

Corollary 2.5 *The closure* $\mathrm{clos}_u \Pi$ *of the set of all polynomials in the space* $\mathbf{C}_u$ *is given by*

$$\mathrm{clos}_u \Pi = \{ f \in \mathbf{C}_u : (fu)(x_i) = 0 \textit{ for all } i \} \,.$$

Proof. In case $u \equiv 1$ this is the well-known theorem of Weierstraß. In case $u \not\equiv 1$ we can use Lemma 2.3: If $(fu)(x_i) = 0$ for all i, then $E^u_{n^k}(f)$ tends to zero (since $\lim_{n\to\infty} E_n(fu) = 0$ in view of Weierstraß' theorem), i.e., $f \in \mathrm{clos}_u \Pi$. On the other hand, $f \in \mathrm{clos}_u \Pi$ means that fu is the uniform limit of weighted polynomials $g_k = P_k u$ ($P_k \in \Pi$). This implies $(fu)(x_i) = 0$ for all i, since $g_k(x_i) = 0$ for all i.

■

Now we are able to prove the main result of the present section, which asserts that the elements f of $\mathbf{C}^0_u$ can be characterized with the help of the classical modulus of continuity of $g = fu$. We recall that this modulus is defined by

$$\omega(g, h) := \sup_{x,y \in [-1,1],\, |x-y| \le h} |g(x) - g(y)| \,, \quad h > 0 \,.$$

Theorem 2.6 *$f \in \mathbf{C}_u$ belongs to $\mathbf{C}_u^0$ if and only if*

$$(2.4) \qquad (fu)(x_i) = 0 \text{ for all } i \quad \text{and} \quad \int_0^1 \omega(fu,h)\,\frac{dh}{h} < \infty\,.$$

Moreover, the expression $\|f\|_{u,0}^* := \|f\|_u + \int_0^1 \omega(fu,h)\,\frac{dh}{h}$ defines an equivalent norm in $\mathbf{C}_u^0$.

Proof. The proof of the norm properties of $\|\,.\,\|_{u,0}^*$ is left to the reader. We need the well-known equivalence

$$(2.5) \qquad \sum_{n=1}^{\infty} \frac{E(n)}{n} \sim \sum_{j=0}^{\infty} E(2^j) \quad \text{for all decreasing } E : [1,\infty) \to [0,\infty)$$

(which follows from $\sum_{n=1}^{\infty} \ldots = \sum_{j=0}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} \ldots = \sum_{j=1}^{\infty} \sum_{n=2^{j-1}}^{2^j-1} \ldots$). (2.5) implies

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{E(n)}{n} &\sim E(1) + \sum_{j=1}^{\infty} E(2^j) \int_{2^{-j}}^{2^{-j+1}} \frac{dh}{h} \\
&\le E(1) + \sum_{j=1}^{\infty} \int_{2^{-j}}^{2^{-j+1}} E(h^{-1})\,\frac{dh}{h} = E(1) + \int_0^1 E(h^{-1})\,\frac{dh}{h} \qquad \text{and} \\
\sum_{n=1}^{\infty} \frac{E(n)}{n} &\sim \sum_{j=0}^{\infty} E(2^j) \int_{2^{-j-1}}^{2^{-j}} \frac{dh}{h} \ge \sum_{j=0}^{\infty} \int_{2^{-j-1}}^{2^{-j}} E(h^{-1})\,\frac{dh}{h} = \int_0^1 E(h^{-1})\,\frac{dh}{h}\,.
\end{aligned}$$

The substitution $h = t^\theta$ shows that the last integral can be replaced by $\int_0^1 E(t^{-\theta})\frac{dt}{t}$, where θ is an arbitrary fixed positive number. So it follows

$$(2.6) \qquad \sum_{n=1}^{\infty} \frac{E(n)}{n} \sim E(1) + \int_0^1 E(t^{-\theta})\,\frac{dt}{t} \quad \text{for all decreasing } E : [1,\infty) \to [0,\infty)\,.$$

Now, let k be an arbitrary fixed natural number and let $f \in \mathbf{C}_u$. If we set $\theta = 1/k$ and $E(x) = E_{[x^k]}^u(f)$, then it follows

$$(2.7) \qquad \sum_{n=1}^{\infty} \frac{E_{n^k}^u(f)}{n} \sim E_1^u(f) + \int_0^1 E_{[t^{-1}]}^u(f)\,\frac{dt}{t} \quad \text{for all } f \in \mathbf{C}_u\,.$$

The right hand side does not depend on k. Consequently, the space $\mathbf{C}_u^0$ does not change (in the sense of equivalent norms) if we define its norm with $E_{n^k}^u(f)$ instead of $E_n^u(f)$. Moreover, all elements f of $\mathbf{C}_u^0$ satisfy $(fu)(x_i) = 0$ $(i = 1, \ldots, N)$, since $\|f\|_{u,0} < \infty$ implies $\inf_n E_n^u(f) = \lim_{n\to\infty} E_n^u(f) = 0$, i.e., $f \in \operatorname{clos}_u \Pi$ (see Corollary 2.5). Together with Lemma 2.3 it follows

$$(2.8) \qquad f \in \mathbf{C}_u^0 \text{ if and only if } fu \in \mathbf{C}^0 \text{ and } (fu)(x_i) = 0 \text{ for all } i\,,$$

where the corresponding norms are equivalent. So it remains to consider the space $\mathbf{C}^0$, i.e., to prove the assertion for $u \equiv 1$. For this aim, let $f \in \mathbf{C}$ and $P_n \in \Pi_n$ such that $E_n(f) = \|f - P_n\|$. From Markov's inequality it follows, for all $n \in \mathbb{N}$ and all $x, t \in [-1, 1]$,

$$\begin{aligned} |f(x) - f(t)| &\le |f(x) - P_n(x)| + |P_n(x) - P_n(t)| + |P_n(t) - f(t)| \\ &\le 2E_n(f) + \|P_n'\|\,|x-t| \le 2E_n(f) + n^2\|P_n\|\,|x-t| \\ &\le 2E_n(f) + 2n^2\|f\|\,|x-t|\,. \end{aligned}$$

For $|x - t| \le 1$ and $n = \left[|x-t|^{-1/4}\right]$ we obtain

$$|f(x) - f(t)| \le 2E_{[|x-t|^{-1/4}]}(f) + 2\|f\|\,|x-t|^{1/2}\,.$$

Consequently, $\omega(f,h) \le 2E_{[h^{-1/4}]}(f) + 2\|f\|\,h^{1/2}$ for all $h \in (0,1]$. Together with (2.6) (applied with $E(x) = E_{[x]}(f)$ and $\theta = 1/4$) it follows

$$\int_0^1 \omega(f,h)\,\frac{dh}{h} \le 2\int_0^1 E_{[h^{-1/4}]}(f)\,\frac{dh}{h} + 4\|f\| \le c\,\|f\|_0\,. \tag{2.9}$$

Thus, the integral on the left hand side is finite if f belongs to $\mathbf{C}^0$. The counterdirection follows from (2.6) and Jackson's theorem:

$$\sum_{n=1}^{\infty} \frac{E_n(f)}{n} \le c\sum_{n=1}^{\infty} \frac{\omega(f,n^{-1})}{n} \sim \omega(f,1) + \int_0^1 \omega(f,h)\,\frac{dh}{h}\,. \tag{2.10}$$

Obviously, (2.9) and (2.10) imply $\|f\|_0 \sim \|f\| + \int_0^1 \omega(f,h)\frac{dh}{h}$.

■

The following corollary shows that, in many cases, the approximation space $\mathbf{C}_u^0$ does not change if we approximate with weighted polynomials instead of usual polynomials.

Corollary 2.7 *Let v be a power weight with positive exponents and set*

$$\widetilde{\mathbf{C}}_u^0 := \left\{ f \in \mathbf{C}_u : \|f\|_{u,0}^{\sim} := \sum_{n=0}^{\infty} \frac{E^u(f;(v/u)\Pi_n)}{n+1} < \infty \right\},$$

where $E^u(f;(v/u)\Pi_n) = \inf_{P_n \in \Pi_n} \|f - (v/u)P_n\|_u$. Then we have

$$\mathbf{C}_u^0 \cap \operatorname{clos}_u (v/u)\Pi = \widetilde{\mathbf{C}}_u^0 \cap \operatorname{clos}_u \Pi$$

in the sense of equivalent norms. (Remark that, in view of Corollary 2.5, $\operatorname{clos}_u (v/u)\Pi = \{f \in \mathbf{C}_u : fu = 0$ in the zeros of $v\}$ and $\operatorname{clos}_u \Pi = \{f \in \mathbf{C}_u : fu = 0$ in the zeros of $u\}$.) Especially, $\mathbf{C}_u^0 = \widetilde{\mathbf{C}}_u^0$ if u and v have the same zeros.

Proof. Obviously, $E^u(f;(v/u)\Pi_n) = E_n^v(fu/v)$, i.e., $\widetilde{\mathbf{C}}_u^0 = \{f : fu/v \in \mathbf{C}_v^0\}$ and $\|f\|_{u,0}^{\sim} = \|fu/v\|_{v,0}$. Now the assertion follows from (2.8).

■

Remark 2.8 *Theorem 2.6 says that $\mathbf{C}_u^0$ can be characterized with the help of $\mathbf{C}^0$ (i.e., (2.8) holds true) and that $\mathbf{C}^0$ is nothing else than the well-known Dini space of all functions $f \in \mathbf{C}$, those moduli of continuity are integrable w.r.t. dh/h. This is somewhat surprising, since usually the classical modulus $\omega(f,h)$ is not appropriated to characterize equivalently the behaviour of the errors of best approximation by algebraic polynomials. For example, the behaviour $E_n(f) = O(n^{-s})$ ($s > 0$ fixed) cannot be formulated in terms of $\omega(f,h)$. But in Theorem 2.6 we do not consider such a classical order $E_n(f) = O(n^{-s})$: The condition $\sum_{n=1}^{\infty} E_n(f)/n < \infty$ is much weaker and does not change if we replace $E_n(f)$ by $E_{n^k}(f)$ ($k \in \mathbb{N}$ fixed). This last fact is not given for classical behaviours, but it is used in the proof of the theorem.*

At the end of this section we want to point out that the classical modulus of continuity is not the only modulus which is appropriated to characterize the elements of $\mathbf{C}_u^0$. For the sake of simplicity, we restrict ourselves to the unweighted space $\mathbf{C}^0$. This is justified by (2.8).

It is well-known that, for $\varphi(x) = \sqrt{1-x^2}$, the φ-modulus of smoothness

$$\omega_\varphi^r(f,t) = \sup_{0<h\le t} \|\Delta_{h\varphi}^r f\|_{\mathbf{L}^\infty(D(\Delta_{h\varphi}^r f))}\,, \quad (\Delta_{h\varphi}^r f)(x) = \sum_{k=0}^{r} (-1)^k \begin{pmatrix} r \\ k \end{pmatrix} f\Big(x + \Big(\frac{r}{2} - k\Big)\, h\varphi(x)\Big)$$

($r \in \mathbb{N}$, $D(\Delta_{h\varphi}^r f) = \{x \in (-1,1) : x \pm \frac{rh}{2}\varphi(x) \in (-1,1)\}$) is well appropriated to characterize the behaviour of polynomial best approximation errors. So it is not surprising that the approximation space $\mathbf{C}^0$ can be described in terms of $\omega_\varphi^r(f,t)$:

$$\|f\|_0 \sim \|f\| + \int_0^1 \omega_\varphi^r(f,t)\,\frac{dt}{t} \quad \text{for all } f \in \mathbf{C} \tag{2.11}$$

([12], Theorem 2.3). This can be viewed as a corollary of Theorem 2.6: The right side of (2.11) can be estimated by the corresponding expression with $\omega(f,t)$. This is a consequence of $\omega_\varphi^r(f,t) \le c\,\omega_\varphi^1(f,t)$, $t \le t_0$ ([7], Theorem 4.1.3) and $\omega_\varphi^1(f,t) \le \omega(f,t)$ (since $\Delta_{h\varphi}^1 f(x) \le \omega(f,h)$). The other part of the equivalence (2.11) follows from the Jackson type theorem

$$E_n(f) \le c\,\omega_\varphi^r(f,n^{-1}) \quad (n \ge n_0\,,\ c \ne c(n,f)) \tag{2.12}$$

([7], Theorem 7.2.1) and (2.6). With the same arguments one can show that in (2.11) the modulus $\omega_\varphi^r(f,t)$ can be replaced by any other modulus of smoothness which satisfies a Jackson type theorem (where even $E_{n^k}(f)$, $k \in \mathbb{N}$ fixed, may appear on the left hand side; see (2.7)) and which is weaker than $\omega(f,t)$.

We finish this section with a short consideration of more general weights u:

Remark 2.9 *If $u(x) = B(x)\prod_{i=1}^N |x-x_i|^{\alpha_i}$ ($x_i \in [-1,1]$, $\alpha_i > 0$) with some non-vanishing real-valued function $B \in \bigcap_i \mathbf{C}^0[x_{i-1}, x_i]$, i.e.,*

$$B = B_0\,\chi_{[-1,x_1)} + B_1\,\chi_{[x_1,x_2)} + \ldots + B_N\,\chi_{[x_N,1]} \quad \text{with} \quad B_i \in \mathbf{C}^0 \quad \text{and} \quad B_i \neq 0 \ \text{on} \ [-1,1]\,,$$

then we have (in the sense of equivalent norms)

$$\mathbf{C}_u^0 = \mathbf{C}_{u/B}^0 = \{f \in \mathbf{C}_u : fu \in \mathbf{C}^0 \ \text{and} \ (fu)(x_i) = 0 \ \text{for all} \ i\}\,. \tag{2.13}$$

If only $u(x) = B(x)r(x)$ with some Hölder continuous function $r : [-1,1] \to \mathbb{R}$ satisfying $r(x) \geq c\prod_{i=1}^N |x-x_i|^{\alpha_i}$ ($0 < c \neq c(x)$) and $r(x_i) = 0$ for all i, then we have at least

$$\mathbf{C}_u^0 = \mathbf{C}_r^0 \hookrightarrow \{f \in \mathbf{C}_u : fu \in \mathbf{C}^0 \ \text{and} \ (fu)(x_i) = 0 \ \text{for all} \ i\}\,. \tag{2.14}$$

("$\hookrightarrow$" means continuous embedding.)

Proof. Obviously, $u = B\,r$ is a continuous function with zeros in the points x_i and the set $\{f \in \mathbf{C}_u : (fu)(x_i) = 0 \text{ for all } i\}$ is equal to $\{f \in \mathbf{C}_r : (fu)(x_i) = 0 \text{ for all } i\}$. Only elements of this set can belong to $\mathbf{C}_u^0$ and $\mathbf{C}_r^0$, respectively, since $f \in \mathbf{C}_u^0$ ($f \in \mathbf{C}_r^0$) implies $E_n^u(f) \to 0$ ($E_n^r(f) \to 0$) and, consequently, $(fu)(x_i) = 0$ (see the proof of Corollary 2.5). Now, to prove $\mathbf{C}_u^0 = \mathbf{C}_r^0$, it remains to show $\|f\|_{u,0} \sim \|f\|_{r,0}$ for all $f \in \mathbf{C}_u$ with $(fu)(x_i) = 0$. But this follows from $\|f\|_u \sim \|f\|_r$ which implies $E_n^u(f) \sim E_n^r(f)$. In view of (2.8), the second identity of (2.13) holds true if we replace $fu \in \mathbf{C}^0$ by $fu/B \in \mathbf{C}^0$. If we write

$$\begin{aligned} \frac{fu}{B} &= \frac{fu}{B_0}\,\chi_{[-1,x_1)} + \frac{fu}{B_1}\,\chi_{[x_1,x_2)} + \ldots + \frac{fu}{B_N}\,\chi_{[x_N,1]} \qquad \text{and} \\ fu &= B_0\frac{fu}{B}\,\chi_{[-1,x_1)} + B_1\frac{fu}{B}\,\chi_{[x_1,x_2)} + \ldots + B_N\frac{fu}{B}\,\chi_{[x_N,1]} \end{aligned}$$

and take into account that $\mathbf{C}^0$ is a Banach algebra which is inversely closed in $\mathbf{C}$ (see [3], Theorems 1 and 2, or use the characterization (2.4) of $\mathbf{C}^0$ together with the estimates $\omega(fg,h) \leq \|g\|\,\omega(f,h) + \|f\|\,\omega(g,h)$ and $\omega(g^{-1},h) \leq \|g\|^{-2}\omega(g,h)$), then it is easy to prove that the assertions $fu/B \in \mathbf{C}^0$ and $fu \in \mathbf{C}^0$ (as well as the corresponding $\mathbf{C}^0$-norms) are equivalent if $(fu)(x_i) = 0$. (Remark that $\omega(\chi_i fu/B_i,h) \leq \omega(fu/B_i,h)$ and $\omega(\chi_i B_i fu/B,h) \leq \omega(B_i fu/B,h)$.) To prove the embedding (2.14), we only have to remark that Schur's inequality (2.1) for the power weight $\prod_{i=1}^N |x-x_i|^{\alpha_i}$ implies the same inequality for the weight r. Thus, the first assertion of Lemma 2.3 can be proved with r instead of a power weight. In view of (2.7), this gives $\|fr\|_0 \leq c\,\|f\|_{r,0} \sim \|f\|_{u,0}$ for all $f \in \mathbf{C}_u^0 = \mathbf{C}_r^0$ and it remains to mention that, in the same way as above, $\|fu\|_0 \leq c\,\|fu/B\|_0 = c\,\|fr\|_0$ for all $f \in \mathbf{C}_r^0$.

■

3 The operator SwI on the space $\mathbf{C}^0_u$

In all what follows, $u(x) = \prod_{i=1}^N |x - x_i|^{\alpha_i}$ and $v(x) = \prod_{j=1}^M |x - y_j|^{\beta_j}$ are power weights with

$$\alpha_i > 0 \text{ for all } i \quad \text{and} \quad 0 < \beta_j < 1 \text{ for all } j\,. \tag{3.1}$$

(In other words: $u, v \in \mathbf{L}^\infty(-1,1)$ and $v^{-1} \in \mathbf{L}^1(-1,1)$.) Moreover, we fix a function $w : \operatorname{supp} v \to \mathbb{R}$ with $|w| = u/v$ which may change its sign in the zeros of u and v:

$$w(x) = \frac{\prod_{i=1}^N [\operatorname{sign}(x - x_i)]^{\rho_i}\, |x - x_i|^{\alpha_i}}{\prod_{j=1}^M [\operatorname{sign}(x - y_j)]^{\tau_j}\, |x - y_j|^{\beta_j}} =: \frac{w_u(x)}{w_v(x)} \quad \text{with} \quad \rho_i, \tau_j \in \{1, 2\}\,. \tag{3.2}$$

The following theorem is the main result of the present paper. It shows that SwI is a bounded linear operator from $\mathbf{C}^0_u$ into $\mathbf{C}_v$ (shortly, $SwI \in \mathcal{L}(\mathbf{C}^0_u, \mathbf{C}_v)$) if u vanishes in all inner zeros of v and in those of the points ± 1 which are no zeros of v. In all other cases, $\mathbf{C}_v$ has to be replaced by some bigger space.

Theorem 3.1 *Let u, v and w satisfy the above conditions. Then*

$$SwI \in \mathcal{L}(\mathbf{C}^0_u, \mathbf{PC}_{\widetilde{v}})\,, \quad \textit{where} \quad \widetilde{v}(x) = \frac{v(x)}{1 + (uv)(-1)\,|\ln(1+x)| + (uv)(1)\,|\ln(1-x)|}\,.$$

Particularly, for $f \in \mathbf{C}^0_u$, the Cauchy principle value integral $(Swf)(x)$ exists in all $x \in \operatorname{supp} \widetilde{v}$. In the common zeros y of fu and v (especially, in all common zeros of u and v), the limits $\lim_{x \to y} (vSwf)(x)$ are zero.

The factors $(uv)(\pm 1)$ are introduced in $\widetilde{v}$ to indicate that $\ln(1 \mp x)$ disappears if $(uv)(\pm 1) = 0$. If uv does not vanish in ± 1, then $\widetilde{v} \neq v$ and one may ask for which functions $f \in \mathbf{C}^0_u$ the images Swf belong to $\mathbf{PC}_v$ in spite of this. Moreover, it is of interest to know whether all of the conditions $(fu)(x_i) = 0$ which appear in

$$\mathbf{C}^0_u = \{f \in \mathbf{C}_u : fu \in \mathbf{C}^0 \text{ and } (fu)(x_i) = 0 \text{ for all } i\}$$

(see (2.8)) are really needed in Theorem 3.1. (If not, then $\mathbf{C}^0_u$ can be replaced by a bigger space.) The following corollary completely answers these questions.

Corollary 3.2 *Let $\mathbf{C}^0(w_u) = \{f \in \mathbf{C}(\operatorname{supp} w_u) : fw_u \in \mathbf{C}^0\}$, endowed with $\|f\|_{\mathbf{C}^0(w_u)} = \|fw_u\|_0$, and define the following subspaces of $\mathbf{C}^0(w_u)$:*

$$\mathbf{C}^0_+(w_u) = \{f \in \mathbf{C}^0(w_u) : (fw_u)(1) = 0\}\,, \quad \mathbf{C}^0_-(w_u) = \{f \in \mathbf{C}^0(w_u) : (fw_u)(-1) = 0\}\,,$$
$$\mathbf{C}^0_\pm(w_u) = \{f \in \mathbf{C}^0(w_u) : (fw_u)(-1) = (fw_u)(1) = 0\}\,.$$

Moreover, denote by v^+, v^-, and $v^\pm$ those "logarithmic" modifications of v which vanish in $+1$, -1, and ± 1, respectively, i.e.,

$$v^+(x) = \frac{v(x)}{1 + v(1)\,|\ln(1-x)|}\,, \quad v^-(x) = \frac{v(x)}{1 + v(-1)\,|\ln(1+x)|}\,,$$
$$v^\pm(x) = \frac{v(x)}{1 + v(1)\,|\ln(1-x)| + v(-1)\,|\ln(1+x)|}\,.$$

Then $SwI \in \mathcal{L}(\mathbf{C}^0(w_u), \mathbf{PC}_{v^\pm})$, $SwI \in \mathcal{L}(\mathbf{C}^0_+(w_u), \mathbf{PC}_{v^-})$, $SwI \in \mathcal{L}(\mathbf{C}^0_-(w_u), \mathbf{PC}_{v^+})$, and $SwI \in \mathcal{L}(\mathbf{C}^0_\pm(w_u), \mathbf{PC}_v)$. Moreover, in all common zeros y of fw_u and v ($f \in \mathbf{C}^0(w_u)$), the limits $\lim_{x\to y}(vSwf)(x)$ are zero.

Proof. This follows from the proof of Theorem 3.1. Alternatively, one also can conclude it directly from Theorem 3.1: In view of (2.8), the assertions $f \in \mathbf{C}^0(w_u)$, $f \in \mathbf{C}^0_+(w_u)$, $f \in \mathbf{C}^0_-(w_u)$, and $f \in \mathbf{C}^0_\pm(w_u)$ are equivalent to

$$fw_u \in \mathbf{C}^0\,, \quad \frac{fw_u}{v^{1,0}} \in \mathbf{C}^0_{v^{1,0}}\,, \quad \frac{fw_u}{v^{0,1}} \in \mathbf{C}^0_{v^{0,1}}\,, \quad \text{and} \quad \frac{fw_u}{v^{1,1}} \in \mathbf{C}^0_{v^{1,1}}\,, \quad \text{respectively}$$

(with corresponding equivalent norms). Hence, we only have to write

$$Swf = S\frac{1}{w_v}\,(fw_u) = S\frac{v^{1,0}}{w_v}\left(\frac{fw_u}{v^{1,0}}\right) = S\frac{v^{0,1}}{w_v}\left(\frac{fw_u}{v^{0,1}}\right) = S\frac{v^{1,1}}{w_v}\left(\frac{fw_u}{v^{1,1}}\right)$$

and to apply Theorem 3.1 with 1, $v^{1,0}$, $v^{0,1}$, and $v^{1,1}$ instead of u. ∎

Before we prove Theorem 3.1, we remark that the results also hold true for more general weights w and u (for example, for generalized Jacobi weights):

Remark 3.3 *Let $w(x) = A(x)w_1(x)/w_2(x)$, where $A, w_1, w_2 : [-1,1] \to \mathbb{R}$ belong to $\mathbf{C}^0$,*

$$w_1 \neq 0 \ \text{ a.e. on } [-1,1] \ \text{ and } \ |w_2(x)| = \prod_{j=1}^{M} |x - y_j|^{\beta_j} =: v(x) \ \text{ with } \ y_j \in [-1,1],\ \beta_j \in (0,1).$$

Then the assertions of Corollary 3.2 remain true with w_1 instead of w_u. (Together with the embedding (2.14) this implies that, in case of a weight $w_1(x) = u(x) = B(x)\,r(x)$ as in Remark 2.9, the assertions of Theorem 3.1 also remain true.)

Proof. Let f belong to one of the spaces $\mathbf{C}^0(w_1)$, $\mathbf{C}^0_+(w_1)$, $\mathbf{C}^0_-(w_1)$, or $\mathbf{C}^0_\pm(w_1)$. Then fw_1A belongs to the corresponding unweighted space $\mathbf{C}^0$, $\mathbf{C}^0_+$, $\mathbf{C}^0_-$, or $\mathbf{C}^0_\pm$, where $\|fw_1A\|_0 \leq c\,\|f\|_{\mathbf{C}^0(w_1)}$ (since $\mathbf{C}^0$ is a Banach algebra; see the proof of Remark 2.9). If we write $Swf = Sw_2^{-1}(fw_1A)$, then the assertion follows from Corollary 3.2, applied with 1 instead of w_u. ∎

Now we come to the proof of Theorem 3.1. First we mention that, in view of the characterization (2.13) and the equality $Swf = Sw_v^{-1}(fw_u)$, it is clear that we have to deal with $Sw_v^{-1}I$ on the unweighted space $\mathbf{C}^0$ (or subspaces of it). For this, we use the following decomposition of $Sw_v^{-1}f$:

$$(3.3) \qquad (Sw_v^{-1}f)(x) = \int_{-1}^{1} \frac{f(t)-f(x)}{t-x}\, w_v^{-1}(t)\,dt + f(x)\int_{-1}^{1} \frac{w_v^{-1}(t)}{t-x}\,dt\,.$$

We will see that the first addend has better properties than the second one:

Lemma 3.4 *$Sw_v^{-1}I - S(w_v^{-1})\cdot I \in \mathcal{L}(\mathbf{C}^0, \mathrm{clos}_v\,\Pi)$. ($\mathrm{clos}_v\,\Pi$ is given in Corollary 2.5 and $Sw_v^{-1}I - S(w_v^{-1})\cdot I$ denotes the operator which is defined by the first addend of (3.3).)*

Proof. First we will show that, for all $x \in \mathrm{supp}\, v$,

$$(3.4) \qquad \int_{-1}^{1} \left|\frac{f(t)-f(x)}{t-x}\right| v^{-1}(t)\,dt \le c\, v^{-1}(x) \left(\|f\| + \int_{-1}^{1} \left|\frac{f(t)-f(x)}{t-x}\right| dt \right).$$

For this, we may assume $M > 0$. Moreover, it is sufficient to deal with $|x-y|^{-\beta}$ ($y \in [-1,1]$ and $\beta \in (0,1)$ fixed) instead of $v^{-1}(x)$, since

$$v^{-1}(x) \sim |x-y_1|^{-\beta_1} + \ldots + |x-y_M|^{-\beta_M}\,.$$

(Write the right hand side as a fraction or consider the cases $x \in I_j$, where I_j are neighborhoods of the points y_j.) Thus, (3.4) is proved if we can show that, for $x \in [-1,1]\setminus\{y\}$,

$$(3.5) \qquad \int_{-1}^{1} \left|\frac{f(t)-f(x)}{t-x}\right| |t-y|^{-\beta}\,dt \le c\,|x-y|^{-\beta} \left(\|f\| + \int_{-1}^{1} \left|\frac{f(t)-f(x)}{t-x}\right| dt \right).$$

For this aim, we first consider the case $x > y$. In this case, the left hand side of (3.5) can be estimated by

$$2\|f\| \left[\int_{-1}^{y-\frac{x-y}{2}(y+1)} \frac{(y-t)^{-\beta}}{x-t}\,dt + \int_{y-\frac{x-y}{2}(y+1)}^{\frac{x+y}{2}} \frac{|t-y|^{-\beta}}{x-t}\,dt \right] + \int_{\frac{x+y}{2}}^{1} \left|\frac{f(t)-f(x)}{t-x}\right| (t-y)^{-\beta} dt$$
$$=: 2\|f\|\,[I_1 + I_2] + I_3\,.$$

If $y = -1$, then I_1 vanishes. Otherwise we use that $x - t \ge y - t$ in the first integral:

$$I_1 \le \int_{-1}^{y-\frac{x-y}{2}(y+1)} (y-t)^{-\beta-1} dt \le c\,(x-y)^{-\beta}\,.$$

In integral I_2 we have $x - t \ge (x-y)/2$ and it follows

$$I_2 \le 2(x-y)^{-1} \int_{y-\frac{x-y}{2}(y+1)}^{\frac{x+y}{2}} |t-y|^{-\beta}\,dt \le c\,(x-y)^{-\beta}\,.$$

For $t \geq (x+y)/2$, $(t-y)^{-\beta}$ can be estimated by $2^\beta (x-y)^{-\beta}$. It follows

$$I_3 \leq c\,(x-y)^{-\beta} \int_{-1}^{1} \left| \frac{f(t)-f(x)}{t-x} \right| dt\,.$$

Thus, (3.5) is proved in case $x > y$. If $x < y$, then one can proceed in a similar way or one uses the substitution $\tau = -t$ which makes it possible to apply what we have already proved (with $-x$ and $-y$ instead of x and y). Now, (3.4) is proved. The last integral in (3.4) can be estimated as follows:

$$\begin{aligned} \int_{-1}^{1} \left| \frac{f(t)-f(x)}{t-x} \right| dt \ &\leq\ \int_{-1}^{1} \frac{\omega(f,|t-x|)}{|t-x|}\, dt = \int_{-1-x}^{1-x} \frac{\omega(f,|h|)}{|h|}\, dh \\ &\leq\ 2\int_0^2 \frac{\omega(f,h)}{h}\, dh \leq c\,\|f\|_0 \end{aligned}$$

(see Theorem 2.6). So we have proved

$$\begin{aligned} &Sw_v^{-1}I - S(w_v^{-1})\cdot I \in \mathcal{L}(\mathbf{C}^0, \mathbf{B}_v)\,, \quad \text{where} \\ &\mathbf{B}_v = \{g : \operatorname{supp} v \to \mathbb{C} \ \text{ such that } \ \|g\|_v = \sup\nolimits_{x\in \operatorname{supp} v} |g(x)v(x)| < \infty\}\,. \end{aligned} \tag{3.6}$$

Now we remark that the first addend in (3.3) is a polynomial if f is a polynomial. Hence, $Sw_v^{-1}I - S(w_v^{-1})\cdot I$ maps Π into Π. Since Π is dense in $\mathbf{C}^0$ (Proposition 2.2), it follows that $\mathbf{C}^0$ is mapped into the closure of Π in $\mathbf{B}_v$ which is equal to $\operatorname{clos}_v \Pi$.

■

To obtain properties of the second addend of (3.3) in case of a Jacobi weight v, we need the following well-known result ([17], Theorem 9.9).

Proposition 3.5 *Let $\alpha, \beta \in (-1,1)\setminus\{0\}$ such that $\alpha+\beta \in \{-1,0,1\}$. Then, the operator*

$$A_{\alpha,\beta} = av^{\alpha,\beta}I + bSv^{\alpha,\beta}I \quad \text{with} \quad a,b\in\mathbb{R} \ \text{ such that } \ a - \mathrm{i}\pi b = \mathrm{e}^{\mathrm{i}\pi\alpha} \tag{3.7}$$

maps Π_n into $\Pi_{n+\alpha+\beta}$ for all $n \in \mathbb{N}$.

Now we are able to treat the last integral in (3.3) for weights v with only one zero:

Lemma 3.6 *Let $y \in [-1,1]$ and $\beta \in (0,1)$ be fixed and set $v_y(x) = |x-y|^\beta$. Moreover, let $w_y(x) = [\operatorname{sign}(x-y)]^k\, v_y(x)$ ($k \in \{1,2\}$). Then $Sw_y^{-1} \in \mathbf{PC}_{v_y^\pm}$. ($v_y^\pm$ is defined in Corollary 3.2.)*

Proof. If $y=1$, then $w_y = Cv_y$ ($C=-1$ or $C=1$), $v_y^\pm = v_1/[1+2^\beta|\ln(1+\,.\,)|\,]$, and we may use Proposition 3.5 (with $n=1$) to show that $v_1 S v_1^{-1}$ is continuous on $[0,1]$:

$$\begin{aligned} (1+x)^\beta \int_{-1}^{1} \frac{(1-t)^{-\beta}}{t-x}\, dt \ &=\ \int_{-1}^{1} \frac{(1+x)^\beta - (1+t)^\beta}{t-x}\, \frac{dt}{(1-t)^\beta} + \int_{-1}^{1} \frac{v^{-\beta,\beta}(t)}{t-x}\, dt \\ &=\ \int_{-1}^{1} \frac{(1+x)^\beta - (1+t)^\beta}{t-x}\, \frac{dt}{(1-t)^\beta} + \frac{(A_{-\beta,\beta}1)(x)}{b} - \frac{a}{b}\, v^{-\beta,\beta}(x). \end{aligned}$$

In view of Lemma 3.4 and Proposition 3.5, the last term belongs to $\mathbf{C}_{v_1}$. To prove the continuity of $v_1^{\pm} S v_1^{-1}$ on $[-1,0]$, we write

$$
\begin{aligned}
&(1-x)\int_{-1}^{1}\frac{(1-t)^{-\beta}}{t-x}\,dt = \\
&\int_{-1}^{1}\frac{(1-x)-(1-t)}{t-x}\,\frac{dt}{(1-t)^{\beta}}+\int_{-1}^{1}\frac{(1-t)^{1-\beta}-(1-x)^{1-\beta}}{t-x}\,dt+\int_{-1}^{1}\frac{(1-x)^{1-\beta}}{t-x}\,dt \\
&=\int_{-1}^{1}\frac{dt}{(1-t)^{\beta}}+\int_{-1}^{1}\frac{(1-t)^{1-\beta}-(1-x)^{1-\beta}}{t-x}\,dt+(1-x)^{1-\beta}\ln\frac{1-x}{1+x}\,.
\end{aligned}
\tag{3.8}
$$

The right hand side, divided by $1+2^{\beta}|\ln(1+x)|$, is continuous on $[-1,0]$ (Lemma 3.4 with $v\equiv 1$). Analogously one can prove the assertion in case $y=-1$. By the way, the sum of the last two addends in (3.8) defines a function $g(x)$ which is Hölder continuous with exponent $1-\beta$ on $[0,1]$ (see, e.g., [17], Remark 9.4). Hence, the absolute value $|g(x)-g(1)|$ of (3.8) can be estimated by $c\,(1-x)^{1-\beta}$ for $x\geq 0$. In this way one can prove the boundedness (but not the continuity) of $v_1^{\pm} S v_1^{-1}$ without using Proposition 3.5. Similar considerations also lead to the boundedness of $v_y^{\pm} S w_y^{-1}$ in case $y\neq\pm 1$. But we go another way, since we want to prove that $(v_y^{\pm} S w_y^{-1})(x)$ has no discontinuities, excepting a possible jump in $x=y$. For this aim, let $y\in(-1,1)$ and write

$$
\int_{-1}^{1}\frac{w_y^{-1}(t)}{t-x}\,dt=\int_{-1}^{1}\frac{w_y^{-1}(t)-w_y^{-1}(x)}{t-x}\,dt+w_y^{-1}(x)\ln\frac{1-x}{1+x}\,.
$$

The second addend is an element of $\mathbf{PC}_{v_y^{\pm}}$ and for the first addend we use the substitution $t-y=\tau(x-y)$:

$$
\begin{aligned}
&\int_{-1}^{1}\frac{w_y^{-1}(t)-w_y^{-1}(x)}{t-x}\,dt=w_y^{-1}(x)\int_{-\frac{1+y}{x-y}}^{\frac{1-y}{x-y}}\frac{(\operatorname{sign}\tau)^k|\tau|^{-\beta}-1}{\tau-1}\,d\tau \\
&=w_y^{-1}(x)\left(\int_{-\frac{1+y}{x-y}}^{\frac{1-y}{x-y}}\frac{(\operatorname{sign}\tau)^k|\tau|^{-\beta}-2(1+\tau^2)^{-1}}{\tau-1}\,d\tau+\int_{-\frac{1+y}{x-y}}^{\frac{1-y}{x-y}}\frac{2(1+\tau^2)^{-1}-1}{\tau-1}\,d\tau\right).
\end{aligned}
$$

The first integrand belongs to $\mathbf{L}^1(\mathbb{R})$. Consequently, the first integral defines a continuous function on $\mathbb{R}\backslash\{y\}$ for which the limits $x\to y\pm 0$ exist. The second integral can be computed explicitly:

$$
\begin{aligned}
\int_{-\frac{1+y}{x-y}}^{\frac{1-y}{x-y}}\frac{2(1+\tau^2)^{-1}-1}{\tau-1}\,d\tau &= -\left[\arctan\tau+\frac{1}{2}\ln(1+\tau^2)\right]_{\tau=-\frac{1+y}{x-y}}^{\tau=\frac{1-y}{x-y}} \\
&= \frac{1}{2}\ln\frac{(x-y)^2+(1+y)^2}{(x-y)^2+(1-y)^2}-\arctan\frac{1-y}{x-y}-\arctan\frac{1+y}{x-y}\,.
\end{aligned}
$$

Again we have a continuous function on $\mathbb{R}\setminus\{y\}$ for which the limits $x\to y\pm 0$ exist. ■

We have seen that the last integral in the decomposition (3.3) of $Sw_v^{-1}f$ $(f \in \mathbf{C}^0)$ may contain logarithmic singularities in ± 1. The following lemma shows that such singularities can be deleted by the factor $f(x)$ if f has a zero in the corresponding point $y = \pm 1$.

Lemma 3.7 *For all $f \in \mathbf{C}^0$ and all $x, y \in [-1, 1]$ with $0 < |x-y| \le 1$ we have*

$$\left|[f(x) - f(y)] \ln |x-y|\right| \le 2 \int_{|x-y|}^{\sqrt{|x-y|}} \omega(f,h)\, \frac{dh}{h} \le c\, \|f\|_0\,, \tag{3.9}$$

where $c \ne c(f, x, y)$.

Proof. We have $\int_{|x-y|}^{\sqrt{|x-y|}} h^{-1} dh = \left|\frac{1}{2} \ln |x-y|\right|$. Hence,

$$\left|[f(x) - f(y)] \ln |x-y|\right| \le 2\, \omega(f, |x-y|) \int_{|x-y|}^{\sqrt{|x-y|}} \frac{dh}{h} \le 2 \int_{|x-y|}^{\sqrt{|x-y|}} \omega(f,h)\, \frac{dh}{h}\,.$$

The second part of (3.9) follows from Theorem 2.6.

■

Now we have all tools which we need for the

Proof of Theorem 3.1. Let $g \in \mathbf{C}_u^0$ and write $Swg = Sw_v^{-1}f$ with $f := g\, w_u$. In view of (2.13), we have $f \in \mathbf{C}^0$, $\|f\|_0 \sim \|g\|_{u,0}$, and $f(x_i) = 0$ for all i. So it remains to prove the assertions of Corollary 3.2 for the case $w_u \equiv 1$. This can be done with the help of (3.3): For the first addend of this decomposition we have

$$Sw_v^{-1}f - fSw_v^{-1} \in \mathrm{clos}_v\, \Pi \quad \text{and} \quad \|Sw_v^{-1}f - fSw_v^{-1}\|_v \le c\, \|f\|_0$$

(Lemma 3.4). Especially, the product of this addend and v vanishes in all zeros of v (see Corollary 2.5). We still have to prove

$$\begin{aligned} &fSw_v^{-1} \in \mathbf{PC}_{v^*}\,, \quad \text{where } v^* = v^\pm, v^-, v^+, v \text{ corresponds to } f \in \mathbf{C}^0, \mathbf{C}^0_+, \mathbf{C}^0_-, \mathbf{C}^0_\pm\,, \\ &\|fSw_v^{-1}\|_{v^*} \le c\, \|f\|_0\,, \quad \text{and} \quad \lim_{x\to x_0} (vfSw_v^{-1})(x) = 0 \text{ for all zeros } x_0 \text{ of } f\,. \end{aligned} \tag{3.10}$$

If $w_v \equiv 1$, then this follows from (1.1) and Lemma 3.7, taking into account that the second term of (3.9) goes to zero for $x \to y$ (because of Theorem 2.6) and, consequently,

$$\lim_{x \to \pm 1} f(x) \ln(1 \mp x) = 0 \text{ and } \|f \ln(1 \mp\ .)\| \le c\, \|f\|_0 \text{ for all } f \in \mathbf{C}^0 \text{ with } f(\pm 1) = 0. \tag{3.11}$$

(Remark that, for example in case $f(1) = 0$, this limit relation $(fS1)(1-0) = 0$ is in accordance with our decomposition (3.3) in which the second term does not appear for

$x = 1$.) Now we prove (3.10) in case $M > 0$. For this aim, set $v_j(x) = |x - y_j|^{\beta_j}$, $w_j(x) = [\mathrm{sign}(x - y_j)]^{\tau_j}\, v_j(x)$, and write

$$w_v^{-1} = \frac{g_1}{w_1} + \ldots + \frac{g_M}{w_M} \quad \text{with} \quad g_k(x) = \frac{\prod_{j \neq k} [\mathrm{sign}(x - y_j)]^{\tau_j} |x - y_j|}{\prod_{j \neq 1} |x - y_j|^{\beta_j + 1} + \ldots + \prod_{j \neq M} |x - y_j|^{\beta_j + 1}} \,.$$

Then, the function $S w_v^{-1}$ can be decomposed as follows:

$$\int_{-1}^{1} \frac{w_v^{-1}(t)}{t - x}\, dt = \sum_{k=1}^{M} \left(\int_{-1}^{1} \frac{g_k(t) - g_k(x)}{t - x}\, w_k^{-1}(t)\, dt + g_k(x) \int_{-1}^{1} \frac{w_k^{-1}(t)}{t - x}\, dt \right).$$

From Lemma 3.4 it follows that the first integral belongs to $\mathbf{C}_{v_k} \subseteq \mathbf{C}_v$ for all k (since g_k is Lipschitz continuous). In view of Lemma 3.6, the second integral is an element of $\mathbf{PC}_{v_k^{\pm}}$. This implies that its product with $g_k(x)$ belongs to $\mathbf{PC}_{v^{\pm}}$, since $g_k(x)$ contains the factor $1 \mp x$ if ± 1 is a zero of v and $y_k \neq \pm 1$. Together with (3.11) it follows (3.10). It remains to mention that, for example in case $f(1) = 0$, $v(1) \neq 0$, we get no problem with the value of the integral $(S w_v^{-1} f)(1)$: In this case the second addend in (3.3) does not appear for $x = 1$ and this is in accordance with the limit relation $(f S w_v^{-1})(1 - 0) = 0$.

■

We finish this section with two remarks about operators related to SwI.

Remark 3.8 *Theorem 3.1 can be used to obtain mapping properties of Cauchy singular integral operators with kernels which have, in addition to the strong singularity on the diagonal $x = t$, a finite number of further strong singularities on lines $t = t_i$. For example, the operator $St^{-1}I$ (which may be defined by the sum of principle value integrals over $[-|x/2|, |x/2|]$ and $[-1, 1] \setminus [-|x/2|, |x/2|]$) maps $\mathbf{C}^0_{\pm}$ into $\mathbf{C}_x$. This follows from Corollary 3.2, applied to the addends of the decomposition*

$$\int_{-1}^{1} \frac{f(t)}{t - x} \frac{dt}{t} = \frac{1}{x} \left(\int_{-1}^{1} \frac{f(t)}{t - x}\, dt - \int_{-1}^{1} \frac{f(t)}{t}\, dt \right) = \frac{(Sf)(x) - (Sf)(0)}{x} \,.$$

Another operator which is closely connected with S is the Cauchy singular integral operator $S_{[a,b]}$ on another interval $[a, b]$ $(-\infty < a < b < \infty)$:

$$(S_{[a,b]} f)(x) = \int_a^b \frac{f(t)}{t - x}\, dt \,.$$

Of course, all what we have proved until now can be transformed onto $[a, b]$, i.e., the following assertions hold true if w is a weight of the form (3.2), where w_u and w_v correspond to power weights $u(x) = \prod_{i=1}^{N} |x - x_i|^{\alpha_i}$ and $v(x) = \prod_{j=1}^{M} |x - y_j|^{\beta_j}$ with $x_i, y_j \in \mathbb{R}$ and $\alpha_i > 0$, $0 < \beta_j < 1$. (Remark that the restriction $x_i, y_j \in [a, b]$ is not necessary, since the differences between u, v, w_u, w_v, w and the corresponding products $u^*, v^*, w_u^*, w_v^*, w^*$ taken over i, j with $x_i, y_j \in [a, b]$ are factors which are non-zero and $\mathbf{C}^{\infty}$ on $[a, b]$.)

(i) The approximation space $\mathbf{C}_u^0[a,b]$ (or shortly $\mathbf{C}^0[a,b]$ if $u \equiv 1$), i.e.,

$$\mathbf{C}_u^0[a,b] := A(\mathbf{C}_u[a,b],\, \mathbf{l}^1(\{(n+1)^{-1}\});\, \{\Pi_n\})$$

$(\mathbf{C}_u[a,b] = \{f \in \mathbf{C}(\{x \in [a,b] : u(x) \neq 0\}) : fu \in \mathbf{C}[a,b]\},\ \|f\|_{u,[a,b]} = \|fu\|_{\mathbf{C}[a,b]})$ can be defined equivalently by

$$\begin{aligned}\mathbf{C}_u^0[a,b] &= \{f \in \mathbf{C}_u[a,b] : fu \in \mathbf{C}^0[a,b] \text{ and } (fu)(x_i) = 0 \text{ for all } x_i \in [a,b]\}\\ &= \{f \in \mathbf{C}_{w_u}[a,b] : fw_u \in \mathbf{C}^0[a,b] \text{ and } (fw_u)(x_i) = 0 \text{ for all } x_i \in [a,b]\}\\ &= \{f \in \mathbf{C}_u[a,b] : \textstyle\int_0^1 \omega_{[a,b]}(fu,h)\, h^{-1}dh < \infty \text{ and } (fu)(x_i) = 0 \text{ for all } x_i \in [a,b]\}.\end{aligned}$$

In other words: $\|f\|_{\mathbf{C}_u^0[a,b]} \sim \|fu\|_{\mathbf{C}^0[a,b]} \sim \|fw_u\|_{\mathbf{C}^0[a,b]}$ for all $f \in \operatorname{clos}_{\mathbf{C}_u[a,b]} \Pi = \{f \in \mathbf{C}_u[a,b] : (fu)(x_i) = 0,\ x_i \in [a,b]\} = \{f \in \mathbf{C}_{w_u}[a,b] : (fw_u)(x_i) = 0,\ x_i \in [a,b]\}$ and $\|g\|_{\mathbf{C}^0[a,b]} \sim \|g\|_{\mathbf{C}[a,b]} + \int_0^1 \omega_{[a,b]}(g,h)\, h^{-1}dh$.

(ii) $S_{[a,b]}\, wI \in \mathcal{L}(\mathbf{C}_u^0[a,b], \mathbf{PC}_{\widetilde{v}}[a,b])$, where $\mathbf{PC}_{\widetilde{v}}[a,b]$ is defined similarly to $\mathbf{PC}_{\widetilde{v}}$ (only $[-1,1]$ has to be replaced by $[a,b]$) and

$$\widetilde{v}(x) = \frac{v(x)}{1 + (uv)(a)\,|\ln(x-a)| + (uv)(b)\,|\ln(b-x)|}\,.$$

It seems to be natural that the image space $\mathbf{PC}_{\widetilde{v}}[a,b]$ in assertion (ii) consists of functions on $[a,b]$. But on the other hand, also for $x \notin [a,b]$, $(S_{[a,b]}\, wf)(x)$ is well-defined (as a usual Lebesgue integral). We will see that, in case $(uv)(a) = (uv)(b) = 0$, one can take a bigger interval for the image space. For the sake of simplicity, we will restrict on subintervals of $[-1,1]$. This makes it possible to use our standard notation and assumptions from the beginning of this section and from Corollary 3.2.

Remark 3.9 *Let $-1 \le a < b \le 1$ and take u, v, w as in the beginning of this section.*

$$\text{If } (uv)(a) = (uv)(b) = 0, \text{ then } S_{[a,b]}\, wI \in \mathcal{L}(\mathbf{C}_u^0[a,b], \mathbf{PC}_v)\,. \tag{3.12}$$

In the cases $a = -1 < b < 1$ and $-1 < a < b = 1$, (3.12) can be generalized:

$$\begin{aligned}&\text{If } (uv)(b) = 0, \text{ then } S_{[-1,b]}\, wI \in \mathcal{L}(\mathbf{C}_u^0[-1,b], \mathbf{PC}_{v^-}).\\ &\text{If } (uv)(a) = 0, \text{ then } S_{[a,1]}\, wI \in \mathcal{L}(\mathbf{C}_u^0[a,1], \mathbf{PC}_{v^+}).\end{aligned} \tag{3.13}$$

Proof. Let $f \in \mathbf{C}_u^0[a,b]$. Then $fw_u \in \mathbf{C}^0[a,b]$ can be extended to a $\mathbf{C}^0$-function by setting $(fw_u)(-1) = 0$ (if $a > -1$), $(fw_u)(1) = 0$ (if $b < 1$) and connecting $(fw_u)(-1)$ and $(fw_u)(a)$ as well as $(fw_u)(b)$ and $(fw_u)(1)$ by lines. Obviously, this yields an extension $f \in \mathbf{C}^0(w_u)$ with $\|fw_u\|_0 \le c\|f\|_{\mathbf{C}_u^0[a,b]}$. Now we define $\widetilde{w}$ by $\widetilde{w} = -w$ on $[a,b]$ and $\widetilde{w} = w$ on $[-1,1]\backslash[a,b]$. In other words, $\widetilde{w}(x) = \operatorname{sign}(x-a)\operatorname{sign}(x-b)\, w(x)$, and this means that $\widetilde{w}$ (or $-\widetilde{w}$ if $b = 1$)

is again a weight of the form (3.2), where only the signs have to be chosen different from those of w. Now we write $2\,S_{[a,b]}\,wf = Swf - S\widetilde{w}f$ and the assertions follow from Corollary 3.2, since fw_u vanishes in 1 if $b<1$ or $u(1)=0$ and in -1 if $a>-1$ or $u(-1)=0$.

■

We mention that the above remark also leads to generalizations of Theorem 3.1 if we transform $[a,b]$ onto $[-1,1]$. For example, the transformation $[a,b]=[-1/2,1/2]\to[-1,1]$, $[-1,1]\to[-2,2]$ of (3.12) yields the following generalization in case $(uv)(\pm1)=0$:

$$\text{If } (uv)(-1)=(uv)(1)=0\,,\text{ then } SwI\in\mathcal{L}(\mathbf{C}_u^0,\mathbf{PC}_v[-2,2])\,. \tag{3.14}$$

Of course, if $u(-1)=u(1)=0$, then this is not surprising, since then $f\in\mathbf{C}_u^0$ can be extended by zero to a $\mathbf{C}_u^0[-2,2]$-function, so that the result for $S_{[-2,2]}$ can be applied. But in case $u(\pm1)\neq0$, (3.14) is a non-trivial corollary of Theorem 3.1.

4 SwI on spaces of piecewise $\mathbf{C}_u^0$-functions

Take the notation and assumptions of Section 3. In Remark 3.9 we have seen that the possible sign changes in (3.2) can be used to consider SwI on piecewise $\mathbf{C}_u^0$-functions of the form $f\chi_{[a,b]}$ ($f\in\mathbf{C}_u^0[a,b]$), where a and b are either endpoints of $[-1,1]$ or zeros of v. (In principle we can also consider zeros a or b of u. But, if for example $u(a)=0$, then also $(fu)(a)=0$ for $f\in\mathbf{C}_u^0[a,b]$, i.e., $f\chi_{[a,b]}\in\mathbf{C}_u^0[-1,b]$.) This makes it possible to consider SwI on piecewise $\mathbf{C}_u^0$-functions w.r.t. the partition

$$[-1,1]=[y_0,y_1]\cup[y_1,y_2]\cup\ldots\cup[y_M,y_{M+1}]\,,\quad y_0:=-1\,,\ \ y_{M+1}:=1$$

(i.e., with possible jumps in those zeros of v which are no zeros of u):

Proposition 4.1 *Let $\mathbf{PC}_u^0(y_1,\ldots,y_M)$ be the space of all $f:\operatorname{supp}u\setminus\{y_1,\ldots,y_M\}\to\mathbb{C}$ with $f_{|(y_i,y_{i+1})}\in\mathbf{C}_u^0[y_i,y_{i+1}]$, $i=0,\ldots,M$, endowed with $\|f\|_{\mathbf{PC}_u^0(y_1,\ldots,y_M)}=\sum\limits_{i=0}^{M}\|f\|_{\mathbf{C}_u^0[y_i,y_{i+1}]}$. Then,*

$$SwI\in\mathcal{L}(\mathbf{PC}_u^0(y_1,\ldots,y_M),\mathbf{PC}_{\widetilde{v}})\,. \tag{4.1}$$

($\mathbf{C}_u^0[y_i,y_{i+1}]$ is defined before Remark 3.9.)

Proof. Write $SwI=\sum_{i=0}^{M}S_{[y_i,y_{i+1}]}\,wI$ and apply (3.12) (for all i with $(uv)(y_i)=(uv)(y_{i+1})$ $=0$) and (3.13) (for $i=0$ if $(uv)(-1)\neq0$ and for $i=M$ if $(uv)(1)\neq0$).

■

As in the proof of Corollary 3.2 one can generalize (4.1) to the space

$$\mathbf{PC}^0(w_u;y_1,\ldots,y_M)=\{f\in\mathbf{C}(\operatorname{supp}u\setminus\{y_1,\ldots,y_M\}):fw_u\in\mathbf{PC}^0(y_1,\ldots,y_M)\}$$

($\mathbf{PC}^0(y_1,\dots,y_M)$ means the unweighted space of piecewise $\mathbf{C}^0$-functions with jumps in y_j) and its subspaces $\mathbf{PC}^0_+(w_u;y_1,\dots,y_M)$, $\mathbf{PC}^0_-(w_u;y_1,\dots,y_M)$, $\mathbf{PC}^0_\pm(w_u;y_1,\dots,y_M)$ (which are defined in the same way as the corresponding spaces in Corollary 3.2). Thus, the assertions of Corollary 3.2 remain true if we replace $\mathbf{C}^0(w_u)$, $\mathbf{C}^0_+(w_u)$, $\mathbf{C}^0_-(w_u)$, and $\mathbf{C}^0_\pm(w_u)$ by the above spaces. (The second assertion remains true, since we may apply the Corollary with $w_u(x)\prod_{i\neq k}|x-y_i|^\varepsilon$ and $v(x)\prod_{i\neq k}|x-y_i|^\varepsilon$ instead of w_u and v if $y=y_k$ is a common zero of fw_u and v.)

Using this result, we are even able to deal with piecewise $\mathbf{C}^0_u$-functions having jumps in arbitrary points (where we may restrict on inner points of $[-1,1]$, since jumps in the endpoints are not really jumps):

Proposition 4.2 *Let $-1<\xi_1<\dots<\xi_m<1$ and let $v(\xi_1,\dots,\xi_m)$ be the "logarithmic" modification of v which vanishes in all ξ_i (defined similarly to the modification $v^\pm=v(-1,1)$ in Corollary 3.2). Then*

$$\begin{aligned}
SwI &\in \mathcal{L}\left(\mathbf{PC}^0(w_u;\xi_1,\dots,\xi_m),\,\mathbf{PC}_{v(-1,\xi_1,\dots,\xi_m,1)}\right),\\
SwI &\in \mathcal{L}\left(\mathbf{PC}^0_+(w_u;\xi_1,\dots,\xi_m),\,\mathbf{PC}_{v(-1,\xi_1,\dots,\xi_m)}\right),\\
SwI &\in \mathcal{L}\left(\mathbf{PC}^0_-(w_u;\xi_1,\dots,\xi_m),\,\mathbf{PC}_{v(\xi_1,\dots,\xi_m,1)}\right),\\
SwI &\in \mathcal{L}\left(\mathbf{PC}^0_\pm(w_u;\xi_1,\dots,\xi_m),\,\mathbf{PC}_{v(\xi_1,\dots,\xi_m)}\right).
\end{aligned}$$

Proof. Let J denote the set of those indices i for which $v(\xi_i)\neq 0$. We have already proved

$$SwI=S\frac{w_u\prod_{i\in J}|\,.\,-\xi_i|^\varepsilon}{w_v\prod_{i\in J}|\,.\,-\xi_i|^\varepsilon}I\in\mathcal{L}\left(\mathbf{PC}^0_{(+,-,\pm)}(w_u;\xi_1,\,\dots\,,\xi_m),\,\mathbf{PC}_{(v\prod_{i\in J}|\,.\,-\xi_i|^\varepsilon)^{\pm(-,+,\;)}}\right)$$

(since $\mathbf{PC}^0(w_u;\xi_1,\dots,\xi_m)\hookrightarrow\mathbf{PC}^0(w_u\prod_{i\in J}|\,.\,-\xi_i|^\varepsilon;y_1,\dots,y_M)$). Thus, the proposition is proved, if we can show that $v(\xi_1,\dots,\xi_m)Swf$ ($f\in\mathbf{PC}^0(w_u;\xi_1,\dots,\xi_m)$) is continuous in neighborhoods N_i of the points ξ_i ($i\in J$), where the corresponding $\mathbf{C}(N_i)$-norm can be estimated by the norm of f. (This is even more than the assertion of the proposition.) For this aim, let $i\in J$ and set $h_i=[(fw_u)(\xi_i+0)-(fw_u)(\xi_i-0)]/w_v(\xi_i)$. Then, the first addend of the decomposition

$$Swf=Sw\left(f-h_i\,w^{-1}\chi_{[\xi_i,1]}\right)+h_i\,S\chi_{[\xi_i,1]}$$

belongs to $\mathbf{PC}_{(v\prod_{j\in J\setminus\{i\}}|\,.\,-\xi_j|^\varepsilon)^\pm}$ (apply the first assertion of this proof with the space $\mathbf{PC}^0(w_u;\xi_1,\dots,\xi_{i-1},\xi_{i+1},\dots,\xi_m)$ instead of $\mathbf{PC}^0(w_u;\xi_1,\,\dots\,,\xi_m)$), where the norm in this space can be estimated by the norm of f. This yields the corresponding assertions with $\mathbf{C}(N_i)$. The second addend, divided by $1+|\ln|x-\xi_i||$, is also continuous on N_i. This follows from $(S\chi_{[\xi_i,1]})(x)=\ln(1-x)-\ln|x-\xi_i|$.

■

5 The operator $awI + SbwI$

In many applications equations of the type $(aI + SbI)f\,(+\dots) = g$ have to be solved, where a, b, and g are given functions on $(-1,1)$ and f is looked for. For the numerical solution of such equations, for example by projection methods, one may look for weighted polynomials wP_n as approximations of f, where w is an appropriated weight. (Later we will see which weights are appropriated.) Equivalently, one can first transform the equation by setting $f = w\widetilde{f}$ and looking for $\widetilde{f}$ instead of f. Then the operator $awI + SbwI$ appears and unweighted polynomials P_n are sought as approximations of $\widetilde{f}$. (This approach seems to be better, since known results in the theory of approximation by polynomials are usually formulated for unweighted polynomials.) A similar transformation leads to operators of the type $aw_0I + bSw_0I$ (w_0: some weight) if the equation $(aI + bSI)f\,(+\dots) = g$ is considered.

In the following two sections we study the mapping properties of $awI + SbwI$ and $aw_0I + bSw_0I$ (more precisely, $\varrho^{-1}(aw_0I + bSw_0I)$ with $\varrho^{-1} \in b^{-1}\Pi$ and $w_0 = w\varrho$) in scales of weighted approximation spaces of continuous functions, where we consider approximation by unweighted polynomials (according to the above approach). The basis is the theory of Sections 3 and 4. Hence, in all what follows we consider again some fixed weight $w = w_u/w_v$ of the form (3.2), i.e., w corresponds to power weights $u(x) = \prod_{i=1}^N |x - x_i|^{\alpha_i}$ and $v(x) = \prod_{j=1}^M |x - y_j|^{\beta_j}$ which satisfy (3.1).

For the coefficient functions a and b we give the following assumptions which have to be satisfied in all what follows (without further mentioning):

$$a\,v^{-1} \in \mathbf{PC}_{\widetilde{v}\,l[x_1,\dots,x_N]} \quad \text{with} \quad l[x_1,\dots,x_N](x) := \left(1 + \sum_{i=1}^N \big|\ln|x - x_i|\,\big|\right)^{-1}, \tag{5.1}$$

$$b \in \mathbf{PC}^0(\{x_1,\dots,x_N,y_1,\dots,y_M\} \setminus \{-1,1\})\,, \tag{5.2}$$

where $\widetilde{v}$ is defined in Theorem 3.1 and $\mathbf{PC}^0(\{x_1,\dots,x_N,y_1,\dots,y_M\} \setminus \{-1,1\})$ denotes the space of all piecewise $\mathbf{C}^0$-functions with possible jumps in the inner zeros of uv. (The exact definition of this space and its norm is given in Proposition 4.1.) We will see that these assumptions ensure that all functions and all images of operators which we consider in Sections 5 and 6 are well-defined and continuous on $(-1,1) \setminus \{x_1,\dots,x_N,y_1,\dots,y_M\}$. In all statements in which continuity in other points is claimed, this has to be understood in the sense of limits.

In the present section we only consider the operator $A := awI + SbwI$.

Proposition 5.1 $A \in \mathcal{L}(\mathbf{C}_u^0, \mathbf{PC}_{\widetilde{v}})$.

Proof. The part awI of A belongs to $\mathcal{L}(\mathbf{C}_u^0, \mathbf{PC}_{\widetilde{v}})$, since $fw_u/l[x_1,\dots,x_N] \in \mathbf{C}$ ($f \in \mathbf{C}_u^0$) vanishes in all x_i (Lemma 3.7) and, consequently, the last factor on the right hand side of

$$\widetilde{v}\,awf = (l[x_1,\dots,x_N]\,\widetilde{v}\,a\,v^{-1}) \cdot (v\,w_v^{-1}) \cdot (f w_u\,l[x_1,\dots,x_N]^{-1})$$

turns the jumps in x_i of the first factor (which is piecewise continuous with possible jumps in the zeros of uv) into zeros. Hence, $awf \in \mathbf{PC}_{\widetilde{v}}$ and, obviously (by Lemma 3.7), $\|awf\|_{\widetilde{v}} \leq c\,\|f\|_{u,0}$. If we want to prove that also the second part $SbwI$ of A belongs to $\mathcal{L}(\mathbf{C}_u^0, \mathbf{PC}_{\widetilde{v}})$, then, in view of (4.1), it remains to show that $bI \in \mathcal{L}(\mathbf{C}_u^0, \mathbf{PC}_u^0(y_1, \ldots, y_M))$. But this is easy, since we can use again that jumps in x_i are transformed into zeros if we multiply by fu ($f \in \mathbf{C}_u^0$): $(bfu)(x_i) = 0$ for all i. Together with the algebra property of $\mathbf{C}^0$, this gives the assertion (by similar considerations as in the proof of Remark 2.9).

■

We will use this proposition to obtain mapping properties of the operator A, restricted on the spaces of the following scale $\mathbf{C}_u^{\gamma,\delta}$, $\gamma > 0$, $\delta \in \mathbb{R}$.

Definition 5.2 *Let* $0 < \gamma < \infty$ *and* $\delta \in \mathbb{R}$. *The space* $\mathbf{C}_u^{\gamma,\delta}$ *is defined by*

$$\mathbf{C}_u^{\gamma,\delta} := \left\{ f \in \mathbf{C}_u : \|f\|_{u,\gamma,\delta} := \sup_{n=0,1,\ldots} E_n^u(f)\,(n+1)^\gamma \ln^\delta(n+2) < \infty \right\} .$$

In other words: $\mathbf{C}_u^{\gamma,\delta}$ is the approximation space $A(\mathbf{C}_u, \mathbf{l}^\infty(\{(n+1)^\gamma \ln^\delta(n+2)\}); \{\Pi_n\})$ (see the considerations after Definition 2.1). Especially, $\mathbf{C}_u^{\gamma,\delta}$ is a Banach space which is compactly embedded into $\mathbf{C}_u^0$ (see [2], Theorems 3.12 and 3.33). Moreover, it is well-known (at least in case of a Jacobi weight u, but newer results also deal with power weights) that $\mathbf{C}_u^{\gamma,\delta}$ can be described in terms of smoothness properties of its elements. Of course, this fact is of great practical importance, but in the present section we do not need it. Later (in Section 7) we come back to this characterization.

We will see that a decomposition of A into a multiplication operator and an operator which maps polynomials into polynomials is very useful. In case $w_u \equiv 1$ we will take the analogue of (3.3), since in Section 3 we already made good experience with this decomposition. In case $w_u \not\equiv 1$ we cannot go back to this decomposition by writing $Af = (aw_v^{-1} + Sbw_v^{-1}I)(w_u f)$, since the nice property (2.13) of $\mathbf{C}_u^0$ does not hold similarly for $\mathbf{C}_u^{\gamma,\delta}$. To obtain an appropriated splitting of A in all cases, we introduce the monic polynomial $p = p_w$ with the following property:

(5.3) $|w/p|$ is a power weight with exponents in $(-1, 0]$, i.e.:
If $|w(x)| = \prod_{i=1}^L |x - z_i|^{\mu_i}$ with $\mu_i \in (k_i - 1, k_i]$, then $p(x) = \prod_{i=1}^L (x - z_i)^{k_i}$.

Now, $Af = awf + Sbwf$ can be written in the following form:

$$(5.4)\qquad \begin{aligned} (Af)(x) &= \int_{-1}^1 \frac{f(t)p(t) - f(x)p(x)}{t - x}\, b(t) \frac{w(t)}{p(t)}\, dt + f(x)p(x)(Ap^{-1})(x) \\ &=: \left([A - (pAp^{-1}) \cdot I]\, f \right)(x) + [(pAp^{-1}) \cdot f]\,(x) \end{aligned}$$

The first operator $A-(pAp^{-1})\cdot I$ maps polynomials into polynomials. If we can show that $A-(pAp^{-1})\cdot I \in \mathcal{L}(\mathbf{C}_u^0,\mathbf{B}_v)$ ($\mathbf{B}_v$ is defined in (3.6)), then this implies $A-(pAp^{-1})\cdot I \in \mathcal{L}(\mathbf{C}_u^{\gamma,\delta},\mathbf{C}_v^{\gamma,\delta-1})$. (Later we give the details.) First we prove $A-(pAp^{-1})\cdot I \in \mathcal{L}(\mathbf{C}_u^0,\mathbf{PC}_{\widetilde{v}})$, which is (in view of Proposition 5.1) equivalent to the assertion of the following lemma.

Lemma 5.3 *$(pAp^{-1})\cdot I \in \mathcal{L}(\mathbf{C}_u^0,\mathbf{PC}_{\widetilde{v}})$.*

Proof. We have $(pAp^{-1})\cdot I = awI + (pSbwp^{-1})\cdot I$. The properties of

$$Sbwp^{-1} = \left(S\frac{1}{|p/w|}I\right)\left(\frac{|p/w|}{p/w}\,b\right)$$

can be concluded from Proposition 4.2, applied with 1 and $|p/w|$ instead of w_u and w_v: $S|w/p|I \in \mathcal{L}\left(\mathbf{PC}^0(\{x_1,\dots,x_N,y_1,\dots,y_M\}\setminus\{-1,1\}),\,\mathbf{PC}_{|p/w|(-1,x_1,\dots,x_N,y_1,\dots,y_M,1)}\right)$. Thus, $Sbwp^{-1} \in \mathbf{PC}_{|p/w|(-1,x_1,\dots,x_N,y_1,\dots,y_M,1)}$. We remark that $|p/w|(-1,x_1,\dots,x_N,y_1,\dots,y_M,1)$ $= |p/w|(-1,x_1,\,\dots,x_N,1)$, since all $y_j \in \operatorname{supp} u$ are zeros of $|p/w|$. For the same reason, $|p/w|(-1,x_1,\,\dots,x_N,1)$ does not contain logarithmic terms w.r.t. ±1 if $v(\pm1)=0$ and $u(\pm1)\neq 0$. It follows

$$\frac{w^{-1}p\;l[x_1,\dots,x_N]}{1+(uv)(-1)\,|\ln(1+\;.\,)|+(uv)(1)\,|\ln(1-\;.\,)|}\cdot Sbwp^{-1} \in \mathbf{PC}(x_1,\dots,x_N,\,y_1,\dots,y_M)\,.$$

In other words: $(w^{-1}pSbwp^{-1})\cdot v^{-1} \in \mathbf{PC}_{\widetilde{v}\;l[x_1,\dots,x_N]}$. Thus, for $w^{-1}pSbwp^{-1}$ we have the same continuity properties as for a, so that the assertion $awI \in \mathcal{L}(\mathbf{C}_u^0,\mathbf{PC}_{\widetilde{v}})$ (see the proof of Proposition 5.1) also holds for $(pSbwp^{-1})\cdot I$.

■

Proposition 5.1 and Lemma 5.3 imply $A-(pAp^{-1})\cdot I \in \mathcal{L}(\mathbf{C}_u^0,\mathbf{PC}_{\widetilde{v}})$. But we can even prove that the images, multiplied by v (and not by $\widetilde{v}$), are bounded:

$$A-(pAp^{-1})\cdot I \in \mathcal{L}(\mathbf{C}_u^0,\mathbf{B}_v)\,. \tag{5.5}$$

Indeed, if for example $(uv)(1)\neq 0$, then $u\sim v\sim |w|\sim |p|\sim 1$ on $[1-2\varepsilon,1]$ and $f_{|[1-\varepsilon,1]}$ ($f\in\mathbf{C}_u^0$) can be extended to a $\mathbf{C}^0$-function $\widetilde{f}$ with $\widetilde{f}_{|[-1,1-2\varepsilon]}\equiv 0$ such that, with some positive and Hölder continuous extension $\widetilde{u}$ of $u_{|[1-2\varepsilon,1]}$,

$$\|\widetilde{f}p\|_0 = \|\widetilde{f}u\cdot p/\widetilde{u}\|_0 \le c\,\|fu\|_0 \sim \|f\|_{u,0}$$

(since $\mathbf{C}^0$ is a Banach algebra; see the proof of Remark 2.9). Thus, for $x\in[1-(\varepsilon/2),1]$, we may estimate

$$\begin{aligned}&\left|\int_{-1}^{1}\frac{f(t)p(t)-f(x)p(x)}{t-x}\,b(t)\frac{w(t)}{p(t)}\,dt\right|\\ &\qquad\le c\int_{1-\varepsilon}^{1}\left|\frac{(\widetilde{f}p)(t)-(\widetilde{f}p)(x)}{t-x}\right|dt+\frac{2}{\varepsilon}\int_{-1}^{1-\varepsilon}|(fp)(t)-(fp)(x)|\,b(t)\frac{w(t)}{p(t)}\,dt\,.\end{aligned}$$

The first addend is bounded by $c\,\|\widetilde{f p}\|_0 \le c\,\|f\|_{u,0}$ (see the proof of Lemma 3.4) and the second by $c\,\|f\|_u$ (since $|fw|(t) \le \|f\|_u v^{-1}(t)$ and $|fp|(x) \le c\,|fu|(x)$ for $x \in [1-(\varepsilon/2), 1]$). Hence, $Af - (pAp^{-1}) \cdot f$ is bounded in a neighborhood of 1 if $(uv)(1) \ne 0$ (analogously with -1 if $(uv)(-1) \ne 0$) and (5.5) is proved.

From (5.5) we can conclude $A - (pAp^{-1}) \cdot I \in \mathcal{L}(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1})$. For this, we only have to apply the following lemma.

Lemma 5.4 *Let $k \in \mathbb{N}_0$ be fixed and let $B \in \mathcal{L}(\mathbf{C}_u^0, \mathbf{B}_v)$. If $B(\Pi_n) \subseteq \Pi_{n+k}$ for all n, then $B \in \mathcal{L}(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1})$ for all $\gamma > 0$ and all $\delta \in \mathbb{R}$.*

Proof. Let $f \in \mathbf{C}_u^{\gamma,\delta}$ and $f_n \in \Pi_n$ such that $E_n^u(f) = \|f - f_n\|_u$. The we have $Bf_n \in \Pi_{n+k}$ and, consequently,

$$\begin{aligned}
E_{n+k}^v(Bf) &\le \|B(f-f_n)\|_v \le c \sum_{m=0}^{\infty} \frac{E_m^u(f-f_n)}{m+1} \\
&\le c\,\|f-f_n\|_u \sum_{m=0}^{n-1} \frac{1}{m+1} + c \sum_{m=n}^{\infty} \frac{E_m^u(f)}{m+1} \\
&\le c\,\frac{\|f\|_{u,\gamma,\delta}}{(n+1)^\gamma \ln^\delta(n+2)} \sum_{m=0}^{n-1} \frac{1}{m+1} + c\,\|f\|_{u,\gamma,\delta} \sum_{m=n}^{\infty} \frac{1}{(m+1)^{1+\gamma} \ln^\delta(m+2)}\,.
\end{aligned}$$

The first sum can be estimated by $c\ln(n+1)$ and the second sum by $c\left[(n+1)^\gamma \ln^\delta(n+2)\right]^{-1}$ (use that $(m+1)^\varepsilon \ln^\delta(m+2)$ is increasing for $m \ge m_0$). Thus, $Bf \in \mathbf{C}_v$ (since $vBf_n \in \mathbf{C}$ converges uniformly to vBf) and

$$\|Bf\|_{v,\gamma,\delta-1} \sim \|Bf\|_v + \sup_{m=k,k+1,\ldots} (m+1-k)^\gamma \ln^{\delta-1}(m+2-k) E_m^v(Bf) \le c\,\|f\|_{u,\gamma,\delta}\,.$$

(Use the substitution $n = m - k$ in the supremum.)

■

Corollary 5.5 $A - (pAp^{-1}) \cdot I \in \mathcal{L}(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1})$ *for all $\gamma > 0$ and all $\delta \in \mathbb{R}$.*

Proof. The assertion follows from (5.5) and Lemma 5.4, since $(A - (pAp^{-1}) \cdot I)(\Pi_n) \subseteq \Pi_{n+\deg p-1}$.

■

Remark 5.6 *The coefficient function a does not appear in $A - (pAp^{-1}) \cdot I$. One may ask why we did not set $a \equiv 0$ in the proof of $A - (pAp^{-1}) \cdot I \in \mathcal{L}(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1})$. The reason is that indirectly the multiplication operator awI with a satisfying $a\,v^{-1} \in \mathbf{PC}_{\widetilde{v}\;l[x_1,\ldots,x_N]}$ has to be considered if we want to show $(pSbwp^{-1}) \cdot I \in \mathcal{L}(\mathbf{C}_u^0, \mathbf{PC}_{\widetilde{v}})$ (see the proof of Lemma 5.3). Now one may ask why we do not allow $a\,v^{-1} \in \mathbf{B}_{\widetilde{v}\;l[x_1,\ldots,x_N]}$, since all considerations are also possible if we replace weighted spaces of piecewise continuous functions by weighted* *spaces of bounded functions. The reason is that finally we want to obtain criteria which ensure* $A \in \mathcal{L}(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1})$ *(at least for some* γ, δ*), i.e.* $(pAp^{-1}) \cdot I \in \mathcal{L}(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1})$*. But the assumptions on* b *(which cannot be weakened, since we apply (4.1) in the proof of Proposition 5.1) ensure* $(pSbwp^{-1}) \cdot I \in \mathcal{L}(\mathbf{C}_u^0, \mathbf{PC}_{\widetilde{v}})$*, so that we must have* $awI \in \mathcal{L}(\mathbf{C}_u^0, \mathbf{PC}_{\widetilde{v}})$ *if* $A \in \mathcal{L}(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1})$*. Of course,* $awI \in \mathcal{L}(\mathbf{C}_u^0, \mathbf{PC}_{\widetilde{v}})$ *is only possible if* a *is continuous on* $(-1,1) \setminus \{x_1, \ldots, x_N, y_1, \ldots, y_M\}$*, where the singularities of* $a\widetilde{v}/v$ *in* $[\{y_j\} \cup \{\pm 1\}] \setminus \{x_i\}$ *can only be jumps. Hence, only in the points* x_i *the assumptions on* a *can be slightly weakened. For example, the existence of the one-sided limits of* $a\, l[x_1, \ldots, x_N]$ *in the points* x_i *is not necessary (see the proof of Proposition 5.1). However, to avoid difficult notation it seems to be better not to weaken the assumptions on* a *(which are general enough in our opinion).*

In the following main theorem of this section we state the meaning of Corollary 5.5 for the validity of the mapping property $A \in \mathcal{L}(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1})$, and we give a sufficient criterion which implies this property.

Theorem 5.7 *Let* $\gamma > 0$ *and* $\delta \in \mathbb{R}$ *be fixed, and let* $p = p_w$ *be defined in (5.3). For* $A = awI + SbwI$ *(*a, b *satisfying (5.1),(5.2)), the following assertions are equivalent:*

(i) $A \in \mathcal{L}(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1})$.

(ii) $(pAp^{-1}) \cdot I \in \mathcal{L}(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1})$.

A sufficient condition which ensures the validity of (ii) (and (i)) is given by

$$Ap^{-1} \in \mathbf{C}_{|p/w|}^{\gamma,\delta-1}\,. \tag{5.6}$$

Remark 5.8 *In the proof of Lemma 5.3 we have seen that* $v^{-1}Ap^{-1} \in \mathbf{PC}_{|p/w|\,\widetilde{v}\, l[x_1,\ldots,x_N]}$ *and that this property of* Ap^{-1} *implies* $(pAp^{-1}) \cdot I \in \mathcal{L}(\mathbf{C}_u^{\gamma,\delta}, \mathbf{PC}_{\widetilde{v}})$*. It seems to be natural that the stronger mapping property (ii) can only be expected if the corresponding stronger property (5.6) of* Ap^{-1} *holds true. But, unfortunately, we are not able to prove this, i.e., the validity of the implication "(ii)⇒(5.6)" is left as an open problem. Only in case* $u \equiv 1$ *(i.e.,* $v = |1/w| = |p/w|$*) it is clear that (ii) implies (5.6). If we apply this with* $Ap^{-1}I = a(w/p)I + Sb(w/p)I$ *instead of* A*, supposed that* p/w *is continuous (which ensures that* w/p *is a weight of the form (3.2)), then we see:*

> *If* $p/w \in \mathbf{C}$*,* $a\,|w/p| \in \mathbf{PC}_{\widetilde{|p/w|}}$ *and* $b \in \mathbf{PC}^0$ *with jumps in the zeros of* p/w*, then (5.6) is satisfied if and only if* $Ap^{-1}I \in \mathcal{L}(\mathbf{C}_1^{\gamma,\delta}, \mathbf{C}_{|p/w|}^{\gamma,\delta-1})$*.*

This means that, under the above conditions, the implication "(ii)⇒(5.6)" is equivalent to the implication "$A \in \mathcal{L}(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1}) \Rightarrow Ap^{-1}I \in \mathcal{L}(\mathbf{C}_1^{\gamma,\delta}, \mathbf{C}_{|p/w|}^{\gamma,\delta-1})$*". Now we get some doubts with respect to the validity of "(ii)⇒(5.6)" in case* $u \not\equiv 1$*. (Instead of this, we think that* *(ii) is equivalent to some smoothness property of Ap^{-1} in which jumps in the zeros of u are allowed.) However, it is remarkable that the weights u and v do not appear on the right hand side of (5.6). This means that we have to check (5.6) if we want to find weights w which are well-appropriated to transform the unweighted operator $aI + SbI$ into a weighted operator $A = awI + SbwI$ which has good properties in polynomial approximation spaces, independent of the possible choices for the weights of these spaces.*

Proof of Theorem 5.7. The equivalence of (i) and (ii) is already proved (Corollary 5.5) and it remains to show that (5.6) implies (ii). For this, let $f \in \mathbf{C}_u^{\gamma,\delta}$ and take best approximations $f_n \in \Pi_n$ and $g_n \in \Pi_n$ of f and Ap^{-1}, respectively, i.e.,

$$E_n^u(f) = \|f - f_n\|_u \quad \text{and} \quad E_n^{|p/w|}(Ap^{-1}) = \|Ap^{-1} - g_n\|_{|p/w|}\,.$$

Then we obtain, for all $n \in \mathbb{N}$,

$$\begin{aligned} E^v_{2n-1+\deg p}(fpAp^{-1}) &\le \|(fpAp^{-1} - f_n p g_n)v\| \\ &\le \|(f-f_n)u \cdot (p/w)Ap^{-1}\| + \|f_n u \cdot (p/w)(Ap^{-1} - g_n)\| \\ &\le c\,\|(f-f_n)u\| + \|f_n\|_u \|(p/w)(Ap^{-1} - g_n)\| \le \frac{c\,\|f\|_{u,\gamma,\delta}}{(n+1)^\gamma \ln^{\delta-1}(n+2)}\,. \end{aligned}$$

For $n = 0$ we have $f_n = g_n = 0$ and the above estimate yields $\|fpAp^{-1}\|_v \le c\,\|f\|_u$. Now it is easy to show that $E_m^v(fpAp^{-1}) \le c\,\|f\|_{u,\gamma,\delta}\,(m+1)^{-\gamma}\ln^{1-\delta}(m+2)$ for all $m \in \mathbb{N} \cup \{0\}$ (since $E_m \le E_{2n-1+\deg p}$ for $m \in \{2n-1+\deg p, 2n+\deg p\}$), i.e. $(pAp^{-1}) \cdot I \in \mathcal{L}(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1})$. ■

We finish this section with the consideration of some special cases in which (5.6) is satisfied:

Proposition 5.9 *Let $\gamma > 0$ and $\delta \in \mathbb{R}$ be fixed. In all of the following cases, the operator A is bounded from $\mathbf{C}_u^{\gamma,\delta}$ into $\mathbf{C}_v^{\gamma,\delta-1}$.*

(i) $A = A_{\alpha,\beta} = av^{\alpha,\beta}I + Sbv^{\alpha,\beta}I$ with constants a, b, α, β as in Proposition 3.5. (For generalizations of this case we refer to [17], Section 9.5 and Theorem 9.9.)

(ii) b arbitrary (i.e., satisfying only (5.2)) and $a = -w^{-1}pSbwp^{-1}$. (Remember that this a satisfies (5.1); see the end of the proof of Lemma 5.3.)

(iii) a and b such that

$$\begin{aligned} &awp^{-1} \in \mathbf{C}^{\gamma,\delta-1}_{|p/w|} \quad \textit{and} \\ &\varphi^{-r}bwp^{-1} \in \mathbf{C}^{\gamma,\delta}_{|p/w|} \;\; \textit{for some } r > \gamma \ (\textit{or } r = \gamma \textit{ if } \delta \le 0), \;\; \textit{where } \varphi(x) := \sqrt{1-x^2}. \end{aligned}$$

Especially, a and b have to vanish in the zeros and jumps of p/w and b also in ± 1.

(iv) $awp^{-1} \in \mathbf{C}^{\gamma,\delta-1}_{|p/w|}$ *and* $bwp^{-1}\chi_{[-1,1]} \in \mathbf{C}^{\gamma,\delta}_{|p/w|}[-2,2]$.

Proof. In case (i) we have $p = -\mathrm{sign}(\alpha)\, v^{r,s}$, where r and s are 1 or 0 in dependence of the sign 1 or -1 of α and β, respectively. Hence, $w/p = -\mathrm{sign}(\alpha)\, v^{\widetilde{\alpha},\widetilde{\beta}}$ with $\widetilde{\alpha} = \alpha - r$ and $\widetilde{\beta} = \beta - s$. $A_{\widetilde{\alpha},\widetilde{\beta}} = -\mathrm{sign}(\alpha)\,(av^{\widetilde{\alpha},\widetilde{\beta}}I + bSv^{\widetilde{\alpha},\widetilde{\beta}}I)$ is again an operator as in Proposition 3.5, where $\widetilde{\alpha} + \widetilde{\beta} = -1$. Thus, $Ap^{-1} = A_{\widetilde{\alpha},\widetilde{\beta}}1 = 0$, i.e., (5.6) is satisfied (and, even more, we have a special case of situation (ii)).

The assumption in (ii) is only a reformulation of $Ap^{-1} = 0$ (i.e., A is equal to the operator $A - (pAp^{-1}) \cdot I$ from Corollary 5.5).

To prove the assertion in case (iii), we first apply Corollary 5.5 in the non-weighted case with $|p/w|$ instead of w_u and w_v:

$$\ln \frac{1+\,.}{1-\,.} \cdot I + S \in \mathcal{L}(\mathbf{C}^{\gamma,\delta}_{|p/w|}, \mathbf{C}^{\gamma,\delta-1}_{|p/w|})\,. \tag{5.7}$$

If we take into account that $\varphi^r \ln(1+\,.\,)/(1-\,.\,)$ belongs to $\mathbf{C}^{r,-1}_1 \subseteq \mathbf{C}^{\gamma,\delta-1}_1$ ([7], Section 8.5), then it is easy to prove that

$$bwp^{-1} \ln \frac{1+\,.}{1-\,.} = \frac{bwp^{-1}}{\varphi^r} \cdot \varphi^r \ln \frac{1+\,.}{1-\,.} \in \mathbf{C}^{\gamma,\delta-1}_{|p/w|}\,. \tag{5.8}$$

Moreover, the function φ^r belongs to $\mathbf{C}^{r,0}_1 \subseteq \mathbf{C}^{\gamma,\delta}_1$ ([7], Section 8.5) and this implies $bwp^{-1} = \varphi^r \cdot \varphi^{-r} bwp^{-1} \in \mathbf{C}^{\gamma,\delta}_{|p/w|}$. Hence, application of (5.7) to the function bwp^{-1} yields, together with (5.8), $Sbwp^{-1} \in \mathbf{C}^{\gamma,\delta-1}_{|p/w|}$. By assumption, the other part of Ap^{-1} also belongs to $\mathbf{C}^{\gamma,\delta-1}_{|p/w|}$, i.e., (5.6) is satisfied.

In case (iv) we only have to mention that, after a linear transformation $[-1,1] \to [-2,2]$, (5.7) holds similarly for the singular integral operator $S_{[-2,2]}$ on $[-2,2]$. Thus,

$$\ln \frac{2+\,.}{2-\,.} \cdot I + S_{[-2,2]} \in \mathcal{L}\big(\mathbf{C}^{\gamma,\delta}_{|p/w|}[-2,2], \mathbf{C}^{\gamma,\delta-1}_{|p/w|}[-2,2]\big) \tag{5.9}$$

and we get $Sbwp^{-1} = S_{[-2,2]} bwp^{-1}\chi_{[-1,1]} \in \mathbf{C}^{\gamma,\delta-1}_{|p/w|}$ (since $\ln(2+\,.\,)/(2-\,.\,) \in \mathbf{C}^\infty[-1,1]$). ∎

Of course, the conditions in (iii) and (iv) are very restrictive and further investigations are necessary to find weaker assumptions which imply $(pAp^{-1}) \cdot I \in \mathcal{L}(\mathbf{C}^{\gamma,\delta}_u, \mathbf{C}^{\gamma,\delta-1}_v)$. In view of the length of this paper we give up further considerations in this direction. We only mention that probably one cannot give much weaker assumptions on a and b if one wants to prove the stronger assertion (5.6) without supposing any further connection between the parts awp^{-1} and $Sbwp^{-1}$ of Ap^{-1} (as in (ii)). In other words: If the transformation $aI + SbI \to awI + SbwI$ shall lead to an operator with good properties in pairs of polynomial approximation spaces, where the upper bound for the parameter γ of these spaces shall be large, independent of the possible choices for the weights of these spaces, then one should try to determine wp^{-1} in such a way that the "bad parts" of awp^{-1} and $Sbwp^{-1}$ disappear in the sum $awp^{-1} + Sbwp^{-1} = Ap^{-1}$. Now the question is, under which conditions on a and b is it possible to find such a weight wp^{-1} (for which $|wp^{-1}|$ is an integrable power weight with negative exponents)? In literature (see e.g., [17], Chapter 9) one often considers operators with Hölder continuous coefficients, for which it is possible to find an appropriated weight w such that the weighted operator A maps polynomials into polynomials (see the next section for more details). For such operators one can show that $Ap^{-1} \in \Pi$, i.e. (5.6) holds for all γ and δ. Maybe it is possible to generalize this known construction of the weight w to the case of operators with piecewise Hölder continuous coefficients. We leave this as an open question.

6 The operator $\rho^{-1}(aw\varrho I + bSw\varrho I)$

Take the notation and assumptions from the beginning of the preceding section and choose some function $\varrho : (-1,1) \setminus \{x_1, \ldots, x_N, y_1, \ldots, y_M\} \to \mathbb{C}$ such that

$$P := b\,\varrho^{-1} \in \Pi \quad \text{and} \quad \varrho\, v^{-1} \in \mathbf{L}^1(-1,1)\,.$$

Now we consider the operator

$$B := awI + PSw\varrho I = \varrho^{-1}(aw\varrho I + bSw\varrho I)\,.$$

For example, we may take $\varrho = b$, but in this case we get the operator

$$A := awI + SbwI$$

which we have already studied. The operator B is of interest if one wants to study integral equations in which an operator of the type $aw_0I + bSw_0I$, $w_0 \in \mathbf{L}^1$, appears. In this case one can look for some ϱ such that $b/\varrho \in \Pi$ and $w = w_0/\varrho$ is a weight of the form (3.2). Then, $\varrho^{-1}(aw_0I + bSw_0I)$ is our operator B. We remark that the operators

$$a\sigma I + \pi^{-1} Sb\sigma I \quad \text{and} \quad a\sigma I + PS\sigma_0 I = c\left(a\sigma_0 I + \pi^{-1} bS\sigma_0 I\right) \tag{6.1}$$

with Hölder continuous coefficients $a, b : [-1,1] \to \mathbb{R}$ (satisfying $a^2 + b^2 > 0$) and appropriated weight $\sigma = \sigma(a,b)$ which are usually considered in literature (see [17], Chapter 9) fit in our theory: For these operators we have $\sigma = v^{\alpha,\beta}h$ with certain $\alpha, \beta \in (-1,1)$ and some Hölder continuous function $h \neq 0$. Moreover, it is usually supposed that $c^{-1}v^{-|\alpha|,-|\beta|}$ is a generalized Jacobi weight. Particularly, $c^{-1}v^{-|\alpha|,-|\beta|} \in \mathbf{L}^1$. Thus, the operators in (6.1) correspond to A and B if we take $w = v^{\alpha,\beta}$, $\widetilde{a} = ah$, $\widetilde{b} = bh/\pi$ and $\rho = c^{-1}h$.

If we look at the difference

$$(6.2)\qquad (Af - Bf)(x) = \int_{-1}^{1} \frac{P(t) - P(x)}{t - x}\, \varrho(t) w(t) f(t)\, dt\,,$$

which belongs to $\Pi_{\deg \mathrm{P}}$ for all $f \in \mathbf{C}_u$, then it becomes clear that, for all $\gamma > 0$ and $\delta \in \mathbb{R}$,

$$(6.3)\qquad B \in \mathcal{L}(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1}) \ \text{ if and only if } \ A \in \mathcal{L}(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1})\,.$$

(Remark that $A - B \in \mathcal{L}(\mathbf{C}_u, (\Pi_{\deg \mathrm{P}}, \|\,.\,\|))$ and, hence, $A - B \in \mathcal{L}(\mathbf{C}_u, \mathbf{C}_v^{\gamma,\delta})$ for all γ, δ.) (6.3) means that we can use Theorem 5.7 to check whether B belongs to $\mathcal{L}(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta-1})$.

Remark 6.1 *If we do not suppose that P is a polynomial, but a continuous and sufficiently smooth function, then usually (6.3) remains true up to a certain upper bound for the value of γ. For example, if $\varrho \in \mathbf{L}^\infty(-1,1)$, then, for fixed $f \in \mathbf{C}_u$, the right hand side of (6.2) can be viewed as an operator A_f applied to P, and the proof of Lemma 3.4 shows that $A_f \in \mathcal{L}(\mathbf{C}^0, \mathbf{B}_v)$ with $\|A_f\| \le c\,\|f\|_u$. But A_f maps Π_n into Π_n and it follows $A_f \in \mathcal{L}(\mathbf{C}_1^{\gamma_0,\delta_0}, \mathbf{C}_v^{\gamma_0,\delta_0-1})$, again with $\|A_f\| \le c\,\|f\|_u$. This means that $A-B \in \mathcal{L}(\mathbf{C}_u, \mathbf{C}_v^{\gamma_0,\delta_0-1})$ if $\varrho \in \mathbf{L}^\infty(-1,1)$ and $P \in \mathbf{C}_1^{\gamma_0,\delta_0}$, i.e., in this case (6.3) is true for $0 < \gamma < \gamma_0$ and, if $\delta \le \delta_0$, also for $\gamma = \gamma_0$.*

7 Remarks and Generalizations

In the preceding sections we have studied the mapping properties of Cauchy singular integral operators $A = awI + SbwI$ and $B = \varrho^{-1}(aw\varrho I + bSw\varrho I)$, $\varrho^{-1} \in b^{-1}\Pi$, with an arbitrary fixed weight w of the form

$$w(x) = \prod_{i=1}^{L} \left[\mathrm{sign}(x - z_i)\right]^{\eta_i} (x - z_i)^{\mu_i}\,, \quad -1 \le z_1 < \ldots < z_L \le 1, \ \ \eta_i \in \{1,2\}, \ \ \mu_i > -1$$

in pairs of approximation spaces based on $\mathbf{C}_u$ and $\mathbf{C}_v$, respectively, where the power weights $u = |w_u| \in \mathbf{L}^\infty(-1,1)$ and $v = |w_v| \in \mathbf{L}^\infty(-1,1) \cap \{v : v^{-1} \in \mathbf{L}^1(-1,1)\}$ correspond to a representation $w = w_u/w_v$ of the type (3.2). (We remark that w has a jump in z_i if $\mu_i = 0$ and $\eta_i = 1$. In this case we have $\alpha(z_i) = \beta(z_i)$ in all admissible representations (3.2) of w.) As singularities of the coefficient functions a and b we have admitted jumps (or even logarithmic singularities for a). Unfortunately, the corresponding assumptions (5.1) and (5.2) depend on u and v. In other words: In dependence on the singularities of a and b, there are restrictions on the possible choices of the weights w_u and w_v in (3.2). One may ask for mapping properties in pairs of spaces those weights do not satisfy these restrictions. Thus, let us consider a representation (3.2) of w which corresponds to arbitrary fixed power weights $u(x) = \prod_{i=1}^{N} |x - x_i|^{\alpha_i}$ and $v(x) = \prod_{j=1}^{M} |x - y_j|^{\beta_j}$ satisfying (3.1), and let

$$(7.1)\qquad a \in \mathbf{PC}_{l[\xi_1,\ldots,\xi_R]}\,, \ \ b \in \mathbf{PC}^0(\xi_1, \ldots, \xi_R)\,, \ \ \text{where } \ -1 = \xi_1 < \xi_2 < \ldots < \xi_R = 1$$

(without supposing any connection between the singularities ξ_i and the zeros of u and v). Then it is clear that $awI \in \mathcal{L}(\mathbf{C}^0_u, \mathbf{PC}_{v\, l[\xi_1,\dots,\xi_R]})$, $bI \in \mathcal{L}(\mathbf{C}^0_u, \mathbf{PC}^0(w_u;\xi_1,\dots,\xi_R))$ (since $f \in \mathbf{C}^0_u$ implies $fw_u \in \mathbf{C}^0$), and $SwbI \in \mathcal{L}(\mathbf{C}^0_u, \mathbf{PC}_{v(\xi_1,\dots,\xi_R)})$ (Proposition 4.2). Hence,

$$A \in \mathcal{L}(\mathbf{C}^0_u, \mathbf{PC}_{v\, l[\xi_1,\dots,\xi_R]})\,. \tag{7.2}$$

Of course (since the continuity in x_i of the part $w_u f$ of $awf = (a/w_v)w_u f$ has to be understood in the sense of the limit $\lim_{x\to x_i}(w_u f)(x) = 0$), we must mention that, from now on, the images of operators have to be viewed as continuous functions on $(-1,1) \setminus \{x_1,\dots,x_N,y_1,\dots,y_M,\xi_1,\dots,\xi_R\}$ (i.e., only on this set they are well-defined) and if continuity in other points is stated, then this has to be understood in the sense of limits.

For the part awI of A we have more than (7.2):

$$\frac{aw}{l[x_1,\dots,x_N]}\, I \in \mathcal{L}(\mathbf{C}^0_u, \mathbf{PC}_{v\, l[\xi_1,\dots,\xi_R]}) \tag{7.3}$$

because of Lemma 3.7, even if only $a \in \mathbf{PC}_{l[\xi_1,\dots,\xi_R]}(x_1,\dots,x_N,y_1,\dots,y_M)$ (which means, by definition, $a\, l[\xi_1,\dots,\xi_R] \in \mathbf{PC}(\xi_1,\dots,\xi_R,x_1,\dots,x_N,y_1,\dots,y_M)$). If we apply this with $w^{-1}p\, l[x_1,\dots,x_N]\, Sbwp^{-1}$ instead of a (p from (5.3)), taking into account that, by Proposition 4.2 (applied with 1 and $|p/w|$ instead of w_u and w_v; see also the proof of Lemma 5.3),

$$w^{-1}p\, l[x_1,\dots,x_N]\, Sbwp^{-1} \in \mathbf{PC}_{l[\xi_1,\dots,\xi_R]}(x_1,\dots,x_N,y_1,\dots,y_M)\,,$$

then it follows $(pAp^{-1})\cdot I \in \mathcal{L}(\mathbf{C}^0_u, \mathbf{PC}_{v\, l[\xi_1,\dots,\xi_R]})$ and, consequently,

$$A - (pAp^{-1})\cdot I \in \mathcal{L}(\mathbf{C}^0_u, \mathbf{PC}_{v\, l[\xi_1,\dots,\xi_R]})\,. \tag{7.4}$$

As in Corollary 5.5 this implies

$$A - (pAp^{-1})\cdot I \in \mathcal{L}\big(\mathbf{C}^{\gamma,\delta}_u, \mathbf{C}^{\gamma,\delta-1}_{v\, l[\xi_1,\dots,\xi_R]}\big) \quad \text{for all } \gamma > 0 \text{ and } \delta \in \mathbb{R}\,. \tag{7.5}$$

Now we want to come back to the weight v in the image space. For this we remark that, for some sufficiently small constant $C > 0$ and for $-1 = t_1 < t_2 < \ldots < t_S = 1$, $\delta_1,\dots,\delta_S \ge 0$ defined by

$$\{t_1,\dots,t_S\} = \{\xi_1,\dots,\xi_R\} \cup \{y_1,\dots,y_M\}\,,\quad \delta_j = \begin{cases} \beta_i & \text{if } t_j = y_i \\ 0 & \text{if } t_j \notin \{y_1,\dots,y_M\} \end{cases}$$

the following estimate of the norm $\|P_n v\|$, $P_n \in \Pi_n$ $(n \in \mathbb{N})$, holds true:

$$\begin{aligned} \|P_n v\| &\sim \max_{j=1,\dots,S-1} \|P_n (t_{j+1} - \,.\,)^{\delta_{j+1}}(\,.\, - t_j)^{\delta_j}\|_{\mathbf{C}[t_j,t_{j+1}]} \\ &\le c \max_{j=1,\dots,S-1} \|P_n (t_{j+1} - \,.\,)^{\delta_{j+1}}(\,.\, - t_j)^{\delta_j}\|_{\mathbf{C}[t_j+Cn^{-2},t_{j+1}-Cn^{-2}]} \end{aligned}$$

(see [7], Theorem 8.4.8). On $[t_j + Cn^{-2}, t_{j+1} - Cn^{-2}]$ we have $l[\xi_1, \ldots, \xi_R] \geq c/\ln(n+1)$ and $v \sim (t_{j+1} - \,.\,)^{\delta_{j+1}}(\,.\, - t_j)^{\delta_j}$. Hence,

$$\|P_n v\| \leq c\,\|P_n v\, l[\xi_1, \ldots, \xi_R]\| \ln(n+1) \quad \text{for } P_n \in \Pi_n \text{ and } n \in \mathbb{N}.$$

($c \neq c(n, P_n)$). This implies that $\mathbf{C}^0_{v\,l[\xi_1,\ldots,\xi_R]}$ is continuously imbedded into $\mathbf{C}_v$ (see [12], Theorem 4.2 and Remark 4.3, or [2], Theorem 4.9). Now we apply Lemma 5.4 with $v\,l[\xi_1, \ldots, \xi_R]$ instead of u and with the embedding operator $B \in \mathcal{L}(\mathbf{C}^0_{v\,l[\xi_1,\ldots,\xi_R]}, \mathbf{C}_v)$. It follows that $\mathbf{C}^{\gamma,\delta}_{v\,l[\xi_1,\ldots,\xi_R]}$ is continuously imbedded into $\mathbf{C}^{\gamma,\delta-1}_v$ for all $\gamma > 0$ and $\delta \in \mathbb{R}$. Together with (7.5) we obtain $A - (pAp^{-1}) \cdot I \in \mathcal{L}(\mathbf{C}^{\gamma,\delta}_u, \mathbf{C}^{\gamma,\delta-2}_v)$ and similarly to Theorem 5.7 and assertion (6.3) we get the following result:

Theorem 7.1 *Let $\gamma > 0$ and $\delta \in \mathbb{R}$ be fixed, and let $p = p_w$ be defined in (5.3). For $A = awI + SbwI$ and $B = \varrho^{-1}(aw\varrho I + bSw\varrho I)$ with a, b satisfying (7.1) and $\varrho = b/P \in v\,\mathbf{L}^1(-1,1)$ ($P \in \Pi$), the following assertions are equivalent:*

(i) $A \in \mathcal{L}(\mathbf{C}^{\gamma,\delta}_u, \mathbf{C}^{\gamma,\delta-2}_v)$.

(ii) $B \in \mathcal{L}(\mathbf{C}^{\gamma,\delta}_u, \mathbf{C}^{\gamma,\delta-2}_v)$.

(iii) $(pAp^{-1}) \cdot I \in \mathcal{L}(\mathbf{C}^{\gamma,\delta}_u, \mathbf{C}^{\gamma,\delta-2}_v)$.

A sufficient condition which ensures the validity of (iii) (and (i),(ii)) is given by

$$Ap^{-1} \in \mathbf{C}^{\gamma,\delta-2}_{|p/w|}\,.$$

Now we consider the question whether Theorems 5.7 and 7.1 can be generalized to other scales of approximation spaces based on $\mathbf{C}_u$ and $\mathbf{C}_v$, respectively. In other words: We are looking for generalizations of Lemma 5.4 and of the implication "$Ap^{-1} \in \mathbf{C}^{\gamma,\delta}_{|p/w|} \Rightarrow (pAp^{-1}) \cdot I \in \mathcal{L}(\mathbf{C}^{\gamma,\delta}_u, \mathbf{C}^{\gamma,\delta}_v)$". Generalizations of Lemma 5.4 can be obtained with the help of the reiteration theorem for approximation spaces ([10], Theorem 6.2 and Corollaries 6.3, 6.4). Another way is described in [11] (even in a general framework in which $\mathbf{C}_u$, $\mathbf{B}_v$, and Π_n can be replaced by arbitrary Banach spaces X, Y, and nested subspaces $X_n \subseteq X \cap Y$):

Lemma 7.2 ([11], Theorem 2.3) *Let $\mathcal{B} = \{b_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that, for some $C, \varepsilon > 0$ and some $n_0 \in \mathbb{N}$,*

$$b_{n+1} \leq C\, b_n \;\; (n \in \mathbb{N}) \quad \textit{and} \quad \left\{b_n^{-1} \ln^{1+\varepsilon}(n+2)\right\}_{n=n_0}^{\infty} \;\; \textit{is decreasing}$$

Moreover, let $\mathbf{C}^{\mathcal{B}}_u := A(\mathbf{C}_u, \mathbf{l}^\infty(\mathcal{B}); \{\Pi_n\})$ (see the considerations after Definition 2.1) and set $\mathcal{B}/\log := \{b_n/\ln(n+2)\}$. The space $\mathbf{C}^{\mathcal{B}}_u$ is continuously embedded into $\mathbf{C}^0_u$ and

$$B \in \mathcal{L}(\mathbf{C}^0_u, \mathbf{B}_v)\,, \;\; B(\Pi_n) \subseteq \Pi_{n+k} \;\; (k\text{: constant}) \quad \textit{imply} \quad B \in \mathcal{L}(\mathbf{C}^{\mathcal{B}}_u, \mathbf{C}^{\mathcal{B}/\log}_v)\,.$$

(The last assertion follows from $b_n \geq c\,b_{n+k}$ and [11], estimate (2.2), which asserts that $E^v_{n+k}(Bf) \leq c\,\|f\|_{\mathbf{C}^{\mathcal{B}}_u}\, b_n^{-1} \ln(n+2)$ for all $n \in \mathbb{N}$ and all $f \in \mathbf{C}^{\mathcal{B}}_u$.)

Lemma 7.2 shows that Corollary 5.5 (which corresponds to $b_n = (n+1)^\gamma \ln^\delta(n+2)$) can be generalized to the pair $(\mathbf{C}^{\mathcal{B}}_u, \mathbf{C}^{\mathcal{B}/\log}_v)$ if $\mathcal{B}$ satisfies the above conditions. Thus, for such $\mathcal{B}$, the first part of Theorem 5.7 (and also (6.3)) remains true with $\mathcal{L}(\mathbf{C}^{\mathcal{B}}_u, \mathbf{C}^{\mathcal{B}/\log}_v)$ instead of $\mathcal{L}(\mathbf{C}^{\gamma,\delta}_u, \mathbf{C}^{\gamma,\delta-1}_v)$. If in addition $b_{2n} \leq c\,b_n$, then, by a slight modification of the proof, also the last assertion of Theorem 5.7 can be generalized. Let us summarize:

Theorem 7.3 *Let $\mathcal{B}$ satisfy the assumptions of Lemma 7.2 and suppose that $b_{2n} \leq c\,b_n$ for all $n \in \mathbb{N}$ ($c \neq c(n)$). Moreover, let a, b satisfy the conditions (5.1),(5.2), and let $p = p_w$ be defined in (5.3). Then, for $A = awI + SbwI$ and $B = \varrho^{-1}(aw\varrho I + bSw\varrho I)$ with $\varrho = b/P \in v\,\mathbf{L}^1(-1,1)$ ($P \in \Pi$), the following assertions are equivalent:*

(i) $A \in \mathcal{L}(\mathbf{C}^{\mathcal{B}}_u, \mathbf{C}^{\mathcal{B}/\log}_v)$.

(ii) $B \in \mathcal{L}(\mathbf{C}^{\mathcal{B}}_u, \mathbf{C}^{\mathcal{B}/\log}_v)$.

(iii) $(pAp^{-1}) \cdot I \in \mathcal{L}(\mathbf{C}^{\mathcal{B}}_u, \mathbf{C}^{\mathcal{B}/\log}_v)$.

A sufficient condition which ensures the validity of (iii) (and (i),(ii)) is given by

$$Ap^{-1} \in \mathbf{C}^{\mathcal{B}/\log}_{|p/w|}\,.$$

Now we want to compare our results with the known result from [9], Section 9.10, in which it is shown that, for any $\alpha \in (0,1)$ and any power weight $\mu(x) = \prod_{i=1}^R |x - t_i|^{\gamma_i}$ with $-1 = t_1 < t_2 < \ldots < t_R = 1$ and $\alpha < \gamma_i < \alpha + 1$ for all i, the following result is true:

$$\begin{aligned} &\text{If } a, b \in \mathbf{PC}(t_1, \ldots, t_R) \text{ with } a_{\,|\,(t_i,t_{i+1})}, b_{\,|\,(t_i,t_{i+1})} \in \mathbf{H}^\alpha([t_i, t_{i+1}])\,, \text{ then} \\ &aI + bS \in \mathcal{L}\big(\mathbf{H}^\alpha_0(\mu), \mathbf{H}^\alpha_0(\mu)\big) \text{ and } aI + SbI \in \mathcal{L}\big(\mathbf{H}^\alpha_0(\mu), \mathbf{H}^\alpha_0(\mu)\big)\,, \text{ where} \\ &\mathbf{H}^\alpha_0(\mu) = \{f \in \mathbf{C}(\operatorname{supp}\mu) : f\mu \in \mathbf{H}^\alpha([-1,1]) \text{ and } (f\mu)(t_i) = 0 \text{ for all } i\}\,. \end{aligned} \tag{7.6}$$

Here we denote by $\mathbf{H}^\alpha(I)$ the space of all functions on I which are Hölder continuous with exponent α. Obviously, $aI, bI \in \mathcal{L}(\mathbf{H}^\alpha_0(\mu), \mathbf{H}^\alpha_0(\mu))$ and so it is sufficient to deal with S instead of $aI + bS$ and $aI + SbI$. First we will consider the weaker assertion

$$S \in \mathcal{L}\big(\mathbf{H}^\alpha_0(\mu), \mathbf{H}^\alpha(\mu)\big) \qquad (\,\mathbf{H}^\alpha(\mu) := \{f \in \mathbf{C}(\operatorname{supp}\mu) : f\mu \in \mathbf{H}^\alpha([-1,1])\}\,)\,. \tag{7.7}$$

Since every $f \in \mathbf{H}^\alpha_0(\mu)$ can be written as $f = \sum_i f_i$ with $f_i \in \mathbf{H}^\alpha_0(|x-t_i|^{\gamma_i})$ and $f_i \equiv 0$ outside some closed neighborhood $N(t_i)$ of t_i which contains no other t_j, this mapping property is proved if one can show that, for $\mu_{t,\gamma}(x) := |x - t|^\gamma$, $t = t_i$ and $\gamma = \gamma_i$ fixed,

$$S \in \mathcal{L}\big(\widetilde{\mathbf{H}}^\alpha_0(\mu_{t,\gamma}), \mathbf{H}^\alpha(\mu_{t,\gamma})\big)\,, \quad \widetilde{\mathbf{H}}^\alpha_0(\mu_{t,\gamma}) := \{f \in \mathbf{H}^\alpha_0(\mu_{t,\gamma}) : f \equiv 0 \text{ outside } N(t)\}\,. \tag{7.8}$$

With our theory it is not possible to obtain exactly this result, since we consider other pairs of spaces. However, we are able to conclude "almost" the same result: First we remark that, obviously, $\mathbf{H}_0^\alpha(\mu_{t,\gamma})$ is continuously embedded into $\mathbf{L}^\infty_{t,\gamma-\alpha} := \{f : f\mu_{t,\gamma-\alpha} \in \mathbf{L}^\infty(-1,1)\}$. Together with $\mu_{t,\gamma}S - S\mu_{t,\gamma}I \in \mathcal{L}(\mathbf{L}^\infty_{t,\gamma-\alpha}, \mathbf{C}^{\gamma,-1}_{t,\gamma-\alpha})$ (see Remark 6.1; $\mathbf{C}^{\gamma,-1}_{t,\gamma-\alpha} := \mathbf{C}^{\gamma,-1}_u$ with $u = \mu_{t,\gamma-\alpha}$) we obtain $\mu_{t,\gamma}S - S\mu_{t,\gamma}I \in \mathcal{L}(\mathbf{H}_0^\alpha(\mu_{t,\gamma}), \mathbf{C}^{\gamma,-1}_{t,\gamma-\alpha})$. After transformation onto $[-2,2]$, taking into account that $f \in \widetilde{\mathbf{H}}_0^\alpha(\mu_{t,\gamma})$ implies $f\mu_{t,\gamma}\chi_{[-1,1]} \in \mathbf{H}^\alpha[-2,2]$, we get

$$\mu_{t,\gamma}S - S\mu_{t,\gamma}I \in \mathcal{L}\big(\widetilde{\mathbf{H}}_0^\alpha(\mu_{t,\gamma}), \mathbf{C}^{\gamma,-1}_{t,\gamma-\alpha}[-2,2]\big)\,.$$

In [14], (7.27) and proof of (7.22), it is shown, for $t \neq \pm 1$, $\|P_n\,\mu_{t,\gamma-\alpha}\| \sim \|P_n(\mu_{t,\gamma-\alpha} + n^{\alpha-\gamma})\|$, $P_n \in \Pi_n$, $n \in \mathbb{N}$. After transformation onto $[-2,2]$ it follows (since $t \neq \pm 2$)

$$\|P_n\|_{\mathbf{C}[-2,2]} \le c\,n^{\gamma-\alpha}\|P_n\,\mu_{t,\gamma-\alpha}\|_{\mathbf{C}[-2,2]} \quad \text{for } P_n \in \Pi_n \text{ and } n \in \mathbb{N}\,.$$

($c \neq c(n,P_n)$). This implies that $\mathbf{C}^{\gamma,-1}_{t,\gamma-\alpha}[-2,2]$ is continuously imbedded into $\mathbf{C}^{\alpha,-1}_1[-2,2] \subseteq \mathbf{H}^{\alpha-\varepsilon} := \mathbf{H}^{\alpha-\varepsilon}([-1,1])$ ([2], Theorem 4.9 and [10], Example 6.5). Consequently,

$$\mu_{t,\gamma}S - S\mu_{t,\gamma}I \in \mathcal{L}\big(\widetilde{\mathbf{H}}_0^\alpha(\mu_{t,\gamma}), \mathbf{H}^{\alpha-\varepsilon}\big)\,.$$

From (5.9), applied with 1 instead of $|p/w|$, it follows $S\mu_{t,\gamma}I \in \mathcal{L}(\widetilde{\mathbf{H}}_0^\alpha(\mu_{t,\gamma}), \mathbf{H}^{\alpha-\varepsilon})$. Thus,

$$\mu_{t,\gamma}S \in \mathcal{L}\big(\widetilde{\mathbf{H}}_0^\alpha(\mu_{t,\gamma}), \mathbf{H}^{\alpha-\varepsilon}\big)\,, \quad \text{i.e.,} \quad S \in \mathcal{L}\big(\widetilde{\mathbf{H}}_0^\alpha(\mu_{t,\gamma}), \mathbf{H}^{\alpha-\varepsilon}(\mu_{t,\gamma})\big)\,,$$

which is weaker than (7.8). So we only get $S \in \mathcal{L}(\mathbf{H}_0^\alpha(\mu), \mathbf{H}^{\alpha-\varepsilon}(\mu))$ instead of (7.7). We can also prove that, for $f \in \mathbf{H}_0^\alpha(\mu)$, μSf vanishes in all t_i: One can show that $f \in \mathbf{H}_0^\alpha(\mu)$ implies $f \in \mathbf{H}_0^\delta(\widetilde{\mu})$, where $\widetilde{\mu}(x) = \prod_{i=1}^R |x-t_i|^{\gamma_i-\alpha+\delta}$ and $0 < \delta \le \alpha$ such that $\gamma_i - \alpha + \delta < 1$ for all i. (This is only needed if $\max\gamma_i \ge 1$. Otherwise we can take $\delta = \alpha$.) From (5.9), now applied with $\widetilde{\mu}$ instead of $|p/w|$, it follows, with some $\eta > 0$, $Sf = S_{[-2,2]}f\chi_{[-1,1]} \in \mathbf{C}^{\eta,-1}_{\widetilde{\mu}}$, since, by Lemma 2.3, $H_0^\delta(\widetilde{\mu})$ (considered as space of functions on $[-2,2]$: $f \to f\chi_{[-1,1]}$) is continuously embedded into $\mathbf{C}^{\eta,0}_{\widetilde{\mu}}[-2,2]$ ($\eta = \delta/k$). This implies $(\widetilde{\mu}Sf)(t_i) = 0$ and, consequently, $(\mu Sf)(t_i) = 0$ for all i. (Alternatively, we can take $\delta < \alpha$ and then this follows from $S \in \mathcal{L}(\mathbf{H}_0^\delta(\widetilde{\mu}), \mathbf{H}^{\delta-\varepsilon}(\widetilde{\mu}))$.) So we can conclude the following result, which is a little bit weaker than (7.6):

$$aI + bS,\, aI + SbI \in \mathcal{L}\big(\mathbf{H}_0^\alpha(\mu), \mathbf{H}_0^{\alpha-\varepsilon}(\mu)\big)\,.$$

The reason for this loss of ε is that, in our theory, we consider $\mathcal{L}(\mathbf{C}^{\gamma,\delta}_u, \mathbf{C}^{\gamma,\delta-1}_v)$ instead of $\mathcal{L}(\mathbf{C}^{\gamma,\delta}_u, \mathbf{C}^{\gamma,\delta}_v)$. Thus, a loss of one power of $\ln n$ for the convergence order of the best approximation errors of the image functions is admitted. We conjecture that in reality we have no such loss. In the following special case this is already known, but the proof is so hard that, in the framework of this paper, we give up further considerations in this direction.

Remark 7.4 ([13], Proof of Theorem 3.1) *Let $\beta \in (0,1)$ and let $A_{-\beta,\beta}$ be the operator from Proposition 3.5. Then, for $u = v^{0,\beta}$ and $v = v^{\beta,0}$, $A_{-\beta,\beta} \in \mathcal{L}(\mathbf{C}_u^{\gamma,\delta}, \mathbf{C}_v^{\gamma,\delta})$ ($\gamma > 0$, $\delta \in \mathbb{R}$). A similar result is true for $A_{\beta,-\beta}$.*

We mention that in [13] it is proved more (end of the proof of [13], Theorem 3.1):

$$E_{2n}^v(A_{-\beta,\beta}f) \le c \sum_{i=0}^{\infty} E_{2^in}^u(f) \quad \text{for } f \in \mathbf{C}_u^0 \text{ and } n \in \mathbb{N} \quad (c \ne c(n,f)) .$$

We finish this paper with two remarks about the characterization of $\mathbf{C}_u^{\gamma,\delta}$ ($\gamma > 0$, $\delta \in \mathbb{R}$, u a power weight with positive exponents) in terms of smoothness properties of its elements. For this aim, let $r \in \mathbb{N}$ and define the modulus of smoothness

$$\begin{aligned}\omega_\varphi^r(f,t)_u \;:=\; & \sup_{0<h\le t} \|u\,\Delta_{h\varphi}^r f\|_{\mathbf{C}\left([-1+4r^2h^2,1-4r^2h^2]\setminus \bigcup_{x_i\in(-1,1)}(x_i-4rh,\,x_i+4rh)\right)} \\ & + \inf_{P\in\Pi_r} \|(f-P)u\|_{\mathbf{C}[-1,-1+4r^2t^2]} + \inf_{P\in\Pi_r} \|(f-P)u\|_{\mathbf{C}[1-4r^2t^2,1]} \\ & + \sum_{x_i\in(-1,1)} \inf_{P\in\Pi_r} \|(f-P)u\|_{\mathbf{C}[x_i-4rt,x_i+4rt]}\end{aligned}$$

([5]). Here we denote by $\Delta_h^r f$ the r-th central difference of f (i.e., $(\Delta_h^1 f)(x) = (\Delta_h f)(x) := f(x+\frac{h}{2})-f(x-\frac{h}{2})$, $\Delta_h^r f := \Delta_h(\Delta_h^{r-1}f)$ for $r > 1$), and $\Delta_{h\varphi}^r f$ means that, in $(\Delta_h^r f)(x)$, h has to be replaced by $h\varphi(x)$, where $\varphi(x) := \sqrt{1-x^2}$. For $h \ge (2r)^{-1}$ we set $[-1+4r^2h^2, 1-4r^2h^2] := \emptyset$ and $\|\,.\,\|_{\mathbf{C}(\emptyset)} := 0$.

Remark 7.5 ([5], Theorem 3.1) *$f \in \mathbf{C}_u$ belongs to $\mathbf{C}_u^{\gamma,\delta}$ if and only if, for some arbitrary fixed $r > \gamma$, $\omega_\varphi^r(f,t)_u \le c\,t^\gamma \ln^{-\delta}(1+t^{-1})$ for all $t \in (0,1]$. Moreover, the expression*

$$\|f\|_u + \sup_{t\in(0,1]} \frac{\omega_\varphi^r(f,t)_u}{t^\gamma} \ln^\delta(1+t^{-1})$$

defines an equivalent norm in $\mathbf{C}_u^{\gamma,\delta}$.

One can also use properties of derivatives of f to estimate the behaviour of $E_n^u(f)$:

Remark 7.6 ([5], Corollary 3.1) *Let $\alpha_i \notin \{1,\dots,r\}$ for all $x_i \in (-1,1)$ and let $f \in \mathbf{C}^{(r-1)}\big((-1,1)\setminus\{x_1,\dots,x_N\}\big) \cap \mathbf{C}^{(r-s-1)}\big((-1,1)\big)$, where $s := \min\left\{r, \min_{x_i\in(-1,1)}[\alpha_i]\right\}$ and $\mathbf{C}^{(-1)}\big((-1,1)\big) := \mathbf{C}((-1,1)\setminus\{x_1,\dots,x_N\})$. If $f^{(r-1)} \in \mathbf{AC}_{\mathrm{loc}}\big((-1,1)\setminus\{x_1,\dots,x_N\}\big)$ and, in case $s < r$, $f^{(r-s-1)} \in \mathbf{AC}_{\mathrm{loc}}\big((-1,1)\big)$, then*

$$E_n^u(f) \le c\,n^{-r} \inf_{P\in\Pi_n} \left(\|(f-P)u\| + \|(f^{(r)}-P^{(r)})\varphi^r u\|\right) \quad \textit{for all } n \in \mathbb{N}$$

($c \ne c(n,f)$). Here we denote by $\mathbf{AC}_{\mathrm{loc}}(M)$ the set of all functions which are absolutely continuous on every compact interval $I \subseteq M$.

References

[1] J. M. Almira and U. Luther, Compactness and generalized approximation spaces, *Numer. Funct. Anal. and Optimiz.* **23(1&2)**, 1-38 (2002).

[2] J. M. Almira and U. Luther, Generalized approximation spaces and applications, Accepted for publication in *Math. Nachr.*

[3] J. M. Almira and U. Luther, Approximation algebras and applications, In: K. Kopotun, T. Lyche, M. Neamtu (eds.), *Trends in Approximation Theory*, Vanderbilt Univ. Press, Nashville, 2001.

[4] J. M. Almira, N. Del Toro, and U. Luther, A survey on a general theory of approximation spaces and its applications, Colloquium del Depto. Análisis Matemático, Univ. Complutense Madrid, 2001.

[5] M. C. De Bonis, G. Mastroianni, and M. G. Russo, Polynomial approximation with special doubling weights, Submitted.

[6] R. A. DeVore and G. G. Lorentz, *Constructive Approximation*, Springer-Verlag, 1993.

[7] Z. Ditzian and V. Totik, *Moduli of Smoothness*, Springer-Verlag, 1987.

[8] Z. Ditzian and V. Totik, Remarks on Besov spaces and best polynomial approximation, *Proc. Amer. Math. Soc.* **104** (1988), 1059-1066.

[9] I. Gohberg and N. Krupnik, *One-Dimensional Linear Singular Integral Equations*, Vol. II, Birkhäuser Verlag, 1992.

[10] U. Luther, Representation, interpolation, and reiteration theorems for generalized approximation spaces, Accepted for publication in *Ann. Mat. Pura Appl.*

[11] U. Luther, Approximation spaces in the numerical analysis of operator equations, Submitted to *Advances in Comp. Math.*

[12] U. Luther and M. G. Russo, Boundedness of the Hilbert transformation in some weighted Besov spaces, *Integr. Equ. Oper. Theory* **36** (2000), 220-240.

[13] G. Mastroianni, M. G. Russo, and W. Themistoclakis, The boundedness of the Cauchy singular integral operator in weighted Besov type spaces with uniform norms, *Integr. Equ. Oper. Theory* **42** (2002), no. 1, 57-89.

[14] G. Mastroianni and V. Totik, Weighted polynomial inequalities with doubling and A_∞ weights, *Constr. Approx.* **16** (2000), 37-71.

[15] S. G. Mikhlin and S. Prössdorf, *Singuläre Integraloperatoren*, Akademie-Verlag, Berlin, 1980.

[16] A. Pietsch, Approximation spaces, *J. Approx. Theory* **32** (1981), 115-134.

[17] S. Prössdorf and B. Silbermann, *Numerical Analysis for Integral and Related Operator Equations*, Akademie-Verlag, Berlin, 1991.

Uwe Luther
Technische Universität Chemnitz
Fakultät für Mathematik
D-09107 Chemnitz, Germany
uwe.luther@mathematik.tu-chemnitz.de

Keywords and phrases: Cauchy singular integral operators, Weighted spaces of continuous functions, Approximation spaces
MSC (AMS) Classification: 44A15, 46E15