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Abstract. The topic of the paper is the study of modified finite sections of Toeplitz operators and their singular values. We prove the splitting property for the singular values and consider two important consequences. We show that the kernel dimension of a Fredholm Toeplitz operator with piecewise continuous matrix-valued generating function can be extracted from the singular values behavior of the modified sections. Secondly, we generalize the results on asymptotic Moore-Penrose invertibility of Heinig and Hellinger to piecewise continuous generating functions.

Key words. Toeplitz operators, singular values, finite sections

AMS subject classifications. 45E10, 65F20

1. Introduction. Let PC denote the C^*-algebra of all piecewise continuous functions defined on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, and let $PC_{N\times N}$ be the C^*-algebra of all $N \times N$ matrices with entries from PC. We shall mainly deal with the question of how the singular values of matrices A_n approximating the Toeplitz operator $T(a)$ acting on the space l^2_N are distributed, where $a \in PC_{N\times N}$ and the operator $T(a)$ is supposed to be Fredholm. Of course one expects that the answer depends strongly on the kind of the matrices A_n. There are many possible approximations A_n; here we restrict ourselves to the so-called modified finite sections. If the approximations are the familiar finite sections $T_n(a)$, (which are square matrices) the complete answer was obtained by S. Roch and the author in [R/S 2]. It was shown that the set Λ_n of the singular values of the finite sections $T_n(a)$ of a Fredholm Toeplitz operator is subject to the splitting property: We say that the singular values (computed via $A_n^* A_n$) of a sequence (A_n) of $k(n) \times l(n)$ matrices A_n have the splitting property if there exist a sequence $c_n \longrightarrow 0$ $(c_n \geq 0)$ and a number $d > 0$ such that

$$\Lambda_n \subset [0, c_n] \cup [d, \infty) \quad \text{for all} \quad n\,,$$

and the singular values of A_n are said to meet the k-splitting property if, in addition, for all sufficiently large n exactly k singular values of A_n lie in $[0, c_n]$.

The mentioned result reads now as follows: If $T(a)$ is Fredholm, $a \in PC_{N\times N}$, then the sequence $(T_n(a))$ has the k-splitting property with

$$k = \dim\ \ker T(a) + \dim\ \ker T(\tilde{a})\,,$$

where $\tilde{a}(t) := a(1/t)$.

Thus, if we would know the number dim ker $T(\tilde{a})$, then we would know the kernel dimension of $T(a)$ provided that we would be able to compute the set $\Lambda_n \cap [0, c_n]$.

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As a rule, we know the number dim ker $T(\tilde{a})$ only in very special cases. So the question arises whether the operator can be approximated by matrices A_n such that the splitting property still holds with some operator $\tilde{A}$ instead of $T(\tilde{a})$ and such that the kernel dimension of $\tilde{A}$ is available. In other words, we try to design approximations A_n to $T(a)$ having prescribed properties. We shall show that the so-called modified finite sections are good candidates for our aim. Besides we show that our approach is intimely related to the approximation of the Moore-Penrose inverse of the Toeplitz operator $T(a)$. In the course of the paper we do not only recover the results of Heinig/Hellinger [H/H]) for Toeplitz operators $T(a)$ with a from the Wiener class $W_{N\times N}$, but extend them to operators $T(a)$ with $a \in PC_{N\times N}$. Notice that the methods of [H/H] do not work in this more general situation. Our main tool is a C^*-algebra approach mainly developed by S. Roch and the author in the last years (see for instance the book [H/R/S]). Let us mention two results proved in the Sections 3 and 4. Define block matrices $T_{n,0,r}(a)$ and $T_{n,r,0}(a)$ (whose entries are $N \times N$-matrices) by

$$T_{n,0,r}(a) = (a_{i-j}) \ , \ \ 0 \le i \le n, \, 0 \le j \le n - r \, ,$$

$$T_{n,r,0}(a) = (a_{i-j}) \ , \ \ 0 \le i \le n - r, \, 0 \le j \le n \, ,$$

respectively, where $a_k \, (k \in \mathbb{Z})$ are the Fourier coefficients of $a \in PC_{N\times N}$. The following theorems are consequences of the main results obtained in Sections 3 and 4, respectively.

Theorem 1.1. Let the Toeplitz operator $T(a) : l^2_N \longrightarrow l^2_N$ be Fredholm, $a \in PC_{N\times \mathbf{N}}$. Then the singular values of the sequence $(T_{n,0,r}(a))$ enjoy the k-splitting property, where k dependes on r. Moreover,

$$k = \dim\ker T(a)$$

for r large enough.

Examples will be presented in an appendix.

In what folows, let A^+ denote the Moore-Penrose inverse of an operator A.

Theorem 1.2. Let $T(a)$ be Fredholm, $a \in PC_{N\times N}$.

(a) If $T(a)$ is left invertible, then there is an r_0 such that the Moore-Penrose inverses $(T^+_{n,0,r}(a))$ converge strongly to $T^+(a)$ for all $r \ge r_0$ as n goes to infinity.
(b) If $T(a)$ is right invertible, then there is an r_0 such that the Moore-Penrose inverses $(T^+_{n,r,0}(a))$ converge strongly to $T^+(a)$ for all $r \ge r_0$ as n goes to infinity.

2. Toeplitz operators and the algebra generated by familiar finite sections. We shall see in Sectoin 3 that the sequences $(T_{n,0,r}(a))$ and $(T_{n,r,0}(a))$ can be identifyed with some sequences of square matrices which belong to the algebra $\mathcal{A}$ generated by all sequences of familiar finite sections of Toeplitz operators with generating functions from $PC_{N\times N}$. This observation already shows that it would certainly be of importance to have as much knowledge on $\mathcal{A}$ as possible. Fortunately, the algebra $\mathcal{A}$

was intensively studied in the past. Here we merely recall definitions and some non-trivial facts needed in what follows. Let l_N^2 denote the Hilbert space of all sequences $(x_i)_{i\in\mathbb{Z}^+}, \mathbb{Z}^+ := \{k \in \mathbb{Z} : k \geq 0\}$, where $x_i \in \mathbb{C}^N$ and

$$\| (x_i) \| := \left(\sum_{i=0}^{\infty} \| x_i \|^2 \right)^{\frac{1}{2}} < \infty$$

($\| x_i \|$ refers to the familiar euclidean norm in $\mathbb{C}^N$.)

Given an $N \times N$ matrix-valued function $a \in L^\infty_{N\times N}$ (where L^∞ means the essentially bounded functions defined on $\mathbb{T}$) denote the sequence of its Fourier coefficients by $(a_n)_{n\in\mathbb{Z}}$. The Toeplitz operator $T(a) : l_N^2 \longrightarrow l_N^2$ is defined by $(x_i) \mapsto (y_i)$, where

$$y_i = \sum_{j=0}^{\infty} a_{i-j} x_j \ (i \in \mathbb{Z}^+) .$$

The Toeplitz operator $T(a)$ with generating function $a \in L^\infty(\mathbb{T})_{N\times N}$ is bounded, that is $T(a) \in \mathcal{L}(l_N^2)$ and moreover $\| T(a) \| = \| a \|_\infty$ (see for instance [B/S 2]). Here, for a Hilbert space $\mathcal{H}$, we denote by $\mathcal{L}(\mathcal{H})$ the C^*-algebra of all bounded linear operators acting on $\mathcal{H}$. Further, let $\mathcal{K}(\mathcal{H})$ stand for the closed two-sided ideal of all compact operators. Now introduce operators P_n and W_n on l_N^2 by

$$\begin{aligned} P_n(x_0, x_1, \ldots, x_n, \ldots) &= (x_0, \ldots, x_n, 0, \ldots) , \\ W_n(x_0, x_1, \ldots, x_n, \ldots) &= (x_n, x_{n-1}, \ldots, x_0, 0, \ldots) . \end{aligned} \tag{2.1}$$

Obviously, $P_n, W_n \in \mathcal{L}(l_N^2)$ and

$$P_n^2 = P_n, W_n^2 = P_n .$$

In what follows we will identify operators acting on im P_n or on l^2 ($N = 1$) with their matrix representation with respect to the standard basis of im P_n or l^2, respectively. We proceed analogously in the case $N > 1$. For $n \in \mathbb{Z}^+$ the (familiar) finite section $T_n(a)$ of $T(a)$ is defined by

$$T_n(a) := P_n T(a) P_n .$$

The finite section $T_n(a)$ is related to the operator W_n by

$$W_n T_n(a) W_n = T_n(\tilde{a}) ,$$

where $\tilde{a}$ is defined by $\tilde{a}(t) := a(1/t)$.

The matrix representation of $T_n(a)$ is given by the finite block Toeplitz matrix

$$(a_{i-j})_{i,j=0}^n ,$$

whereas the underlying matrix representation of $T(a)$ is given by the infinite Toeplitz matrix

$$(a_{i-j})_{i,j=0}^\infty .$$

Let $\mathcal{F}$ denote the collection of all operator sequences $(A_n)_{n\in Z^+}$ with $A_n \in \mathcal{L}(\text{im } P_n)$ and

(2.2) $$\| (A_n) \| := \sup_n \| A_n \| < \infty .$$

With the operations $(A_n) + (B_n) := (A_n + B_n)$, $(A_n)(B_n) := (A_n B_n)$, $(A_n)^* := (A_n^*)$ and the norm (2.2), $\mathcal{F}$ actually becomes a C^*-algebra. First of all, recall that a sequence $(A_n) \in \mathcal{F}$ is called norm stable if the operators $A_n : \text{im } P_n \longrightarrow \text{im } P_n$ are invertible for n large enough (say for $n \geq n_0$) and

$$\sup_{n\geq n_0} \| A_n^{-1} \| < \infty .$$

If, in addition, A_n converges strongly to some invertible operator A, then the sequence $(A_k^{-1})_{k\geq n_0}$ converges strongly to A^{-1}. We shall write s-lim $A_n = A$ if the sequence A_n tends strongly to A. Let $\mathcal{G}$ denote the collection of all sequences $(A_n) \in \mathcal{F}$ with $\|A_n\| \to 0$. Clearly, $\mathcal{G}$ actually forms a closed two-sided ideal in $\mathcal{F}$.

Note the following

Proposition 2.1. (see [B/S 2] or [B/S 3]) $(A_n) \in \mathcal{F}$ is norm stable if and only if the coset $(A_n) + \mathcal{G}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{G}$.

Now consider the smallest C^*-subalgebra $\mathcal{A} \subset \mathcal{F}$ containing all sequences $(T_n(a))$ with $a \in PC_{N\times N}$. The algebra $\mathcal{A}$ has a lot of remarkable properties which will be of decisive importance in studying the problems formulated in the introduction.

Proposition 2.2. ([B/S 2], [B/S 3]) Let $K_1, K_2 \in \mathcal{K}(l_N^2), (C_n) \in \mathcal{G}$. Then

(2.3) $$(B_n) := (P_n K_1 P_n + W_n K_2 W_n + C_n) \in \mathcal{A} .$$

Moreover, all sequences of the form (2.3) form a closed two-sided ideal in $\mathcal{A}$.

Proposition 2.3. ([B/S 2], [B/S 3]) For each sequence $(A_n) \in \mathcal{A}$ there exist the strong limits

$$\begin{aligned} \mathcal{W}_1 (A_n) &:= s - \lim A_n , \\ \mathcal{W}_2 (A_n) &:= s - \lim W_n A_n W_n . \end{aligned}$$

Moreover, $\mathcal{W}_1 : \mathcal{A} \to \mathcal{L}(l_N^2)$ and $\mathcal{W}_2 : \mathcal{A} \to \mathcal{L}(l_N^2)$ are $*$-homomorphisms that act as follows:

$$\begin{aligned} &\mathcal{W}_1 (T_n(a)) = T(a) , && \mathcal{W} (T_n(a)) = T(\tilde{a}) , \\ &\mathcal{W}_1 (B_n) = K_1 , && \mathcal{W}_2 (B_n) = K_2 , \end{aligned}$$

where (B_n) is the sequence (2.3).

Now we formulate a theorem, which is completely proved in [B/S 1] and [S 1]. This theorem was, however, not explicitly stated there, but it is a direct consequence of Theorems 1 and 2 in [B/S 1]. The first explicit formulation was published in [S 2].

Theorem 2.1. Let $(A_n) \in \mathcal{A}$ be arbitrarily given.

(a) The sequence (A_n) is norm stable if and only if the operators $\mathcal{W}_1(A_n)$ and $\mathcal{W}_2(A_n)$ are invertible.
(b) The operator $\mathcal{W}_1(A_n)$ is a Fredholm operator, if and only if the operator $\mathcal{W}_2(A_n)$ is a Fredholm operator.

We call a sequence $(A_n) \in \mathcal{A}$ a Fredholm sequence if $\mathcal{W}_1(A_n)$ is a Fredholm operator.

This theorem is a faregoing extension of classic results (see for instance [G/F]). It is easy to see, that the mapping

$$\text{smb} : (A_n) \mapsto (\mathcal{W}_1(A_n), \mathcal{W}_2(A_n))$$

is a $*$-homomorphism of the C^*-algebra $\mathcal{A}$ into the C^*-algebra $\mathcal{L}_2 := \mathcal{L}(l^2_N) \oplus \mathcal{L}(l^2_N)$, the direct sum of two copies of $\mathcal{L}(l^2_N)$, with norm $\| (B, C) \| = \max\{\| B \|, \| C \|\}$. The image of $\mathcal{A}$ under this homomorphism is denoted by smb $\mathcal{A}$. The element smb (A_n) is called the stability symbol of (A_n).

Theorem 2.2. [H/R/S], [S 2]) The algebras $\mathcal{A}/\mathcal{G}$ and smb $\mathcal{A}$ are isometrically isomorphic. The isomorphism is given by

$$(A_n) + \mathcal{G} \mapsto \text{ smb } (A_n) \, .$$

This theorem shows that $\mathcal{A}/\mathcal{G}$ can be represented in a very nice way. Moreover, it says that all properties of a sequence $(A_n) \in \mathcal{A}$ which do not depend on the first members of (A_n), should be stored in the operators $\mathcal{W}_1(A_n)$ and $\mathcal{W}_2(A_n)$. In other words the asymptotic properties of (A_n) should be reflected in the mentioned operators. The following theorem makes this precise for the asymptotic behavior of the singular values.

Theorem 2.3. (see [H/R/S] or [R/S 1]) Let $(A_n) \in \mathcal{A}$ be a Fredholm sequence and let Λ_n denote the set of all singular values of A_n. Then (A_n) is subject to the k-splitting property with

$$k = \dim \ \ker \mathcal{W}_1(A_n) + \dim \ \ker \mathcal{W}_2(A_n) \, .$$

One can show, that the k-splitting property is also necessary for $\mathcal{W}_1(A_n)$ being Fredholm (see [H/R/S]). We will make use of these theorems in the next sections.

3. Modified finite sections and the splitting property. For each multiindex $\alpha = (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbf{Z}^+ (i = 1, \ldots, N)$ and for each operator $A \in \mathcal{L}(l^2)$ we define an operator $A^\alpha \in \mathcal{L}(l^2_N)$ by

$$\text{diag } (A^{\alpha_1}, \ldots, A^{\alpha_N})$$

Further, let e_1 and e_{-1} stand for the functions $e_1, e_{-1} : \mathbb{T} \longrightarrow \mathbb{T}$ defined by $t \mapsto t$ and $t \mapsto t^{-1}$, respectively. The Toeplitz operator $T(e_1) : l^2 \longrightarrow l^2$ will be also denoted by V. Then it follows that $V^* = T(e_{-1})$. Notice that for any multiindices α and β and any function $a \in L^\infty_{N \times N}(\mathbb{T})$ the property

$$V^{*\beta} T(a) V^\alpha = T\left(e^\beta_{-1} a e^\alpha_1\right) \tag{3.1}$$

is fulfilled. We shall also use the projections

$$P_\alpha := \text{ diag } (P_{\alpha_1}, \ldots, P_{\alpha_N}) \, , \tag{3.2}$$

where P_{α_i} is defined by (2.1) for $N = 1$. The multiindex $(n, n, \ldots, n)$ will also be denoted by n. In each case the meaning will become clear from the context. Notice also the relations (α, β,-multiindices)

$$V^{\alpha} P_{\beta} = P_{\beta+\alpha} V^{\alpha} \, , \; P_{\beta} V^{*\alpha} = V^{*\alpha} P_{\beta+\alpha} \tag{3.3}$$

and

$$P_{\beta} V^{\alpha} = P_{\beta} V^{\alpha} P_{\beta} \, , \; V^{*\alpha} P_{\beta} = P_{\beta} V^{*\alpha} P_{\beta} \, . \tag{3.4}$$

It is sufficient to prove these assertions in the case $N = 1$. Recall that the projection P_m ($m \in \mathbb{Z}^+$) can be written as $P_m = I - V^m V^{*m}$. Now it follows that

$$V^k P_m = V^k - V^{m+k} V^{*m+k} V^k = P_{m+k} V^k \, .$$

By taking the adjoint we get (3.3). The proof of (3.4) is also very simple.

In what follows we shall consider modified finite sections of the Toeplitz operator $T(a)$ of the form

$$T_{n,\alpha,\beta}(a) := P_{n-\alpha} T(a) P_{n-\beta} \, , \tag{3.5}$$

where $n = (n, \ldots, n)$ and $P_{n-\alpha}(P_{n-\beta})$ is the zero operator if $n - \alpha$ $(n - \beta)$ is not a multiindex, that is if it has negative components. With help of (3.3) and (3.4) the finite sections (3.5) can be rewritten ($n - \alpha \geq 0$, $n - \beta \geq 0$) as

$$P_{n-\alpha} T(a) P_{n-\beta} = P_n V^{*\alpha} P_n T(e_{-1}^{-\alpha} a e_1^{-\beta}) P_n V^{\beta} P_n \, . \tag{3.6}$$

This simple observation is crucial: it shows that the sequence of the finite sections (3.5) belongs to the algebra $\mathcal{A}$ for $a \in PC_{N\times N}$. In what follows we will identify (3.5) with (3.6), which are square matrices and can be assumed to be extensions of the matrices (3.5) by zeros. The following theorem is a direct consequence of Theorem 2.3.

Theorem 3.1. Let the Toeplitz operator $T(a)$ be Fredholm and $a \in PC_{N\times N}$. Then the singular values of $(T_{n,\alpha,\beta}(a))$ meet the k-splitting property with

$$k = \dim\ \ker T(a) + \dim\ \ker \tilde{T}_{\alpha,\beta}(a) \, , \tag{3.7}$$

where

$$\tilde{T}_{\alpha,\beta}(a) := V^{\alpha} T(e_{-1}^{\alpha} \tilde{a} e_1^{\beta}) V^{*\beta} \, .$$

Proof: It is easy to see that

$$\begin{aligned} \mathcal{W}_1(T_{n,\alpha,\beta}(a)) &= T(a) \, , \\ \mathcal{W}_2(T_{n,\alpha,\beta}(a)) &= \tilde{T}_{\alpha,\beta}(a) \, . \end{aligned}$$

Now it remains to apply Theorem 2.3.

We would like to employ this theorem in order to compute the kernel dimension of a Fredholm Toeplitz operator $T(a)$ with $a \in PC_{N\times N}$. To this aim we introduce the notion of generalized factorization for $p = 2$ (see [L/S]): a right factorization in $L^2(\mathbb{T})$ of a matrix function $G : \mathbb{T} \longrightarrow \mathbb{C}_{N\times N}$ is by definition a representation of the form

$$G(t) = G_-(t)\Lambda(t)G_+(t)\,, \tag{3.8}$$

where $G_+^{\pm 1} \in H^2$ (H^2 is the Hardy space), $G_-^{\pm 1} \in \overline{H}^2$, $\Lambda(t) =$ diag $(t^{\kappa_1}, \ldots, t^{\kappa_N})$, and $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_N$ are integers. It is known, that the numbers κ_i, $i = 1, 2, \ldots, N$, are uniquely determined if the representation (3.8) exists. They are called the right partial indices. Analogously one defines a left factorization:

$$G(t) = \hat{G}_+(t)\Omega(t)\hat{G}_-(t)\,,$$

where $\hat{G}_+, \hat{G}_-$ and Ω fulfill the same conditions as above. Even if for a given matrix function G a left and a right factorization exists, then the right and left partial indices do not necessarily coincide. A simple but useful example is provided by the matrix function

$$G(t) = \begin{pmatrix} t & 1 \\ 0 & t^{-1} \end{pmatrix} :$$

In fact, the left and right factorizations are given by

$$\begin{aligned} G(t) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & t^{-1} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ t^{-1} & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

This circumstance causes difficulties in the theory of Toeplitz operators with matrix valued generating functions. From the last example it follows that $T(G)$ is invertible but $T(\tilde{G})$ is not (contrary to scalar valued generating functions). Clearly, if G possesses a right (left) factorization, then G^{-1} possesses a left (right) factorization, too.

The following fact will be used in what follows (see [L/S]): If $T(a)$, $a \in L^\infty_{N\times N}(\mathbb{T})$, is Fredholm in l^2_N, then a possesses a right factorization and

$$\dim\ \ker T(a) = \sum_{i=1}^{N} \max\{-\kappa_i, 0\}\,.$$

Now we specify Theorem 3.1.

Theorem 3.2. Let $T(a)$ be Fredholm, $a \in PC_{N\times N}$. Then there is an $r_0 \in \mathbb{Z}_+$ such that for all $r := (r, \ldots, r) \geq (r_0, \ldots, r_0)$ the operator $T(\tilde{a}e_1^r)$ has trivial kernel and all statements of Theorem 3.1 with respect to the modified section $T_{n,0,r}(a)$ hold and the kernel dimension of $\tilde{T}_{0,r}(a)$ equals $N \cdot r$.

Proof: Theorem 3.1 ensures that the sequence $(T_{n,0,r}(a))$ has the k-splitting property with

$$k = \dim\ker T(a) + \dim\ker \tilde{T}_{0,r}(a)\,.$$

Since $T(\tilde{a})$ is Fredholm too, the function $\tilde{a}$ possesses a right factorization $\tilde{a}(t) = F_-(t)\Lambda(t)F_+(t), \Lambda(t) =$ diag $\{t^{s_1}, \cdots, t^{s_N}\}$ and $s_1 \geq s_2 \geq \cdots \geq s_N$. Then there exists a number r_0 such that for all multiindices $r = (r, \cdots, r) \geq (r_0, \cdots, r_0)$ the operator $T(\tilde{a}e_1^r)$ has trivial kernel. Indeed, take $r_0 = \max\{-s_1, \cdots, -s_N, 0\}$. Obviously, $\tilde{a}e_1^r$ is subject to the factorization

$$\tilde{a}e_1^r = F_-\Lambda F_+ e_1^r = F_-\Lambda e_1^r F_+ \, ,$$

and

$$\dim \ker T(\tilde{a}e_1^r) = \sum_{i=1}^{N} \max\{-s_i - r, 0\} = 0 \, , \, r \geq r_0 \, .$$

Thus, dim ker $\tilde{T}_{0,r}(a) = \dim \ker V^{*r} = N \cdot r$, and we are done.

Remark 3.1. In order to compute the kernel dimension of $T(a)$ one has to determine the singular values for $T_{n,0,r}(a)$ lying in $[0, c_n]$ and to subtract $N \cdot r$ (n, r large enough). How large r must be chosen? The following observation is useful: if r is replaced by $r + 1$ and the number of singular values in the respective set $[0, c_n]$ increases exactly by N then a correct r is found, that is $r \geq r_0$. Indeed, if $r < r_0$ then the difference of the kernel dimensions

$$\dim \ker \tilde{T}_{0,r+1}(a) - \dim \ker \tilde{T}_{0,r}(a) \tag{3.9}$$

is less than N because dim ker $T(\tilde{a}e_1^r) - \dim \ker T(\tilde{a}e_1^{r+1}) > 0$.

Remark 3.2. If the kernel dimensions of the operators $T(ae_1^r)$ ($r = (r, \ldots, r)$), can be computed, then the right partial indices κ_i of a can also be computed.

Remark 3.3. The described procedure offers a way to compute the kernel dimension of a Fredholm Toeplitz operator $T(a)$ with $a \in PC_{N\times N}$. This might seem strange because the kernel dimension of a Fredholm operator A is not stable under small pertubations (it is however upper semi-continuous). Nevertheless the proposal method of kernel computation is stable under small pertuabations. The reason is at least the following: We compute the number of singular values of the related matrices lying in $[0, c_n]$ (n large enough), that is something like the sum of kernel dimensions, where the related singular values are far from the remaining part of the singular values. More precisely, we have to show that for given $\varepsilon > 0$ the singular values of $T_{n,0,r}(b), b \in PC_{n\times N}$, lie in the set $[0, c_n + \varepsilon] \cup [d - \varepsilon, \infty)$ if only $||T(a) - T(b)||$ is small enough; moreover, we have to show that the number of the singular values of $T_{n,0,r}(b)$ lying in $[0, c_n + \varepsilon]$ equals $\dim \ker T(a) + \dim \ker \tilde{T}_{0,r}(a)$. Further, the computation of the singular values of $T_{n,0,r}(b)$ leads again to computational errors. If they are small enough, we will get the same statement as above. How one can see this?

First observe that the uniform limiting set of the sets Λ_n equals

$$\begin{aligned} \Lambda(a) &= \text{sp } (\mathcal{W}_1^*(T_{n,0,r})\mathcal{W}_1(T_{n,0,r}))^{\frac{1}{2}} \cup \text{ sp } (\mathcal{W}_2^*(T_{n,0,r})\mathcal{W}_2(T_{n,0,r}))^{\frac{1}{2}} \\ &= \text{sp } (\text{smb } (T_{n,0,r})^* \text{ smb } (T_{n,0,r}))^{\frac{1}{2}} \end{aligned} \tag{3.10}$$

(see [R/S 2], Theorem 4.14).

If $0 \notin \Lambda(a)$ there is nothing to prove. Indeed, the property $0 \notin \Lambda(a)$ implies for a Fredholm operator $T(a)$, $a \in PC_{N\times N}$, the stability of $(T_{n,0,r})$ by Theorem 2.1. But stability is stable under small pertubations; this is a direct consequence of Proposition 2.1. Suppose $0 \in \Lambda(a)$. Then the point 0 is an isolated point in $\Lambda(a)$ (by Proposition 4.2 below), moreover, the multiplicity of 0 equals

$$(3.11)\, \dim \ \ker \ (\mathrm{smb}\ (T_{n,0,r})^* \ \mathrm{smb}\ T_{n,0,r})^{\frac{1}{2}} = \dim \ \ker T(a) + \dim \ \ker \tilde{T}_{0,r}(a)\,.$$

If we approximate the Toeplitz operator $T(a)$ (in the class of Toeplitz operators with $PC_{N\times N}$ generating functions), then if $T(b)$ is close enough to $T(a)$ the point 0 can split into a finite number of points which lie in $[0,\varepsilon) \cap \Lambda(b)$, ($\varepsilon > 0$ given and sufficiently small), and their number (counted with respect to their multiplicity) equals again (3.11) (see Theorem 6.27 (d) in [H/R/S]). Now one has to use Theorem 7.12 in [H/R/S] (recall that $\mathcal{A}$ is a standard algebra in the sense of [H/R/S]). Hence, the number of singular values of $T_{n,0,r}(b)$ lying in $[0, c_n + \varepsilon]$ equals again (3.7) for n large enough. This shows that the described procedure is as stable as it can be.

Remark 3.4. Many programs such as Matlab use immediately the rectangular form of the matrix $T_{n,0,r}(a)$ (that is they drop down the r last columns consisting of zero matrices) for computing the singular values. In this case the singular values of $(T_{n,0,r}(a))$ have the k-splitting property with

$$k = \dim \ \ker T(a)\,,$$

if r is large enough. The above mentioned criterion reads now as follows: a correct r is found if $(T_{n,0,r}(a))$ and $(T_{n,0,r+1}(a))$ have the same k-splitting property. This fact is reflected in Theorem 1.1.

Remark 3.5. If K is compact and $a \in PC_{N\times N}$, then the described methods can also be used to compute dim ker $(T(a) + K)$, where $T(a)$ is Fredholm. One has only additionally to take into account Proposition 2.2.

4. Asymptotic Moore-Penrose invertibility. The splitting property proved in the last section is closely related to the asymptotic Moore-Penrose invertibility. Let H be a Hilbert space and let us recall that an operator $A \in \mathcal{L}(H)$ is called Moore-Penrose invertible if there is an operator $B \in \mathcal{L}(H)$ such that

$$(4.1) \qquad ABA = A\,,\ BAB = B\,,\ (AB)^* = AB\,,\ (BA)^* = BA\,.$$

It is well-known that an operator is Moore-Penrose invertible if and only if its range is closed (such operators are also called normally solvable). Moreover, the operator B is uniquely determined and will be called the Moore-Penrose inverse of A (also written as $B = A^+$). If $A \in \mathcal{L}(H)$ is Moore-Penrose invertible, then A^+y is the pseudosolution of the equation $Ax = y$, that is the element with smallest norm of all the elements x for which $\| Ax - y \|$ is minimal.

We will heavily use the following (and well-known) characterization.

Proposition 4.1. The following statements are equivalent:

(i) The operator $A \in \mathcal{L}(H)$ is Moore-Penrose invertible.

(ii) The operator $A^*A + P_{\ker A}$ is invertible.
(iii) The operator $AA^* + P_{\ker A^*}$ is invertible.
Moreover, if this is fulfilled then

$$A^+ = (A^*A + P_{\ker A}A)^{-1}A^* = A^*\left(AA^* + P_{\ker A^*}\right)^{-1} ,$$

where P_M denotes the orthogonal projection onto the closed subspace $M \subset H$.

Sketch of the proof: That (i) is equivalent to (ii) is proved for instance in [H/R/S]. The equivalence of (i) to (iii) can be proved analogously.

Via the axioms (4.1) one can define Moore-Penrose invertibility for elements in arbitrary C^*-algebras. Again, the Moore-Penrose inverse of a given element is unique provided it exists, which can be easily seen by representing the C^*-algebra as an algebra of operators.

Notice the following result.

Proposition 4.2. (see [R/S 1], [H/R/S])

(i) An element a of a C^*-algebra with identity is Moore-Penrose invertible if and only if the element a^*a is invertible or if 0 is an isolated point of the spectrum of a^*a. If this condition is fulfilled, then $\|a^+\| = \min(\mathrm{sp}\ a^*a \setminus \{0\})$.
(ii) C^*-subalgebras of C^*-algebras with identity are inverse closed with respect to Moore-Penrose invertibility, that is, if an element of a C^*-subalgebra C of a C^*-algebra B has a Moore-Penrose inverse in B, then this Moore-Penrose inverse necessarily belongs to C.

The C^*-algebras to which we will apply this proposition are $\mathcal{F}/\mathcal{G}$ and some C^*-subalgebras of it ($\mathcal{F}$ and $\mathcal{G}$ are introduced in Section 2). A sequence $(A_n) \in \mathcal{F}$ is said Moore-Penrose stable if

$$\sup_{n\geq 1} \| A_n^+ \| < \infty$$

(recall that A_n^+ exists for all n because dim im $P_n < \infty$). We are mainly interested in sequences belonging to $\mathcal{A} \subset \mathcal{F}$ ($\mathcal{A}$ defined also in Section 2). It is not hard to find examples of sequences $(T_n(a))$ for which $(T_n^+(a))$ is not bounded, but $T(a)$ is Fredholm. Moreover, for $a \in PC \setminus C$ $(N = 1)$ the sequence $(T_n^+(a))$ is not bounded if $T(a)$ is Fredholm but not invertible (see [B/S 3]). If one allows modified finite sections, the picture changes dramatically. We will use an approach which first occured in [S 2] and study temporarily a weaker problem:

Theorem 4.1. (see [S 2] or [H/R/S]) The following assertions are equivalent for a sequence $(A_n) \in \mathcal{A}$:

(i) The operators $\mathcal{W}_1(A_n)$ and $\mathcal{W}_2(A_n)$ are normally solvable (that is they have closed range).
(ii) There is a sequence $(B_n) \in \mathcal{A}$ such that

$$\| A_nB_nA_n - A_n \| \longrightarrow 0 \, , \| B_nA_nB_n - B_n \| \longrightarrow 0 \, ,$$
$$\| (A_nB_n)^* - A_nB_n \| \longrightarrow 0 \, , \| (B_nA_n)^* - B_nA_n \| \longrightarrow 0$$

as $n \longrightarrow \infty$.

If one of the conditions is fulfilled then (B_n) is unique up to sequences in the ideal $\mathcal{G}$ (even in $\mathcal{F}$) and (B_n) tends strongly to $\mathcal{W}_1^+(A_n)$.

If $\mathcal{W}_1(A_n)$ (and therefore also $\mathcal{W}_2(A_n)$) is Fredholm, then the assertion is a consequence of Theorem 2.2.

This theorem can be accomplished by the following Proposition.

Proposition 4.3. Let the situation be as in the preceding theorem, and let $(A_n) \in \mathcal{A}$. If the operator $\mathcal{W}_1(A_n)$ is Fredholm (therefore, $\mathcal{W}_2(A_n)$ is also Fredholm), then the sequences $(D_n), (D_n')$,

$$\begin{aligned} D_n &= A_n^* A_n + P_n P_{\ker \mathcal{W}_1(A_n)} P_n + W_n P_{\ker \mathcal{W}_2(A_n)} W_n \,, \\ D_n' &= A_n A_n^* + P_n P_{\ker \mathcal{W}_1(A_n^*)} P_n + W_n P_{\ker \mathcal{W}_2(A_n^*)} W_n \,, \end{aligned}$$

belong to $\mathcal{A}$ and are stable, and the sequences $(B_n), (B_n')$ given by

$$\begin{aligned} B_n &= D_n^+ A_n^* \,, \\ B_n' &= A_n^* D_n'^+ \end{aligned} \tag{4.2}$$

are subject to condition (ii) of Theorem 4.1 (whence, it follows that $(B_n) - (B_n') \in \mathcal{G}$).

The proof can be carried out as the proof of Theorem 6.4 in [H/R/S].

Now one might think that the Moore-Penrose inverses A_n^+ for a Moore-Penrose stable sequence $(A_n) \in \mathcal{A}$ have something to do with the operators (4.2). Under some additionally given conditions this is indeed the case. These conditions are summarized in the next proposition which is a special case of a general statement (Proposition 6.5, Theorem 6.7 in [H/R/S]).

Proposition 4.4. Let $(A_n) \in \mathcal{A}$ and let $\mathcal{W}_1(A_n)$ be Fredholm.
Set $B_n := P_n P_{\ker \mathcal{W}_1(A_n)} P_n \,, C_n := W_n P_{\ker \mathcal{W}_2(A_n)} W_n$.

(a) If $A_n B_n = A_n C_n = 0$ for n large enough and
(b) B_n and C_n are projections and $B_n C_n = 0$ for n large enough,

then the sequence (A_n) is Moore-Penrose stable and

$$P_{\ker A_n} = B_n + C_n$$

for n sufficiently large.

The connection of this result with the k-splitting property is almost obvious: We have (n large enough)

$$\dim \ker A_n = \dim \ker \mathcal{W}_1(A_n) + \dim \ker \mathcal{W}_2(A_n) \,.$$

This observation already implies the Moore-Penrose stability of (A_n).

Theroem 4.2. Let $a \in PC_{N \times N}$ and let the operator $T(a)$ be Fredholm. Consider $(T_{n,\alpha,\beta}(a)) \in \mathcal{A}$ with given multiindices α, β. If there is an n_0 such that

$$\ker T(a) \subset \text{ im } P_{n_0} \text{ and } \ker \tilde{T}_{\alpha,\beta}(a) \subset \text{ im } P_{n_0} \tag{4.3}$$

or

$$\ker T^*(a) \subset \text{ im } P_{n_0} \text{ and } \ker \tilde{T}_{\alpha,\beta}(a)^* \subset \text{ im } P_{n_0},$$

then the sequence $(T_{n,\alpha,\beta})$ is Moore-Penrose stable and $(T^+_{n,\alpha,\beta}(a))$ converges strongly to $T^+(a)$. Moreover, for $n \geq n_0 + \alpha$ we have

$$\begin{aligned} P_{\ker T_{n,\alpha,\beta}(a)} &= P_n P_{\ker T(a)} P_n + W_n P_{\ker \tilde{T}_{\alpha,\beta}(a)} W_n, \\ P_{\ker T^*_{n,\alpha,\beta}(a)} &= P_n P_{\ker T^*(a)} P_n + W_n P_{\ker \tilde{T}^*_{\alpha,\beta}(a)} W_n, \end{aligned}$$

respectively.

Proof. We have to check the conditions (a) and (b) of Proposition 4.4. First consider the case where the first condition in (4.3) is fulfilled.

(a): For $n \geq n_0 + \beta$ we get

$$T_{n,\alpha,\beta}(a) P_n P_{\ker T(a)} P_n = P_{n-\alpha} T(a) P_{\ker T(a)} P_n = 0,$$

and

$$\begin{aligned} &T_{n,\alpha,\beta} W_n P_{\ker V^\alpha T(e^\alpha_{-1} \tilde{a} e^\beta_1) V^{*\beta}} W_n = \\ &= W_n (W_n T_{n,\alpha,\beta} W_n P_{\ker V^\alpha T(e^\alpha_{-1} \tilde{a} e^\beta_1) V^{*\beta}}) W_n = \\ &= W_n (V^\alpha P_n T(e^\alpha_{-1} \tilde{a} e^\beta_1) P_n V^{*\beta} P_{\ker V^\alpha T(e^\alpha_{-1} \tilde{a} e^\beta_1) V^{*\beta}}) W_n = \\ &= W_n (V^\alpha P_n T(e^\alpha_{-1} \tilde{a} e^\beta_1) V^{*\beta} P_{\ker V^\alpha T(e^\alpha_{-1} \tilde{a} e^\beta_1) V^{*\beta}}) W_n . \end{aligned}$$

Because the first condition of (4.3) is valid, the operator inside the brackets is zero (notice that $P_n V^{*\beta} P_{n_0} = P_n P_{n_0} V^{*\beta} P_{n_0} = V^{*\beta} P_{n_0}$ for $n \geq n_0$). Thus (a) is fulfilled, (b) is obvious, and the sequence $(T_{n,\alpha,\beta}(a))$ is Moore-Penrose stable and the Moore-Penrose inverses converge strongly to $T^+(a)$. If the second condition is fulfilled in (4.3) then $(T^{*+}_{n,\alpha,\beta}(a))$ tends strongly to $T^{*+}(a)$. Taking adjoints we get the claim.

Conjecture 4.1. Let $T(a)$ be Fredholm, $a \in PC_{N \times N}$, and the sequence $(T_{n,\alpha,\beta}(a))$ be Moore-Penrose stable. Then one of the conditions (4.3) is fulfilled.

Remark 4.1. For $N = 1$ and $\alpha = \beta = 0$ this was proved by Heinig and Hellinger in [H/H]. A more general conjecture is the following:

Conjecture 4.2. Let the first condition of Conjecture 4.1 be fulfilled. Then there is an n_0 such that for $n \geq n_0$

$$\dim \ker T_{n,\alpha,\beta}(a) = \max\{\gamma, \gamma^*\},$$

where

$$\gamma = \dim (\text{im } P_{n_0} \cap \ker T(a)) + \dim \left(\text{im } P_{n_0} \cap \ker V^\alpha T(e^\alpha_{-1} \tilde{a} e^\beta_1) V^{*\beta}\right)$$

and

$$\gamma^* = \dim (\text{im } P_n \cap \ker T^*(a)) + \dim \left(\text{im } P_{n_0} \cap \ker V^\beta T^*(e^\alpha_{-1} \tilde{a} e^\beta_1) V^{*\alpha}\right) .$$

Next we describe a sufficiently large class of Fredholm operators, $a \in PC_{N \times N}$, for which $\ker T(a) \subset \text{ im } P_{n_0}$ for some n_0. Of course each left invertible Toeplitz operator owns this property. If a is such that $(a^{-1})_m = 0$ for all sufficiently large m (here $(a^{-1})_j$ denotes the jth Fourier coefficient of a^{-1}), then $T(a)$ has the mentioned property, too. This can be easily seen by factorization.

By specifying Theorem 4.2 we get the following theorem.

Theorem 4.3. Let $a \in PC_{N\times N}$ and $T(a)$ be Fredholm.

(a) If $T(a)$ is left invertible or $(a^{-1})_m = 0$ for m large enough, then there is r_0 such that $(T^+_{n,0,r}(a))$ converges strongly to $T^+(a)$ for all $r \geq r_0$.
(b) If $T(a)$ is right invertible or $(a^{-1})_{-m} = 0$ for m large enough, then there is a r_0 such that $(T^+_{n,r,0}(a))$ converges strongly to $T^+(a)$ for all $r \geq r_0$.

Proof. (a): If r is large enough then the kernel of $T(\tilde{a}e_1^r)V^{*r}$ is contained in im P_r. Now it follows that the conditions of Theorem 4.3 are fulfilled, whence the claim follows.
(b): Can be reduced to (a) by taking adjoints. The theorem is completely proved.

Remark 4.2. The same results are true if one replaces $PC_{N\times N}$ by $QC_{N\times N}$ or more generally by $PQC_{N\times N}$. QC stands here for the algebra of all quasicontinuous functions and PQC for the algebra of all piecewise quasicontinuous functions defined on $\mathbb{T}$. The reason is that all results of Section 2 again hold.

Remark 4.3. One can expect that analogous results are also true for further operator classes and their approximations. This will be considered in a forthcoming paper.

5. Appendix. Here we present two examples which show that at least for smooth generating functions the kernel dimension of Fredholm Toeplitz operators can be computed effectively. These examples are given via a randomly chosen factors of the Wiener-Hopf factorization:

1^0

$$\begin{aligned} a(t) &= \begin{pmatrix} t^2+3t+1+\frac{7}{2}t^{-1} & t^3+t+\frac{1}{2}t^{-1}+2t^{-2} \\ t+4 & t^2+1+4t^{-1} \end{pmatrix} \\ &= \begin{pmatrix} t^{-2}+1 & \frac{1}{2}t^{-1} \\ t^{-1} & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} t+3 & t^2 \\ t & t+4 \end{pmatrix} \end{aligned}$$

Therefore the kernel dimension of the Toeplitz operator $T(a)$ equals 1.

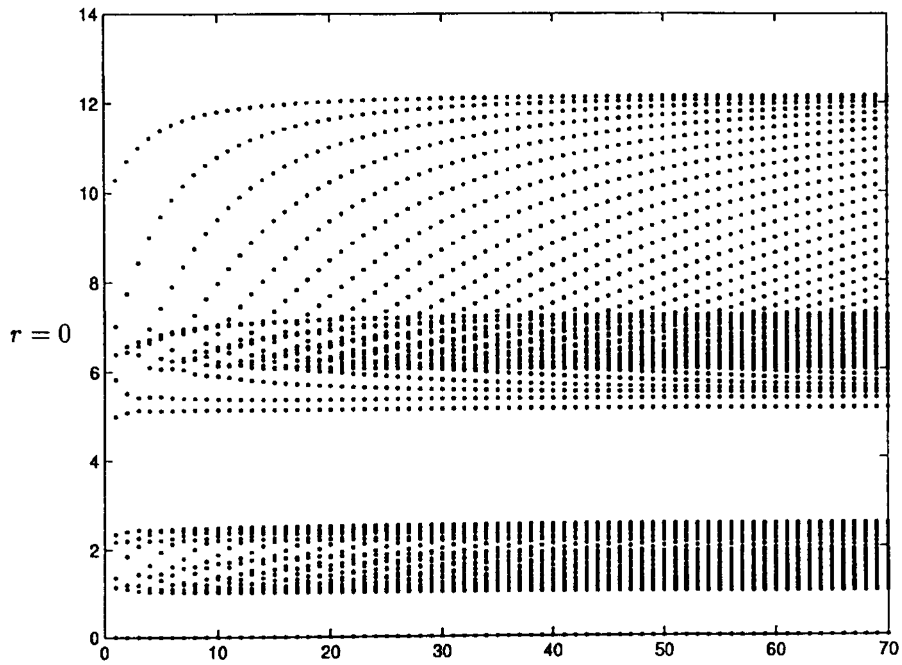
2^0

$$\begin{aligned} a(t) &= \begin{pmatrix} 2t^2+7t+3+\frac{1}{2}t^{-1} & \frac{1}{2}t^{-2} \\ t+3+t^{-1} & t^{-2} \end{pmatrix} \\ &= \begin{pmatrix} t^{-1}+2 & \frac{1}{2} \\ t^{-1} & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-2} \end{pmatrix} \begin{pmatrix} t+3 & 0 \\ t & 1 \end{pmatrix}. \end{aligned}$$

Thus, $T(a)$ is Fredholm with dim ker $T(a) = 2$.

In Figures 1 - 4 we plotted the singular values $s_j(T_{n,0,r}(a))$ versus $1 \leq n \leq 70$ for the generating functions a given in Examples 1^0 and 2^0 and for $r = 0, 1$, respectively. The computations showed that in all cases d can be chosen about $\frac{1}{4}$. The number of the lower singular values which approach to zero cannot be seen because to the computer they are equal zero. However, the computer allows also to determine their number.

Example 1^0.

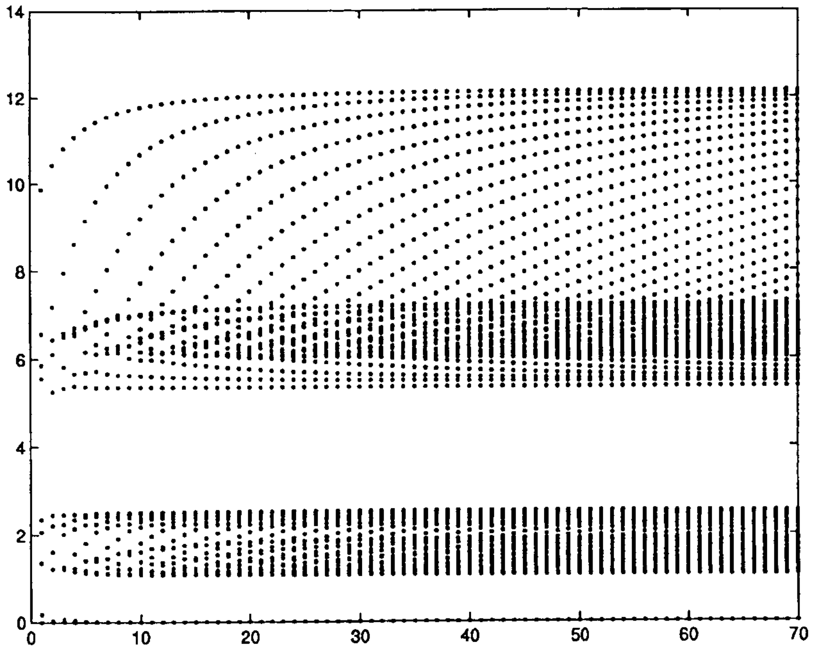


– Fig. 1 –

The computations show that the sequence $(T_{n,0,0}(a))$ is subject to the 1-splitting property.

The next figure is devoted to the case $r = 1$. In this case the sequence $(T_{n,0,1}(a))$ is subject to the 3-splitting property and we observe already stabilization in the sense of the remark made after Theorem 3.2. Thus, the computations lead to $\dim \ker T(a) = 1$ (Recall that $N = 2$).

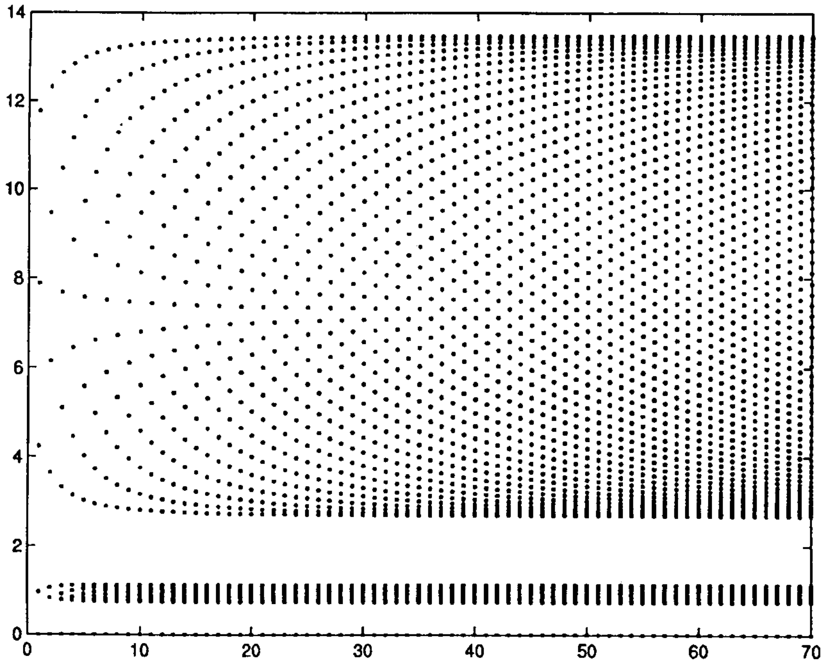
$r = 1$:



– Fig. 2 –

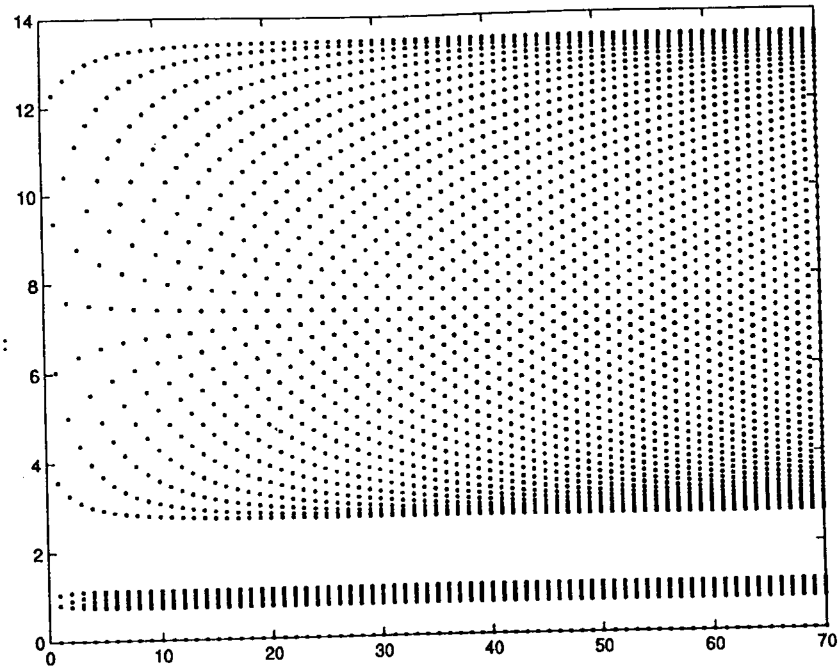
Example 2^0.

$r = 0$:



– Fig. 3 –

$r = 1$:



– Fig. 4 –

The computations give that $(T_{n,0,0}(a))$ and $(T_{n,0,1}(a))$ have the 2- and 4-splitting property, respectively. Thus, the developed theory gives dim ker $T(a) = 2$.

The examples show that the values c_n can be taken converging very fast to zero if the generating functions are smooth. For $N = 1$ and the familiar finite sections this result is already proved in [B/S 3]. It would be of interest to have a proof in the general case.

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