An approach to the Finite Section Method in the space of essentially bounded functions using Limit Operators

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ABSTRACT. We present an approach to the finite section method for so called band-dominated operators on $L^\infty(\mathbb{R}^n)$. These are the elements of an algebra generated by multiplication and shift operators. Here, the multiplication operators under consideration do not need to have limits at infinity. One of our main tools are so called limit operators. In many parts of this paper we follow ideas of [3] where the case $L^2(\mathbb{R}^n)$ is studied using C^*-algebra techniques. It is however not surprising that some things are much more sophisticated in $L^\infty(\mathbb{R}^n)$ but we will show how the L^2 results can also be retrieved to our case.

1 Main Results

Our main result concerning the finite section method $A_\tau = P_{[-\tau,\tau]}AP_{[-\tau,\tau]}$ on the real axis is the following theorem (presented in Section 5) where $\sigma^{\mathsf{op}}_\pm(A)$ refers to the set of limit operators of A at $+\infty$ and $-\infty$, respectively.

Theorem 5.2 *If A is some rich operator with finite section sequence $(A_\tau)_{\tau>0}$ in $\mathcal{C}+\mathcal{N}$ then the following statements are equivalent.*

(i) The finite section method (A_τ) is applicable to A.

(ii) The finite section sequence (A_τ) is stable.

(iii) The set consisting of

  a) the operator A itself,

  b) all $QA_hQ + P$ with A_h running through $\sigma^{\mathsf{op}}_+(A)$ and

  c) all $PA_hP + Q$ with A_h running through $\sigma^{\mathsf{op}}_-(A)$

is uniformly invertible.

2 Some basic notions

2.1 Operators of shift and multiplication

Definition 2.1 *For every function $b \in L^\infty(\mathbb{R}^n)$ let $M_b \in \mathcal{L}(L^\infty(\mathbb{R}^n))$ denote the operator of multiplication by b*

$$(M_b u)(x) = b(x)u(x) \qquad \forall x \in \mathbb{R}^n.$$

Sometimes we call b the symbol of the multiplicator M_b.

Fix some Banach space $\mathbf{X}$ and consider some sequence $c = (c_\alpha)_{\alpha\in\mathbb{Z}^n}$ with values c_α in $\mathbf{X}$. By $\ell^\infty(\mathbb{Z}^n, \mathbf{X})$ we denote the space of all such sequences with

$$\|c\| := \sup_{\alpha\in\mathbb{Z}^n} \|c_\alpha\|_{\mathbf{X}}$$

being finite.

If $b = (B_\alpha)_{\alpha\in\mathbb{Z}^n}$ is a bounded sequence of operators $B_\alpha \in \mathcal{L}(\mathbf{X})$ then by $\hat{M}_b$ we will denote the generalized multiplication operator

$$(\hat{M}_b u)_\alpha = B_\alpha u_\alpha \qquad \forall \alpha \in \mathbb{Z}^n$$

on the space of bounded sequences $\ell^\infty(\mathbb{Z}^n, \mathbf{X})$. Clearly, $\|\hat{M}_b\| = \sup_{\alpha\in\mathbb{Z}^n} \|B_\alpha\|$.

Definition 2.2 *For $\alpha \in \mathbb{Z}^n$ let V_α denote the so called shift operator on $\ell^\infty(\mathbb{Z}^n, \mathbf{X})$ acting by the rule*

$$(V_\alpha u)_\beta = u_{\beta-\alpha}$$

i.e. shifting the whole sequence $u \in \ell^\infty(\mathbb{Z}^n, \mathbf{X})$ forward by α components.

Many approximation methods are based on so called "cut-off projections" which will be defined right now.

Definition 2.3 *For every set $U \subset \mathbb{R}^n$ put $P_U := M_{\chi_U}$ where χ_U is the characteristic function of U. Clearly, P_U is a projector. We will refer to its complementary projector $I - P_U$ by Q_U.*

2.2 Band- and band-dominated operators

For a quite natural introduction to these classes of operators on $\ell^\infty(\mathbb{Z}^n, \mathbf{X})$ with some $n \in \mathbb{N}$ and some Banach space $\mathbf{X}$ we will have a short recourse to the case $\ell^\infty(\mathbb{Z})$ ($n = 1$ and $\mathbf{X} = \mathbb{C}$).

One usually regards a linear bounded operator A on $\ell^\infty(\mathbb{Z})$ as *band operator* if its matrix representation $(a_{ij})_{i,j\in\mathbb{Z}}$ with respect to the canonical basis in $\ell^\infty(\mathbb{Z})$ is a band matrix, which means the entries a_{ij} vanish if $|i-j|$ is larger than some number b which is then called bandwidth of A. A little thought shows that this are exactly those operators A which can be written as a finite sum

$$\sum_{|i|\le b} \hat{M}_{b_i} V_i$$

with $b_i \in \ell^\infty(\mathbb{Z})$ and V_i being the operator of shifting forward by i components.

With $\alpha \in \mathbb{Z}^n$, $|\alpha| := \max\limits_{i=1}^{n} |\alpha_i|$ and $b_\alpha \in \ell^\infty(\mathbb{Z}^n, \mathcal{L}(\mathbf{X}))$ for every α we can carry over this notation to $\ell^\infty(\mathbb{Z}^n, \mathbf{X})$ by saying that every finite sum of the form

$$\sum_{|\alpha|\le b} \hat{M}_{b_\alpha} V_\alpha \tag{1}$$

be a *band operator* on $\ell^\infty(\mathbb{Z}^n, \mathbf{X})$. Moreover, every norm limit of a sequence of band operators is referred to as a *band-dominated operator* on $\ell^\infty(\mathbb{Z}^n, \mathbf{X})$.

This was the discrete case ℓ^∞. Since we will finally deal with the function case we will export the notion of band- and band-dominated operators to $L^\infty(\mathbb{R}^n)$. This can be done by a rather easy and natural construction:

Let therefore H denote the (hyper-)cube $[0,1)^n$ and put $H_\alpha := \alpha + H\ \forall \alpha \in \mathbb{Z}^n$. Clearly, $\{H_\alpha\}_{\alpha\in\mathbb{Z}_n}$ is a disjoint decomposition of $\mathbb{R}^n$. Let G denote the operator that assigns to every function $f \in L^\infty(\mathbb{R}^n)$ the sequence of its restrictions to the cubes H_α

$$G : f \mapsto (f|_{H_\alpha})_{\alpha\in\mathbb{Z}^n}.$$

By identifying $L^\infty(H_\alpha)$ with $L^\infty(H)$ we get that G is an isometrical isomorphism from $L^\infty(\mathbb{R}^n)$ to the $L^\infty(H)$-valued $\ell^\infty(\mathbb{Z}^n)$ which we will denote by $\ell^\infty(\mathbb{Z}^n, L^\infty(H))$ (i.e. the Banach space $\mathbf{X}$ is nothing but $L^\infty(H)$).

Consequently, every operator A on $L^\infty(\mathbb{R}^n)$ can be identified with the operator

$$A^G := GAG^{-1} \tag{2}$$

on $\ell^\infty(\mathbb{Z}^n, L^\infty(H))$. For illustration, if $n = 1$ then A^G essentially is a matrix whose entries a_{ij} are not numbers but operators on $L^\infty(H) = L^\infty[0, 1)$.

Without introducing a new symbol we will say that V_τ is the shift operator in $\mathcal{L}(L^\infty(\mathbb{R}^n))$ shifting by $\tau \in \mathbb{R}^n$, i.e. acting by the rule

$$(V_\tau u)(x) \;=\; u(x - \tau), \qquad x \in \mathbb{R}^n.$$

If $M_f \in \mathcal{L}(L^\infty(\mathbb{R}^n))$ is the operator of multiplication by $f \in L^\infty(\mathbb{R}^n)$ then its discretization $(M_f)^G$ is a generalized multiplication operator $\hat{M}_b$ where the components of the sequence $b = (B_\alpha)$ are the (usual) multiplication operators $B_\alpha = M_{f|_{H_\alpha}}$ on $L^\infty(H)$. In this case we will write $\hat{f}$ for the sequence b, i.e.

$$\hat{M}_{\hat{f}} \;:=\; \hat{M}_b \;=\; (M_f)^G.$$

Sometimes we will also pass from the discrete case ℓ^∞ to L^∞ and we therefore introduce the abbreviation B^{-G} for the operator $G^{-1}BG \in \mathcal{L}(L^\infty(\mathbb{R}^n))$ with $(B^{-G})^G = B$. And conversely we have $(A^G)^{-G} = A$.

Definition 2.4 *We say that an operator $A \in \mathcal{L}(L^\infty(\mathbb{R}^n))$ is band-dominated if A^G is band-dominated in the sense of $\ell^\infty(\mathbb{Z}^n, L^\infty(H))$ and we call A band operator if A^G is a band operator. The class of band-dominated operators on $L^\infty(\mathbb{R}^n)$ will be denoted by* BDO.

Here is an important fact about band- and band-dominated operators.

Proposition 2.5 **a)** *If $A \in \mathcal{L}(L^\infty(\mathbb{R}^n))$ is a band operator then there is a number $b > 0$ such that*

$$P_U A P_V = 0 \qquad \forall U, V \subset \mathbb{R}^n : \; \mathrm{dist}(U, V) > b.$$

b) *If $A \in \mathcal{L}(L^\infty(\mathbb{R}^n))$ is band-dominated then*

$$P_U A P_V \rightrightarrows 0 \qquad \textit{as} \quad \mathrm{dist}(U, V) \to \infty, \tag{3}$$

in detail, $\forall \varepsilon > 0$ there is a $d > 0$ such that $\|P_U A P_V\| < \varepsilon$ for all subsets $U, V \subset \mathbb{R}^n$ with $\mathrm{dist}(U, V) > d$.

Proof. **Part a)** If A is band operator then A^G can be written in the form (1). Then

$$\begin{aligned}(P_U A P_V)^G &= G(P_U(G^{-1}A^G G)P_V)G^{-1} \\ &= GP_U G^{-1}\Big(\sum_{|\alpha|\le b} M_{b_\alpha} V_\alpha\Big) GP_V G^{-1} \\ &= \sum_{|\alpha|\le b} P_U^G M_{b_\alpha} V_\alpha P_V^G.\end{aligned}$$

But clearly $P_U^G M_{b_\alpha} V_\alpha P_V^G = M_{b_\alpha} P_U^G P_{\alpha+V}^G V_\alpha = 0$ if $\mathrm{dist}(U,V) > |\alpha|$ and hence $(P_U A P_V)^G = 0$ which yields that $P_U A P_V = 0$ if $\mathrm{dist}(U,V)$ exceeds the band-width b of A^G.

Part b) Take a sequence (A_m) of band operators with $A_m \rightrightarrows A$ as $m \to \infty$. Given an arbitrary $\varepsilon > 0$ there is some m such that $\|A - A_m\| < \varepsilon$. From part a) we know that there is a $b > 0$ such that $P_U A_m P_V$ vanishes for all sets U, V with distance greater than b. Fix our desired $d := b$. Indeed,

$$\begin{aligned}\|P_U A P_V\| &\le \|P_U A_m P_V\| + \|P_U(A - A_m)P_V\| \\ &< \|P_U A_m P_V\| + \varepsilon \\ &= \varepsilon \qquad \forall U, V \subset \mathbb{R}^n : \mathrm{dist}(U,V) > d\end{aligned}$$

■

Remark: Operators $A \in \mathcal{L}(L^\infty(\mathbb{R}^n))$ that are subject to (3) are referred to as *operators of local type* and the set of all these operators is denoted by $\mathfrak{A}$. It turns out that $\mathfrak{A}$ is a Banach subalgebra of $\mathcal{L}(L^\infty(\mathbb{R}^n))$ and by Proposition 2.5 b) we have BDO $\subset \mathfrak{A}$. □

2.3 The algebra $\mathcal{S}$.

Of course L^1 and L^∞ are two closely related extreme cases of L^p, each at an other end of the scale and both resist to most of the usual L^p arguments and techniques. We will mainly deal with $L^\infty := L^\infty(\mathbb{R}^n)$ but will frequently pass to $L^1 := L^1(\mathbb{R}^n)$ by several duality arguments. Here some caution is in place:
Given a bounded linear operator A on L^∞ its adjoint operator A^* clearly acts on the dual space $(L^\infty)'$ of L^∞ which contains L^1 as a **proper** subspace.

But on the other hand, the dual of L^1 **equals** L^∞. We will therefore restrict ourselves to operators $A \in \mathcal{L}(L^\infty)$ whose adjoint maps L^1-functions to L^1-functions, i.e. $A^*(L^1) = \mathrm{im}(A^*|_{L^1}) \subseteq L^1$. Let us denote this set of operators by $\mathcal{S}$,

$$\mathcal{S} := \Big\{ \; A \in \mathcal{L}(L^\infty) \;\; : \;\; A^*(L^1) \subseteq L^1 \; \Big\}.$$

Now if $A \in \mathcal{S}$, the restriction $B := A^*|_{L^1}$, seen as operator in $\mathcal{L}(L^1)$ has the property $B^* = A$, whence we regard B as the *pre-adjoint* operator of A – the operator whose adjoint equals A – and we will frequently denote it by ${}^*\!A$. So for $A \in \mathcal{S}$ we can pass from L^∞ to L^1 by $A \mapsto {}^*\!A$ and back to L^∞ by $B \mapsto B^*$.

In what follows the sign $\rightrightarrows$ denotes norm convergence of an operator sequence.

Proposition 2.6 *$\mathcal{S}$ is an inverse closed Banach subalgebra of $\mathcal{L}(L^\infty)$.*

Proof. If $A, B \in \mathcal{S}$ then also $AB \in \mathcal{S}$. Indeed, $(AB)^* = B^*A^*$ maps L^1 to L^1 since A^* and B^* do so. For the proof of closedness take some sequence (A_n) in $\mathcal{S}$ with $A_n \rightrightarrows A$ as $n \to \infty$. From $({}^*\!A_n)^* = A_n$ we conclude $\|{}^*\!A_n\| = \|A_n\|$. Consequently, $({}^*\!A_n)$ is a Cauchy sequence since (A_n) is one. Denote the uniform limit of ${}^*\!A_n$ by B,

$${}^*\!A_n \rightrightarrows B \quad \text{as } n \to \infty.$$

This and $\mathrm{im}{}^*\!A_n \subseteq L^1 \; \forall n$ show that $\mathrm{im}B \subseteq L^1$. But from

$$A_n = ({}^*\!A_n)^* \rightrightarrows B^* \quad \text{as } n \to \infty$$

and $A_n \rightrightarrows A$ we conclude $B^* = A$, i.e. B is the pre-adjoint of A. So $A \in \mathcal{S}$ which shows that $\mathcal{S}$ is closed.
To see that $\mathcal{S}$ is inverse closed, take some arbitrary $A \in \mathcal{S}$ being invertible as an operator on L^∞. We have to show that $(A^{-1})^*$, restricted to L^1 only maps to L^1-functions. Since A is invertible (Fredholm with kernel and cokernel dimension zero) A^* is invertible as well and $(A^*)^{-1} = (A^{-1})^*$. As above, put $B := {}^*\!A = A^*|_{L^1}$, seen as an operator in $\mathcal{L}(L^1)$. Clearly, B is injective. If we can show that B is also surjective, we are finished since then its inverse $B^{-1} = (A^*|_{L^1})^{-1}$ (the inverse of the restriction of A^*) clearly coincides with the restriction of the inverse of A^* which is $(A^*)^{-1}|_{L^1} = (A^{-1})^*|_{L^1}$ and hence $A^{-1} \in \mathcal{S}$. But the missing argument for B being invertible (Fredholm with kernel and cokernel dimension zero) is that $B^* = ({}^*\!A)^* = A$ is invertible and so we are finished. ■

2.4 The algebras $\mathcal{B}$ and $\mathcal{B}_{\mathcal{S}}$.

Definition 2.7 *By* BC *we denote the set of bounded and continuous complex-valued functions on* $\mathbb{R}^n$ *and let* BUC *refer to those* BC *functions that are even uniformly continuous on* $\mathbb{R}^n$.

Further let $[A\,,\,B]$ refer to the commutator of two operators A and B, that is

$$[A\,,\,B] = AB - BA.$$

Given a function $\varphi \in L^\infty$ and a vector $t = (t_1, ..., t_n) \in \mathbb{R}^n$ we define the function φ_t at the point $x = (x_1, ..., x_n) \in \mathbb{R}^n$ as $\varphi_t(x) := \varphi(t_1x_1, ..., t_nx_n)$ and $\varphi_{t,k}(x) := \varphi_t(x - k)$ for every $k \in \mathbb{R}^n$.

Definition 2.8 *By* $\mathcal{B}$ *we denote the set of all operators* $A \in \mathcal{L}(L^\infty)$ *for which*

$$[\,M_{\varphi_{t,k}}\,,\,A] \rightrightarrows 0 \qquad \textit{as } t \to 0$$

holds for all functions $\varphi \in \mathrm{BUC}$, *the convergence being uniform with respect to* $k \in \mathbb{R}^n$. *By* $\mathcal{B}_{\mathcal{S}}$ *we will denote the intersection* $\mathcal{B} \cap \mathcal{S}$.

Clearly, for $A \in \mathcal{B}$ also $[\,M_{\varphi_t}\,,\,A]$ tends to 0 as $t \to 0$.

Proposition 2.9 $\mathcal{B}$ *and* $\mathcal{B}_{\mathcal{S}}$ *are inverse closed Banach subalgebras of* $\mathcal{L}(L^\infty)$.

Proof. The identities

$$[M\,,\,A+B] = [M\,,\,A] + [M\,,\,B] \text{ and } [M\,,\,AB] = [M\,,\,A]B + A[M\,,\,B]$$

show that $\mathcal{B}$ is an algebra. Its norm-closedness follows from

$$\begin{aligned} \|\,[M_{\varphi_{t,k}}\,,\,A]\,\| &\le \|\,[M_{\varphi_{t,k}}\,,\,A_m]\,\| + \|\,[M_{\varphi_{t,k}}\,,\,A - A_m]\,\| \\ &\le \|\,[M_{\varphi_{t,k}}\,,\,A_m]\,\| + 2\|\varphi\|_\infty\,\|A - A_m\| \end{aligned}$$

and if $A \in \mathcal{B}$ is invertible, then by

$$[M\,,\,A^{-1}] = A^{-1}[A\,,\,M]A^{-1}$$

we get that also $A^{-1} \in \mathcal{B}$. From Proposition 2.6 it follows that the same is true for $\mathcal{B}_{\mathcal{S}}$. ■

Here is the somewhat surprising connection between the two algebras $\mathcal{B}_{\mathcal{S}}$ and BDO:

Proposition 2.10 *Every $A \in \mathcal{B}_{\mathcal{S}}$ is band-dominated, i.e. $\mathcal{B}_{\mathcal{S}} \subset$ BDO.*

Proof. Pick some arbitrary $A \in \mathcal{B}_{\mathcal{S}}$ and put $B := A^G \in \mathcal{L}(\ell^\infty(\mathbb{Z}^n, L^\infty(H)))$. We have to show that B can be approximated (in the operator norm) by operators of the form (1).

For $x = (x_1, ..., x_n) \in \mathbb{R}^n$ define

$$\varepsilon(x) := \exp(i \sum_{j=1}^{n} x_j).$$

Clearly, $\varepsilon \in$ BUC and consequently, $[M_{\varepsilon_t}, A] \rightrightarrows 0$ as $t \to 0$. Now we will pass to the discretization:

$$[\hat{M}_{\hat{\varepsilon}_t}, B] \rightrightarrows 0 \quad \text{as } t \to 0 \tag{4}$$

Since $\varepsilon_t(x) = \exp i(t, x)$ with $(.,.)$ referring to the usual inner product of $\mathbb{R}^n$ one has $\varepsilon_s \varepsilon_t = \varepsilon_{s+t}$ for all $s, t \in \mathbb{R}^n$. Consequently,

$$\|\hat{M}_{\hat{\varepsilon}_t} B \hat{M}_{\hat{\varepsilon}_{-t}} - \hat{M}_{\hat{\varepsilon}_s} B \hat{M}_{\hat{\varepsilon}_{-s}}\| \leq \|\hat{M}_{\hat{\varepsilon}_{t-s}} B - B \hat{M}_{\hat{\varepsilon}_{t-s}}\|,$$

which together with (4) implies the continuity of the function

$$\varphi : \mathbb{R}^n \to \mathcal{L}(\ell^\infty(\mathbb{Z}^n, L^\infty(H))), \qquad \varphi : t \mapsto \hat{M}_{\hat{\varepsilon}_t} B \hat{M}_{\hat{\varepsilon}_{-t}}. \tag{5}$$

Moreover, the functions $t \mapsto \hat{M}_{\hat{\varepsilon}_t}$ and $t \mapsto \hat{M}_{\hat{\varepsilon}_{-t}}$ are 2π-periodic in each variable t_j, and so is the function φ from (5). So we can approximate φ by its Fejer polynomials $\varphi^{(k)}$, i.e.

$$\sup_{t \in [0,2\pi]^n} \|\varphi(t) - \varphi^{(k)}(t)\| \to 0 \quad \text{as } k \to \infty \tag{6}$$

where

$$\varphi^{(k)}(t) = \sum_{|\alpha| \leq k} B^{(\alpha)} e^{i(t,\alpha)}, \quad \alpha \in \mathbb{Z}^n \tag{7}$$

and

$$B^{(\alpha)} = c^{(\alpha)} \int_{[0,2\pi]^n} \varphi(\tau) e^{-i(\tau,\alpha)} d\tau \quad \in \mathcal{L}(\ell^\infty(\mathbb{Z}^n, L^\infty(H))) \tag{8}$$

with certain constants $c^{(\alpha)}$.

The operators $B^{(\alpha)}V_{-\alpha}$ commute with each of the (generalized) multiplication operators $\hat{M}_{\hat{\varepsilon}_s}$ where $s \in \mathbb{R}^n$. Indeed,

$$\hat{M}_{\hat{\varepsilon}_s} B^{(\alpha)} V_{-\alpha} \hat{M}_{\hat{\varepsilon}_{-s}} = c^{(\alpha)} \int_{[0,2\pi]^n} (\hat{M}_{\hat{\varepsilon}_{s+t}} B \hat{M}_{\hat{\varepsilon}_{-s-t}}) e^{-i(s+t,\alpha)} dt \; V_{-\alpha} = B^{(\alpha)} V_{-\alpha}.$$

Since $A \in \mathcal{S}$, B also has the $\mathcal{S}$-property as well as $B^{(\alpha)}V_{-\alpha}$. Theorem 2.1 in [4] shows that such operators are continuous in the (L^∞, L^1) topology. Moreover, Proposition 5.1.11 in [1] says that an (L^∞, L^1)-continuous operator M which commutes with every $\hat{M}_{\hat{\varepsilon}_s}$ even commutes with every scalar multiplicator, that is $\hat{M}_b$ with $b = (B_\alpha)$ where all components B_α are multiples of the identity, $B_\alpha = \lambda_\alpha I,\ \lambda_\alpha \in \mathbb{C}$. For every $\beta \in \mathbb{Z}^n$ we put $b^{(\beta)} := (B_\alpha^{(\beta)})$ where $B_\alpha^{(\beta)} = \delta_{\alpha,\beta} I$ (Kronecker symbol), in other words, $\hat{M}_{b^{(\beta)}} = (P_{H_\beta})^G$.
The equation $M\hat{M}_{b^{(\beta)}} = \hat{M}_{b^{(\beta)}}M$ shows that M maps sequences $(c_\alpha)_{\alpha\in\mathbb{Z}^n}$ that are only supported in component number β to exactly such sequences again, i.e. M acts on every component c_α independently from all other components. Consequently, M is a generalized multiplication operator.

From this discussion it follows that every operator $B^{(\alpha)}V_{-\alpha}$ is of the form $\hat{M}_{a_\alpha}$ with some $a_\alpha \in \ell^\infty(\mathbb{Z}^n, L^\infty(H))$ and hence

$$B^{(\alpha)} = \hat{M}_{a_\alpha} V_\alpha.$$

This shows that the value $\varphi^{(k)}(0) = \sum_{|\alpha|\le k} B^{(\alpha)}$ of the k-th Fejer polynomial (7) is a band operator! From (6) we infer that

$$\varphi^{(k)}(0) \rightrightarrows \varphi(0) = B \quad \text{as } k \to \infty$$

and hence, $B(= A^G)$ is band dominated. And so is A. ■

Remark: The reverse inclusion is not true. To see this take some generalized multiplication operator $\hat{M}_b$ where the sequence $b = (B_\alpha)_{\alpha\in\mathbb{Z}^n}$ contains operators $B_\alpha \in \mathcal{L}(L^\infty(H))$ whose adjoint $B_\alpha^* \in \mathcal{L}(L^\infty(H)')$ does not map $L^1(H)$ to $L^1(H)$. Then $(\hat{M}_b)^{-G}$ is not in $\mathcal{S}$ whence BDO $\not\subset \mathcal{B}_{\mathcal{S}}$. □

2.5 Invertibility at infinity

Usually in the theory of approximation methods on L^p the ideal $\mathcal{K}$ of compact operators plays an important role being the set of perturbations which don't affect things essentially. Invertibility modulo such perturbations is then called Fredholmness. We will show quite shortly why $\mathcal{K}$ is **not** the appropriate tool in our case at hand, $p = \infty$.

For the unit cube $C = [-1, 1]^n$ the sequence of (outside) cut-off projections $(Q_{mC})_{m=1}^{\infty}$ strongly converges to 0 on every space L^p, $p < \infty$ (where functions need to vanish more or less slowly towards infinity). But not so in L^∞.

One crucial property of operators $T \in \mathcal{K}$ is that they turn strong convergence into norm convergence if they are applied to the convergent sequence from the right (and, under additional assumptions on the adjoint sequence, also from the left). Consequently, if $1 < p < \infty$, then $TQ_{mC} \rightrightarrows 0$ and $Q_{mC}T \rightrightarrows 0$ as $m \to \infty$ for every compact operator T. But for $p = \infty$ (and the same would be necessary for $p = 1$) we will introduce a new ideal $\mathcal{J}$ which will serve as a substitute of the ideal $\mathcal{K}$ of compact operators.

Definition 2.11 *By $\mathcal{J}$ we denote the set of all operators $T \in \mathcal{L}(L^\infty)$ with*

$$TQ_{[-R,R]^n} \rightrightarrows 0 \quad \textit{and} \quad Q_{[-R,R]^n}T \rightrightarrows 0 \quad \textit{as} \quad R \to \infty$$

and by $\mathcal{J}_{\mathcal{S}}$ we will denote the intersection $\mathcal{J} \cap \mathcal{S}$.

Proposition 2.12 *$\mathcal{J}$ is a closed ideal in $\mathcal{B}$ and $\mathcal{J}_{\mathcal{S}}$ is a closed ideal in $\mathcal{B}_{\mathcal{S}}$.*

Proof. Let T be some arbitrary operator in $\mathcal{J}$. Take some arbitrary $\varphi \in$ BUC and define $\psi := \varphi - \varphi(0) \in$ BUC. Then

$$[M_{\varphi_t}, T] = [M_{\psi_t}, T]. \tag{9}$$

For every $R > 0$

$$M_{\psi_t}T = M_{\psi_t}P_{[-R,R]^n}T + M_{\psi_t}Q_{[-R,R]^n}T$$

holds. The second term on the right can be made as small as desired by choosing R large enough and the first term evidently tends to 0 as $t \to 0$. The same argumentation for TM_{ψ_t} shows that

$$M_{\psi_t}T \rightrightarrows 0 \quad \text{and} \quad TM_{\psi_t} \rightrightarrows 0 \quad \text{as} \quad t \to \infty. \tag{10}$$

Now (9) shows that $T \in \mathcal{B}$ and hence $\mathcal{J} \subset \mathcal{B}$. Clearly, $\mathcal{J}$ is closed and we are left with the proof of the ideal property.

Let therefore $A \in \mathcal{B}$ and choose $\varphi \in \mathrm{BUC}$ such that φ vanishes on $[-1,1]^n$ and is equal to 1 outside $[-2,2]^n$. Let $\Phi_R(x) := \varphi_{(1/R,\ldots,1/R)}(x) = \varphi(x/R)$ $\forall x \in \mathbb{R}^n$. Then

$$\begin{aligned} Q_{[-2R,2R]^n}(AT) &= Q_{[-2R,2R]^n} M_{\Phi_R} AT \\ &= Q_{[-2R,2R]^n}([M_{\Phi_R}\,,\,A]T + AM_{\Phi_R}T) \\ &= Q_{[-2R,2R]^n}(\underbrace{[M_{\Phi_R}\,,\,A]}_{\rightrightarrows 0}T + AM_{\Phi_R}\underbrace{Q_{[-R,R]^n}T}_{\rightrightarrows 0}) \rightrightarrows 0 \end{aligned}$$

as $R \to \infty$. The convergence $(AT)Q_{[-R,R]^n} \rightrightarrows 0$ as $R \to \infty$ is obvious and hence $AT \in \mathcal{J}$. Analogously one shows that also $TA \in \mathcal{J}$.

Our second claim is then trivial by the definition of $\mathcal{B}_\mathcal{S}$ and $\mathcal{J}_\mathcal{S}$. ∎

In analogy to Fredholmness we will introduce the notion of invertibility at infinity where $\mathcal{K}$ is substituted by $\mathcal{J}$.

Definition 2.13 *An operator $A \in \mathcal{L}(L^\infty)$ is said to be* invertible at infinity *if there exists an operator $B \in \mathcal{L}(L^\infty)$ such that*

$$AB = I + T_1 \quad \textit{and} \quad BA = I + T_2$$

hold with operators $T_1, T_2 \in \mathcal{J}$.

Proposition 2.14 *If $A \in \mathcal{B}$ is invertible at infinity then the operator B from Definition 2.13 is in $\mathcal{B}$ as well.*

Proof. Take some arbitrary $\varphi \in \mathrm{BUC}$ and define $\psi := \varphi - \varphi(0) \in \mathrm{BUC}$. Then

$$[M_{\varphi_t}\,,\,B] = [M_{\psi_t}\,,\,B] = B[A\,,\,M_{\psi_t}]B + C_t$$

holds with $C_t = T_2 M_{\psi_t} B - B M_{\psi_t} T_1 \rightrightarrows 0$ by (10). ∎

An immediate consequence of this proposition is the following

Corollary 2.15 *An operator $A \in \mathcal{B}$ is invertible at infinity if and only if the coset $A + \mathcal{J}$ is invertible in the factor algebra $\mathcal{B}/\mathcal{J}$.*

2.6 Limit operators

On the preceding pages we introduced a way to evaluate an operator's behaviour "at infinity" – i.e. out there where perturbations by operators like $T \in \mathcal{J}$ don't matter. Here we will present a completely different approach to watch an operator's "contours at the horizon", so called *limit operators*.

Definition 2.16 *Let $A \in \mathcal{L}(L^\infty)$ and $h = (h_m)$ be some sequence in $\mathbb{Z}^n$ tending to infinity. If there exists some operator $B \in \mathcal{L}(L^\infty)$ such that*

$$P_U(V_{-h_m}AV_{h_m} - B) \rightrightarrows 0 \quad and \quad (V_{-h_m}AV_{h_m} - B)P_U \rightrightarrows 0$$

as $m \to \infty$ for all bounded sets $U \subset \mathbb{R}^n$ then B is unique and we call B the limit operator *of A with respect to the sequence h and denote it by A_h. In this case we will also say that the sequence h* leads to a limit operator *of A.*

If $h \subset \mathbb{Z}^n$ is a sequence tending to infinity such that A_h exists and if g is an infinite subsequence of h then also g leads to a limit operator of A and $A_g = A_h$.

Limit operators are compatible with addition, composition, passing to the norm limit and to the (pre-)adjoint operator.[1] Here we summarize these basic properties.

Proposition 2.17 *Let A, B and $A^{(m)}$ $(m = 1, 2, ...)$ be arbitrary band-dominated operators on L^∞ and let $h \subset \mathbb{Z}^n$ be some sequence tending to infinity.*

- **a)** *If A_h and B_h exist, then $(A + B)_h$ exists and is equal to $A_h + B_h$.*
- **b)** *If A_h and B_h exist, then $(AB)_h$ exists and is equal to A_hB_h.*
- **c)** *If $A = \lim_{m\to\infty} A^{(m)}$ and the limit operators $(A^{(m)})_h$ exist for all sufficiently large m then A_h exists and equals $\lim_{m\to\infty}(A^{(m)})_h$.*
- **d)** *If $^*A$ and A_h exist, then also $(^*A)_h$ exists and equals $^*(A_h) =: {}^*A_h$.*
- **e)** *If $A \in \mathcal{B}$ and A_h exists, then also $A_h \in \mathcal{B}$.*
- **f)** *If A_h exists, then $\|A_h\| \le \|A\|$.*

The proof is straightforward.

[1] Note that in $\mathcal{L}(L^1)$ limit operators are defined exactly as in Definition 2.16.

Definition 2.18 *By $\mathcal{B}_\$$ we denote the set of all operators $A \in \mathcal{B}_\mathcal{S}$ with the following property: Every sequence $h = (h_m) \subset \mathbb{Z}^n$ with $h_m \to \infty$ possesses a subsequence g which leads to a limit operator of A, i.e. A_g exists.*
We will regard operators in $\mathcal{B}_\$$ as rich operators *in terms of having limit operators en masse.*

Proposition 2.19 **a)** *$\mathcal{B}_\$$ is a Banach subalgebra of $\mathcal{L}(L^\infty)$.*
b) *Every operator $A \in \mathcal{B}_\$$ can be approximated (in the norm topology) by a sequence of band operators from $\mathcal{B}_\$$.*

Proof. **Part a)** Let $A, B \in \mathcal{B}_\$$ and take some arbitrary sequence $h = (h_m) \subset \mathbb{Z}^n$ tending to infinity. Since $A \in \mathcal{B}_\$$ there is a subsequence g of h which leads to a limit operator of A. But since $B \in \mathcal{B}_\$$ we can once more choose a subsequence f of g for which B_f exists and clearly $A_f = A_g$ and by Proposition 2.17 a) and b) f leads to a limit operator of $A + B$ and of AB. The closedness of $\mathcal{B}_\$$ is verified by an analoguous construction of a subsequence of h and finally applying Proposition 2.17 c).

Part b) Pick some arbitrary $A \in \mathcal{B}_\$$. By Proposition 2.10 we have $A \in \mathcal{B}_\$ \subset \mathcal{B}_\mathcal{S} \subset$ BDO and consequently A can be norm-approximated by a sequence $(A^{(k)})$ of band operators. But what has to be shown is that this approximating sequence can be chosen from $\mathcal{B}_\$$ again.
We will therefore use the construction from the proof of Proposition 2.10 and put $A^{(k)} := (\varphi^{(k)}(0))^{-G}$ where $\varphi^{(k)}$ (see (7)) is the k-th Fejer polynomial of the function φ given by (5) and $B = A^G$. If we can show that this choice of the approximating sequence $(A^{(k)})$ is already in $\mathcal{B}_\$$ we are done.

Since multiplicators with BUC symbol are in $\mathcal{B}_\$$ (which will be shown in Proposition 5.7) and $A = B^{-G} \in \mathcal{B}_\$$ all operators $(\varphi(t))^{-G}$ are in $\mathcal{B}_\$$. Since all occuring functions are continuous the integral (8) is a Riemann integral and hence, can be norm approximated by sums. From part a) we conclude that also all $(B^{(\alpha)})^{-G}$ are in $\mathcal{B}_\$$. Finally, $A^{(k)} := (\varphi^{(k)}(0))^{-G}$ is in $\mathcal{B}_\$$ for every k since $\varphi^{(k)}(0)$ is only a finite sum of operators of the form $B^{(\alpha)}$. ■

At some later point we will see that $\mathcal{B}_\$$ is indeed a proper subset of $\mathcal{B}_\mathcal{S}$, i.e. there are some not so rich operators among those in $\mathcal{B}_\mathcal{S}$.

Definition 2.20 *Let $A \in \mathcal{L}(L^\infty)$. The set of all limit operators of A is denoted by $\sigma^{\mathsf{op}}(A)$ and we refer to it as the* operator spectrum *of A.*

The operator spectrum includes all limit operators A_h of A, regardless of the direction in which h tends to infinity. But sometimes this information is needed and so we will split $\sigma^{\mathsf{op}}(A)$ into many sets, the so called local operator spectra. They are defined as follows.

Definition 2.21 *Let S^{n-1} denote the unit sphere (with respect to the Euclidian norm $|.|_E$) of $\mathbb{R}^n$. Let $s \in S^{n-1}$. Then we say that a sequence $h = (h_m) \subset \mathbb{Z}^n$ tends to infinity in the direction s if for every $R > 0$ and every neighborhood $U \subset S^{n-1}$ of s there is a m_0 such that*

$$|h_m|_E > R \qquad \textit{and} \qquad h_m/|h_m|_E \in U \qquad \textit{for all } m > m_0.$$

The local operator spectrum $\sigma_s^{\mathsf{op}}(A)$ is defined as the set of all limit operators A_h with h tending to infinity in the direction s.

Then it is not surprising that

Proposition 2.22 *For every operator A the identity*

$$\sigma^{\mathsf{op}}(A) = \bigcup_{s \in S^{n-1}} \sigma_s^{\mathsf{op}}(A)$$

holds.

Proof. Clearly, every local operator spectrum is contained in the operator spectrum $\sigma^{\mathsf{op}}(A)$. For the reverse inclusion take some $A_h \in \sigma^{\mathsf{op}}(A)$. The sequence $h_m/|h_m|_E$ need not converge to a point s in S^{n-1} but since this unit sphere is compact, there is a subsequence g of h that has this property and hence g tends to infinity in the direction s. But $A_h = A_g \in \sigma_s^{\mathsf{op}}(A)$. ∎

3 Limit operators vs. invertibility at infinity

As we have seen, invertibility at infinity is a property that only depends on an operator's behaviour in a neighborhood of infinity. For rich operators all necessary information about this behaviour is accurately stored in the operator spectrum $\sigma^{\mathsf{op}}(A)$ and it is the topic of this section to show the interplay between these two concepts.

The ideal $\mathcal{J}$ is an appropriate item do describe this relationship: On the one hand we have seen that invertibility at infinity is nothing but invertibility modulo $\mathcal{J}$. On the other hand the ideal $\mathcal{J}$ is exactly what becomes "factored out" by considering limit operators:

Proposition 3.1 *Let, for a moment, $\mathcal{L}_\$$ denote the set of all operators $A \in \mathcal{L}(L^\infty)$ for which every sequence h tending to infinity has a subsequence that leads to a limit operator of A (so $\mathcal{B}_\$ = \mathcal{L}_\$ \cap \mathcal{B}_\mathcal{S}$). Then*

$$\mathcal{J} \;=\; \Big\{T \in \mathcal{L}_\$ \;:\; \sigma^{\mathsf{op}}(T) = \{0\}\Big\} \quad \text{and} \quad \mathcal{J}_\mathcal{S} \;=\; \Big\{T \in \mathcal{B}_\$ \;:\; \sigma^{\mathsf{op}}(T) = \{0\}\Big\}.$$

Proof. From $V_\alpha P_U = P_{\alpha+U} V_\alpha$ and V_α being an isometry we conclude that for a sequence $h = (h_m) \to \infty$ and an operator T the limit operator T_h equals 0 if and only if for all bounded subsets $U \subset \mathbb{R}^n$

$$\begin{aligned}
\|P_U(V_{-h_m}TV_{h_m} - 0)\| &= \|P_U V_{-h_m} T V_{h_m}\| \\
&= \|V_{-h_m} P_{h_m+U} T V_{h_m}\| \\
&= \|P_{h_m+U} T\| \quad \text{and} \\
\|(V_{-h_m}TV_{h_m} - 0)P_U\| &= \|V_{-h_m} T V_{h_m} P_U\| \\
&= \|V_{-h_m} T P_{h_m+U} V_{h_m}\| \\
&= \|T P_{h_m+U}\|
\end{aligned}$$

tend to zero as $m \to \infty$.

If $T \in \mathcal{J}$ then clearly, $\|P_{h_m+U}T\|$ and $\|TP_{h_m+U}\|$ tend to zero for all sequences $h = (h_m) \to \infty$, so $T \in \mathcal{L}_\$$ and $\sigma^{\mathsf{op}}(T) = \{0\}$.
Conversely, if an operator $T \in \mathcal{L}_\$$ is not in $\mathcal{J}$ then by Definition 2.11 $\|Q_{[-R,R]^n}T\|$ or $\|TQ_{[-R,R]^n}\|$ does not tend to zero as $R \to \infty$. A little thought shows that then there is a sequence $h = (h_m)$ tending to infinity and a bounded set U such that $\|P_{h_m+U}T\|$ or $\|TP_{h_m+U}\|$ remains bounded away from zero as m goes to ∞. Since $T \in \mathcal{L}_\$$ we can choose a subsequence g of h which leads to a limit operator of T. But if T_g were 0 then $\|P_{g_m+U}T\|$ and $\|TP_{g_m+U}\|$ would both tend to zero which they don't. So $\sigma^{\mathsf{op}}(T) \neq \{0\}$ and we have shown the first equality. On both sides taking the intersection with $\mathcal{B}_\mathcal{S}$ gives the second claim since $\mathcal{J} \subset \mathcal{B}$ and $\mathcal{B}_\$ = \mathcal{L}_\$ \cap \mathcal{B}_\mathcal{S}$. ■

This observation is the key motivation for – roughly speaking – identifying the coset $A + \mathcal{J}$ with $\sigma^{\mathsf{op}}(A)$ if A is some rich operator. The essence of this identification is Theorem 3.3 which is formulated and proven on the following pages.

3.1 Invertibility at infinity in terms of limit operators

Definition 3.2 *We say that a family* $\{A_i\}_i$ *of operators* A_i *is uniformly invertible if all elements* A_i *are invertible and*

$$\sup_i \|A_i^{-1}\| < \infty.$$

The purpose of this section is the proof of the following theorem which is closing the gap between invertibility at infinity and uniform invertibility of the set of limit operators.

Theorem 3.3 *A rich operator* $A \in \mathcal{B}_\$$ *is invertible at infinity if and only if its set of limit operators* $\sigma^{\mathsf{op}}(A)$ *is uniformly invertible.*

To prove this we first need some propositions.

Proposition 3.4 *Let* $A \in \mathcal{L}(L^\infty)$ *and suppose a limit operator* A_h *of* A *is invertible. Then for every function* $\xi \in L^\infty$ *with bounded support there exists a* m_0 *such that for every* $m > m_0$ *there are operators* B_m *and* C_m *with* $\|B_m\|, \|C_m\| < 2\|A_h^{-1}\|$ *and*

$$B_m A M_{V_{h_m}\xi} = M_{V_{h_m}\xi} = M_{V_{h_m}\xi} A C_m.$$

Proof. Take such a function ξ and put $U := \operatorname{supp} \xi$. Then

$$V_{-h_m} A V_{h_m} P_U = A_h P_U + T_m$$

with some sequence $T_m \rightrightarrows 0$ as $m \to \infty$. Multiplying the latter from the right by $M_\xi V_{-h_m}$ and from the left by A_h^{-1} gives

$$A_h^{-1} V_{-h_m} A V_{h_m} M_\xi V_{-h_m} = (I + A_h^{-1} T_m) M_\xi V_{-h_m}. \tag{11}$$

Now take m_0 large enough that $\|A_h^{-1} T_m\| < 1/2 \ \forall m > m_0$. Then $I + A_h^{-1} T_m$ is invertible with $\|(I + A_h^{-1} T_m)^{-1}\| < 2$. Finally multiply (11) from the left by $V_{h_m}(I + A_h^{-1} T_m)^{-1}$ and consider that $V_{h_m} M_\xi V_{-h_m} = M_{V_{h_m}\xi}$ to see that

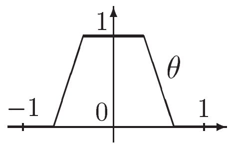
$$\underbrace{V_{h_m}(I + A_h^{-1} T_m)^{-1} A_h^{-1} V_{-h_m}}_{=:B_m} A M_{V_{h_m}\xi} = M_{V_{h_m}\xi}$$

which is our first claim. The second one is shown completely symmetric. ■

In what follows by $|x|$ we denote the maximum norm in $\mathbb{R}^n$, i.e. $|x| = \max_{i=1}^{n} |x_i|$ for all $x = (x_1, ..., x_n) \in \mathbb{R}^n$. We will now define a set of continuous functions each of them being supported in a small cube only with the property that their sum is 1 everywhere in $\mathbb{R}^n$. Such a set will be regarded as a partition of unity.

By $\theta : \mathbb{R} \to [0, 1]$ we denote the continuous function

$$\theta(x) \;:=\; \begin{cases} 3x + 2 & \text{for} \quad x \in [-2/3\,,\, -1/3) \\ 1 & \text{for} \quad x \in [-1/3\,,\, 1/3] \\ -3x + 2 & \text{for} \quad x \in (1/3\,,\, 2/3) \\ 0 & \text{otherwise} \end{cases} \tag{12}$$



with support in $[-2/3\,,\, 2/3]$. Let $\omega : \mathbb{R}^n \to [0, 1]$ be the continuous function with support in $[-2/3\,,\, 2/3]^n$ which is given by

$$\omega(x_1, ..., x_n) \;:=\; \sqrt{\theta(|(x_1, ..., x_n)|)}.$$

For $\alpha \in \mathbb{Z}^n$ we put

$$\omega_\alpha(x) \;:=\; \omega(x - \alpha) \qquad \forall x \in \mathbb{R}^n.$$

Note that ω is chosen such that

$$\sum_{\alpha \in \mathbb{Z}^n} \omega_\alpha^2(x) \;=\; 1 \qquad \forall x \in \mathbb{R}^n$$

holds. In this sense we will say that $\{\omega_\alpha^2\}_\alpha$ is a partition of unity. Note that for every x the latter sum is actually a finite sum.

For every $R > 0$ we define the function $\omega_{\alpha,R}(x) := \omega_\alpha(x/R) = \omega(x/R - \alpha)$ $\forall x \in \mathbb{R}^n$. Obviously $\{\omega_{\alpha,R}^2\}_\alpha$ is a partition of unity as well for every $R > 0$.

Finally let $\psi : \mathbb{R}^n \to [0, 1]$ be some continuous function with bounded support which is equal to 1 on $[-3/4\,,\, 3/4]^n$ (and hence on the whole support of ω). In analogy to ω_α and $\omega_{\alpha,R}$ we define ψ_α and $\psi_{\alpha,R}$, respectively.

Lemma 3.5 *Let $\{A_\alpha\}_{\alpha \in \mathbb{Z}^n}$ be some family of operators on L^∞ with $\|A_\alpha\| \leq M < \infty$ $\forall \alpha \in \mathbb{Z}^n$. Let moreover $\varphi \in L^\infty$ be some function with*

$$\sum_{\alpha \in \mathbb{Z}^n} |\varphi_\alpha(x)| \leq N_\varphi < \infty \qquad \forall x \in \mathbb{R}^n$$

where $\varphi_\alpha(x) = \varphi(x-\alpha)$ *and* $\varphi_{\alpha,R}(x) = \varphi_\alpha(x/R)$. *Then for every* $R > 0$ *by*

$$\sum_{\alpha\in\mathbb{Z}^n} M_{\varphi_{\alpha,R}} A_\alpha$$

a unique operator $A \in \mathcal{L}(L^\infty)$ *is given whose norm does not exceed* MN_φ.

Proof. Take some arbitrary $u \in L^\infty$ and put $u_\alpha := A_\alpha u \;\; \forall \alpha \in \mathbb{Z}^n$. Then

$$|u_\alpha(x)| \le \|u_\alpha\|_\infty \le \|A_\alpha\|\|u\|_\infty \le M\|u\|_\infty \qquad \forall \alpha, x$$

Thus we get

$$\begin{aligned} |(Au)(x)| &= \Big|\Big(\sum_{\alpha\in\mathbb{Z}^n} M_{\varphi_{\alpha,R}} A_\alpha u\Big)(x)\Big| \\ &\le \sum |\varphi_{\alpha,R}(x)|\,|u_\alpha(x)| \\ &\le M\|u\|_\infty \sum |\varphi_\alpha(x/R)| \\ &\le MN_\varphi\|u\|_\infty \qquad \forall x \in \mathbb{R}^n \end{aligned}$$

Consequently, $\|Au\|_\infty = \sup\limits_{x\in\mathbb{R}^n} |(Au)(x)| \le MN_\varphi\|u\|_\infty$. ∎

Obviously the functions ω and ψ defined previously are subject to the conditions on φ in this lemma. Here is some result from the theory of Banach algebras which we will make several use of in what follows.

Lemma 3.6 *Consider some Banach algebra* $\mathbf{B}$ *with identity* e *and a convergent sequence* $(x_n)_{n=1}^\infty$ *of invertible elements in* $\mathbf{B}$. *If* $\|x_n^{-1}\| < M < \infty \; \forall n$, *then also the limit* $x := \lim\limits_{n\to\infty} x_n$ *is invertible and* $\|x^{-1}\| \le M$.

Proof. From

$$\|x_m^{-1} - x_n^{-1}\| = \|x_m^{-1}(x_n - x_m)x_n^{-1}\| < M^2\|x_n - x_m\|$$

we conclude that also $(x_n^{-1})_{n=1}^\infty$ is a Cauchy sequence in $\mathbf{B}$. Denote its limit by y, then we have $x_n \to x$ and $x_n^{-1} \to y$ as $n \to \infty$ and consequently $e = x_n x_n^{-1} \to xy$ and $e = x_n^{-1}x_n \to yx$ which shows that $y = x^{-1}$. Clearly, $\|y\| \le M$. ∎

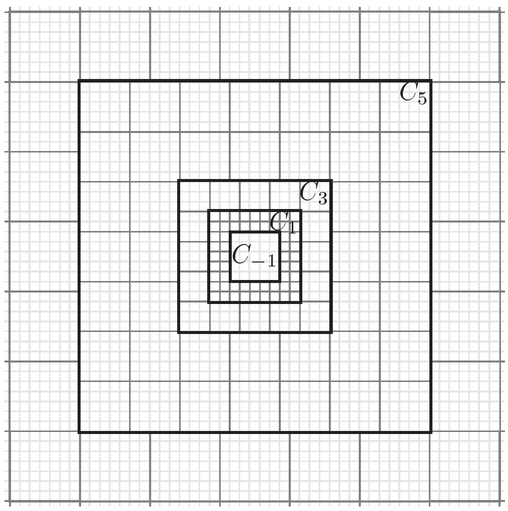
Proposition 3.7 *Let A be some band operator in $\mathcal{B}$. If there is some $M > 0$ such that for every $R > 0$ there exists a $\rho(R)$ such that for every $\alpha \in \mathbb{Z}^n$ with $|\alpha| > \rho(R)$ there are two operators $B_{\alpha,R}$ and $C_{\alpha,R}$ with $\|B_{\alpha,R}\|, \|C_{\alpha,R}\| < M$ and*

$$B_{\alpha,R}AM_{\omega_{\alpha,R}} = M_{\omega_{\alpha,R}} = M_{\omega_{\alpha,R}}AC_{\alpha,R} \tag{13}$$

then A is invertible at infinity.

Proof. Our proof requires some elementar geometry: We will interpret the $\mathbb{R}^n$ as a puzzle of infintely many (hyper-)cubes whose size increases with their distance from the origin. We will choose this decomposition such that, by our premise, on every of these cubes a left- and a right inverse to A exist and by puzzling these operators together (see Lemma 3.5) we will get some left- and right regularizer for A at infinity.

The width of our cubes runs through the odd numbers $U := \{1, 3, 5, ...\}$. To provide us with a frame for this "cubic puzzle" we will in a first step think of an infinite sequence of increasing concentric (centered in 0) cubes $C_{-1} \subset C_1 \subset C_3 \subset C_5 \subset ...$ reaching every point of $\mathbb{R}^n$. The choice of this sequence shall be such that for every $k \in U$ firstly, the width w_k of cube C_k is an odd multiple of $k(k+2)$ and secondly, the inequality $w_{k-2}/2 \geq k\rho(k)$ holds.



Our first constraint guarantees that for every $k \in U$ both w_k and w_{k-2} are divisible by k and hence both C_k and C_{k-2} can be puzzled by cubes of width k. But what we actually do then is puzzling **their difference** $C_k \setminus C_{k-2}$ with cubes of width k. Doing this for all $k \in U$ gives us the proposed (disjoint up to sets of measure zero) decomposition of $\mathbb{R}^n \setminus C_{-1}$ into cubes of increasing width.

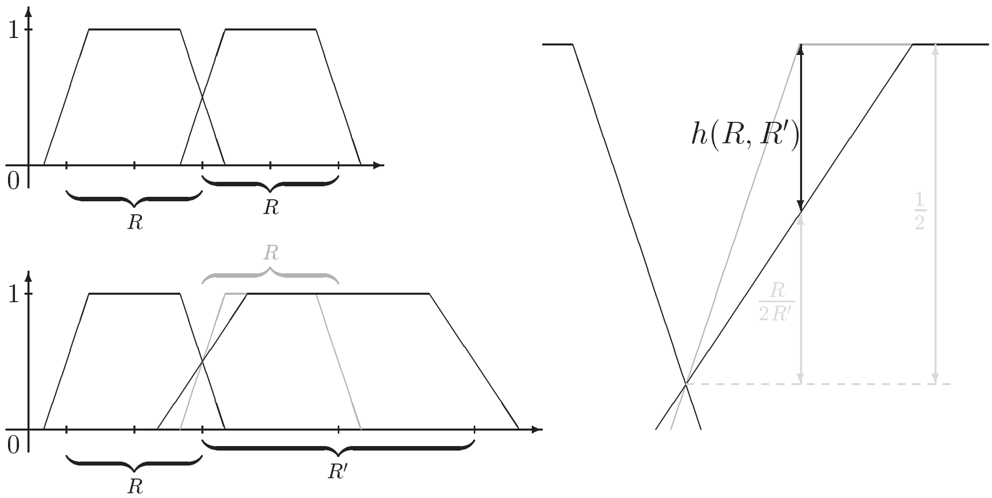
Our second constraint yields that from $k\alpha \in C_k \setminus C_{k-2}$ for some $\alpha \in \mathbb{Z}^n$, we can conclude $|\alpha| > \rho(k)$ and hence the existence of some operators $B_{\alpha,k}$ and $C_{\alpha,k}$ being subject to (13).

For every $k \in U$ we put $D_k := \{\alpha \in \mathbb{Z}^n : k\alpha \in C_k \setminus C_{k-2}\}$ whence $\{\omega^2_{\alpha,k}\}_{\alpha \in D_k}$ is a partition of unity on $C_k \setminus C_{k-2}$ with exception of some points near the boundary of $C_k \setminus C_{k-2}$.

Consequently, the set

$$\bigcup_{k \in U} \left\{\omega^2_{\alpha,k}\right\}_{\alpha \in D_k}$$

is a partition of unity on $\mathbb{R}^n \setminus C_{-1}$ with exception of the boundary of every cube C_k and of some points in the near, i.e. where cubes of different width meet one another. To see what happens at these points we will – for simplicity but without loss of generality – study this situation for the case $n = 1$ where cubes are just intervals and ω^2 equals θ from equation (12).



Two copies of the function θ, stretched and shifted onto adjacent intervals ($\mathbb{R}^1$-cubes) of the same and of different width and a zoom into the second picture showing how much this situation differs from a partition of unity. Seen in the supremum norm this difference is $h(R, R') = (1 - R/R')/2$ which clearly tends to 0 as $R/R' \to 1$.

So where cubes of different width $R = k$ and $R' = k+2$ are adjacent we do not really get a partition of unity. But what we have there is only $h(k, k+2)$ away from unity – measured in the sense of L^∞ – and it is an easy computation to see that the distance $h(R, R')$ equals $(1 - R/R')/2$ and hence, $h(k, k+2)$ tends to zero as $k \to \infty$. From this observation we conclude that

$$\sum_{k \in U} \sum_{\alpha \in D_k} M_{\omega^2_{\alpha,k}} = I - \sum_{\alpha \in C_{-1}} M_{\omega^2_\alpha} + T = I + T' \tag{14}$$

where $T \in \mathcal{J}$ and so is T'. So much about geometry. Now here is the left regularizer B for our operator A: Put

$$B := \sum_{k\in U} \sum_{\alpha\in D_k} M_{\omega_{\alpha,k}} B_{\alpha,k} M_{\omega_{\alpha,k}}$$

which is well defined by Lemma 3.5. Then

$$\begin{aligned}
BA &= \sum_{k\in U} \sum_{\alpha\in D_k} M_{\omega_{\alpha,k}} B_{\alpha,k} M_{\omega_{\alpha,k}} A \\
&= \sum_{k\in U} \sum_{\alpha\in D_k} M_{\omega_{\alpha,k}} \underbrace{B_{\alpha,k} A M_{\omega_{\alpha,k}}}_{M_{\omega_{\alpha,k}}} + \sum_{k\in U} \underbrace{\sum_{\alpha\in D_k} M_{\omega_{\alpha,R}} B_{\alpha,R} [M_{\omega_{\alpha,R}}, A]}_{=:T_k} \\
&= I + T' + \sum_{k\in U} T_k
\end{aligned}$$

where $T' \in \mathcal{J}$ is the operator from equation (14) and by Lemma 3.5 we have $\|T_k\| \le N_\omega \sup_{\alpha\in D_k} M \|[M_{\omega_{\alpha,k}}, A]\|$. From $A \in \mathcal{B}$ we conclude that $T_k \rightrightarrows 0$ as $k \to \infty$ and, again thinking of Lemma 3.5, $\sum_{k\in U} T_k \in \mathcal{J}$ which shows that B is indeed a left regularizer for A in terms of invertibility at infinity.

Recall that A is a band operator. By definition of ψ and ω the distance of supp $(1 - \psi_{\alpha,k})$ and supp $\omega_{\alpha,k}$ tends to infinity as $k \to \infty$. Proposition 2.5 a) tells us that $M_{1-\psi_{\alpha,k}} A M_{\omega_{\alpha,k}} = 0$ if k is large enough, say $k > k_0$ and hence $A M_{\omega_{\alpha,k}} = M_{\psi_{\alpha,k}} A M_{\omega_{\alpha,k}}$ if $k > k_0$. Put $U_0 := \{k \in U : k > k_0\}$. Then

$$C := \sum_{k\in U_0} \sum_{\alpha\in D_k} M_{\omega_{\alpha,k}} C_{\alpha,k} M_{\omega_{\alpha,k}}$$

is a right regularizer for A. Indeed,

$$\begin{aligned}
AC &= \sum_{k\in U_0} \sum_{\alpha\in D_k} M_{\psi_{\alpha,k}} A M_{\omega_{\alpha,k}} C_{\alpha,k} M_{\omega_{\alpha,k}} \\
&= \sum_{k\in U_0} \sum_{\alpha\in D_k} M_{\psi_{\alpha,k}} \underbrace{M_{\omega_{\alpha,k}} A C_{\alpha,k}}_{M_{\omega_{\alpha,k}}} M_{\omega_{\alpha,k}} \\
&\quad + \sum_{k\in U_0} \underbrace{\sum_{\alpha\in D_k} M_{\psi_{\alpha,k}} [A, M_{\omega_{\alpha,k}}] C_{\alpha,k} M_{\omega_{\alpha,k}}}_{=:T_k} \\
&= I + T'' + \sum_{k\in U_0} T_k
\end{aligned}$$

with $T'' \in \mathcal{J}$ and $T_k \rightrightarrows 0$ and as in the left-sided part we see that this right hand side is of the form $I + T$ with some $T \in \mathcal{J}$. ■

Now we are ready for the

Proof of Theorem 3.3

$\Longrightarrow$ Suppose A is invertible at infinity. Then there is some $B \in \mathcal{L}(L^\infty)$ such that $I = BA + T_1$ and $I = AB + T_2$ with $T_1, T_2 \in \mathcal{J}$. For arbitrary elements $u \in L^\infty$ and $v \in (L^\infty)'$ (passing to adjoints in the second equation) this yields to

$$\begin{aligned} \|u\|_\infty &\leq \|B\|\|Au\|_\infty + \|T_1 u\|_\infty \\ \|v\|_{\infty'} &\leq \|B^*\|\|A^* v\|_{\infty'} + \|T_2^* v\|_{\infty'}. \end{aligned}$$

Restricting v to $L^1 \subset (L^\infty)'$ and putting $C := \max(\|B\|, 1)$ we get that the norm inequalities

$$\|u\|_\infty \leq C(\|Au\|_\infty + \|T_1 u\|_\infty) \tag{15}$$

$$\|v\|_1 \leq C(\|{}^*Av\|_1 + \|T_2^* v\|_{\infty'}) \tag{16}$$

hold for all $u \in L^\infty$ and $v \in L^1$.

Let $h = (h_m) \subset \mathbb{Z}^n$ be an arbitrary sequence tending to infinity and leading to a limit operator A_h of A. We will show that A_h is invertible.

Since V_α is an isometry for every $\alpha \in \mathbb{Z}^n$ we get from (15) that

$$\begin{aligned} \|u\|_\infty = \|V_{h_m} u\|_\infty &\leq C(\|AV_{h_m} u\|_\infty + \|T_1 V_{h_m} u\|_\infty) \\ &= C(\|V_{-h_m} AV_{h_m} u\|_\infty + \|V_{-h_m} T_1 V_{h_m} u\|_\infty) \quad \forall u \in L^\infty. \end{aligned}$$

For the function ψ from above and every $R > 0$ we define the function ψ_R by $\psi_R(x) := \psi(x/R)\ \forall x \in \mathbb{R}^n$. Replacing u in the last inequality by the function $M_{\psi_R} u$ and doing all the same steps with inequality (16) we get that

$$\begin{aligned} \|M_{\psi_R} u\|_\infty &\leq C(\|V_{-h_m} AV_{h_m} M_{\psi_R} u\|_\infty + \|V_{-h_m} T_1 V_{h_m} M_{\psi_R} u\|_\infty) \\ \|M_{\psi_R} v\|_1 &\leq C(\|V_{-h_m}{}^*AV_{h_m} M_{\psi_R} v\|_1 + \|V_{-h_m} T_2^* V_{h_m} M_{\psi_R} v\|_{\infty'}) \end{aligned}$$

are true for all $R > 0$.

From $(V_{-h_m} AV_{h_m} - A_h)M_{\psi_R} \rightrightarrows 0$, Prop. 2.17 d), $V_{-h_m} T_1 V_{h_m} M_{\psi_R} \rightrightarrows 0$ and $\|V_{-h_m} T_2^* V_{h_m} M_{\psi_R}\|_{\infty'} = \|(M_{\psi_R} V_{-h_m} T_2 V_{h_m})^*\|_{\infty'} = \|M_{\psi_R} V_{-h_m} T_2 V_{h_m}\|_\infty \to 0$ as $m \to \infty$ we conclude

$$\begin{aligned} \|M_{\psi_R} u\|_\infty &\leq C\|A_h M_{\psi_R} u\|_\infty \leq C(\|M_{\psi_R} A_h u\|_\infty + \|[A_h\,,\, M_{\psi_R}]u\|_\infty) \\ \|M_{\psi_R} v\|_1 &\leq C\|{}^*A_h M_{\psi_R} v\|_1 \leq C(\|M_{\psi_R}{}^*A_h v\|_1 + \|[{}^*A_h\,,\, M_{\psi_R}]v\|_1). \end{aligned}$$

Since $A_h \in \mathcal{B}$ and $[{}^*\!A_h\,,\, M_{\psi_R}] = {}^*[M_{\psi_R}\,,\, A_h]$ holds, letting $R \to \infty$ leads to

$$\begin{aligned} \|u\|_\infty &\le C\|A_h u\|_\infty \qquad \forall u \in L^\infty \\ \|v\|_1 &\le C\|{}^*\!A_h v\|_1 \qquad \forall v \in L^1. \end{aligned}$$

Taking this together we see that ${}^*\!A_h$ and $A_h = ({}^*\!A_h)^*$ are injective, whence ${}^*\!A_h$ is invertible. Consequently, A_h is invertible and $\|A_h^{-1}\| \le C$ where C does not depend on the sequence h.

$\boxed{\Longleftarrow}$ Let $\sigma^{\mathsf{op}}(A)$, the set of limit operators of A, be uniformly invertible. We will show that then A is invertible at infinity. Clearly, $A \in \mathcal{B}_{\$} \subset \mathcal{B}_{\mathcal{S}} \subset \mathrm{BDO}$ is either a band operator or a norm limit of such.

a) If A is a band operator then the following argumentation works: Suppose A is not invertible at infinity. Then by Proposition 3.7 for every $M > 0$ (and so for $M := 2\sup_{A_h \in \sigma^{\mathsf{op}}(A)} \|A_h^{-1}\|$) there is a $R > 0$ such that for all ρ there exists an $\alpha \in \mathbb{Z}^n$ with $|\alpha| > \rho$ such that

$$BAM_{\omega_{\alpha,R}} \ne M_{\omega_{\alpha,R}}$$

for all operators B with $\|B\| < M$. This means there is a sequence $(\alpha_k) \subseteq \mathbb{Z}^n$ tending to infinity such that for **no** $k \in \mathbb{N}$ there is an operator B_k with $\|B_k\| < M$ for which

$$B_k AM_{\omega_{\alpha_k,R}} = M_{\omega_{\alpha_k,R}} \tag{17}$$

holds.

Since $A \in \mathcal{B}_{\$}$ there is a subsequence g of (α_k) such that A_g exists. But $A_g \in \sigma^{\mathsf{op}}(A)$ is invertible by our premise. And so Proposition 3.4 (putting $\xi := \omega_{0,R}$) says that for infinitely many k there is an operator B_k with $\|B_k\| < M$ that is subject to equation (17). This is clearly a contradiction.

b) If $A \in \mathcal{B}_{\$} \subset \mathcal{B}_{\mathcal{S}}$ is not a band operator then, by Proposition 2.19 b) there is a sequence $(A^{(m)}) \subset \mathcal{B}_{\$}$ of rich band operators with $A^{(m)} \rightrightarrows A$ as $m \to \infty$. Put m_0 large enough that

$$\|A^{(m)} - A\| < \frac{1}{2M} \qquad \forall m > m_0$$

with $M = \sup_{A_h \in \sigma^{\mathsf{op}}(A)} \|A_h^{-1}\|$ and fix some $m > m_0$. Now take an arbitrary sequence $h \subset \mathbb{Z}^n$ tending to infinity which leads to a limit operator of $A^{(m)}$.

Since $A \in \mathcal{B}_\$$ there is a subsequence g of h which also leads to a limit operator of A. By our premise A_g is invertible. Since

$$\|A_h^{(m)} - A_g\| = \|A_g^{(m)} - A_g\| = \|(A^{(m)} - A)_g\| \leq \|A^{(m)} - A\| < \frac{1}{2M}$$

also $A_h^{(m)}$ is invertible with $\|(A_h^{(m)})^{-1}\| \leq 2M$.

That means: For all $m > m_0$ the limit operator sets $\sigma^{\mathsf{op}}(A^{(m)})$ are uniformly invertible. Remember that all $A^{(m)}$ are rich band operators. So from part a) we conclude that all $A^{(m)}$ with $m > m_0$ are invertible at infinity (i.e. all cosets $A^{(m)} + \mathcal{J}$ with $m > m_0$ are invertible in the factor algebra $\mathcal{B}/\mathcal{J}$) with norm of the inverses $\|(A^{(m)} + \mathcal{J})^{-1}\|_{\mathcal{B}/\mathcal{J}} \leq 2M \ \forall m > m_0$. Lemma 3.6 now guarantees that also the norm limit $A + \mathcal{J}$ is invertible in $\mathcal{B}/\mathcal{J}$ and consequently, A is invertible at infinity. ■

4 Stability versus invertibility at infinity

4.1 Approximation methods

Let $T \subseteq \mathbb{R}$ be some index set and $(\Omega_\tau)_{\tau \in T}$ refer to a sequence of sets $\Omega_\tau \subset \mathbb{R}^n$ where $\Omega_\tau \subset \Omega_{\tau'}$ for $\tau < \tau'$ and where Ω_τ is reaching every point of $\mathbb{R}^n$ if τ is large enough.

If one has to solve an equation of the form

$$Au = b \tag{18}$$

where $A \in \mathcal{L}(L^\infty(\mathbb{R}^n))$ and $b \in L^\infty(\mathbb{R}^n)$ are given and $u \in L^\infty(\mathbb{R}^n)$ is to be determined, one often tries to approximate the operator A by a sequence $(A_\tau)_{\tau \in T}$ with $A_\tau \in \mathcal{L}(L^\infty(\Omega_\tau))$ and a index set T which is unbounded towards $+\infty$ and to solve the somewhat easier equations

$$A_\tau u_\tau = P_{\Omega_\tau} b \tag{19}$$

in $L^\infty(\Omega_\tau)$ instead, hoping these are uniquely solvable – at least for all sufficiently large τ – and that the solutions u_τ of (19) in some sense tend to a solution u of (18) as τ goes to infinity. If this is the case for every right-hand side $b \in L^\infty(\mathbb{R}^n)$ then we say that the *approximation method* (A_τ) is *applicable* to A.

This is the idea of approximation methods. Since here the convergence process $\tau \to +\infty$ is of interest, we will restrict the index set T to the positive numbers. A quite natural choice of the sequence (A_τ) is

$$A_\tau := P_{\Omega_\tau} A P_{\Omega_\tau}, \qquad \tau > 0$$

which is called the *finite section method* for obvious reasons.

Up to now we did not specify clearly in which way we will expect the operators A_τ and the solutions u_τ to approximate A and u, respectively. It is easily seen that in L^∞ it is not suitable to expect the convergence $A_\tau \to A$ to be strongly as one usually does in L^p and also the solutions u_τ will not converge to u in the norm of L^∞ as it is the case in L^p for $p < \infty$ (also see the short discussion at the beginning of section 2.5).

Instead, by $\mathcal{P}$ we will denote the collection of projectors $\{P_U\}$ with U running through all bounded subsets of $\mathbb{R}^n$ and say that (A_τ) *$\mathcal{P}-$converges to A* and u_τ *$\mathcal{P}-$converges to u* if

$$\begin{gathered} \|P_U(A_\tau - A)\| \to 0, \qquad \|(A_\tau - A)P_U\| \to 0 \\ \text{and} \quad \|P_U(u_\tau - u)\| \to 0 \quad \text{as } \tau \to \infty \end{gathered}$$

for all $P_U \in \mathcal{P}$.[2]

It turns out that the question whether this method is applicable or not heavily depends on the *stability* of the sequence $(A_\tau)_{\tau>0}$, that is A_τ is invertible (as operator on $L^\infty(\Omega_\tau)$) for all τ being sufficiently large, say $\tau > \tau_0$ and $\sup\limits_{\tau>\tau_0} \|A_\tau^{-1}\| < \infty$.

Definition 4.1 *By $\mathcal{F}$ we denote the set of all operator sequences $(A_\tau)_{\tau\in T}$ with $A_\tau \in \mathcal{L}(L^\infty(\Omega_\tau))$ and*

$$\|(A_\tau)\|_{\mathcal{F}} := \sup_{\tau\in T} \|A_\tau\| < \infty.$$

Let $\mathcal{N}$ denote the set of all sequences $(N_\tau)_{\tau\in T} \in \mathcal{F}$ with $N_\tau \rightrightarrows 0$ as $\tau \to \infty$.

Furnished with elementwise operations (addition, composition, multiplication by scalars), $\mathcal{F}$ becomes a Banach algebra. Clearly, $\mathcal{N}$ is a closed two-sided ideal in $\mathcal{F}$.

[2]Note that we are already familiar with such constructions from the definition of limit operators which are – in this notion – just $\mathcal{P}-$limits of sequences $(V_{-h_m} A V_{h_m})_{m=1}^\infty$.

An important result by A. KOZAK says that a sequence $(A_\tau) \in \mathcal{F}$ is stable if and only if its coset $(A_\tau) + \mathcal{N}$ is invertible in the factor algebra $\mathcal{F}/\mathcal{N}$.

The following POLSKI type theorem is proven in [2, 4.41f]. [3]

Theorem 4.2 *If $A \in \mathfrak{A}$ and $(A_\tau) \in \mathcal{F}$ is $\mathcal{P}-$convergent to A then the approximation method (A_τ) is applicable to A if and only if A is invertible and (A_τ) is stable.*

From Proposition 2.5 b) we know that BDO $\subset \mathfrak{A}$ and hence for band-dominated operators A we can restrict ourselves to the study of the stability of (A_τ) and be aware that we hereby essentially answer the question whether (A_τ) – as an approximation method – is applicable or not to the operator A.

4.2 Stacked operators

Definition 4.3 *Given a function $u \in L^\infty(\mathbb{R}^{n+1})$ we will define the sequence $(u_\tau)_{\tau\in\mathbb{R}}$ of functions in $L^\infty(\mathbb{R}^n)$ by*

$$u_\tau(x) := u(x,\tau) \qquad \forall x \in \mathbb{R}^n$$

for all $\tau \in \mathbb{R}$ and regard u_τ as the τ-th layer of u.

Clearly, it makes no sense to talk about single layers u_τ since they are defined on sets of measure zero in $\mathbb{R}^{n+1}$. But if we treat the whole sequence $(u_\tau)_{\tau\in\mathbb{R}}$ as an object in the $L^\infty(\mathbb{R}^n)$-valued $L^\infty(\mathbb{R})$ this makes sense indeed. We will use this layer construction (in the reverse direction) to associate an operator on $L^\infty(\mathbb{R}^{n+1})$ with our operator sequence $(A_\tau)_{\tau\in T}$ by "stacking" this sequence to an "operator pile".

Definition 4.4 *If $(A_\tau)_{\tau\in T}$ is some sequence in $\mathcal{F}$ then by*

$$(Bu)(x,\tau) := \begin{cases} (A_\tau u_\tau)(x) & \text{if } \tau \in T \text{ and } x \in \Omega_\tau \\ u(x,\tau) & \text{otherwise} \end{cases} \qquad \forall x \in \mathbb{R}^n,\ \tau \in \mathbb{R}$$

an operator B on $L^\infty(\mathbb{R}^{n+1})$ is given which acts like A_τ (on $L^\infty(\Omega_\tau)$) in the τ-th layer and is the identity operator outside. In this sense we will regard

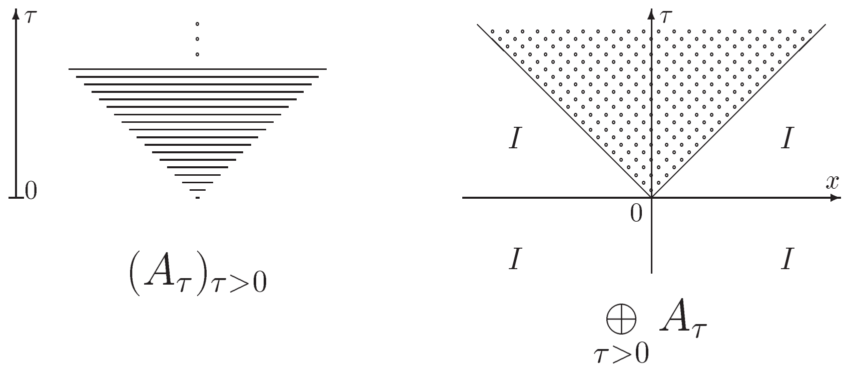
[3] Note that the notion of a $\mathcal{P}-$converging operator sequence is even more general there than in our situation.

A_τ as the τ-th layer of B. We will refer to B as stacked operator of the sequence $(A_\tau)_{\tau \in T}$ and denote it by

$$\bigoplus_{\tau \in T} A_\tau$$

or just $\oplus A_\tau$ if it is clear which index set T we are dealing with. Clearly, this is a bounded operator on $L^\infty(\mathbb{R}^{n+1})$ with

$$\| \bigoplus_{\tau \in T} A_\tau \| \; = \; \sup_{\tau \in T} \|A_\tau\| \; = \; \|(A_\tau)\|_{\mathcal{F}}.$$



An illustration of the mapping $(A_\tau)_{\tau>0} \mapsto \bigoplus_{\tau>0} A_\tau$ for $n = 1$ and $\Omega_\tau = [-\tau, \tau]$.

Our intuition now tells us that stablility of the operator sequence is very closely related with the stacked operator's invertibility at infinity. But in general these two properties are not equivalent since a perturbation of $\oplus A_\tau$ in countably many layers does not affect this operator while it can essentially change the stability of (A_τ). We will illustrate this fact in a very trivial example:

Example 1: $(A_\tau) = (P_{\Omega_\tau})$ is stable and $\oplus A_\tau = I$ is clearly invertible at infinity. But perturbing this sequence in the integer components

$$A'_\tau \; = \; \begin{cases} 0 & \text{if } \tau \in \mathbb{N} \\ P_{\Omega_\tau} & \text{otherwise} \end{cases}$$

completely disturbes the stability of the sequence while it does not change anything in the stacked operator which is still I. So how can we get along with these difficulties?

Escape 1: We might slightly soften the notion of stability by saying that two sequences (A_τ) and (B_τ) are "equivalent" if the set $\{\tau \in T : A_\tau \neq B_\tau\}$ has measure zero in $\mathbb{R}$ and call a sequence "essentially stable" if it is "equivalent" to a stable one.

Indeed, one can show that (A_τ) is "essentially stable" if and only if $\oplus A_\tau$ is invertible at infinity. But what is the practical relevance of this property we called "essential stability"? The physicist who came up with the equation (18) will choose a subsequence $(A_{\tau_k})_{k=1}^{\infty}$ out of $(A_\tau)_{\tau>0}$ and hopes to approximate the solution u of (18) by the solutions $u_k \in L^\infty(\Omega_{\tau_k})$ of

$$A_{\tau_k} u_k \;=\; P_{\Omega_{\tau_k}} b, \qquad k = 1, 2, ...$$

But however, if (A_τ) is only what we called "essentially stable" we can not guarantee him that this works with every subsequence (A_{τ_k}) – only with **almost** every one. For example, (A'_τ) from Example 1 is "essentially stable". But τ_k must not contain any integers for $k > k_0$ to provide us with an applicable approximation method (A'_{τ_k}).

Escape 2: We might expect the operators A_τ to depend continuously on τ in some sense. This could prevent effects as in Example 1. We are going to follow this idea. $\square$

4.3 Continuous operator sequences

Let Ω be some convex subset of $\mathbb{R}^n$ containing the origin in its interior. In what follows we will simplify the choice of the sets Ω_τ to the most usual case being $\Omega_\tau := \tau\Omega \; \forall \tau > 0$ and we will abbreviate $P_{\tau\Omega}$ to P_τ.

Remember the idea from Escape 2. But obviously it is not very fruitful to expect the mapping

$$\tau \mapsto A_\tau$$

to be continuous from $\mathbb{R}_+ := (0, \infty)$ to $\mathcal{L}(L^\infty(\mathbb{R}^n))$ (where A_τ is identified with the operator $P_\tau A_\tau P_\tau$ on $L^\infty(\mathbb{R}^n)$) since even the sequence (P_τ) itself does not enjoy this property. The problem is that $\|P_\tau - P_{\tau'}\| = 1$ as soon as τ' slightly differs from τ because then the domains $\tau'\Omega$ and $\tau\Omega$ differ in a set of measure greater than zero. We will therefore first transform all operators A_τ to the same domain and then require continuous dependence on τ.

Therefore let $\Phi_\tau : L^\infty(\Omega) \to L^\infty(\tau\Omega)$ denote the operator acting by

$$(\Phi_\tau u)(x) \;=\; u(x/\tau), \qquad x \in \tau\Omega$$

that is, stretching (having $\tau > 1$ in mind) the argument by the factor τ. Its inverse $\Phi_\tau^{-1} : L^\infty(\tau\Omega) \to L^\infty(\Omega)$ with

$$(\Phi_\tau^{-1} u)(x) \;=\; u(\tau x), \qquad x \in \Omega$$

shrinks (again think of $\tau > 1$) the argument by the factor τ. Now we can use Φ_τ and Φ_τ^{-1} to transform all operators A_τ to the same domain:

Definition 4.5 *We will say that $(A_\tau)_{\tau>0}$ with $A_\tau \in \mathcal{L}(L^\infty(\tau\Omega))$ is a continuous sequence if the mapping*

$$\tau \;\mapsto\; A_\tau^\Omega := \Phi_\tau^{-1} A_\tau \Phi_\tau$$

is continuous from $\mathbb{R}_+$ to $\mathcal{L}(L^\infty(\Omega))$. The set of all continuous sequences will be denoted by $\mathcal{C}$.

Remark: For the finite section method $(A_\tau) = (P_\tau A P_\tau)$ one has

$$A_\tau^\Omega = P_\Omega \Phi_\tau^{-1} A \Phi_\tau P_\Omega$$

since $\Phi_\tau^{-1} P_\tau = P_\Omega \Phi_\tau^{-1}$ and $P_\tau \Phi_\tau = \Phi_\tau P_\Omega$. $\square$

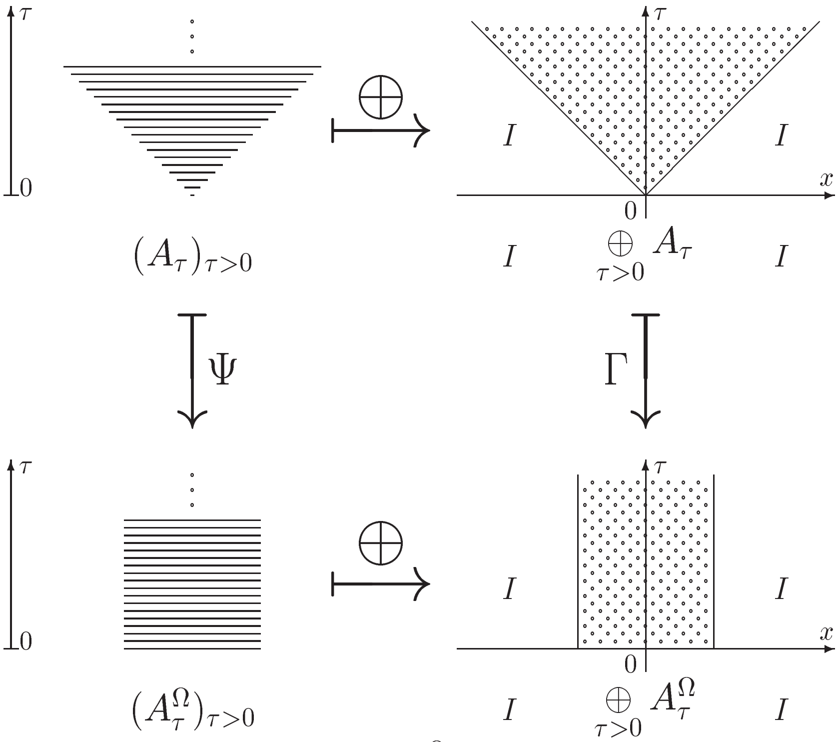
Clearly, the mapping $A_\tau \mapsto A_\tau^\Omega$ from $\mathcal{L}(L^\infty(\tau\Omega))$ to $\mathcal{L}(L^\infty(\Omega))$ is an isometrical isomorphism and consequently

$$\Psi : \; (A_\tau)_{\tau>0} \;\mapsto\; (A_\tau^\Omega)_{\tau>0} = (\Phi_\tau)_\tau^{-1}(A_\tau)_\tau(\Phi_\tau)_\tau$$

is an isometrical isomorphism (from $\mathcal{F}$ to $\mathcal{F}$) as well. Instead of the stability of (A_τ) we can study that of (A_τ^Ω). Finally, let $\Gamma : \mathcal{L}(L^\infty(\mathbb{R}^{n+1})) \to \mathcal{L}(L^\infty(\mathbb{R}^{n+1}))$ refer to the mapping

$$\Gamma : \bigoplus_{\tau>0} A_\tau \mapsto \bigoplus_{\tau>0} A_\tau^\Omega$$

(where also $\oplus A_\tau^\Omega$ is defined by 4.4 but with $T = \mathbb{R}_+$ and $\Omega_\tau \equiv \Omega$). A little thought shows that $\oplus A_\tau$ is invertible at infinity if and only if $\oplus A_\tau^\Omega$ is so.



The interplay between $(A_\tau)_{\tau>0}$, $(A_\tau^\Omega)_{\tau>0}$ and their stacked operators.

The following proposition shows that indeed, everything works fine in the case of continuous sequences. Its proof makes extensive use of Lemma 3.6 and this simple corollary.

Corollary 4.6 *Let $(x_n)_{n=1}^\infty$ be some convergent sequence of invertible elements in a Banach algebra B with unity element. If $\lim\limits_{n\to\infty} x_n$ is not invertible then the sequence $(\|x_n^{-1}\|)_{n=1}^\infty$ is unbounded.*

Proposition 4.7 *If $\tau \mapsto A_\tau^\Omega$ is continuously mapping $\mathbb{R}_+ \to \mathcal{L}(L^\infty(\Omega))$ then $(A_\tau^\Omega)_{\tau>0}$ is stable if and only if $\bigoplus\limits_{\tau>0} A_\tau^\Omega$ is invertible at infinity.*

Proof. $\boxed{\Longrightarrow}$ Let (A_τ^Ω) be stable and take $\tau_0 > 0$ large enough that A_τ^Ω is invertible for all $\tau > \tau_0$ and $\sup\limits_{\tau>\tau_0} \|(A_\tau^\Omega)^{-1}\| < \infty$. Put

$$B_\tau^\Omega := \begin{cases} P_\tau, & 0 < \tau \le \tau_0 \\ (A_\tau^\Omega)^{-1}, & \tau > \tau_0 \end{cases} .$$

Then $\oplus B_\tau^\Omega$ is inverse to $\oplus A_\tau^\Omega$ at infinity.

$\boxed{\Longleftarrow}$ Let $\oplus A_\tau^\Omega$ be invertible at infinity.
Suppose (A_τ^Ω) were not stable. Then we have either

a) All A_τ^Ω with necessarily large τ are invertible but $(\|(A_\tau^\Omega)^{-1}\|)_{\tau>\tau_0}$ is unbounded for all τ_0.

or

b) There is a sequence $(\tau_k)_{k=1}^\infty \to \infty$ such that $A_{\tau_k}^\Omega$ is not invertible for $k = 1, 2, ...$

In case a) we have a sequence $(\tau_k)_{k=1}^\infty \to \infty$ with $\|(A_{\tau_k}^\Omega)^{-1}\| > k$ for every k. Due to the continuity of the sequence $(A_\tau^\Omega)_\tau$ there is a neighborhood of each τ_k where the norms of the inverses $(A_\tau^\Omega)^{-1}$ are still larger than $k/2$ and this contradicts the existence of a bounded inverse of $\oplus A_\tau^\Omega$ at infinity.

So suppose we are in case b).
If not only $A_{\tau_k}^\Omega$ itself but all A_τ^Ω with τ in some neighborhood of τ_k were not invertible ($k = 1, 2, ...$) this would clearly contradict the existence of an inverse of $\oplus A_\tau^\Omega$ at infinity. Hence, for every k there is a sequence $(\tau_{k,i})_{i=1}^\infty \to \tau_k$ where all the $A_{\tau_{k,i}}^\Omega$ are invertible. And with them all A_u^Ω with u in some neighboorhood $U_{k,i}$ of $\tau_{k,i}$ are invertible. These neighborhoods can be supposed to be chosen such that all sequences $(u_{k,i})_{i=1}^\infty$ with $u_{k,i} \in U_{k,i}$ tend to τ_k as $i \to \infty$.

Now choose $\tilde{u}_{k,i} \in U_{k,i}$ such that

$$\|(A_{\tilde{u}_{k,i}}^\Omega)^{-1}\| \leq \inf_{u \in U_{k,i}} \|(A_u^\Omega)^{-1}\| + 1.$$

Remember that $A_{\tilde{u}_{k,i}}^\Omega \rightrightarrows A_{\tau_k}^\Omega$ as $i \to \infty$, all $A_{\tilde{u}_{k,i}}^\Omega$ being invertible but the limit $A_{\tau_k}^\Omega$ is not. From Corollary 4.6 we conclude that the sequence $(\|(A_{\tilde{u}_{k,i}}^\Omega)^{-1}\|)_{i=1}^\infty$ is unbounded. Hence there is a subsequence (passing from index i to i_m) growing as fast as the integers $(m)_{m=1}^\infty$. From the choice of $\tilde{u}_{k,i}$ it follows that for all $u \in U_{k,i_m}$ the norm of $(A_u^\Omega)^{-1}$ is larger than $m-1$, i.e. $\|(A_u^\Omega)^{-1}\|$ becomes as large as desired on whole sets $U_{k,i} \ni u$ near τ_k having measure greater than zero. Since this is the case for all $k = 1, 2, ...$ and $\tau_k \to \infty$ the operator $\oplus A_\tau^\Omega$ can not have a bounded inverse at infinity, which contradicts our assumption. ∎

Clearly, not only continuous sequences but also perturbations of such by sequences in $\mathcal{N}$ enjoy the desired property since perturbations with $\mathcal{N}$ do not affect stability and, on the other side, stacked operators of sequences from $\mathcal{N}$ are in $\mathcal{J}$ and such perturbations do not affect invertibility at infinity. We will sum up our achievements by the following theorem.

Theorem 4.8 *An operator sequence $(A_\tau)_{\tau>0} \in \mathcal{C} + \mathcal{N}$ is stable if and only if its stacked operator $\bigoplus\limits_{\tau>0} A_\tau$ is invertible at infinity.*

Proof. If $(A_\tau) \in \mathcal{C} + \mathcal{N}$ then $(A_\tau^\Omega) = (C_\tau^\Omega) + (N_\tau^\Omega)$ with C_τ^Ω continuously depending on $\tau \in \mathbb{R}_+$ and $N_\tau^\Omega \rightrightarrows 0$ as $\tau \to \infty$. Then

$$\begin{array}{lcl} (A_\tau) \text{ is stable} & \Longleftrightarrow & (A_\tau^\Omega) = (C_\tau^\Omega) + (N_\tau^\Omega) \text{ is stable} \\ & \Longleftrightarrow & (C_\tau^\Omega) \text{ is stable} \\ & \Longleftrightarrow & \oplus C_\tau^\Omega \text{ is invertible at } \infty \\ & \Longleftrightarrow & \oplus C_\tau^\Omega + \oplus N_\tau^\Omega \text{ is invertible at } \infty \\ & \Longleftrightarrow & \oplus A_\tau^\Omega \text{ is invertible at } \infty \\ & \Longleftrightarrow & \oplus A_\tau \text{ is invertible at } \infty. \end{array}$$

■

5 The Finite Section Method: Stability versus Limit operators

The two preceding sections suggest that – by combining Theorems 3.3 and 4.8 – the stability of a sequence (A_τ) is closely related to the operator spectrum of the stacked operator $\oplus A_\tau$ being uniformly invertible, provided both theorems are applicable, i.e. $(A_\tau) \in \mathcal{C} + \mathcal{N}$ and $\oplus A_\tau \in \mathcal{B}_\$$.

We will demonstrate how these concepts fit together for the finite section method in $L^\infty(\mathbb{R}^1)$ where we are able to give quite explicite results. So in what follows the operator sequence (A_τ) always stands for $(P_\tau A P_\tau)$, with $\tau \in T = \mathbb{R}_+$, n equals 1 and $\Omega = [-1, 1]$.

5.1 Limit operators of stacked operators

To finally apply Theorem 3.3 we will suppose that A is some rich operator, $A \in \mathcal{B}_\$$ which means $A \in \mathcal{B}_\mathcal{S} \subset$ BDO and every sequence $h = (h_m) \subset \mathbb{Z}^n$ tending to infinity has a subsequence that leads to a limit operator of A.

From the pictures in Section 4 we have some intuition how $\oplus A_\tau$ acts on $L^\infty(\mathbb{R}^2)$. Put

$$\begin{array}{rcll} H_{NE} & := & \{(x,\tau)\in\mathbb{R}^2 \ : \ \tau \ge -x\}, & \\ H_{NW} & := & \{(x,\tau)\in\mathbb{R}^2 \ : \ \tau \ge x\} \text{ and} & \\ V & := & \{(x,\tau)\in\mathbb{R}^2 \ : \ \tau \ge |x|\} \ = \ H_{NE}\cap H_{NW}. & \end{array}$$

Then clearly,

$$\bigoplus_{\tau>0} A_\tau \ = \ P_V(\bigoplus_{\tau\in\mathbb{R}} A)P_V \ + \ Q_V \tag{20}$$

(where $\oplus A$ is defined by 4.4 with $T=\mathbb{R}$, $A_\tau\equiv A$ and $\Omega_\tau\equiv\mathbb{R}$). We will sometimes use the notations $\mathcal{S}$, $\mathcal{B}$, $\mathcal{B}_{\mathcal{S}}$ and $\mathcal{B}_{\$}$ for operators on $L^\infty(\mathbb{R})$ and sometimes on $L^\infty(\mathbb{R}^2)$ without introducing new notations for that since from the context it will be clear in which setting the symbol is used.

When examining $\oplus A_\tau$ for limit operators we will have to distinguish between the different directions into which infinity can be approached. By Proposition 2.22, for an operator $B\in\mathcal{L}(L^\infty(\mathbb{R}^2))$ the operator spectrum $\sigma^{\mathsf{op}}(B)$ is the union of all $\sigma^{\mathsf{op}}_{(\cos\varphi,\,\sin\varphi)}(B)$ with φ running through $[0,2\pi)$. There are some especially interesting and some not so interesting directions in our case at hand. We therefore put

$$\begin{array}{rcl} \sigma^{\mathsf{op}}_N(B) & := & \displaystyle\bigcup_{\frac{\pi}{4}<\varphi<\frac{3}{4}\pi} \sigma^{\mathsf{op}}_{(\cos\varphi,\,\sin\varphi)}(B) \\ \sigma^{\mathsf{op}}_{NE}(B) & := & \sigma^{\mathsf{op}}_{(\cos\frac{\pi}{4},\,\sin\frac{\pi}{4})}(B) \\ \sigma^{\mathsf{op}}_{NW}(B) & := & \sigma^{\mathsf{op}}_{(\cos\frac{3}{4}\pi,\,\sin\frac{3}{4}\pi)}(B) \\ \sigma^{\mathsf{op}}_S(B) & := & \sigma^{\mathsf{op}}(B)\setminus(\sigma^{\mathsf{op}}_N(B)\cup\sigma^{\mathsf{op}}_{NE}(B)\cup\sigma^{\mathsf{op}}_{NW}(B)) \end{array}$$

and for $A\in\mathcal{L}(L^\infty(\mathbb{R}))$ we abbreviate $\sigma^{\mathsf{op}}_+(A) := \sigma^{\mathsf{op}}_{+1}(A)$ and $\sigma^{\mathsf{op}}_-(A) := \sigma^{\mathsf{op}}_{-1}(A)$. Finally put $P_{NE}:=P_{H_{NE}}$, $Q_{NE}:=I-P_{NE}$, $P_{NW}:=P_{H_{NW}}$ and $Q_{NW}:=I-P_{NW}$.

Proposition 5.1 *If $A\in\mathcal{L}(L^\infty(\mathbb{R}))$ is in $\mathcal{B}_{\$}$ then also $\oplus A_\tau\in\mathcal{B}_{\$}$ and a limit operator of $\oplus A_\tau$ is either*

1) I,
2) $\oplus(V_{-c}AV_c)$ *with some* $c\in\mathbb{Z}$,
3) $\oplus A_h$ *with* $A_h\in\sigma^{\mathsf{op}}(A)$,
4) $P_{NW}\,(\oplus A_h)\,P_{NW} \ + \ Q_{NW}$ *with some* $A_h\in\sigma^{\mathsf{op}}_+(A)$ *or*
5) $P_{NE}\,(\oplus A_h)\,P_{NE} \ + \ Q_{NE}$ *with some* $A_h\in\sigma^{\mathsf{op}}_-(A)$

all the $\oplus$ in 1) - 5) being over $T=\mathbb{R}$ and $\Omega_\tau\equiv\mathbb{R}$.

Proof. Let $A \in \mathcal{B}_{\$}$. By equation (20) and Proposition 2.19 a) we have to show that $\oplus A$, P_V and Q_V are in $\mathcal{B}_{\$}$.
From $^*(\oplus A) = \oplus^* A$ we conclude that $\oplus A \in \mathcal{S}$. For $b \in \mathrm{BUC}(\mathbb{R}^2)$ let b_τ refer to the $\tau-$th layer of b as in Definition 4.3. Then $\|[M_b, \oplus A]\| = \|[\oplus M_{b_\tau}, \oplus A]\| = \| \oplus [M_{b_\tau}, A]\| = \sup_\tau \|[M_{b_\tau}, A]\|$. If $\varphi_{t,k}$ is in place of b this tends to zero as $t \to 0$ (note that $\tau \mapsto b_\tau$ is in $\mathrm{BUC}(\mathbb{R})$) and hence, $\oplus A \in \mathcal{B}$. As all multiplicators, P_V and Q_V are in $\mathcal{B}_{\mathcal{S}}$ and hence, $\oplus A_\tau \in \mathcal{B}_{\mathcal{S}}$.

To see that $\oplus A_\tau \in \mathcal{B}_{\$}$ we will take an arbitrary sequence $h = (h_m)$ with $h_m = (h_m^{(1)}, h_m^{(2)}) \in \mathbb{Z}^2$ tending to infinity. We think of equation (20) and first look for

Limit operators of $\oplus A$:
Clearly, here the component $h_m^{(2)}$ has no influence at all. So we have a look at the first component.

If $h_m^{(1)}$ remains bounded then there is some infinite subsequence g of h and a $c \in \mathbb{Z}$ such that $g_m^{(1)} = c \;\forall m$ and $(\oplus A)_g = \oplus(V_{-c} A V_c)$.

If $h_m^{(1)}$ is unbounded, then
a) for some subsequence g of h the first component $g_m^{(1)}$ tends to $+\infty$ and since $A \in \mathcal{B}_{\$}$ we can choose a subsequence f of g such that $f^{(1)}$ leads to a limit operator of A and hence, $(\oplus A)_f = \oplus A_{f^{(1)}}$, or
b) for some subsequence g of h the first component $g_m^{(1)}$ tends to $-\infty$ and as in a) we can choose a subsequence f of g such that $(\oplus A)_f = \oplus A_{f^{(1)}}$.

Limit operators of P_V:
Clearly, $\sigma_S^{\mathsf{op}}(P_V) = \{0\}$ whence $\sigma_S^{\mathsf{op}}(\oplus A_\tau) = \{I\}$ and $\sigma_N^{\mathsf{op}}(P_V) = \{I\}$ whence $\sigma_N^{\mathsf{op}}(\oplus A_\tau) = \{\oplus(V_{-c} A V_c)\ ,\ \oplus A_h\ :\ c \in \mathbb{Z},\, A_h \in \sigma^{\mathsf{op}}(A)\}$. So the interesting cases remaining are $\sigma_{NE}^{\mathsf{op}}(P_V)$ and $\sigma_{NW}^{\mathsf{op}}(P_V)$ which are symmetric to each other.

Let $h_m = (h_m^{(1)}, h_m^{(2)}) \in \mathbb{Z}^2$ tend to infinity into direction $(\cos\frac{\pi}{4}, \sin\frac{\pi}{4})$. The difference $h_m^{(1)} - h_m^{(2)}$ can

- **a)** remain bounded,
- **b)** contain a subsequence tending to $+\infty$ or
- **c)** contain a subsequence tending to $-\infty$.

In either case we can pass to a subsequence g of h for which the difference $g_m^{(1)} - g_m^{(2)}$

a) is constant, say $g_m^{(1)} - g_m^{(2)} = d \in \mathbb{Z}$,
b) tends to $+\infty$, e.g. $g_m = (m^2 + m, m^2)$ or
c) tends to $-\infty$, e.g. $g_m = (m^2 - m, m^2)$.

In case b) we clearly have $(P_V)_g = 0$ and hence $(\oplus A_\tau)_g = I$. In case c) we have $(P_V)_g = I$ and hence $(\oplus A_\tau)_f = \oplus A_{f^{(1)}}$ for a suitable subsequence f of g with $f^{(1)}$ leading to a limit operator of A. And in case a) the equation

$$V_{-g_m} P_V V_{g_m} = V_{-(g_m^{(2)}+d,\, g_m^{(2)})} P_V V_{(g_m^{(2)}+d,\, g_m^{(2)})} = V_{-\binom{d}{0}} V_{-(g_m^{(2)},\, g_m^{(2)})} P_V V_{(g_m^{(2)},\, g_m^{(2)})} V_{\binom{d}{0}}$$

shows that

$$(P_V)_g = V_{-\binom{d}{0}} P_{NW} V_{\binom{d}{0}} = P_{H_{NW} - \binom{d}{0}} = P_{H_{NW} + \binom{0}{d}}.$$

From this sequence g a subsequence f shall be chosen such that $A_{f^{(1)}}$ exists, then $(\oplus A)_f = \oplus A_{f^{(1)}}$. Consequently,

$$\begin{aligned} (\oplus A_\tau)_f &= P_{H_{NW}+\binom{0}{d}} \left(\oplus A_{f^{(1)}}\right) P_{H_{NW}+\binom{0}{d}} + Q_{H_{NW}+\binom{0}{d}} \\ &\cong P_{NW} \left(\oplus A_{f^{(1)}}\right) P_{NW} + Q_{NW}. \end{aligned}$$

So much about $\sigma^{\mathsf{op}}_{NE}(\oplus A_\tau)$. And by a completely symmetric argumentation $\sigma^{\mathsf{op}}_{NW}(\oplus A_\tau)$ can be described and we are finished. ■

This proposition – in combination with the two main theorems above – leaves us with a stability criterion for the finite section method of rich operators $A \in \mathcal{B}_\$$ (provided their sequence of finite sections is in $\mathcal{C} + \mathcal{N}$) namely, (A_τ) is stable if and only if the set of all operators listed in Proposition 5.1 1)-5) is uniformly invertible.

But there is some little disappointment in this answer since we asked for the stability of some operator on $L^\infty(\mathbb{R})$ and got the answer in terms of invertibility on $L^\infty(\mathbb{R}^2)$. Well, this can be repaired easily since an operator

$$\bigoplus_{\tau \in \mathbb{R}} B_\tau$$

is (boundedly) invertible if and only if the sequence $(B_\tau)_{\tau\in\mathbb{R}}$ is invertible in $\mathcal{F}$ that is, $\{B_\tau\}_{\tau\in\mathbb{R}}$ is uniformly invertible as a set of operators.

Before concluding this discussion with the main theorem we will introduce the abbreviations P and Q for the complementary projectors $P := P_{[0,+\infty)}$ and $Q := I - P$ in $\mathcal{L}(L^\infty(\mathbb{R}))$.

Theorem 5.2 *If $A \in \mathcal{B}_{\$}$ is some rich operator with finite section sequence $(A_\tau)_{\tau>0}$ in $\mathcal{C}+\mathcal{N}$ then the following statements are equivalent.*

(i) *The finite section method $(A_\tau) = (P_\tau A P_\tau)$ is applicable to A.*

(ii) *The finite section sequence (A_τ) is stable.*

(iii) *The set consisting of*

a) *the operator A itself,*

b) *all $QA_hQ + P$ with A_h running through $\sigma_+^{\mathsf{op}}(A)$ and*

c) *all $PA_hP + Q$ with A_h running through $\sigma_-^{\mathsf{op}}(A)$*

is uniformly invertible.

Proof. We first show that (ii) and (iii) are equivalent: By Proposition 5.1 and Theorems 3.3 and 4.8 we know that (ii) is equivalent to the set of operators listed in Proposition 5.1 1) - 5) being uniformly invertible. For 1) this is trivial. 2) is given if and only if A is invertible. But then 3) is automatically fulfilled since $\sigma^{\mathsf{op}}(A)$ is uniformly invertible if A is invertible (even if only at infinity).

Concerning 4) we remark that if A_h is a limit operator of A then $V_{-c}A_hV_c$ is the limit operator of A with respect to the sequence $h + c := (h_m + c)_m$ which tends to infinity in the same direction as h does. Consequently,

$$\begin{aligned} P_{NW}(\bigoplus_{\tau\in\mathbb{R}} A_h)P_{NW} + Q_{NW} &= \bigoplus_{\tau\in\mathbb{R}} (P_{(-\infty,\tau)}A_hP_{(-\infty,\tau)} + P_{[\tau,+\infty)}) \\ &= \bigoplus_{\tau\in\mathbb{R}} (V_\tau P_{(-\infty,0)}V_{-\tau}A_hV_\tau P_{(-\infty,0)}V_{-\tau} \\ &\quad + V_\tau P_{[0,+\infty)}V_{-\tau}) \\ &= \bigoplus_{\tau\in\mathbb{R}} (V_\tau QA_{h+\tau}QV_{-\tau} + V_\tau PV_{-\tau}) \\ &= \bigoplus_{\tau\in\mathbb{R}} (V_\tau(QA_{h+\tau}Q + P)V_{-\tau}) \end{aligned}$$

is invertible if and only if $QA_{h+\tau}Q + P$ is invertible for all τ. Hence, 4) is equivalent to b) and in the same way we see that 5) and c) correspond to each other.

Finally, Theorem 4.2 says that (i) is equivalent to (ii) plus A being invertible. But since (ii)$\Leftrightarrow$(iii), the invertibility of A is already guaranteed in (ii) and so we are finished. ■

So this is the proposed theorem. Clearly, we must confess that this result is quite useless if one has no idea about the local operator spectra $\sigma_+^{\mathsf{op}}(A)$

and $\sigma_-^{\mathsf{op}}(A)$. To give some information about these sets we will add one more subsection.

5.2 Limit operators of a multiplicator

Theorem 5.2 leaves us with an effective stability criterion – provided one knows the local operator spectra $\sigma_+^{\mathsf{op}}(A)$ and $\sigma_-^{\mathsf{op}}(A)$. By Proposition 2.10 we know that the operator $A \in \mathcal{B}_{\$} \subset \mathcal{B}_{\mathcal{S}} \subset \text{BDO}$ has the property that its discretization A^G is the norm limit of operators of the form (1). In many applications the generalized multiplicators $\hat{M}_b$ are of the form $\hat{M}_{\hat{f}}$ which means they are the discretization of some (usual) multiplicator $M_f \in \mathcal{L}(L^\infty(\mathbb{R}))$.

Proposition 2.17 tells us that limit operators are compatible with addition, composition and passing to the norm-limit. Hence,

$$\begin{aligned} A_h &= \Big(\lim_{n\to\infty} \sum_{|\alpha|\le b^{(n)}} M_{f_\alpha^{(n)}} V_\alpha \Big)_h \\ &= \lim_{n\to\infty} \sum_{|\alpha|\le b^{(n)}} (M_{f_\alpha^{(n)}})_h (V_\alpha)_h \\ &= \lim_{n\to\infty} \sum_{|\alpha|\le b^{(n)}} (M_{f_\alpha^{(n)}})_h V_\alpha \end{aligned}$$

and the only question remaining is how limit operators of multiplicators M_f behave.

5.2.1 Rich functions

In contrast to what follows we will first prove a result from the discrete case ℓ^∞. Note that limit operators have pretty much the same definition there as we gave for L^∞ in section 2.6.

Proposition 5.3 *Let $\mathbf{X}$ refer to some finite dimensional vector space. Every generalized multiplication operator $\hat{M}_b$ on $\ell^\infty(\mathbb{Z}^n, \mathbf{X})$ is rich and every limit operator of $\hat{M}_b$ is a generalized multiplication operator again.*

Proof. Take some bounded sequence $b = (B_\alpha)_{\alpha\in\mathbb{Z}^n}$ of operators $B_\alpha \in \mathcal{L}(\mathbf{X})$ and some sequence $h = (h_m) \subset \mathbb{Z}^n$ tending to infinity. For every $u = (u_\alpha) \in$

$\ell^\infty(\mathbb{Z}^n, \mathbf{X})$ we have

$$(V_{-h_m}\hat{M}_b V_{h_m} u)_\alpha \; = \; B_{\alpha+h_m} u_\alpha, \qquad \forall \alpha \in \mathbb{Z}^n.$$

For simplicity enumerate the points of $\mathbb{Z}^n$ by $\alpha_1,\ \alpha_2, ...$ By our premise, $\mathcal{L}(\mathbf{X})$ is finite dimensional. That's why the bounded set $\{B_{\alpha_1+h_m}\}_{m=1}^\infty \subset \mathcal{L}(\mathbf{X})$ has some partial limit (pre-compactness, i.e. Bolzano-Weierstrass Theorem) in $\mathcal{L}(\mathbf{X})$ which we will denote by C_{α_1}. So there is some subsequence $h^{(1)}$ of h such that

$$B_{\alpha_1+h_m^{(1)}} \rightrightarrows C_{\alpha_1} \qquad \text{as } m \to \infty.$$

Moreover, the bounded set $\{B_{\alpha_2+h_m^{(1)}}\}_{m=1}^\infty$ has some partial limit C_{α_2} in $\mathcal{L}(\mathbf{X})$ and we choose $h^{(2)}$ out of $h^{(1)}$ such that

$$B_{\alpha_2+h_m^{(2)}} \rightrightarrows C_{\alpha_2} \qquad \text{as } m \to \infty.$$

Continuing this way we can define a sequence $g := (h_1^{(1)}, h_2^{(2)}, h_3^{(3)}, ...) \subset \mathbb{Z}^n$ which is clearly a subsequence of h.

It is easy to see that g leads to a limit operator of $\hat{M}_b$ which is again a generalized multiplicator, say $\hat{M}_c$ where $c = (C_\alpha)_{\alpha\in\mathbb{Z}^n}$ is bounded again. Indeed, for every finite set $U \subset \mathbb{Z}^n$ we have

$$\begin{aligned} \|(V_{-g_m}\hat{M}_b V_{g_m} - \hat{M}_c)P_U\| \; &= \; \max_{\alpha\in U} \|B_{\alpha+g_m} - C_\alpha\| \\ &= \; \max_{k\in K} \|B_{\alpha_k+h_m^{(m)}} - C_{\alpha_k}\| \to 0 \quad \text{as } m \to \infty \end{aligned}$$

where $K = \{k \in \mathbb{N} : \alpha_k \in U\}$ is just a finite set. Of course the same is true for P_U coming from the left and we have shown that $\hat{M}_b$ is rich with $(\hat{M}_b)_g = \hat{M}_c$, the symbol c arising from our partial limit construction.

To see that every limit operator of $\hat{M}_b$ is a generalized multiplicator take some arbitrary limit operator $(\hat{M}_b)_h$. Now, by the same construction as above, choose a subsequence g out of h and discover that $(\hat{M}_b)_g$ is a generalized multiplicator. But since $(\hat{M}_b)_g = (\hat{M}_b)_h$ we are finished. ■

Unfortunately, this proof is not portable to the case of (usual) multiplication operators with $b \in L^\infty(\mathbb{R}^n)$.

1.) $\mathbb{R}^n$ fails to become enumerated by $\alpha_1, \alpha_2, ...$
2.) By trying some workaround to 1.) and writing $L^\infty(\mathbb{R}^n) \cong \ell^\infty(\mathbb{Z}^n, \mathbf{X})$ with $\mathbf{X} = L^\infty(H),\ H = [0,1)^n$ we have an infinite dimensional space $\mathcal{L}(\mathbf{X})$ and the Bolzano-Weierstrass argument is not applicable.

There is some very good reason for this proof being not portable to $L^\infty(\mathbb{R}^n)$: Proposition 5.3 is not true there. The only fact that can be saved is that the limit operator of M_b ($b \in L^\infty(\mathbb{R})$) with respect to a sequence $h = (h_m) \to \infty$ is always another operator of multiplication, say M_c with $c \in L^\infty(\mathbb{R})$ and

$$b|_{h_m+U} \rightrightarrows c|_U, \qquad m \to \infty \tag{21}$$

(uniform convergence on U) for all finite intervals $U \subset \mathbb{R}$ where we identify $b|_{h_m+U}$ with the restriction of $V_{-h_m}b$ to U.

Definition 5.4 *We call a function $b \in L^\infty(\mathbb{R})$* rich *if M_b is what we called a rich operator, i.e. from every sequence of integers $h = (h_m) \to \infty$ a suitable subsequence g can be chosen such that M_b possesses a limit operator with respect to g. Otherwise we call b* ordinary. *There are even functions $b \in L^\infty(\mathbb{R})$ for which no sequence of integers $h = (h_m) \to \infty$ possesses a subsequence that leads to a limit operator of M_b. Such functions will be referred to as* poor *functions.*

We will denote the set of rich functions by **\$**.

For example

$$b_1(x) = (-1)^{[x/\pi]},$$

where $[y]$ denotes the integer part of $y \in \mathbb{R}$, is a poor function. b_1 is 2π-periodic and has a jump at every multiple of π. That's why the sequence $(b_1|_{h_m+U})_m$ cannot be a Cauchy sequence in $L^\infty(U)$ for any sequence of integers (h_m). Here the cause for b_1 being poor is clearly the condition that all h_m have to be integers.

Another example of a poor function is

$$b_2(x) = \sin(x^2).$$

One can easily show that no sequence (of integers or reals) tending to infinity leads to a limit operator of M_{b_2}.

Definition 5.5 *Take some positive real number l. A function $f \in L^\infty(\mathbb{R})$ is called* step function with steps of length l *if there is an $x_0 \in \mathbb{R}$ with the property that f is constant on all intervals $(x_0 + kl, x_0 + (k+1)l)$, $k \in \mathbb{Z}$. The set of these functions will be denoted by T_l. Finally put*

$$T_{\mathbb{Q}} := \bigcup_{p,q \in \mathbb{N}} T_{p/q}.$$

Our example b_1 is obviously in T_π and has proven to be ordinary (even poor). We will see that this is a consequence of the irrationality of π.

Proposition 5.6 *Step functions with rational step length are rich,* $T_{\mathbb{Q}} \subset \$$.

Proof. Pick some arbitrary $p, q \in \mathbb{N}$. Since $T_{p/q} \subset T_{1/q}$ it remains to show that all functions $b \in T_{1/q}$ are rich. So put $l = 1/q$ and take arbitrary $b \in T_l$ and $h = (h_m) \subset \mathbb{Z}$ with $h_m \to \infty$. The intervals $h_m + U$ $(m = 1, 2, ...)$ differ by an integer translation. Since $lq = 1$ the question whether $(b|_{h'_m+U})_m$ is a Cauchy sequence in $L^\infty(U)$ for some subsequence (h'_m) of h can be identified with the discrete case $\ell^\infty(\mathbb{Z}, \mathbb{C}^q)$. In this discrete case we have Proposition 5.3 and so we are done with $b \in T_{1/q}$ as well. ■

Since the mapping $A \mapsto A_h$ is compatible with addition, composition and passing to the limit in the operator norm (with eventually passing to a suitable subsequence of h, see the proof of Proposition 2.19) also $\$$ is closed under addition, multiplication and supremum norm, i.e. $\$$ is a Banach subalgebra of $L^\infty(\mathbb{R})$.

If $\mathbf{X}$ is a normed space, $x \in \mathbf{X}$ and $\varepsilon > 0$ we will introduce the notation

$$U_\varepsilon(x) \ := \ \{y \in \mathbf{X} \, : \, \|x - y\|_{\mathbf{X}} < \varepsilon\}.$$

Proposition 5.7 BUC $\subset \$$

Proof. Pick some arbitrary $f \in$ BUC. To every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$f\Big(U_\delta(x)\Big) \subset U_\varepsilon\Big(f(x)\Big) \quad \text{holds for all } x \in \mathbb{R}.$$

So this is true for some $\delta' \in \mathbb{Q}$ with $0 < \delta' \leq \delta$ as well. Consequently, there is a step function $g \in T_{\delta'}$ with $\|f - g\|_\infty < \varepsilon$ which tells that $T_{\mathbb{Q}}$ is dense in BUC. Proposition 5.6 and the norm-closedness of $\$$ show the assertion. ■

So all uniformly continuous $L^\infty(\mathbb{R})$ functions are rich. But are there any more rich functions among the others in BC? To answer this question first have a look at an example of such a function in BC $\setminus$ BUC:

Take a real-valued continuous function f which is only supported in the intervals $(m^2 - \frac{1}{m}, m^2 + \frac{1}{m})$, $m = 1, 2, ...$ and $0 \leq f(x) \leq f(m^2) = 1$ for all $x \in \mathbb{R}$. M_f clearly has no limit operator with respect to any subsequence of $h = (m^2)$. (But there are sequences like $(m^2 + m)$ which lead to a limit operator of M_f. So f is not poor.) However, f is just ordinary.

Indeed one can show that every function in $\mathrm{BC} \setminus \mathrm{BUC}$ is a little bit like f and thus ordinary. Here is the main result of this subsection:

Proposition 5.8 $\$ \cap \mathrm{BC} = \mathrm{BUC}$

Proof. Thinking of Proposition 5.7 it remains to show that all functions in $\mathrm{BC} \setminus \mathrm{BUC}$ are ordinary.

So take a bounded and (not uniformly) continuous function b. Since b is uniformly continuous on every finite interval the reason for its nonuniform continuity lies at infinity, i.e. there is an $\varepsilon_0 > 0$ and a sequence $(x_m) \subset \mathbb{R}$ with $x_m \to \infty$ and

$$b\Big(U_{1/m}(x_m)\Big) \not\subset U_{\varepsilon_0}\Big(b(x_m)\Big) \tag{22}$$

for all $m = 1, 2, ...$

Now choose the integer sequence $h = (h_m) := ([x_m])$. We will see that M_b does not possess a limit operator with respect to any subsequence of h. Suppose there is such a limit operator M_c and let $U = [-1, 1]$. Then b is uniformly continuous on every interval $h_m + U$ and (21) shows that c is uniformly continuous on U, i.e. for every $\varepsilon > 0$ there is a $\delta > 0$ such that $c(U_\delta(x)) \subset U_\varepsilon(c(x))$ for all $x \in U = [-1, 1]$. So especially

$$c\Big(U_\delta(x_m - h_m)\Big) \subset U_{\frac{\varepsilon_0}{4}}\Big(c(x_m - h_m)\Big), \qquad m = 1, 2, ...$$

Because of (22) for all sufficiently large m (with $1/m \le \delta$) the functions $b|_{h_m+U}$ and $c|_U$ differ on a δ-neighborhood at least by $\varepsilon_0/2$ which contradicts (21). ■

5.2.2 Slowly oscillating functions

At first we will say what we understand under the (local) essential range of a $L^\infty-$function.

Definition 5.9 *Let Ω be some arbitrary subset of the real line $\mathbb{R}$. The essential range $\mathcal{R}(b)$ of a function $b \in L^\infty(\Omega)$ is the spectrum of the element b*

in the Banach algebra $L^\infty(\Omega)$. In most cases the set Ω is evident from the context and is therefore not mentioned explicitly.

For every $x \in \overline{\mathbb{R}}$ let $U(x)$ denote the set of all neighborhoods of x and put

$$\mathcal{R}_x(b) \;:=\; \bigcap_{U \in U(x)} \mathcal{R}(b|_U)$$

which is referred to as the local essential range *or the* essential cluster points *of the function b at point x.*

Now we can define another interesting class of functions which has the property that limit operators of multiplicators with such symbol behave especially nice:

Definition 5.10 *We will say that a function $f \in L^\infty(\mathbb{R})$ is* slowly oscillating *towards $+\infty$ or $-\infty$ if for every $\varepsilon > 0$ almost all values attained by f on the interval $[x, x+1]$ are contained in a disc of radius ε in the complex plane,*

$$\mathcal{R}(f|_{[x,x+1]}) \subset U_\varepsilon(m_x) \quad \textit{for some } m_x \in \mathbb{C}$$

whenever x is close enough to $+\infty$ or $-\infty$, respectively. We denote the set of functions with that property by SO_+ and SO_-, respectively and finally put $\mathrm{SO} := \mathrm{SO}_+ \cap \mathrm{SO}_-$.

In what follows we will restrict ourselves to SO_+. The other case is completely symmetric. The following lemma will give us an equivalent characterization of this property.

Lemma 5.11 **a)** *The following two conditions are equivalent.*
(i) $f \in \mathrm{SO}_+$
(ii) For every $\varepsilon > 0$ and every finite interval U the essential range of the restriction of f to the interval $x + U$ is contained in some disc with radius ε in the complex plane,

$$\mathcal{R}(f|_{x+U}) \subset U_\varepsilon(m_x) \quad \textit{for some } m_x \in \mathbb{C} \tag{23}$$

whenever x is large enough.

b) *If f is continuously differentiable in some neighborhood of $+\infty$, say $x > x_0$, then the condition that the derivative $f'(x)$ tends to zero with $x \to +\infty$ is sufficient for (i) and (ii) but not necessary.*

Proof. a) $(ii) \Rightarrow (i)$ is trivial since (i) is nothing but (ii) with $U = [0, 1]$.

$(i) \Rightarrow (ii)$: If (i) holds then (23) holds with $U = [0, 1]$ and all subsets of [0,1]. If (23) holds for a set U then it also holds for all sets of the form $t+U$ ($t \in \mathbb{R}$) in place of U. Finally, if it holds for $U = U_1$ and $U = U_2$ with $U_1 \cap U_2 \neq \emptyset$ then the essential ranges of $f|_{x+U_1}$ and $f|_{x+U_2}$ are each contained in a disc D_1, D_2 of radius $\varepsilon/2$ if x is large enough. Since $D_1 \cap D_2 \neq \emptyset$, both discs are contained in one of radius ε, i.e. (23) holds for $U = U_1 \cup U_2$ as well. Taking all this together it is clear that (23) holds for all finite intervals U. This is (ii).

b) Pick some $\varepsilon > 0$. If $f'(x) \to 0$ as $x \to +\infty$ there is some $x_\varepsilon > x_0$ such that $|f'(x)| < \varepsilon$ for all $x > x_\varepsilon$. But from

$$|f(x+u) - f(x)| = |f'(\xi_{x,u})u| < \varepsilon u \quad (\text{some } \xi_{x,u} \in (x, x+u))$$

we conclude that $f(x+u) \in U_\varepsilon(f(x))\ \forall u \in [0, 1]$ if $x > x_\varepsilon$. This is (i).
Check the function $f(x) = \frac{\sin x^2}{x}$ to see that $f'(x)$ need not tend to zero if $f \in \mathrm{SO}_+$. Moreover, slowly oscillating functions need not even be continuous, not to mention differentiability. For instance, look at $f(x) = \frac{(-1)^{[x]}}{x}$. ■

In what follows we will often use property (ii) to characterize the set SO_+. It is easy to observe that SO_+ is a closed subalgebra of $L^\infty(\mathbb{R})$. We will now study the set of limit operators of M_b when b is slowly oscillating.

Proposition 5.12 *If $b \in \mathrm{SO}_+$ then the set of limit operators of M_b is*

$$\sigma_+^{\mathsf{op}}(M_b) \;=\; \{cI : c \in \mathcal{R}_{+\infty}(b)\}.$$

Proof. Pick some $c \in \mathcal{R}_{+\infty}(b)$ and some finite interval U. There is a sequence of points $(x_m) \subset \mathbb{R}$ tending to $+\infty$ and a sequence of complex numbers (c_m) with $c_m \in \mathcal{R}(b|_{x_m+U})$ such that $c_m \to c$ as $m \to \infty$. Since $U' := U + [-1, 1]$ is a bounded interval and $b \in \mathrm{SO}_+$ there is a m_0 such that for every $m > m_0$, $\mathcal{R}(b|_{x_m+U'})$ is contained in the disc $U_{\frac{\varepsilon}{2}}(c_m)$. If m_0 is taken large enough that in addition $|c_m - c| < \frac{\varepsilon}{2}$ then

$$\mathcal{R}(b|_{x_m+U'}) \subset U_\varepsilon(c) \quad \text{if } m > m_0.$$

Now define the sequence of integers $h = (h_m)$ by $h_m := [x_m]$. Then $h_m + U$ is contained in $x_m + U'$ and consequently, $\mathcal{R}(b|_{h_m+U})$ is contained in $U_\varepsilon(c)$ for all $m > m_0$, i.e. cI is the limit operator of M_b with respect to the sequence h.

Conversely, by (21) and Lemma 5.11 it is clear that for all limit operators (towards $+\infty$) M_c of M_b the function c has to be constant and contained in the partial limiting set of b at $+\infty$. ■

Corollary 5.13 *Slowly oscillating functions are rich,* $\mathrm{SO}_+ \subset \$$.

Proof. For every sequence $h = (h_m),\ h_m \to \infty$, the partial limiting set of the local essential ranges of b in $h_1,\ h_2,\ ...$ is nonempty. So pick a complex number c in that partial limiting set and an appropriate subsequence g of h and proceed as in the proof of the previous proposition to show that cI is the limit operator of M_b with respect to $h^{(1)}$. ■

In some sense even the reverse direction of Proposition 5.12 is true.

Proposition 5.14 *If* $b \in \$$ *and every limit operator of* M_b *towards* $+\infty$ *is a multiple of the identity operator* I *then* $b \in \mathrm{SO}_+$.

Proof. Let the conditions of the proposition be fulfilled and suppose that $b \notin \mathrm{SO}_+$. Then there exist a finite interval U, an $\varepsilon_0 > 0$ and a sequence of points $(x_m) \subset \mathbb{R}$ tending to $+\infty$ such that for every m the essential range $\mathcal{R}(b|_{x_m+U})$ is not contained in any disc of radius ε_0. Now let $U' := U + [-1, 1]$ and $h = (h_m) \subset \mathbb{Z}$ with $h_m := [x_m]$. Since $x_m + U \subset h_m + U'$, for every m the essential range $\mathcal{R}(b|_{h_m+U'}) \supset \mathcal{R}(b|_{x_m+U})$ is not contained in any disc with radius ε_0. Consequently for every m and every complex number z

$$\|b|_{h_m+U'} - z\|_\infty \ \ \geq \ \ \varepsilon_0. \tag{24}$$

Since $b \in \$$ there is a subsequence g of h such that the limit operator of M_b exists with respect to $g = (g_m)$. Denote this limit operator by M_c. Since U' is a finite interval this means that

$$\|b|_{g_m+U'} - c|_{U'}\|_\infty \to 0 \quad \text{as } m \to \infty.$$

But comparing this with (24) it is clear that the function c is not constant on the whole of $\mathbb{R}$. So at least one limit operator of M_b is not a multiple of I which contradicts our assumption. ■

6 An algebra of convolution- and multiplication operators

We will apply the results from the previous sections to an algebra that is generated by operators of multiplication with symbols in BUC and convolution operators with symbol in the Wiener algebra. We will do this for the case $n = 1$.

For a function $k \in L^1(\mathbb{R})$ let Fk refer to its Fourier transform

$$(Fk)(x) = \int_{-\infty}^{+\infty} k(t)e^{itx}\ dt$$

and denote the set of functions $\{c + Fk : c \in \mathbb{C},\ k \in L^1(\mathbb{R})\}$ by $\mathbb{C} + FL^1$ which is also called the *Wiener algebra*. It is readily seen that $a = c + Fk$ is continuous on the one-point compactification $\dot{\mathbb{R}}$ of the real line with $a(\infty) = c$.

With every Wiener function $a = c + Fk$ one can associate a *convolution operator* $\overset{\circ}{W}_a$ acting on $L^\infty(\mathbb{R})$ by

$$(\overset{\circ}{W}_a\varphi)(t) \ :=\ c\varphi(t) + \int_{-\infty}^{\infty} k(t-s)\varphi(s)\ ds$$

and say that the function a is the *symbol* of the operator $\overset{\circ}{W}_a$.

We will now study the finite section method for operators in the smallest closed algebra of $\mathcal{L}(L^\infty(\mathbb{R}))$ containing all $\overset{\circ}{W}_a$ with $a \in \mathbb{C} + FL^1$ and all M_b with $b \in \mathrm{BUC}$:

$$\mathcal{A} := \mathbf{alg}_{\mathcal{L}(L^\infty(\mathbb{R}))}\Big\{\overset{\circ}{W}_a,\ M_b\ :\ a \in \mathbb{C} + FL^1,\ b \in \mathrm{BUC}\Big\}$$

Indeed, we are able to show that

Proposition 6.1 *Every operator $A \in \mathcal{A}$ is subject to the conditions on Theorem 5.2 that is, $A \in \mathcal{B}_{\$}$ and the finite section sequence $(A_\tau) \in \mathcal{C} + \mathcal{N}$.*

We will divide the proof into several steps by proving some lemmata.

Lemma 6.2 $\mathcal{A} \subset \mathcal{B}_{\$}$.

Proof. It is easily checked that $\mathring{W}_a$ and M_b are in $\mathcal{B}$ and hence the whole algebra is so. Taking into account that the adjoint operators of $\mathring{W}_a$ and M_b are $\mathring{W}_{\overline{a}}$ and $M_{\overline{b}}$, respectively (just passing to the complex conjugated symbols) we clearly have $\mathring{W}_a, M_b \in \mathcal{B}_{\mathcal{S}}$ and hence $\mathcal{A} \subset \mathcal{B}_{\mathcal{S}}$. So it remains to study limit operators.

Convolution operators are what is usually called "translation invariant" that is, $\mathring{W}_a V_h = V_h \mathring{W}_a$ and hence

$$V_{-h}\mathring{W}_a V_h = \mathring{W}_a \qquad \forall a \in \mathbb{C} + FL^1,\ \forall h \in \mathbb{R}.$$

Consequently, every convolution operator $\mathring{W}_a$ possesses a limit operator with respect to any sequence and all these limit operators are equal to $\mathring{W}_a$ itself.

Proposition 5.7 finally shows that also $M_b \in \mathcal{B}_{\$}$. ■

Lemma 6.3 *For every operator* $A \in \mathcal{A}$ *we have*

$$P_\tau A Q_{2\tau} \rightrightarrows 0 \quad \textit{and} \quad Q_{2\tau} A P_\tau \rightrightarrows 0 \quad \textit{as } \tau \to \infty.$$

1^{st} *Proof.* We show that the set

$$M = \{A \in \mathcal{L}(L^\infty(\mathbb{R}))\ :\ P_{\alpha\tau} A Q_{\beta\tau} \rightrightarrows 0,\ Q_{\beta\tau} A P_{\alpha\tau} \rightrightarrows 0 \quad \forall \beta > \alpha > 0\}$$

is a Banach algebra containing $\mathring{W}_a$ and M_b. Then the claim is obvious.

Additivity and closedness of M are readily seen and by

$$P_{\alpha\tau}(AB)Q_{\beta\tau} = P_{\alpha\tau} A \underbrace{P_{\frac{\alpha+\beta}{2}\tau} B Q_{\beta\tau}}_{\rightrightarrows 0} + \underbrace{P_{\alpha\tau} A Q_{\frac{\alpha+\beta}{2}\tau}}_{\rightrightarrows 0} B Q_{\beta\tau} \rightrightarrows 0$$

M is a Banach algebra.

Clearly, $P_{\alpha\tau} M_b Q_{\beta\tau} = 0 = Q_{\beta\tau} M_b P_{\alpha\tau}$ for all $\beta > \alpha > 0$ and one also easily computes that $\mathring{W}_a \in M$. ■

2^{nd} *Proof.* By Lemma 6.1 we have $\mathcal{A} \subset \mathcal{B}_{\$} \subset \mathcal{B}_{\mathcal{S}}$. Proposition 2.10 says $\mathcal{B}_{\mathcal{S}} \subset$ BDO and from Proposition 2.5 b) we know that BDO $\subset \mathfrak{A}$. Summing this up we see that $\mathcal{A} \subset \mathfrak{A}$ and since the sets $[-\tau, \tau]$ and $\mathbb{R} \setminus [-2\tau, 2\tau]$ are drifting apart with a distance as large as τ which goes to infinity, we are finished. ■

Remember the notations from section 4.3. We agreed upon calling the sequence $(A_\tau) = (P_\tau A P_\tau)$ continuous, $(A_\tau) \in \mathcal{C}$, if the mapping $\tau \mapsto \Phi_\tau^{-1} A_\tau \Phi_\tau$ is continuous from $\mathbb{R}_+$ to $\mathcal{L}(L^\infty[-1,1])$. To finally show that $(A_\tau) \in \mathcal{C} + \mathcal{N}$ for all $A \in \mathcal{A}$ we need some tricky argument which goes like that:

Lemma 6.4 *The set*

$$M := \Big\{ A \in \mathcal{L}(L^\infty(\mathbb{R})) \ : \ \Big(A_\tau\Big),\ \Big((\Phi_2^{-1} A)_\tau\Big),\ \Big((A\Phi_2)_\tau\Big) \in \mathcal{C} + \mathcal{N} \Big\}$$

is a Banach algebra which contains $\mathcal{A}$.

Proof. First note the obvious rules $\Phi_a \Phi_b = \Phi_{ab}$, $\Phi_a^{-1} = \Phi_{1/a}$ and $\Phi_a P_b = P_{ab}\Phi_a$ and hence, $P_b \Phi_a^{-1} = \Phi_a^{-1} P_{ab}$ for all $a, b > 0$. Consequently,

$$\begin{aligned} \Phi_\tau^{-1} A_\tau \Phi_\tau &= \Phi_\tau^{-1} P_\tau A P_\tau \Phi_\tau \\ \Phi_\tau^{-1} (\Phi_2^{-1} A)_\tau \Phi_\tau &= \Phi_\tau^{-1} P_\tau \Phi_2^{-1} A P_\tau \Phi_\tau = \Phi_{2\tau}^{-1} P_{2\tau} A P_\tau \Phi_\tau \\ \Phi_\tau^{-1} (A\Phi_2)_\tau \Phi_\tau &= \Phi_\tau^{-1} P_\tau A \Phi_2 P_\tau \Phi_\tau = \Phi_\tau^{-1} P_\tau A P_{2\tau} \Phi_{2\tau}. \end{aligned}$$

The closedness of M is easy to see. Let $A, B \in M$. It is also clear that $A + B \in M$. Here we show that $AB \in M$:

$\Phi_\tau^{-1}(AB)_\tau \Phi_\tau$

$$\begin{aligned} &= \Phi_\tau^{-1} P_\tau A B P_\tau \Phi_\tau \\ &= (\Phi_\tau^{-1} P_\tau A P_{2\tau} \Phi_{2\tau})(\Phi_{2\tau}^{-1} P_{2\tau} B P_\tau \Phi_\tau) + \Phi_\tau^{-1} P_\tau A Q_{2\tau} B P_\tau \Phi_\tau \\ &= (\Phi_\tau^{-1} (A\Phi_2)_\tau \Phi_\tau)(\Phi_\tau^{-1} (\Phi_2^{-1} B)_\tau \Phi_\tau) + \Phi_\tau^{-1} \underbrace{P_\tau A Q_{2\tau}}_{\rightrightarrows 0} B P_\tau \Phi_\tau \end{aligned}$$

and using Lemma 6.3 we get that $\Big((AB)_\tau\Big)$ is in $\mathcal{C} + \mathcal{N}$. Analogouosly,

$\Phi_\tau^{-1}(\Phi_2^{-1} AB)_\tau \Phi_\tau$

$$\begin{aligned} &= \Phi_\tau^{-1} P_\tau \Phi_2^{-1} A B P_\tau \Phi_\tau \\ &= (\Phi_{2\tau}^{-1} P_{2\tau} A P_{2\tau} \Phi_{2\tau})(\Phi_{2\tau}^{-1} P_{2\tau} B P_\tau \Phi_\tau) + \Phi_{2\tau}^{-1} P_{2\tau} A Q_{2\tau} B P_\tau \Phi_\tau \\ &= (\Phi_{2\tau}^{-1} A_{2\tau} \Phi_{2\tau})(\Phi_\tau^{-1} (\Phi_2^{-1} B)_\tau \Phi_\tau) + \Phi_{2\tau}^{-1} P_{2\tau} A \underbrace{Q_{2\tau} B P_\tau}_{\rightrightarrows 0} \Phi_\tau \end{aligned}$$

and

$$
\begin{aligned}
&\Phi_\tau^{-1}(AB\Phi_2)_\tau\Phi_\tau \\
&\qquad = \Phi_\tau^{-1}P_\tau AB\Phi_2P_\tau\Phi_\tau \\
&\qquad = (\Phi_\tau^{-1}P_\tau AP_{2\tau}\Phi_{2\tau})(\Phi_{2\tau}^{-1}P_{2\tau}BP_{2\tau}\Phi_{2\tau}) + \Phi_\tau^{-1}P_\tau AQ_{2\tau}BP_{2\tau}\Phi_{2\tau} \\
&\qquad = (\Phi_\tau^{-1}(A\Phi_2)_\tau\Phi_\tau)(\Phi_{2\tau}^{-1}B_{2\tau}\Phi_{2\tau}) + \Phi_\tau^{-1}\underbrace{P_\tau AQ_{2\tau}}_{\rightrightarrows 0}BP_{2\tau}\Phi_{2\tau}
\end{aligned}
$$

complete the proof that M is indeed a Banach algebra.

We still have to show that $\mathcal{A}$ is contained in M which is done best by showing that $\mathring{W}_a$ and M_b are contained for all symbols a, b under consideration. We therefore have to study operators $A_\tau^\Omega = \Phi_\tau^{-1}P_\tau AP_\tau\Phi_\tau$ for A being M_b, $M_b\Phi_2$, $\Phi_2^{-1}M_b$, $\mathring{W}_a$, $\mathring{W}_a\Phi_2$ and $\Phi_2^{-1}\mathring{W}_a$. First put $A = M_b$.

$$
\begin{aligned}
(M_b)_\tau^\Omega &= \Phi_\tau^{-1}P_\tau M_bP_\tau\Phi_\tau \\
&= P_1\Phi_\tau^{-1}M_b\Phi_\tau P_1 \\
&= P_1M_{\Phi_\tau^{-1}b}P_1 \\
&= M_{P_1\Phi_\tau^{-1}b}
\end{aligned}
$$

By very similar computations one can complete the following table:

$$
\begin{aligned}
(M_b)_\tau^\Omega &= M_{P_1\Phi_\tau^{-1}b} \\
(M_b\Phi_2)_\tau^\Omega &= M_{P_1\Phi_\tau^{-1}b}\Phi_2 \\
(\Phi_2^{-1}M_b)_\tau^\Omega &= \Phi_2^{-1}M_{P_1\Phi_\tau^{-1}b} \\
(\mathring{W}_a)_\tau^\Omega &= P_1\mathring{W}_{\Phi_\tau a}P_1 \\
(\mathring{W}_a\Phi_2)_\tau^\Omega &= P_1\mathring{W}_{\Phi_\tau a}\Phi_2P_1 \\
(\Phi_2^{-1}\mathring{W}_a)_\tau^\Omega &= P_1\Phi_2^{-1}\mathring{W}_{\Phi_\tau a}P_1
\end{aligned}
$$

Thus, we are finished if we can show that

a) $M_{P_1\Phi_\tau^{-1}b}$ and
b) $\mathring{W}_{\Phi_\tau a}$

are continuously dependent on $\tau \in \mathbb{R}_+$.

Considering **a)**, fix some $b \in \mathrm{BUC}$ and some $\varepsilon > 0$. Then there is some $\delta > 0$ such that $|b(y+r) - b(y)| < \varepsilon$ holds for all $y \in \mathbb{R}$ and all $|r| < \delta$. Consequently, for all $x \in [-1, 1]$ we have

$$|b((\tau + r)x) - b(\tau x)| = |b(\tau x + rx) - b(\tau x)| < \varepsilon \qquad \forall |r| < \delta$$

which shows that $\|M_{P_1\Phi^{-1}_{\tau+r}b} - M_{P_1\Phi^{-1}_{\tau}b}\|$ is smaller than every $\varepsilon > 0$ if $|r|$ is chosen small enough.

Considering **b)**, we fix some $a = c + Fk$ with $c \in \mathbb{C}$ and $k \in L^1(\mathbb{R})$. By $\|\mathring{W}_f\| = \|f\|_\infty$ we have to show that $\sup_{x\in\mathbb{R}} |a(x/\tau') - a(x/\tau)| \to 0$ as $\tau' \to \tau$. Putting $\rho' := 1/\tau'$ and $\rho := 1/\tau$ we have $\tau' \to \tau \iff \rho' \to \rho$ and thus, we will show that

$$\sup_{x\in\mathbb{R}} |a(\rho' x) - a(\rho x)| \to 0 \quad \text{as } \rho' \to \rho. \tag{25}$$

So fix some arbitrary $\varepsilon > 0$ and $\rho > 0$. Since $a(y) \to a(\infty) = c$ as $y \to \pm\infty$ there is some $y_\varepsilon > 0$ such that $|a(y') - a(y)| < \varepsilon$ for all $|y'|, |y| > y_\varepsilon$. Hence, we have (25) for all x with $|x| > y_\varepsilon/\rho$.
That (25) is also true for $x \in [-y_\varepsilon/\delta\,,\, y_\varepsilon/\delta]$ is shown exactly as in part **a)**. ■

Now we are ready for the
Proof of Proposition 6.1
Take some arbitrary operator $A \in \mathcal{A}$. By Lemma 6.2 we have $A \in \mathcal{B}_{\$}$. Lemma 6.4 yields – besides some other things we are no longer interested in – that the finite section method (A_τ) is in $\mathcal{C} + \mathcal{N}$ and this is all we claimed. ■

Having this proposition at our disposal we can now apply Theorem 5.2 on our algebra $\mathcal{A}$ and have some sufficient and necessary criterion for the stability of the finite section method for operators in our algebra generated by multiplication and convolution operators.

An operator $A \in \mathcal{A}$ is either a finite sum-product of the form

$$\sum_{i=1}^{k} \prod_{j=1}^{l} M_{b_{i,j}} \mathring{W}_{a_{i,j}} \tag{26}$$

with $b_{i,j} \in \mathrm{BUC}$ and $a_{i,j} \in \mathbb{C} + FL^1$ for all i, j or it is the norm limit of this sort of operators.

We will give a very explicit result for the case where the multipliers $b_{i,j}$ are all slowly oscillating. For such symbols we have good knowledge about the limit operators due to Proposition 5.12.

For a moment, think of the set of symbols $\{b_{i,j}\}_{i,j=1}^{k,l}$ as a vector $(b_{1,1}, ..., b_{k,l})$ of functions. For a sequence $h = (h_m) \subset \mathbb{Z}$ tending to – say plus – infinity which leads to a limit operator of every $M_{b_{i,j}}$ the vector $(b_{1,1}(h_m), ..., b_{k,l}(h_m))$ tends to some point $(r^{(h)}_{1,1}, ..., r^{(h)}_{k,l})$ with $r^{(h)}_{i,j} \in \mathcal{R}_{+\infty}(b_{i,j})$.

Doing this for all sequences h under consideration we can collect the points $(r_{1,1}^{(h)}, ..., r_{k,l}^{(h)})$ in a set called $\mathcal{R}^+$ and doing the same towards $-\infty$ produces the set $\mathcal{R}^-$.

Remark: Clearly, we have

$$\mathcal{R}^\pm \subseteq \mathcal{R}_{\pm\infty}(b_{1,1}) \times ... \times \mathcal{R}_{\pm\infty}(b_{k,l}) \tag{27}$$

for $+$ and $-$, respectively. In the case that (27) is actually an equality we will say that the functions $b_{1,1}, ..., b_{k,l}$ are *slowly oscillating independently.* □

Proposition 6.5 *Let $A \in \mathcal{A}$ be an operator of the form (26) with $a_{i,j} \in \mathbb{C} + FL^1$ and $b_{i,j} \in \mathrm{SO}$. For every point $r = (r_{1,1}, ..., r_{k,l}) \in \mathcal{R} := \mathcal{R}^+ \cup \mathcal{R}^-$ put*

$$f_r(x) := \sum_{i=1}^{k} \prod_{j=1}^{l} r_{i,j}\ a_{i,j}(x), \qquad x \in \mathbb{R}.$$

Then the following three statements are equivalent.

(i) The finite section method $(A_\tau) = (P_\tau A P_\tau)$ is applicable to A.
(ii) The finite section sequence (A_τ) is stable.
(iii) The operator A itself is invertible and for all functions f_r with $r \in \mathcal{R}$ the image $f_r(\dot{\mathbb{R}})$ is a closed curve in the complex plane which does not contain the origin and has winding number zero.

Proof. By Proposition 6.1 and Theorem 5.2 we have that (i) and (ii) are equivalent to each other and to the uniform invertibility of the set consisting of

a) the operator A itself,
b) all $QA_hQ + P$ with A_h running through $\sigma_+^{\mathsf{op}}(A)$ and
c) all $PA_hP + Q$ with A_h running through $\sigma_-^{\mathsf{op}}(A)$.

We start by examining the operators in c). Since limit operators are compatible with $\sum$ and $\prod$, convolution operators are their own limit operators, the $b_{i,j}$ are slowly oscillating independently from each other and limit operators of SO-multiplicators are given by Proposition 5.12 we conclude that

$$\begin{aligned}\sigma_-^{\mathsf{op}}(A) &= \left\{ \sum_{i=1}^{k} \prod_{j=1}^{l} r_{i,j} \mathring{W}_{a_{i,j}} \ :\ r_{i,j} \in \mathcal{R}_{-\infty}(b_{i,j}) \right\} \\ &= \left\{ \mathring{W}_{f_r} \ :\ r \in \mathcal{R}^- \right\}.\end{aligned}$$

By their definition all functions f_r are in $\mathbb{C} + FL^1$ and we know about those functions that they are continuous on $\dot{\mathbb{R}}$. Hence, their image $f_r(\dot{\mathbb{R}})$ is some closed curve in the complex plane.
The operators $P\mathring{W}_{f_r}P + Q$ from c) are so called *Wiener-Hopf operators* W_{f_r} and such an operator is invertible if and only if the image of its symbol f_r does not contain the origin and has winding number zero with respect to the origin.

Concerning b), we get analoguously, that

$$\sigma_+^{\mathsf{op}}(A) = \left\{ \mathring{W}_{f_r} \,:\, r \in \mathcal{R}^+ \right\}$$

and an operator $Q\mathring{W}_{f_r}Q + P$ is invertible if and only if $W_{f_r} = P\mathring{W}_{f_r}P + Q$ is invertible where the norms of the inverses coincide.

So all operators in b) and c) are invertible if and only if all functions f_r ($r \in \mathcal{R}$) stay away from the origin and have winding number zero. We still have to show that in this case this set of operators is even **uniformly** invertible.
To see this let Ψ refer to the mapping $\Psi : \mathcal{R} \to \mathcal{L}(L^\infty(\mathbb{R}))$ with $r \mapsto W_{f_r}$. Since $\mathcal{R}_{\pm\infty}(b)$ is bounded and closed in $\mathbb{C}$ for every $b \in L^\infty(\mathbb{R})$ we have that $\mathcal{R}$ is bounded and closed and hence compact in $\mathbb{C}^{kl}$. Moreover, it is easily seen that Ψ is continuous. Taking this together we see that the set $\{W_{f_r}\}_{r\in\mathcal{R}}$ is uniformly invertible if and only if it is elementwise invertible and hence the same is true for the set of operators listed under b) and c). ■

Remark: We have seen that in this case the demand for uniform boundedness of the inverses (see Theorem 3.3 and consequently Theorem 5.2) can be omitted since it is naturally given if all the inverses exist. It is conjectured that there are huge classes of operators A for which this can be done.

To be honest, until today we do not know of any operator A where "uniformly" can't be replaced by "elementwise" in Theorem 3.3! But we are optimistic that there are such examples. □

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