A NOTE ON SIMULTANEOUS DIOPHANTINE APPROXIMATION

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ABSTRACT. In this note we solve a Müntz type problem by the use of classical Bernstein polynomials and we use our focus to prove that in certain cases the simultaneous diophantine approximation of functions is possible also when we delete an infinite set of powers $\{x^{\alpha_i}\}$. The proofs are easy and constructive.

1. INTRODUCTION

In this note we give an easy constructive proof that in certain cases it is possible to delete infinitely many powers $\{x^{\alpha_i}\}_{i=0}^{\infty}$ for the simultaneous diophantine approximation of a function and its derivative on intervals away from the origin. The result for the continuous case is neither new nor the best, since a Müntz type theorem holds for the diophantine approximation of functions in $[0,1]$ (see [3]). But the corresponding result for functions of class $\mathbf{C}^{(s)}$ $(s \geq 1)$ is, as far as we know, new. On the other hand the proofs are easy, so that the theorems could be proved at the undergraduate level in a first course on numerical analysis. Concretely, we will solve the following Müntz type problem:

(P) Under which conditions on the sequence of natural numbers $m(n) \leq n$ $(n \in \mathbb{N})$ is it true that every function $f \in \mathbf{C}[a,b]$ is uniformly approximable by a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ with $p_n \in \mathbf{span}\{x^{m(n)}, ..., x^n\}$ for all $n \geq 1$?

and we will use our results for the diophantine approximation of functions.

Problem (**P**) is different from the classical Müntz problem, in which the density of $\mathbf{span}\{x^{\lambda_k} : k \geq 0\}$ is studied for sequences of exponents $\{\lambda_k\}_{k=0}^{\infty} \subset [0, \infty)$. This means that we are interested in a new focus on Müntz type problems while considering our problem (**P**). We will prove a neccesary condition for problem (**P**) (and this result is

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new as far as we know). Moreover, we show, by a constructive proof based on Bernstein polynomials, that this condition is sufficient for certain intervals $[a,b]$.

2. Solution of the Müntz type problem

In all what follows we consider a sequence $\{m(n)\}_{n=1}^{\infty}$ of natural numbers with $m(n)\leq n$ for all n. Moreover, we only consider intervals $[a,b]$ away from the origin (i.e., $0\notin[a,b]$). This is reasonable, since otherwise the solution of $(\mathbf{P})$ is obvious: $m(n)$ should be equal to zero for all $n\geq n_0$ in order to be able to approximate functions f with $f(0)\neq 0$.

Theorem 1. *Let $[a,b]$ be an interval away from the origin and let us assume that every function $f\in\mathbf{C}[a,b]$ is uniformly approximable by a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ with $p_n\in\mathbf{span}\{x^{m(n)},...,x^n\}$ for all $n\geq 1$. Then $\limsup_{n\to\infty}\frac{m(n)}{n}<1$.*

Proof. It is clear that $\limsup_{n\to\infty}\frac{m(n)}{n}\leq 1$, since $m(n)\leq n$ for all n. If $\limsup_{n\to\infty}\frac{m(n)}{n}=1$ then there are natural numbers $n_1<n_2<\cdots$ such that $\log\frac{n_k}{m(n_k)-1}\leq 2^{-k}$ for all $k\geq 1$. But

$$\sum_{j=m(n_k)}^{n_k}\frac{1}{j}\leq\sum_{j=m(n_k)}^{n_k}\int_{j-1}^{j}\frac{dx}{x}=\int_{m(n_k)-1}^{n_k}\frac{dx}{x}=\log\frac{n_k}{m(n_k)-1}$$

holds for all $k\geq 1$, so that

$$\sum_{k=1}^{\infty}\sum_{j=m(n_k)}^{n_k}\frac{1}{j}\leq\sum_{k=1}^{\infty}\log\frac{n_k}{m(n_k)-1}\leq\sum_{k=1}^{\infty}2^{-k}<\infty.$$

By the classical Müntz theorem this implies that in this case the space

$$\mathbf{H}_{\{m(n_k)\}}=\mathbf{span}\{x^h:h\in\bigcup_{k=1}^{\infty}\{m(n_k),....,n_k\}\}$$

is not dense in $\mathbf{C}[a,b]$. Now, let $f\in\mathbf{C}[a,b]$ be such that it is not approximable by polynomials from $\mathbf{H}_{\{m(n_k)\}}$. If $\{p_n\}_{n=1}^{\infty}$ is a sequence of polynomials with $p_n\in\mathbf{span}\{x^{m(n)},...,x^n\}$ for all $n\geq 1$ such that $\lim_{n\to\infty}||f-p_n||_{[a,b]}=0$, then $p_{n_k}\in\mathbf{H}_{\{m(n_k)\}}$ for all k and $\lim_{n\to\infty}||f-p_{n_k}||_{[a,b]}=0$, which is in contradiction with our hypothesis on f. □

Theorem 2. *Let us assume that $\limsup\frac{m(n)}{n}=c<1$. Then for all $c_0\in(c,1)$ and all $f\in\mathbf{C}[c_0,1]$, the sequence of polynomials*

$$\overline{Q}_n(f)=\sum_{k=m(n)}^{n}\overline{f}(k/n)\binom{n}{k}x^k(1-x)^{n-k}$$

converges to f uniformly on $[c_0, 1]$, where $\overline{f}$ denotes an arbitrary extension of f such that $\overline{f} \in \mathbf{C}[0,1]$ and $\overline{f}_{|[0,c+\varepsilon]} = 0$ for some $\varepsilon > 0$. Moreover, if both f and $\overline{f}$ are differentiable functions of class at least $\mathbf{C}^{(s)}$ in their respective domains, then $\lim_{n\to\infty} ||\overline{Q}_n(f) - f||_{\mathbf{C}^{(s)}[c_0,1]} = 0$. (Note that $\overline{Q}_n(f) \in \mathbf{span}\{x^{m(n)}, ..., x^n\}$ for all n).

Proof. Let $\overline{f}$ be an extension of f such that $\overline{f} \in \mathbf{C}[0,1]$ and $\overline{f}_{|[0,c+\varepsilon]} = 0$, and let us consider the sequence of Bernstein polynomials

$$B_n\overline{f}(x) = \sum_{k=0}^{n} \overline{f}(k/n)\binom{n}{k} x^k(1-x)^{n-k}.$$

If $k < m(n)$ then $k/n \leq c+\varepsilon$, so that $\overline{f}(k/n) = 0$ $(n > n_0)$. This means that $B_n\overline{f} = \overline{Q}_n(f)$ for all $n \geq n_0$, so that the sequence of polynomials $\{\overline{Q}_n(f)\}_{n=1}^{\infty}$ converges uniformly to $\overline{f}$ on $[0,1]$. The theorem follows since $\overline{f}_{|[c_0,1]} = f$ and the sequence of Bernstein polynomials is useful not only for the approximation of continuous functions but also for the simultaneous approximation of a function and its derivatives (see [2]). □

Corollary 3. *With the same notation as in Theorem 2, the polynomials with integral coefficients*

$$\widetilde{Q}_n(f) = \sum_{k=m(n)}^{n} \left[\overline{f}(k/n)\binom{n}{k}\right] x^k(1-x)^{n-k}$$

converge to f uniformly on $[c_0, 1]$, whenever $f(1) \in \mathbb{Z}$. (Note that $\widetilde{Q}_n(f) \in \mathbb{Z}[x] \cap \mathbf{span}\{x^{m(n)}, \cdots, x^n\}$ for all n).

Proof. It is well known that if $\overline{f} \in \mathbf{C}_0[0,1] = \{h \in \mathbf{C}[0,1] : h(0), h(1) \in \mathbb{Z}\}$ then the modified Bernstein polynomials

$$\widetilde{B}_n\overline{f}(x) = \sum_{k=0}^{n} \left[\overline{f}(k/n)\binom{n}{k}\right] x^k(1-x)^{n-k}$$

converge to $\overline{f}$ uniformly on $[0,1]$. Now, if $\overline{f}$ satisfies the hypothesis of Theorem 2, then $\widetilde{Q}_n(f) = \widetilde{B}_n\overline{f}$ for $n \geq n_0$ and the proof follows. □

The proof of the Müntz theorem for diophantine approximation of functions $f \in \mathbf{C}_0[0,1]$ is very difficult (see [3]). On the other hand, our proof is easy because we replaced the classical problem by the new problem **(P)**. Of course, our solution can also be used to solve some special cases of the classical problem, at least on a certain subinterval $[c_0, 1]$: If we know that $p_n \in \mathbb{Z}[x] \cap \mathbf{span}\{x^{m(n)}, \ldots, x^n\}$ converges to f uniformly on this interval, then this is also the case for any subsequence p_{n_j}. This means that we may proof density of $\mathbb{Z}[x] \cap \mathbf{span}\,\{x^{k_i}\}$ in $\mathbf{C}_0[c_0, 1] := \{f \in \mathbf{C}[c_0, 1] : f(1) \in \mathbb{Z}\}$ for sets $\{k_i\}$ of exponents that are of the form $\bigcup_j \{m(n_j), \dots, n_j\}$. The disadvantage is that we have to take all natural numbers between $m(n_j)$ and n_j which makes impossible to consider, for example, sets $\{k_i\}$ containing only even numbers or, more general, numbers $Kn + L$, where $K, L > 0$ are fixed. To get only such numbers, one can make the substitution $u(x) = \frac{f(x^{1/K})}{x^L}$ and set $\overline{q}_n(x) = x^L \overline{Q}_n(u; x^K)$ and $\widetilde{q}_n(x) = x^L \widetilde{Q}_n(u; x^K)$. Then we obtain that, if $c = \limsup \frac{m(n)}{n} < 1$ and $f \in \mathbf{C}[c_0, 1]$ (with $f(1) \in \mathbb{Z}$ if we consider $\widetilde{q}_n$) for a certain $c_0 > c^{1/K}$, then

$$\lim_{n\to\infty} ||\overline{q}_n - f||_{[c_0,1]} = \lim_{n\to\infty} ||\widetilde{q}_n - f||_{[c_0,1]} = 0.$$

Hence, if the sequence of real numbers $\{k_i\}_{i=0}^{\infty}$ contains a set of the form

$$\bigcup_{j=1}^{\infty} \{m(n_j)K + L,\, [m(n_j) + 1]K + L, \dots, nK + L\}$$

with $n_j, m(n_j) \in \mathbb{N}$, $m(n_j) \le n_j < n_{j+1}$ (for all j) and $c = \limsup \frac{m(n_j)}{n_j} < 1$, then $\mathbf{span}\{x^{k_i}\}$ is dense in $\mathbf{C}[c_0, 1]$ and $\mathbb{Z}[x] \cap \mathbf{span}\{x^{k_i}\}$ is dense in $\mathbf{C}_0[c_0, 1]$ whenever $c_0 > c^{1/K}$.

Now we consider another interesting question: Fixed the function $f \in \mathbf{C}[0, 1]$ and the sequence of natural numbers $\{m(n)\}_{n=1}^{\infty}$ with $\limsup \frac{m(n)}{n} = c < 1$, we may ask for the set of points in $[0, 1]$ where the sequences of polynomials

$$P_n(f) = \sum_{k=m(n)}^{n} f(k/n) \binom{n}{k} x^k (1 - x)^{n-k}$$

and

$$\widetilde{P}_n(f) = \sum_{k=m(n)}^{n} \left[f(k/n) \binom{n}{k} \right] x^k (1 - x)^{n-k},$$

converge to f. This is clearly equivalent to study the set of points where the differences

$$\Delta_n f(x) = B_n f - P_n(f) = \sum_{k=0}^{m(n)-1} f(k/n) \binom{n}{k} x^k (1 - x)^{n-k}$$

and (under the additional hypothesis that $f(0), f(1) \in \mathbb{Z}$)

$$\widetilde{\Delta}_n f(x) = \widetilde{B}_n f - \widetilde{P}_n(f) = \sum_{k=0}^{m(n)-1} \left[f(k/n) \binom{n}{k} \right] x^k (1 - x)^{n-k}$$

converge to zero.

Theorem 4. *Let $f \in \mathbf{C}[0,1]$ and $\{m(n)\}_{n=1}^{\infty} \subset \mathbb{N}$ with $\limsup \frac{m(n)}{n} = c < 1$ be fixed. The sequence of polynomials*

$$P_n(f) = \sum_{k=m(n)}^{n} f(k/n)\binom{n}{k} x^k (1-x)^{n-k}$$

converges to f uniformly on $[c_0, 1]$ whenever $c_0 > 1 - 2^{-1/(1-c)}$. The same holds for the sequence of polynomials

$$\widetilde{P}_n(f) = \sum_{k=m(n)}^{n} \left[f(k/n)\binom{n}{k} \right] x^k (1-x)^{n-k},$$

under the aditional hypothesis that $f(0), f(1) \in \mathbb{Z}$.

Proof. Let $c_0 \in (0,1)$ and let us assume that $x \in [c_0, 1]$. Moreover, let $c < \alpha < 1$ be fixed and let $n_0(\alpha) \in \mathbb{N}$ be such that the relation $m(n) < \alpha n$ holds for all $n \geq n_0(\alpha)$. If we set $M = \|f\|_{[0,1]} + 1$, then the inequality

$$\begin{aligned}
\max\{|\Delta_n f(x)|, |\widetilde{\Delta}_n f(x)|\} &\leq M \sum_{k=0}^{m(n)-1} \binom{n}{k} x^k (1-c_0)^{n-k} \\
&\leq M(1-c_0)^{n-m(n)+1} \sum_{k=0}^{m(n)-1} \binom{n}{k} x^k \\
&\leq M(1-c_0)^{n-m(n)+1}(1+x)^n \\
&\leq M(1-c_0)^{n(1-\alpha)} 2^n \text{ (since } m(n) < \alpha n)
\end{aligned}$$

holds true for all $n \geq n_0(\alpha)$. Now, $((1-c_0)^{(1-\alpha)}2)^n$ converges to zero whenever $(1-c_0)^{(1-\alpha)}2 < 1$, which is equivalent to say that $c_0 > 2^{-1/(1-\alpha)}$. If $c_0 > 1 - 2^{-1/(1-c)}$ then the relation $c_0 > 1 - 2^{-1/(1-\alpha)}$ holds for α close enough to c. □

The disadvantage of Theorem 4 is that we can not obtain $c_0 \leq 1/2$ even if $m(n)/n$ converges to zero. For this reason, we give the following theorem:

Theorem 5. *If $\limsup \frac{m(n)\log_2 n}{n} = c < 1$, then the assertions of Theorem 4 hold true for all $c_0 > 1 - 2^{-c}$.*

Proof. Without loss of generality we may suppose that $c_0 < 1/2$ (since $1 - 2^{-c} < 1$). From Theorem 4 it follows the uniform convergence on compact subsets of $(1/2, 1]$, since our hyphotesis implies $\limsup \frac{m(n)}{n} = 0$. Thus, we only need to prove uniform convergence on $[c_0, 1/2 + \varepsilon]$ for a certain $\varepsilon > 0$. With this objective in mind, we define $\alpha \in (0,1)$ by

$c_0 = 1-2^{-\alpha}$ and we set $M = ||f||_{[0,1]}+1$. Then for all $x \in [1-2^{-\alpha}, 2^{-\alpha}]$ we can estimate the differences $\Delta_n f(x)$ and $\widetilde{\Delta}_n f(x)$ by

$$\begin{aligned}
\max\{|\Delta_n f(x)|, |\widetilde{\Delta}_n f(x)|\} &\leq M \sum_{k=0}^{m(n)-1} \binom{n}{k} x^k (1-x)^{n-k} \\
&\leq 2^{-\alpha n} M \sum_{k=0}^{m(n)-1} \frac{n^k}{k!} \\
&\leq 2^{-\alpha n} n^{m(n)-1} M \sum_{k=0}^{m(n)-1} \frac{1}{k!} \\
&\leq 2^{-\alpha n} n^{m(n)-1} M \exp(1) \\
&= n^{-1} \left(2^{-\alpha} n^{m(n)/n}\right)^n M \exp(1) \qquad (1)
\end{aligned}$$

but $\limsup \frac{m(n)\log_2 n}{n} = c$ implies that $\frac{m(n)\log_2 n}{n} \leq \alpha$ holds for all $n \geq n_0(\alpha)$ for a certain natural number $n_0(\alpha)$. Hence the relation

$$2^{\frac{m(n)\log_2 n}{n}} = 2^{\log_2 n^{\frac{m(n)}{n}}} = n^{\frac{m(n)}{n}} \leq 2^{\alpha}$$

holds for all $n \geq n_0(\alpha)$. This implies that the upper bound we have already estimated in (1) converges to zero for $n \to \infty$. $\square$

3. Simultaneous diophantine approximation

As it is well known, Bernstein polynomials are useful for the simultaneous approximation of a function and its derivatives (see [2]). We have already used this to prove the second claim of Theorem 2. In this section we will prove that simultaneous approximation of a function and its derivative by polynomials with integral coefficients on compact subsets of $(0,1)$ is possible indeed if we delete a certain type of infinite sets of powers $\{x^{\alpha_i}\}_{i=0}^{\infty}$.

Theorem 6. *Let us assume that* $\limsup \frac{m(n)}{n} = c < 1$. *Then for all intervals* $[a,b] \subset (c,1)$ *and all* $f \in \mathbf{C}^{(s)}[a,b]$, *the sequence of polynomials*

$$\widetilde{Q}_n(f) = \sum_{k=m(n)}^{n} \left[\overline{f}(k/n)\binom{n}{k}\right] x^k (1-x)^{n-k}$$

converges to f *in the norm of* $\mathbf{C}^{(s)}[a,b]$, *where* $\overline{f}$ *denotes an arbitrary extension of* f *such that* $\overline{f} \in \mathbf{C}^{(s)}[0,1]$ *and* $\overline{f}_{|[0,c+\varepsilon]\cup[1-\varepsilon,1]} = 0$ *for some* $\varepsilon > 0$.

Proof. We know that $B_n\overline{f}$ converges to $\overline{f}$ in $\mathbf{C}^{(s)}[0,1]$ and $\widetilde{Q}_n(f) = \widetilde{B}_n\overline{f}$ for $n \geq n_0$, so that we must compare $\widetilde{B}_n\overline{f}$ and $B_n\overline{f}$ in the norm of $\mathbf{C}^{(s)}[a,b]$. Now,

$$\left|(B_n\overline{f})^{(v)}(x) - (\widetilde{B}_n\overline{f})^{(v)}(x)\right| = \left|\sum_{k=1}^{n-1}\left(\overline{f}(k/n)\binom{n}{k} - \left[\overline{f}(k/n)\binom{n}{k}\right]\right)\frac{d^v}{dx^v}(x^k(1-x)^{n-k})\right|,$$

since $\overline{f}(0) = \overline{f}(1) = 0$. If we use the notation

$$n^{(r)} := \begin{cases} 1 & \text{if } r = 0 \\ n(n-1)\cdots(n-r+1) & \text{if } r > 0 \end{cases},$$

and we assume that $x \in [\alpha,\beta] \subset (0,1)$, we obtain that

$$\begin{aligned}
\left|(B_n\overline{f})^{(v)}(x) - (\widetilde{B}_n\overline{f})^{(v)}(x)\right| &\leq \sum_{k=1}^{n-1}\left|\frac{d^v}{dx^v}(x^k(1-x)^{n-k})\right| \\
&= \sum_{k=1}^{n-1}\left|\sum_{s=0}^{v}\binom{v}{s}\frac{d^s}{dx^s}((1-x)^{n-k})\frac{d^{v-s}}{dx^{v-s}}(x^k)\right| \\
&= \sum_{k=1}^{n-1}\left|\sum_{s=0}^{v}\binom{v}{s}(n-k)^{(s)}(1-x)^{n-k-s}(k)^{(v-s)}x^{k-v+s}\right| \\
&\leq \sum_{k=1}^{n-1}\left(\sum_{s=0}^{v}\binom{v}{s}(n-k)^{(s)}(k)^{(v-s)}\right)\rho^{n-v},
\end{aligned}$$

where $\rho = \max\{1-\alpha,\beta\} < 1$. Now,

$$\sum_{s=0}^{v}\binom{v}{s}(n-k)^{(s)}(k)^{(v-s)} \leq \sum_{s=0}^{v}\binom{v}{s}(n-k)^s k^{v-s} = n^v,$$

so that

$$\sum_{k=1}^{n-1}\left(\sum_{s=0}^{v}\binom{v}{s}(n-k)^{(s)}(k)^{(v-s)}\right) \leq n^{v+1}$$

and

$$||(B_n\overline{f})^{(v)}(x) - (\widetilde{B}_n\overline{f})^{(v)}(x)||_{[\alpha,\beta]} \leq n^{v+1}\rho^{n-v},$$

which converges to zero for $n \to \infty$. This proves that $\widetilde{B}_n\overline{f}$ converges to $\overline{f}$ in the norm of $\mathbf{C}^{(s)}[\alpha,\beta]$ for all compact interval $[\alpha,\beta] \subset (0,1)$. The theorem follows since $\overline{f}$ is an extension of f. □

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References

[1] **J. M. Almira, N. Del Toro,** Sobre aproximación con polinomios de coeficientes enteros, La Gaceta de la R.S.M.E., enviado (2001).

[2] **R.A. DeVore, G. G. Lorentz,** *Constructive Approximation*, Springer (1993).

[3] **O. Ferguson, M. v Golitscheck,** Müntz-Szász Theorem with integral coefficients, II, Transactions of the AMS **213** (1975) 115-126 .