Some approach for solving an inverse option pricing problem

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Abstract

In this paper, we are dealt with a generalized version of the Black-Scholes model for option pricing with non-constant volatilities. We formulate and derive the Black-Scholes partial differential equation for the case that the volatility is a function of time and asset price. In the focus of our considerations stands the solution of the inverse option pricing problem for European calls. In this context, a time-dependent local volatility function is to be determined. We study this nonlinear ill-posed inverse problem from the analytic point of view and give a numerical approach for situations of practical data. A case study including regularization completes the paper.

Keywords: Stochastic differential equation, Black–Scholes partial differential equation, inverse problem of option pricing, identification of local volatility, ill-posed problem, regularization.

MSC 2000 Subject Classification: 35R30, 91B28, 65J20, 60H15

1 Introduction

The theory of option pricing started its long history back in 1900 when the French mathematician Louis Bachelier deduced an option pricing formula in his doctoral thesis [2]. He was the first who had the innovative idea of using a stochastic process as a model for the price evolution of stocks. However, option pricing theory really made progress in 1973 (see [3] and [13]) when Fisher Black and Myron Scholes, in-cooperation with Robert Merton, derived an option pricing formula that only depends on the actual stock price, the exercise (strike) price, the risk-free interest rate, the volatility of the stock and the expiration date (maturity).

The Black-Scholes model (see, for example, [9] and [14]) for pricing options at time $t = t^*$ is based on a stochastic differential equation (SDE) of the form

$$dX(t) = X(t)\left[\mu(t, X(t))\, dt + \sigma(t, X(t))\, dB(t)\right], \tag{1}$$

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where $X(t) \geq 0$ denotes the asset price that is a function of time t. In particular, we consider the time interval $t^* \leq t \leq \overline{T}$ with an actual time $t^* \geq 0$ and $\overline{T} > 0$ being the maximum duration of associated European options written on the asset. It is clear that only the actual asset price $X^* := X(t^*)$ and historical data $X(t)$ $(t < t^*)$ are available. The functions $\mu(t, X)$ and $\sigma(t, X)$ are the drift and the local volatility of the stochastic price process $X(t)$ for the asset. The symbol $B(t)$ designates the standard Wiener process.

As we will see, the usually unknown local volatility function $\sigma(t, X)$ $(t^* \leq t \leq \overline{T}, X \geq 0)$ is crucial for pricing an option at time t^*. Provided the volatilities were known, the Black-Scholes partial differential equation (BSPDE) formulated and derived in Section 2 could yield fair option prices. Therefore, the identification and treatment of volatilities play an important role in modern versions of Black-Scholes theory.

The classical case $\sigma(t, X) = \sigma$ of a constant volatility can be handled by using historical asset data for estimating σ (see, e.g., [14]). On the other hand, implied volatilities obtained from actual market prices of options by inverting the classical Black-Scholes formula can be interpreted as an alternative for finding the value σ. However, this value is problematic, since the implied volatility substantially varies in a specific way for different strikes and maturities of options on the same asset. Frequently, this effect is called volatility smile or term structure (see [9]). In the last years the problem of finding local volatility functions was of essential interest in financial theory. Different approaches were developed by several authors (see, for example, [1], [4], [5] and [12]).

In Section 3 of this paper we will formulate and study a variant of the inverse option pricing problem for European calls, i.e., the nonlinear problem of determining the local volatility function $\sigma(t)$ $(0 \leq t \leq \overline{T})$ from known option prices at time $t^* = 0$. Note that in our considerations the volatility does not depend on the asset price X, but only on time t. Nevertheless, the inverse problem is ill-posed. We give an approach for solving the inverse problem for that situation from an analytic point of view. The inverse problem can be decomposed in an outer nonlinear and an inner linear problem, which are both mildly ill-posed. In this context, the prices of a whole familiy of such options with a fixed strike price K and varying maturities T from the interval $(0, \overline{T}]$ are to be exploited. Section 4 presents some remarks on the numerical solution of our inverse problem for discrete and noisy data. A numerical case study including regularization completes the paper. The numerical results indicate the character of ill-posedness of the inverse option pricing problem and the chances of finding appropriate approximations of the local volatility function.

2 Black–Scholes partial differential equation

In this section we derive the Black-Scholes partial differential equation, which is a deterministic equation for the price of an option in terms of maturity T and strike price K. We consider a smooth function $v(t, X(t))$ depending on time t and asset price $X(t)$. According to [11] Itô's formula implies the integral representation

$$
\begin{aligned}
v(t,X(t)) = v(0,X(0)) + \int_0^t \Bigg(&\frac{\partial v(\tau,X(\tau))}{\partial \tau} + \mu(\tau,X(\tau))X(\tau)\frac{\partial v(\tau,X(\tau))}{\partial X} \\
&+ \frac{1}{2}X^2(\tau)\sigma^2(\tau,X(\tau))\frac{\partial^2 v(\tau,X(\tau))}{\partial X^2}\Bigg) d\tau \qquad (2)\\
&+ \int_0^t \sigma(\tau,X(\tau))X(\tau)\frac{\partial v(\tau,X(\tau))}{\partial X} dB(\tau).
\end{aligned}
$$

In differential notation the equation (2) attains the form

$$
\begin{aligned}
dv(t,X) &= \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial X}dX + \frac{1}{2}\frac{\partial^2 v}{\partial X^2}dX\,dX \\
&= \left(\frac{\partial v}{\partial t} + \mu X\frac{\partial v}{\partial X} + \frac{1}{2}X^2\sigma^2\frac{\partial^2 v}{\partial X^2}\right)dt + \sigma X\frac{\partial v}{\partial X}dB.
\end{aligned} \qquad (3)
$$

We denote by $r > 0$ be the risk-free interest rate and consider the situation $\mu(t,X(t)) = r$ based on the absence of arbitrage opportunities. For the function v we make the ansatz

$$
v(t,X) = e^{r(T-t)}u(t,X) \qquad (0 \le t \le T,\ X \ge 0), \qquad (4)
$$

where $u(t,X)$ expresses the actual price at time t of an option with maturity T on the asset with actual price X. Moreover, $h(X(T))$ is the pay-off for the option under consideration. For that ansatz we know from [11] that $v(t,X)$ is a martingale. Then the integral

$$
\int_0^t \left(\frac{\partial v(\tau,X(\tau))}{\partial \tau} + rX(\tau)\frac{\partial v(\tau,X(\tau))}{\partial X} + \frac{1}{2}X(\tau)^2\sigma^2(\tau,X(\tau))\frac{\partial^2 v(\tau,X(\tau))}{\partial X^2}\right) d\tau
$$

must be zero for all considered t. This implies that the integrand is zero and we obtain the corresponding backward Kolmogorov equation

$$
\frac{\partial v(t,X)}{\partial t} + rX\frac{\partial v(t,X)}{\partial X} + \frac{1}{2}X^2\sigma^2(t,X)\frac{\partial^2 v(t,X)}{\partial X^2} = 0 \qquad (0 \le t \le T,\ X \ge 0). \quad (5)
$$

This partial differential equation is combined with the terminal condition

$$
v(T,X) = h(X),
$$

where $h(X)$ is a known continuous function of price $X \ge 0$.

Using the transformation (4) and the differential relations

$$
\frac{\partial v(t,X)}{\partial t} = -re^{r(T-t)}u(t,X) + e^{r(T-t)}\frac{\partial u(t,X)}{\partial t},
$$

$$
\frac{\partial v(t,X)}{\partial X} = e^{r(T-t)}\frac{\partial u(t,X)}{\partial X}
$$

and

$$\frac{\partial^2 v(t,X)}{\partial X^2} = e^{r(T-t)} \frac{\partial^2 u(t,X)}{\partial X^2},$$

the equation (5) can be rewritten as an equation for the option price

$$\frac{\partial u(t,X)}{\partial t} + \frac{1}{2} X^2 \sigma^2(t,X) \frac{\partial^2 u(t,X)}{\partial X^2} + rX \frac{\partial u(t,X)}{\partial X} - ru(t,X) = 0, \tag{6}$$

which is valid for $0 \le t \le T$ and $X \ge 0$. This partial differential equation is known in the literature as the Black-Scholes partial differential equation (BSPDE). For option pricing it must be combined with the terminal condition

$$u(T,X) = h(X), \tag{7}$$

where the pay-off function attains the form

$$h(X) = \begin{cases} (X-K)^+ & \text{for call options} \\ (K-X)^+ & \text{for put options} \end{cases} \tag{8}$$

with strike price K. Frequently, the natural boundary conditions

$$u(t,0) = \begin{cases} 0 \quad (0 \le t \le T) & \text{for call options} \\ K \quad (0 \le t \le T) & \text{for put options} \end{cases}$$

are also imposed.

Now we can formulate the forward problem in the context of option pricing, namely the problem of determining the price of a well-determined option.

Definition 1 (Forward problem of option pricing) *Given the local volatility function $\sigma(t,X)$ in the cylinder $\Omega = [0,T] \times \bar{R}_+$, find the solution $u(t,X)$ to (6) and (7) for $(t,X) \in \Omega$.*

We always assume $\sigma(t,X) \in C^{\lambda,\frac{\lambda}{2}}(\Omega)$, $0 < \lambda \le 1$, i.e., the local volatility is Hölder continuous in Ω with Hölder index λ and satifies for all $(t,X) \in \Omega$ and $(t_0,X_0) \in \Omega$ an inequality of the form

$$|\sigma(t,X) - \sigma(t_0,X_0)| \le L(|t-t_0|^{\frac{\lambda}{2}} + |X - X_0|^{\lambda}). \tag{9}$$

Moreover, we take the volatility function σ strictly positive and bounded with $0 < \sigma_{min} \le \sigma(t,X) \le \sigma_{max} < \infty$.

The partial differential problem (6) – (7) is of non-standard form, since it is a problem backward in time defined on the half-space $X \ge 0$. In order to use the classical theory of parabolic partial differential equations formulated by Friedman [6] for existence and uniqueness assertions of the forward problem of option pricing, we transform the problem (6) – (7) to a Cauchy problem forward in time. For this reason we substitute

$$\tau := T - t, \;\; Y := \ln(X/K), \;\; \tilde{\sigma}(\tau,Y) := \sigma(t,X) \;\text{ and }\; w(\tau,Y) := u(t,X).$$

Then the BSPDE (6) is transformed to the parabolic equation

$$\frac{\partial w(\tau,Y)}{\partial \tau} = \frac{1}{2}\tilde{\sigma}^2(\tau,Y)\frac{\partial^2 w(\tau,Y)}{\partial Y^2} + \left(r - \frac{1}{2}\tilde{\sigma}^2(\tau,Y)\right)\frac{\partial w(\tau,Y)}{\partial Y} - rw(\tau,Y), \qquad (10)$$

which is valid for $0 \leq \tau \leq T$ and $-\infty < Y < \infty$. The terminal condition (7) – (8) becomes an initial condition

$$w(0,Y) = \begin{cases} K\left(e^Y - 1\right)^+ & \text{for call options} \\ K\left(1 - e^Y\right)^+ & \text{for put options.} \end{cases} \qquad (11)$$

Then for the Hölder continuous local volatility functions σ according to (9) corresponding theorems in [6] guarantee that there exists a unique solution of the Cauchy problem (10) – (11) satisfying the growth condition

$$|w(\tau,Y)| \leq c_1 e^{c_2 Y^2} \quad (0 \leq \tau \leq T,\ -\infty < Y < \infty)$$

with positive constants c_1 and c_2. Therefore, the forward problem of option pricing defined above is uniquely solvable under weak growth conditions.

3 The inverse option pricing problem

In order to overcome the fact that the local volatility function $\sigma(t,X)$ is usually unknown and has to be determined, we use a family of option prices at time $t = t^* \geq 0$ with different strikes $K > 0$ and maturities $T > t^*$ for fitting the volatility values. If such options are traded on the market, the required option prices can be obtained from the broker. Moreover, we know the asset price $X^* = X(t^*) > 0$.

We focus our considerations to a set of European call options on the same asset with maturities $T > 0$ and strikes $K > 0$ such that the option prices $u(t,X)$ used in the sequel satisfy the partial differential equation (6) and the terminal condition

$$u(T,X) = (X - K)^+. \qquad (12)$$

Then the option price is always bounded by $0 \leq u(t,X) \leq X$. The last inequality ensures that the price paid for the right to buy an asset cannot exceed the asset price itself.

Obviously, the option prices additionally depend on K and on values of the local volatility function σ. For completeness we should write

$$u(t^*,X^*) = u(t^*,X^*,K,T,\sigma).$$

By $u^*(K,T)$ we denote a variety of fair option prices $u(t^*,X^*,K,T,\sigma)$ depending on strike K and maturity T. Note that all these prices have a uniform underlying local volatility function $\sigma(t,X)$. Since we assume the unique solvability of the forward problem of option pricing, the values $u^*(K,T)$ are well-defined for all $(K,T) \in R^2_+$.

So we can formulate the corresponding inverse problem of option pricing aimed at the reconstruction of a local volatility function.

Definition 2 (Inverse problem of option pricing) *Given for fixed $t^* \geq 0$ and $X^* = X(t^*) > 0$ a function $u^*(K,T)$ of fair option prices depending on strikes $K > 0$ and maturities $T > t^*$, find the underlying local volatility function $\sigma(t,X)$ on an appropriate domain of (t,X).*

Dupire shows in [5] that option prices $u^*(K,T)$ given for all strikes and maturities uniquely determine the volatility function σ. However, in practice options are traded only for a discrete set of strikes and maturities. On the other hand, the formula derived by Dupire does not provide an efficient algorithm to compute the local volatility function numerically.

Assumption 3 *In the sequel we consider the time interval $0 \leq t \leq \overline{T}$ and assume $t^* := 0$ and $X^* := X(0)$. Moreover, let the local volatility σ be a Hölder continuous function that only depends on time, where we assume*

$$0 < \sigma_{min} \leq \sigma(t) \leq \sigma_{max} < \infty \quad (0 \leq t \leq \overline{T}). \tag{13}$$

Let $0 < T \leq \overline{T}$. Provided the local volatility function $\sigma(t)$ $(0 \leq t \leq T)$ is known, we can find an explicit solution of the forward option pricing problem by a simple modification of the Black-Scholes formula as follows (for details see [10]). Let

$$s(T) := \int_0^T \sigma^2(t)\, dt.$$

Then using the fundamental solution of the partial differential equation (6) the fair price $u^*(K,T)$ of a European call can be obtained by

$$u^*(K,T) = X^*\, N(d_1) - K\, e^{-rT} N(d_2), \tag{14}$$

where

$$d_1 := \frac{\ln\left(\frac{X^*}{K}\right) + r\,T + \frac{s(T)}{2}}{\sqrt{s(T)}}, \quad d_2 := d_1 - \sqrt{s(T)} \tag{15}$$

and

$$N(z) := \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}}\, dx \tag{16}$$

is the standard normal distribution function.

Since in view of formula (13) the volatility function σ is strictly positive on the interval $[0,\overline{T}]$, we get a first important result immediatley from the formulas (14) – (16).

Theorem 4 *For a fixed maturity $T > 0$, the option price $u^*(K) := u^*(K,T)$ depends for all $K > 0$ only on the L^2-norm of σ*

$$\|\sigma\|_{L^2(0,T)} = \sqrt{\int_0^T \sigma^2(t)\, dt} = \sqrt{s(T)},$$

but not on the detailed behaviour of the function σ in the interval $[0,T]$.

This theorem has important consequences for the choice of price data $u^*(K,T)$ in the context of the determination of σ. It makes no sense to examine data for the same maturity T and different strikes K. They all give only the integral information $s(T)$ about the parameter σ. Either the data $u^*(K)$ are exact, then we have for any fixed T an infinite set of solutions σ of the inverse problem, or in the case of perturbed data we may have no solutions.

On the other hand, we can try to determine the volatility $\sigma(t)$ on the whole interval $0 \le t \le \overline{T}$ if we have fair option prices $u^*(K,T)$ for all maturities $T \in (0,\overline{T}]$ and a fixed strike price $K > 0$. For this situation we write $u^*(T) := u^*(K,T)$ and can formulate our inverse problem specifically.

Definition 5 (Specific inverse problem of option pricing) *Given for fixed $K > 0$ and $X^* = X(0) > 0$ a function $u^*(T)$ $(0 < T \le \overline{T})$ of fair option prices depending on maturities T, find the underlying local volatility function $\sigma(t)$ $(0 \le t \le \overline{T})$.*

If we have an exact function $u^*(T)$ $(0 < T \le \overline{T})$, i.e., the given option prices are fair and satisfy the associated partial differential equation problem (6) – (7), then we can reconstruct the function σ along the lines of the follwing algorithm.

Algorithm 6

1. *Determine the function $s(T)$ $(0 \le T \le \overline{T})$ by solving the family of nonlinear equations*

$$u(X^*,K,T,s(T)) := X^* N(d_1) - K e^{-rT} N(d_2) = u^*(T) \tag{17}$$

   for all $T \in (0,\overline{T}]$, where d_1, d_2 and $N(\cdot)$ are defined by the formulas (15) and (16).

2. *Determine the function $y(t)$ $(0 \le t \le \overline{T})$ by solving the linear integral equation*

$$\int_0^T y(t)\,dt = s(T) \quad (0 \le T \le \overline{T}). \tag{18}$$

3. *Find*

$$\sigma(t) := \sqrt{y(t)} \quad (0 \le t \le \overline{T}). \tag{19}$$

Now we are going to analyse the algorithm. The determination of the function y from the function u^* via the auxiliary function s by using the first two steps of Algorithm 6 is equivalent to the solution of a nonlinear operator equation

$$F(y) \;=\; G\,(A\,y) \;=\; u^* \tag{20}$$

with the nonlinear operator $F : y \mapsto u^*$. The operator F can be decomposed into an outer nonlinear operator $G : s \mapsto u^*$, where according to the formulas (15) and (16)

$$[G(s)](T) := X^* N(d_1) - K e^{-rT} N(d_2) \quad (0 < T \le \overline{T}),$$

and an inner linear operator $A : y \mapsto s$, where

$$[Ay](T) := \int_0^T y(t)dt \quad (0 \le T \le \overline{T}).$$

All three functions y, s and u^* are defined on the same interval $[0, \overline{T}]$ and should be considered as elements of appropriate Banach spaces between which F, G and A operate.

If a local volatility function σ according to Assumption 3 implies the element $u^* = F(\sigma^2)$ on the right-hand side of (20) as a function of associated fair option prices depending on T, then the nonlinear operator equation (20) has at least one continuous and strictly positive solution y. Algorithm 6 works reasonable when we can show that this solution y of (20) is unique in the set of continuous functions. It suffices to show that the outer operator equation

$$G(s) \;=\; u^* \tag{21}$$

has only one continuously differentiable solution s, since the inner operator equation

$$A\,y \;=\; s \tag{22}$$

is uniquely solvable by a simple differentiation $y(t) = s'(t)$ $(0 \le t \le \overline{T})$. Now, (21) is uniquely solvable with respect to continuous functions s if we determine the values $s(T)$ $(0 \le T \le \overline{T})$ from $u^*(T)$ $(0 < T \le \overline{T})$ in a unique manner. However, for all $T \in (0, \overline{T}]$, the value $[G(s)](T)$ only depends on $s(T)$ and not on $s(t)$ for $t \neq T$. Moreover, $[G(s)](T)$ is strictly increasing with respect to $s(T)$ as a consequence of the following theorem, which can be proven easily from the formulas (15), (16) and (17). This ensures the existence of a unique continuous solution s of (21) and justifies the algorithm.

Theorem 7 *For all fixed $T \in (0, \overline{T}]$ and $s := s(T) > 0$ we have with*

$$u := u(X^*, K, T, s) := X^* N(d_1) - K e^{-rT} N(d_2)$$

positive partial derivatives

$$\frac{\partial u}{\partial s} = N'(d_1)\, X^* \, \frac{1}{2\sqrt{s}} > 0. \tag{23}$$

Moreover, the following limit conditions hold

$$\lim_{s \to 0} \frac{\partial u}{\partial s} = 0, \quad \lim_{s \to 0} u = 0 \quad \textit{and} \quad \lim_{s \to \infty} u = X^*. \tag{24}$$

If the data $u^*(T)$ are perturbed, but satisfy the inequalities $0 < u^*(T) < X^*$ for all $0 < T \le \overline{T}$, then Theorem 7 also ensures the existence of a unique function $s(t)$ $(0 < t \le \overline{T})$ satisfying the equation (17).

Note that the nonlinear operator equation (21) is ill-posed with respect to $C[0,\overline{T}]$-solutions s, because of the first limit condition in (24) and $s(0)=0$. That means, small perturbations in the right-hand side of (21) may lead to large maximum-norm errors in the reconstruction of the function $s(t)$, in particular for small arguments t. However, this ill-posedness is rather mild. If only uniformly bounded monotone solutions s are considered, then the reconstruction of s from u^* is even well-posed in $L^2(0,\overline{T})$.

From the theory of linear ill-posed problems we have learned that the standard problem (22) is mildly ill-posed in $L^2(0,\overline{T})$ as well as in $C[0,\overline{T}]$ (see, e.g., [7]) and can be solved in a stable manner by using any regularization approach. On the other hand, the regularization can also be applied directly to the original problem (20).

4 Numerical realization and a case study

In the preceding section we assumed that we had fair prices $u^*(T)$ on the whole interval $(0,\overline{T}]$. However, options are practically traded only for a discrete set of maturities $0 < T_1 < T_2 < \ldots < T_j < \ldots < T_n = \overline{T}$. Additionally let denote $T_0 := 0$. Moreover, the option prices $\widetilde{u}^*(T_j)$ $(j=1,2,...,n)$ obtained in practice are market prices and can be considered as perturbed fair option prices. In the sequel we assume

$$0 \;<\; \widetilde{u}^*(T_j) \;<\; X^* \quad (j=1,2,...,n). \tag{25}$$

On this basis we will suggest a realistic modification of Algorithm 6.

In correspondence with Step 1 of Algorithm 6 the values $s_j := s(T_j)$ $(j=1,2,...,n)$ of the function s solving the outer equation (21) can be determined approximately by n line searches (e.g. using a simple bisection method). Taking into account the equation (17), the purpose of these line searches is to find minimizers $\widetilde{s}_j > 0$ of the optimization problems

$$|u(X^*,K,T_j,\hat{s}_j) - \widetilde{u}^*(T_j)| \to \min, \quad \text{subject to} \quad \hat{s}_j > 0 \tag{26}$$

for $j=1,2,...,n$, Since $u(X^*,K,T_j,s)$ is strictly monotone with respect to $s>0$ (see formula (23)), all optimization problems (26) are uniquely solvable and have a zero minimum in view of the assumed inequalities (25) and the second and third limit conditions in (24). The approximation errors $|\widetilde{s}_j - s_j|$ can be estimated by the mean value theorem in the form

$$|\widetilde{s}_j - s_j| = \left|\left(\frac{\partial u}{\partial s}\right)_{im}\right|^{-1} |\widetilde{u}^*(T_j) - u^*(T_j)|,$$

where the partial derivative $\left(\frac{\partial u}{\partial s}\right)_{im}$ is to be taken for an intermediate value $s_{im} \in (\min(\widetilde{s}_j,s_j),\max(\widetilde{s}_j,s_j))$. The modulus inverse $\left|\left(\frac{\partial u}{\partial s}\right)_{im}\right|^{-1}$ tends to infinity if T_j goes to zero (see Theorem 7). Such ill-posedness effect is avoided if we choose the T_1 not too close to zero. This approach is a version of regularization by discretization.

For solving the linear integral equation (18) we assume the volatility σ and therefore the function y piecewise constant on the intervals $[T_0,T_1]$ and $(T_{j-1},T_j]$ $(j=2,...,n)$. We set

$$\varphi_1(T) := \chi_{[T_0,T_1]}(T) \quad \text{and} \quad \varphi_j(T) := \chi_{(T_{j-1},T_j]}(T) \quad (j=2,...,n),$$

where

$$\chi_I(T) := \begin{cases} 1 & (T \in I) \\ 0 & (T \notin I) \end{cases}$$

is the characteristic function of an interval I, and approximate $y(T)$ by

$$\widetilde{y}(T) \; = \; \sum_{j=1}^{n} \widetilde{y}_j \, \varphi_j(T) \quad (0 \le T \le \overline{T}).$$

For noisy data this set-up yields with (18) an $n \times n$-system of linear equations

$$\underline{A}\, \underline{\widetilde{y}} = \underline{\widetilde{s}}, \tag{27}$$

where the system matrix is lower triangular

$$\underline{A} := \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_1 & a_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \in R^{n \times n} \text{ with entries } a_j := T_j - T_{j-1} \; (j = 1, 2, ..., n).$$

The vectors of solution and right-hand side are $\underline{\widetilde{y}} := \begin{pmatrix} \widetilde{y}_1 \\ \vdots \\ \widetilde{y}_n \end{pmatrix}$ and $\underline{\widetilde{s}} := \begin{pmatrix} \widetilde{s}_1 \\ \vdots \\ \widetilde{s}_n \end{pmatrix}$.
Since $\underline{A}$ has a well-known bidiagonal inverse, we can solve the system (27) explicitly by the formula

$$\widetilde{y}_j \; = \; \frac{\widetilde{s}_j - \widetilde{s}_{j-1}}{T_j - T_{j-1}} \qquad (j = 1, 2, ..., n), \tag{28}$$

where we set $\widetilde{s}_0 := s(0) = 0$. The obtained finite backward differences in (28) show the differentiation character of Step 2 in Algorithm 6. If the differences $T_j - T_{j-1}$ are close to zero, the ill-posedness of the differentiation problem influences the discretized version (28) in the sense that the vector $\underline{\widetilde{y}}$ is oscillating. Then the errors $|\widetilde{y}_j - y(T_j)|$ may be partially large.

Finally, in Step 3 we set for all $j = 1, 2, ..., n$

$$\widetilde{\sigma}_j := \begin{cases} \sigma_{max} & (\widetilde{y}_j > \sigma^2_{max}) \\ \sigma_{min} & (\widetilde{y}_j < \sigma^2_{min}) \\ \sqrt{\widetilde{y}_j} & (\sigma^2_{min} \le \widetilde{y}_j \le \sigma^2_{max}). \end{cases}$$

Consequently,

$$\widetilde{\sigma}(T) = \sum_{j=1}^{n} \widetilde{\sigma}_j \varphi_j(T) \qquad (0 \le T \le \overline{T})$$

is the obtained approximation of the unknown local volatility function σ.

Because of the ill-posedness of the linear equation (18), a regularization approach for solving this equation in a stable manner can be recommended. We compare the numerical realization of Algorithm 6 prescribed above with a Tikhonov regularization approach. In the regularization context we solve the minimization problem

$$\|\underline{A}\,\underline{\widetilde{y}} - \underline{\widetilde{s}}\|_2^2 + \alpha\,\|\underline{L}\,\underline{\widetilde{y}}\|_2^2 \to \min, \quad \text{subject to} \quad \underline{\widetilde{y}} \in \overline{R}_+^n \tag{29}$$

with a positive regularization parameter α using the Euclidean norm $\|\cdot\|_2$, where $\|\underline{L}\,\underline{y}\|_2^2$ denotes the obvious discretization of the L^2-norm square $\|y''\|^2_{L^2(0,T)}$ of the second derivative of the function y. If we replaced $\|\underline{L}\,\underline{y}\|_2^2$ by the discretized L^2-norm square $\|\underline{y}\|_2^2$ of y itself, the obtained regularized solutions were unsatisfactory in any case.

Now we present a case study concerning our algorithm. We consider as an exact solution the function

$$\sigma(T) := (T - 0.5)^2 + 0.1 \qquad (0 \le T \le 1).$$

Furthermore, we set for the actual asset price $X^* = X(0) := 0.6$, for the exercise price $K := 0.5$ for the interest rate $r := 0.05$. Let the option price be known for $n := 20$ equidistant maturities

$$T_j := \frac{j}{n} \qquad (j = 1, 2, ...n).$$

In order to perform the numerical experiment, we count the fair option price vector $\underline{u}^* = (u^*(T_1), ..., u^*(T_n))^T \in R^n$ according to formula (14). To simulate noisy data $\underline{\widetilde{u}}^* \in R^n$ we generate a random vector $\underline{e} = (e_1, \ldots, e_n)^T \in R^n$ and perturb the data in the form

$$\underline{\widetilde{u}}^* := \underline{u}^* + \delta\,\frac{\|\underline{u}^*\|_2}{\|\underline{e}\|_2}\,\underline{e}$$

for a given relative error $\delta > 0$ with respect to the Euclidean norm.

Now we will discuss some results. All figures show two functions, namely on the one hand the exact solution σ to be determined and on the other hand the obtained numerical approximation. The graphs are painted with a linear interpolation. In Figure 1 we see the reconstruction of the local volatility function when no noise is added. In this case, the unknown parameter function can be estimated very well by using our algorithm. When adding noise the results tend to the worse. As seen in Figure 2, a small relative error of $\delta = 0.0002$ still has only a little effect on the solution. But already a pertubation with $\delta = 0.001$ leads to a strongly oscillating solution. This is demonstrated in Figure 3. In order to smooth this approximate solution, we used for comparison the regularization approach (29). The parameter α was determined by the Hansen's L-curve criterium (see [8]). In this special case we got $\alpha = 7.1263 \cdot 10^{-7}$. Figure 4 shows the regularized solution using the L-curve, which is quite good, but somewhat oversmoothed. Other regularization parameters α do not provide essentially better regularized solutions for that situation.

Acknowledgment

The authors thank PD Dr. Ralf Wunderlich for his helpful comments during the preparation of this paper.

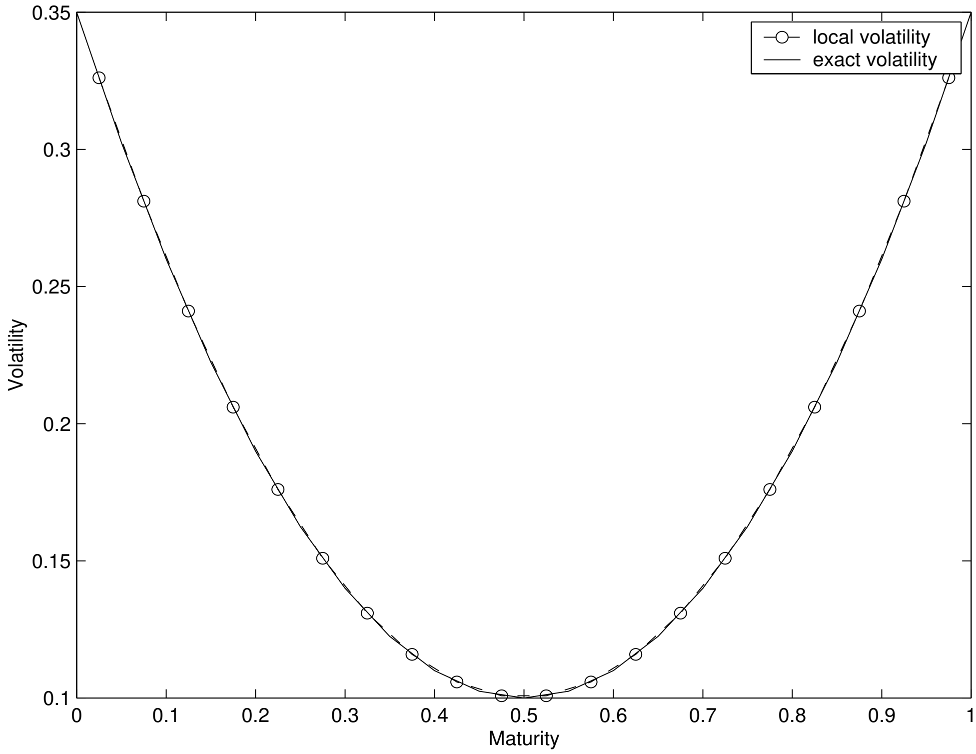

Figure 1: Case without pertubation ($\delta = 0$)

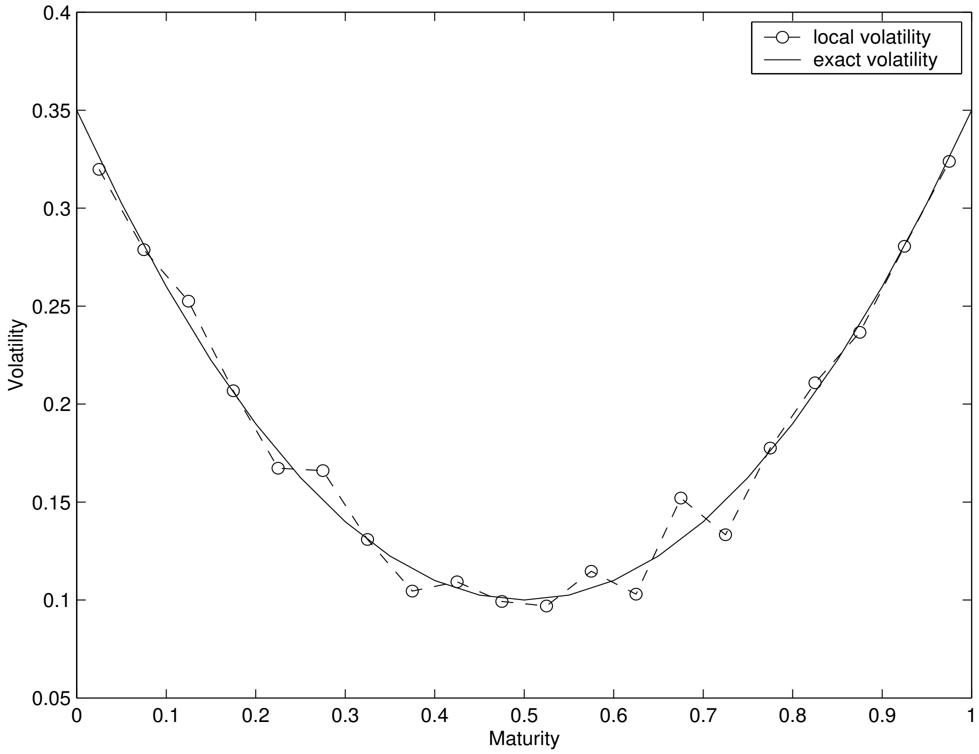

Figure 2: Case of small noise ($\delta = 0.0002$)

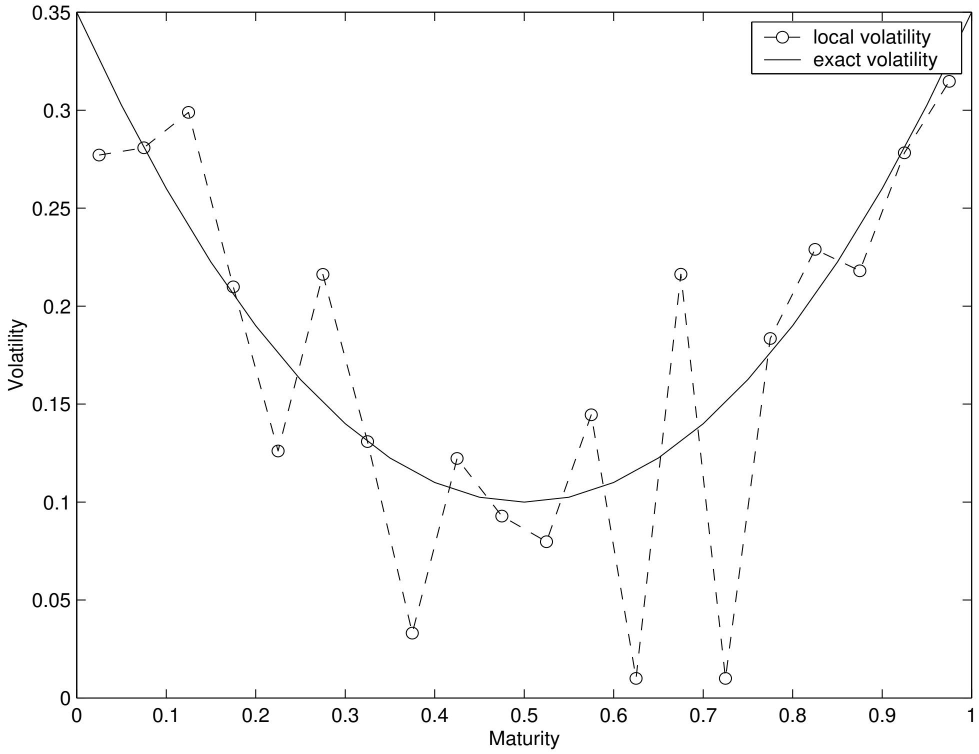

Figure 3: Case of larger noise ($\delta = 0.001$)

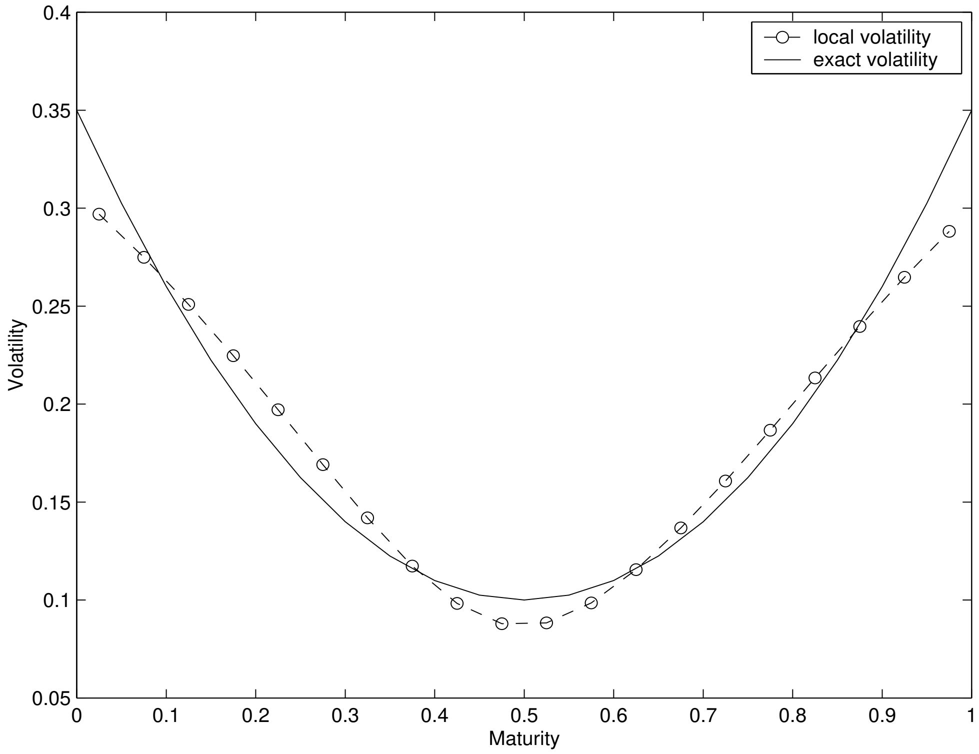

Figure 4: Regularizised solution ($\delta = 0.001$, $\alpha = 7.1263 \cdot 10^{-7}$)

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