Transverse vibrations of a beam with random ε–correlated excitation and model reduction

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Abstract

This paper is devoted to the computation of second-order moment functions of the solution of large-scale systems of linear random ODEs which arise in the modeling of randomly excited linear vibration systems. Model reduction techniques are applied to find approximations of the desired moment functions of the large-scale system by computing them for a suitable low-dimensional system. Numerical results concerning transverse vibrations of thin beams are presented.

1 Introduction

The aim of this paper is to investigate vibration systems with an external random excitation which can be mathematically described by the linear system of first-order ordinary differential equations (ODEs)

$$\dot{\mathbf{z}}(t) = \mathbf{A}\mathbf{z}(t) + \mathbf{B}\mathbf{h}(t,\omega). \tag{1}$$

Here, $\mathbf{z}$ is an n-dimensional vector function and the inhomogeneous term contains the excitation function $\mathbf{h}$ which is assumed to be a centered and wide-sense stationary random vector process with values in $\mathbb{R}^r$, $r, n \in \mathbb{N}$. $\mathbf{A}$ and $\mathbf{B}$ denote an $n \times n$ and an $n \times r$ matrix, respectively. For the sake of greater generality the entries of the matrices $\mathbf{A}$ and $\mathbf{B}$ are supposed to be complex-valued, thus the values of $\mathbf{z}(t)$ are in $\mathbb{C}^n$. Moreover, it is assumed that $\mathbf{h}$ is defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ where $\mathcal{A}$ denotes a σ-algebra of subsets of Ω on which a probability measure $\mathbf{P}$ is defined. The expectation with respect to $\mathbf{P}$ is denoted by $\mathbf{E}\{.\}$

The main tasks are the determination of so-called stationary solutions and the computation of their distribution characteristics, in particular second-order moment functions. These questions arise in the investigation of the long-time behaviour of the response of systems described by Equation (1) to permanently acting random excitations.

The computation of moment functions of $\mathbf{z}$ used in this paper is based on the assumption that the random excitation term $\mathbf{h}$ can be modeled by so-called ε-correlated functions which are introduced in Section 2. The essential property of ε-correlated functions is that the correlation of the random function does not reach far. That means, the values of the function at two points are not correlated if the distance of the two points exceeds a certain quantity $\varepsilon > 0$. The quantity ε is denoted as correlation length. In applications ε is always assumed to be sufficiently small. Hence, ε-correlated functions can also be characterized as functions without "distant effect". In contrast to the white noise model which is frequently used in stochastic analysis ε-correlated functions may possess trajectories of arbitrary smoothness.

The results of this paper can be directly applied to discrete vibration systems with damping and an external excitation which are usually described by systems of N real second-order linear ODEs

$$\mathbf{M}_2\ddot{\mathbf{y}}(t) + \mathbf{M}_1\dot{\mathbf{y}}(t) + \mathbf{M}_0\mathbf{y}(t) = \hat{\mathbf{B}}\mathbf{h}(t,\omega).$$

Here, $\mathbf{y}= (y_1,\dots,y_N)^\tau$, $N \in \mathbb{N}$, is a N-dimensional vector function containing the response variables, $\mathbf{M}_0, \mathbf{M}_1, \mathbf{M}_2$ are some $N\times N$ matrices and $\hat{\mathbf{B}}$ is an $N\times r$ matrix. The transformation of the above system into a first-order system for the state vector function $\mathbf{z} = \binom{\dot{\mathbf{y}}}{\mathbf{y}}$ results in a $2N$-dimensional system of the form (1).

System (1) also arises as a result of the semi-discretization of partial differential equations (PDEs) describing continuous vibration systems. This case is considered in Section 4. Here, the dimension n usually becomes very large and procedures for the computation of moment functions of $\mathbf{z}$ become intractable because of enormous computational problems. To this end, in Section 3 model reduction techniques are investigated which provide approximations of the desired moment functions of the large-scale system by solving a suitable low-dimensional system.

2 Linear systems with stationary excitations

Consider the linear system (1) with a wide-sense stationary random excitation function $\mathbf{h}$. For the sake of shorter notation the term "wide-sense" is supressed throughout this paper. A random function $\mathbf{z}(t,\omega)$ satisfying (1) is called stationary solution of (1) if $\mathbf{z}$ and $\mathbf{h}$ are stationarily related, i.e. $\binom{\mathbf{z}}{\mathbf{h}}$ is a stationary process.

Lemma 1 (Bunke [2], Arnold, Wihstutz [1])
Assuming a centered stationary and a.s. continuous excitation function $\mathbf{h}$ *and a stable matrix* $\mathbf{A}$, *i.e. all eigenvalues of* $\mathbf{A}$ *possess negative real parts, then the linear first-order system (1) possesses the unique stationary solution*

$$\mathbf{z}(t,\omega) = \int_{-\infty}^{t} \mathrm{e}^{\mathbf{A}(t-s)}\mathbf{B}\mathbf{h}(s,\omega)ds = \int_{0}^{\infty} \mathrm{e}^{\mathbf{A}s}\mathbf{B}\mathbf{h}(t-s,\omega)ds. \tag{2}$$

The representation (2) can be used for the probabilistic analysis of the stationary solution $\mathbf{z}(t,\omega)$ and the computation of its moment functions. Since $\mathbf{h}$ is assumed to be centered for the first-order moments it follows $\mathbf{E}\{\mathbf{z}(t)\} = 0$.

For the computation of second-order moments several procedures are known in the literature (see Newland [8], Preumont [9], Soong, Grigoriu [21], vom Scheidt, Fellenberg, Wöhrl [13]). A first group is based on time domain analysis. Here, several averages of the representation (2) are used. For the correlation function of $\mathbf{z}$ it follows

$$\begin{aligned}
\mathbf{R}_{\mathbf{zz}}(\tau) &= \mathbf{E}\{\mathbf{z}(t)\mathbf{z}^*(t+\tau)\} \\
&= \int_0^\infty\int_0^\infty \mathrm{e}^{\mathbf{A}s_1}\mathbf{B}\mathbf{E}\{\mathbf{h}(t-s_1)\mathbf{h}^*(t+\tau-s_2)\}\mathbf{B}^*\mathrm{e}^{\mathbf{A}^*s_2}ds_1ds_2 \\
&= \int_0^\infty\int_0^\infty \mathrm{e}^{\mathbf{A}s_1}\mathbf{B}\mathbf{R}_{\mathbf{hh}}(\tau+s_1-s_2)\mathbf{B}^*\mathrm{e}^{\mathbf{A}^*s_2}ds_1ds_2.
\end{aligned}$$

Other procedures are based on frequency domain analysis. Here, the spectral density of $\mathbf{z}$ which is defined by

$$\mathbf{S}_{\mathbf{zz}}(\alpha) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\mathrm{i}\alpha\tau}\mathbf{R}_{\mathbf{zz}}(\tau)\,d\tau$$

is related to the spectral density of $\mathbf{h}$.

Lemma 2 (Soong, Grigoriu [21], vom Scheidt, Fellenberg, Wöhrl [13])
Let the assumptions of Lemma 1 be fulfilled and assume that the excitation function $\mathbf{h}$ *possesses a spectral density denoted by* $\mathbf{S}_{\mathbf{hh}}(\alpha)$, $\alpha \in \mathbb{R}$. *Then, for the spectral density of* $\mathbf{z}$ *it holds for* $\alpha \in \mathbb{R}$

$$\mathbf{S}_{\mathbf{zz}}(\alpha) = \mathbf{H}(\alpha)\mathbf{S}_{\mathbf{hh}}(\alpha)\mathbf{H}^*(\alpha) \quad \textit{where} \quad \mathbf{H}(\alpha) = -(\mathrm{i}\alpha\mathbf{I}+\mathbf{A})^{-1}\mathbf{B}.$$

The matrix function $\mathbf{H}(\alpha)$ is known as the frequency response function or transition function.

Next, a special class of random processes which will be used to model the random excitation $\mathbf{h}$ is considered.

Definition 3
A centered random process ${}^{\varepsilon}\mathbf{f}$, $\varepsilon > 0$, *with* ${}^{\varepsilon}\mathbf{f} : \mathbb{R}\times\Omega \to \mathbb{R}^r$, $r \in \mathbb{N}$, *possessing the property*

$$\mathbf{E}\left\{ {}^{\varepsilon}\mathbf{f}(s)\, {}^{\varepsilon}\mathbf{f}^{T}(t)\right\} = \mathbf{O}_{rr} \quad \textit{for } |t-s| \geq \varepsilon,$$

where $\mathbf{O}_{rr}$ *denotes the* $r\times r$ *null matrix, is called* ε-correlated *and* ε *its* correlation length.

For example, ${}^{\varepsilon}\mathbf{f}$, $\varepsilon > 0$, can be so-called weakly correlated processes which are introduced in vom Scheidt [12] and vom Scheidt, Purkert [14]. It is noted that the definition of weakly correlated processes contains further conditions to the higher-order moment functions of ${}^{\varepsilon}\mathbf{f}$. The idea of weakly as well as ε-correlated processes goes back to Uhlenbeck and Ornstein [22]. They investigated a differential equation for the velocity of a Brownian particle subjected to the impacts of the molecules of the surrounding medium and assumed this random force to be weakly correlated.

Afterwards, the following families of stationary ε-correlated processes will play an important role.

Definition 4
A family $({}^{\varepsilon}\mathbf{f})_{\varepsilon>0}$ *of* ε*-correlated processes is called* regular, *if*

1) $\forall\varepsilon > 0$ *the processes* ${}^{\varepsilon}\mathbf{f}$ *are stationary with correlation functions*

$$\mathbf{E}\left\{ {}^{\varepsilon}\mathbf{f}(s)\, {}^{\varepsilon}\mathbf{f}^{T}(t)\right\} = \mathbf{R}_{{}^{\varepsilon}\mathbf{f}\,{}^{\varepsilon}\mathbf{f}}(t-s), \quad s,t \in \mathbb{R},$$

2) *the correlation functions* $\mathbf{R}_{{}^{\varepsilon}\mathbf{f}\,{}^{\varepsilon}\mathbf{f}}(.)$, $\varepsilon > 0$, *are generated by a correlation function* $\mathbf{R}(.)$ *of a* 1*-correlated stationary process, i.e.*

$$\mathbf{R}_{{}^{\varepsilon}\mathbf{f}\,{}^{\varepsilon}\mathbf{f}}(\tau) = \mathbf{R}\left(\frac{\tau}{\varepsilon}\right), \quad \tau \in \mathbb{R}, \varepsilon > 0,$$

3) $\forall\varepsilon > 0$ *the processes* ${}^{\varepsilon}\mathbf{f}$ *are a.s. and mean-square continuous on* $\mathbb{R}$.

For a regular family of ε*-correlated processes the quantity*

$$\boldsymbol{\mu} := \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \mathbf{R}_{{}^{\varepsilon}\mathbf{f}\,{}^{\varepsilon}\mathbf{f}}(\tau)d\tau = \int_{-1}^{1} \mathbf{R}(\tau)d\tau$$

is called intensity *of* $({}^{\varepsilon}\mathbf{f})_{\varepsilon>0}$.

Next, random processes possessing an integral functional representation

$$\mathbf{g}(t,\omega) = \int_0^\infty \mathbf{Q}(s)\, {}^\varepsilon\mathbf{f}(t-s,\omega)ds$$

with a deterministic kernel function $\mathbf{Q}$ are considered. For the correlation function $\mathbf{R}_{\mathbf{gg}}(\tau)$ and spectral density $\mathbf{S}_{\mathbf{gg}}(\alpha)$ the papers [15, 16, 17, 23, 25] derive asymptotic expansions in powers of ε. They are based on Taylor expansions of the kernel function $\mathbf{Q}$ and use so-called correlation moments. The leading terms of the expansions which are needed in the subsequent sections are given below.

Lemma 5 [15, 17, 23]

Let $({}^\varepsilon\mathbf{f})_{\varepsilon>0}$ be a regular family of ε-correlated random processes with ${}^\varepsilon\mathbf{f} : \mathbb{R}\times\Omega \to \mathbb{R}^r$ and $\mathbf{Q}$ an absolutely continuous kernel function on $\mathbb{R}_+$ with $\mathbf{Q}, \mathbf{Q}' \in \mathbf{L}_1(\mathbb{R}_+) \cap \mathbf{L}_2(\mathbb{R}_+)$. Then for the correlation function and spectral density of $\mathbf{g}(t,\omega) = \int_0^\infty \mathbf{Q}(s)\, {}^\varepsilon\mathbf{f}(t-s,\omega)ds$ there hold for $\tau \in \mathbb{R}$ and fixed $\alpha \in \mathbb{R}$ the following expansions as $\varepsilon \downarrow 0$

$$\mathbf{R}_{\mathbf{gg}}(\tau) = \varepsilon\, {}^0\mathbf{R}_{\mathbf{gg}}(\tau) + o(\varepsilon) \quad \textit{and} \quad \mathbf{S}_{\mathbf{gg}}(\alpha) = \varepsilon\, {}^0\mathbf{S}_{\mathbf{gg}}(\alpha) + o(\varepsilon)$$

$$\textit{where} \quad {}^0\mathbf{R}_{\mathbf{gg}}(\tau) = \begin{cases} \int_0^\infty \mathbf{Q}(u)\boldsymbol{\mu}\mathbf{Q}^*(u+\tau)du & \textit{for } \tau \geq 0 \\ {}^0\mathbf{R}^*_{\mathbf{gg}}(-\tau) & \textit{for } \tau < 0 \end{cases}$$

$${}^0\mathbf{S}_{\mathbf{gg}}(\alpha) = \frac{1}{2\pi}\mathbf{H}(\alpha)\boldsymbol{\mu}\mathbf{H}^*(\alpha) \ \textit{with} \quad \mathbf{H}(\alpha) = \int_0^\infty e^{\mathrm{i}\alpha s}\mathbf{Q}(s)ds.$$

In the following the excitation function $\mathbf{h}$ is assumed to be a r-dimensional ε-correlated vector process, i.e. the linear system

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\, {}^\varepsilon\mathbf{f} \tag{3}$$

is considered. Let $({}^\varepsilon\mathbf{f})_{\varepsilon>0}$ be a regular family of ε-correlated processes. From Lemma 1 it can be seen that for stable $\mathbf{A}$ the stationary solution of (3) possesses the integral functional representation $z(t,\omega) = \int_0^\infty \mathbf{Q}(s)\, {}^\varepsilon\mathbf{f}(t-s)ds$ with $\mathbf{Q}(s) = \mathrm{e}^{\mathbf{A}s}\mathbf{B}$. If $\mathbf{A}$ is diagonalizable, i.e. it possesses a representation $\mathbf{A} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}$ with a diagonal matrix $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$ and a non-singular matrix $\mathbf{V}$ then the kernel function reads as $\mathbf{Q}(s) = \mathbf{V}\mathrm{e}^{\boldsymbol{\Lambda}s}\mathbf{V}^{-1}\mathbf{B}$ and the function $\mathbf{H}$ as

$$\mathbf{H}(\alpha) = \int_0^\infty \mathrm{e}^{\mathrm{i}\alpha s}\mathbf{Q}(s)ds = -(\mathrm{i}\alpha\mathbf{I} + \mathbf{A})^{-1}\mathbf{B} = -\mathbf{V}(\mathrm{i}\alpha\mathbf{I} + \boldsymbol{\Lambda})^{-1}\mathbf{V}^{-1}\mathbf{B}.$$

If $\mathbf{A}$ is assumed to be stable then $\mathbf{Q}$ satisfies the conditions of Lemma 5 which implies the following expansions for the correlation function and the spectral density of $\mathbf{z}$ (see [23]).

Theorem 6
Let $({}^{\varepsilon}\mathbf{f})_{\varepsilon>0}$ be a regular family of ε-correlated random processes with ${}^{\varepsilon}\mathbf{f} : \mathbb{R} \times \Omega \to \mathbb{R}^r$, $\mathbf{A}$ a stable and diagonalizable $n \times n$ matrix with $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$ where $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, $\mathbf{V}$ is non-singular and $\mathbf{B}$ an $n \times r$ matrix. Then, the system (3) possesses the unique stationary solution

$$\mathbf{z}(t,\omega) = \int\limits_0^\infty e^{\mathbf{A}s}\mathbf{B}\,{}^{\varepsilon}\mathbf{f}(t-s,\omega)ds.$$

For $\varepsilon \downarrow 0$ there hold the following expansions:
for the correlation function for $\tau \in \mathbb{R}$

$$\mathbf{R}_{\mathbf{zz}}(\tau) \;=\; \varepsilon\,{}^0\mathbf{R}_{\mathbf{zz}}(\tau) + o(\varepsilon) \quad \textit{with} \quad {}^0\mathbf{R}_{\mathbf{zz}}(\tau) = \begin{cases} \mathbf{V}\,\mathbf{\Theta}(\tau)\mathbf{V}^* & \tau \geq 0 \\ {}^0\mathbf{R}^*_{\mathbf{zz}}(-\tau) & \tau < 0 \end{cases}$$

where $\Theta_{kl}(\tau) = -\frac{M_{kl}}{\lambda_k+\overline{\lambda}_l}e^{\overline{\lambda}_l\tau}$, $k,l = 1,\ldots,n$, and $\mathbf{M} = \mathbf{V}^{-1}\mathbf{B}\boldsymbol{\mu}\mathbf{B}^*(\mathbf{V}^{-1})^*$;
and for the spectral density with fixed $\alpha \in \mathbb{R}$

$$\begin{aligned} \mathbf{S}_{\mathbf{zz}}(\alpha) &= \varepsilon\,{}^0\mathbf{S}_{\mathbf{zz}}(\alpha) + o(\varepsilon) \\ \textit{where}\quad {}^0\mathbf{S}_{\mathbf{zz}}(\alpha) &= \frac{1}{2\pi}\mathbf{V}\left(-\frac{M_{kl}}{(\mathrm{i}\alpha+\lambda_l)(\mathrm{i}\alpha-\overline{\lambda}_k)}\right)_{k,l=1,\ldots,n}\mathbf{V}^* \end{aligned}$$

Remark 7 The assertions of Lemma 5 and Theorem 6 can be generalized to higher-order expansions as well as to ε-correlated processes with values in $\mathbb{C}^n$ (see [15, 17]) and to ε-correlated random fields ${}^{\varepsilon}\mathbf{f} : \mathbb{R}^d \times \Omega \to \mathbb{C}^N$, $d \in \mathbb{N}$, (see [16]). If, additionally, the ${}^{\varepsilon}\mathbf{f}$, $\varepsilon > 0$ are weakly correlated functions, i.e. further conditions to the higher-order moments are fulfilled, then the asymptotic normality of $\frac{1}{\sqrt{\varepsilon}}\mathbf{g}(t,\omega)$ and $\frac{1}{\sqrt{\varepsilon}}\mathbf{z}(t,\omega)$ can be proved (see [12]).

In this case for small ε the function $\frac{1}{\sqrt{\varepsilon}}\,{}^{\varepsilon}\mathbf{f}(t,\omega)$ can be considered as an approximation of (multivariate) white noise. The leading expansion terms ${}^0\mathbf{R}_{\mathbf{zz}}(\tau)$ and ${}^0\mathbf{S}_{\mathbf{zz}}(\alpha)$ in Theorem 6 correspond to the correlation function and spectral density of $\mathbf{z}$ if in (3) the excitation ${}^{\varepsilon}\mathbf{f}$ is replaced by white noise.

3 Model reduction

For formulating the task of model reduction the first-order system (1) is rewritten in a slightly modified form known from control theory (see e.g. Knobloch, Kwakernaak [5], Zhou, Doyle [26])

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{h}, \qquad \mathbf{z} = \mathbf{T}\mathbf{y}, \tag{4}$$

with $\mathbf{T} = \mathbf{I}_n$, i.e. $\mathbf{z} \equiv \mathbf{y}$. Here, the first equation is called state equation while the second equation is the output equation. Further, $\mathbf{h}$, $\mathbf{y}$ and $\mathbf{z}$ are called input, state and output, respectively. An output $\mathbf{z}$ is called stationary output if $\mathbf{z}$ and $\mathbf{h}$ are stationarily related. The system (4) will be referred to as the original system. To find an approximation of the stationary output $\mathbf{z}$ one tries to find a system possessing a stable state equation with a dimension $m < n$, i.e.

$$\dot{\mathbf{y}}_m = \mathbf{A}_m\,\mathbf{y}_m + \mathbf{B}_m\,\mathbf{h}, \qquad \tilde{\mathbf{z}}^{(m)} = \mathbf{T}_m\,\mathbf{y}_m, \tag{5}$$

with an $m \times m$ matrix $\mathbf{A}_m$, an $m \times r$ matrix $\mathbf{B}_m$, an $n \times m$ matrix $\mathbf{T}_m$ and an m-dimensional vector function $\mathbf{y}_m : \mathbb{R} \times \Omega \to \mathbb{C}^m$. The above system is called reduced-order system to the original system (4) and its n-dimensional stationary output $\tilde{\mathbf{z}}^{(m)}$ is called the reduced-order approximation of $\mathbf{z}$.

There exists a great number of papers concerning the model reduction problem in the context of control theory where $\mathbf{h}$ is considered as input or control function and $\mathbf{z}$ is the output function which has to be controlled. In some papers the input function is allowed to be random - in most cases multi-dimensional white noise - see e.g. Hyland, Bernstein [4], Moore [6], Wilson [24]. In the present paper the construction of a suitable reduced-order system is based on the assumption that the random dynamics of the stationary output $\mathbf{z}$ can be expressed up to a certain degree of accuracy by projections of its values onto a m-dimensional subspace of $\mathbb{C}^n$. A similar approach is used in Sirovich, Knight, Rodriguez [20].

Consider the canonical basis $(\mathbf{e}_k)_{k=1,\dots,n}$ of $\mathbb{C}^n$ and the representation of the state function $\mathbf{y}(t,\omega)$ with respect to this basis $\mathbf{y}(t,\omega) = \sum\limits_{k=1}^{n} y_k(t,\omega)\mathbf{e}_k$. Now, let $(\mathbf{v}_k)_{k=1,\dots,n}$ be a set of linear independent elements of $\mathbb{C}^n$ with $\|\mathbf{v}_k\| = 1$. Denote by $x_1, \dots, x_n$ the new coordinates of $\mathbf{y} \in \mathbb{C}^n$ with respect to the basis $(\mathbf{v}_k)_{k=1,\dots,n}$, i.e.

$$\mathbf{y}(t,\omega) = \sum_{k=1}^{n} x_k(t,\omega)\mathbf{v}_k. \tag{6}$$

Defining the $\mathbb{C}^n$-valued vector function $\mathbf{x}(t,\omega) = (x_1(t,\omega), \dots, x_n(t,\omega))^\tau$ and the matrix $\mathbf{V} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ which is non-singular because of the linear independence of $\mathbf{v}_1, \dots, \mathbf{v}_n$, the transformation (6) and its inverse transformation can be written as $\mathbf{y}(t,\omega) = \mathbf{V}\mathbf{x}(t,\omega)$ and $\mathbf{x}(t,\omega) = \mathbf{V}^{-1}\mathbf{y}(t,\omega)$.

Next, sequences $(\mathcal{S}_m)_{m=1,\dots,n}$ of linear subspaces of $\mathbb{C}^n$ are considered, where $\mathcal{S}_m$ is defined by $\mathcal{S}_m := \operatorname{span}(\mathbf{v}_1, \dots, \mathbf{v}_m)$, $\quad m = 1, \dots, n$. To each sequence

$(\mathcal{S}_m)_{m=1,\ldots,n}$ corresponds a sequence $(\mathcal{P}_m)_{m=1,\ldots,n}$ of projectors, where $\mathcal{P}_m : \mathbb{C}^n \to \mathcal{S}_m$ is defined by

$$\mathcal{P}_m \mathbf{y} \; := \; \sum_{k=1}^{m} x_k \mathbf{v}_k, \quad m = 1, \ldots, n. \tag{7}$$

Defining for $m = 1, \ldots, n-1$ the following subvectors of $\mathbf{x}$

$$\mathbf{x}_m := (x_1, \ldots, x_m)^\tau \quad \text{and} \quad \mathbf{x}^{\mathbf{c}}_m := (x_{m+1}, \ldots, x_n)^\tau$$

and the submatrices $\mathbf{V}_m := (\mathbf{v}_1, \ldots, \mathbf{v}_m)$ of $\mathbf{V}$ the relation (7) can be written as $\mathcal{P}_m \mathbf{y}(t,\omega) = \mathbf{V}_m \, \mathbf{x}_m(t,\omega)$.

A reduced-order system can be derived if an m-dimensional state equation for the m coordinate functions defining the projection $\mathcal{P}_m \mathbf{y}$ is considered. To this end, the original system (4) is written in terms of the transformed coordinates using the transformation $\mathbf{y} = \mathbf{V}\mathbf{x}$. Here, $\mathbf{x} = (x_1, \ldots, x_n)^\tau$ contains the coordinates of $\mathbf{y}$ with respect to the basis $(\mathbf{v}_k)_{k=1,\ldots,n}$. Then (4) reads as

$$\dot{\mathbf{x}} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{h}, \qquad \mathbf{z} = \mathbf{V}\mathbf{x}, \tag{8}$$

with $\mathbf{C} := \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ and $\mathbf{D} := \mathbf{V}^{-1}\mathbf{B}$ and is referred to as the transformed original system. Defining

$$\mathbf{C}_m := \{C_{ij}\}_{i,j=1,\ldots,m}\,, \; \mathbf{C}^{\mathbf{c}}_m := \{C_{ij}\}_{\substack{i=1,\ldots,m \\ j=m+1,\ldots,n}}\,, \; \mathbf{D}_m := \{D_{ij}\}_{\substack{i=1,\ldots,m \\ j=1,\ldots,r}}\,, \tag{9}$$

$$\text{i.e.} \qquad \mathbf{C} = \begin{pmatrix} \mathbf{C}_m & \mathbf{C}^{\mathbf{c}}_m \\ \cdots & \cdots \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} \mathbf{D}_m \\ \cdots \end{pmatrix},$$

for $m = 1, \ldots, n-1$, the truncated system using only the first m state equations of (8) reads as

$$\dot{\mathbf{x}}_m = \mathbf{C}_m \mathbf{x}_m + \mathbf{C}^{\mathbf{c}}_m \mathbf{x}^{\mathbf{c}}_m + \mathbf{D}_m \mathbf{h}, \qquad \mathcal{P}_m \mathbf{z} = \mathbf{V}_m \mathbf{x}_m. \tag{10}$$

The output of the above system is the projection of $\mathbf{z}$ onto the subspace $\mathcal{S}_m = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$. Provided the matrix $\mathbf{C}_m$ is stable, the state equations of (10) form a stable first-order system for $\mathbf{x}_m$ with the stationary random excitation term $\mathbf{C}^{\mathbf{c}}_m \mathbf{x}^{\mathbf{c}}_m + \mathbf{D}_m \mathbf{h}$.

The truncated system (10) possesses not the desired form (5) of a reduced-order system since its excitation term contains $\mathbf{C}^{\mathbf{c}}_m \mathbf{x}^{\mathbf{c}}_m$. The stationary output of system (10) and its moment functions cannot be determined exactly without the knowledge of $x_{m+1}, \ldots, x_n$ which requires the knowledge of the stationary solution of the large-scale state equation of the transformed original system (8). To overcome the mentioned drawback of the non-vanishing term $\mathbf{C}^{\mathbf{c}}_m \mathbf{x}^{\mathbf{c}}_m$

a second approximation step is necessary. It consists in the approximation of the variables $x_{m+1}, \ldots, x_n$ by their zero means and generates an additional error. It results the following system for the m-dimensional state $\mathbf{y}_m$ and the n-dimensional output $\tilde{\mathbf{z}}^{(m)}$

$$\dot{\mathbf{y}}_m = \mathbf{C}_m \mathbf{y}_m + \mathbf{D}_m \mathbf{h}, \qquad \tilde{\mathbf{z}}^{(m)} = \mathbf{V}_m \mathbf{y}_m. \tag{11}$$

The above system may serve as the desired reduced-order system (5) if the settings $\mathbf{A}_m = \mathbf{C}_m$, $\mathbf{B}_m = \mathbf{D}_m$ and $\mathbf{T}_m = \mathbf{V}_m$ are used. If $\mathbf{C}_m$ is stable then system (11) possesses the same qualitative properties as the original system. Both systems are stable and possess stationary excitations. That means, procedures for the computation of moment functions of the original system can be applied to the reduced-order system (11), too.

The relation $\tilde{\mathbf{z}}^{(m)} = \mathbf{V}_m \mathbf{y}_m$ implies that the correlation function $\mathbf{R}_{\tilde{\mathbf{z}}^{(m)}\tilde{\mathbf{z}}^{(m)}}(\tau)$ and the spectral density $\mathbf{S}_{\tilde{\mathbf{z}}^{(m)}\tilde{\mathbf{z}}^{(m)}}(\alpha)$ which are used as approximations of the corresponding moment functions of $\mathbf{z}$ can be determined from the moment functions of $\mathbf{y}_m$ using

$$\mathbf{R}_{\tilde{\mathbf{z}}^{(m)}\tilde{\mathbf{z}}^{(m)}}(\tau) = \mathbf{V}_m \mathbf{R}_{\mathbf{y}_m\mathbf{y}_m}(\tau)\mathbf{V}_m^* \text{ and } \mathbf{S}_{\tilde{\mathbf{z}}^{(m)}\tilde{\mathbf{z}}^{(m)}}(\alpha) = \mathbf{V}_m \mathbf{S}_{\mathbf{y}_m\mathbf{y}_m}(\alpha)\mathbf{V}_m^*. \tag{12}$$

In order to assess the approximation accuracy and to find suitable basis elements $\mathbf{v}_1, \ldots, \mathbf{v}_n$ error criterions have to be defined. Since the main interest is to compute the correlation function $\mathbf{R}_{\mathbf{zz}}(\tau) = \mathbf{E}\{\mathbf{z}(t)\mathbf{z}^*(t+\tau)\}$ and the corresponding spectral density $\mathbf{S}_{\mathbf{zz}}(\alpha)$ we define

$$\begin{aligned} \Delta^{\mathbf{R}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}) &:= \int_{\mathbb{R}} \sum_{i,j=1}^{n} \left| \mathbf{R}_{z_i z_j}(\tau) - \mathbf{R}_{\tilde{z}_i^{(m)}\tilde{z}_j^{(m)}}(\tau) \right| \mathbf{\Psi}_{ij}\, \theta(\tau) d\tau \text{ and} \\ \Delta^{\mathbf{S}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}) &:= \int_{\mathbb{R}} \sum_{i,j=1}^{n} \left| \mathbf{S}_{z_i z_j}(\alpha) - \mathbf{S}_{\tilde{z}_i^{(m)}\tilde{z}_j^{(m)}}(\alpha) \right| \mathbf{\Psi}_{ij} d\alpha \end{aligned}$$

where $\mathbf{\Psi} = \{\Psi_{i,j}\}_{i,j=1,\ldots,n}$ is a weighting matrix with non-negative entries and $\sum\limits_{i,j=1}^{n} \Psi_{ij} = 1$ while θ is a non-negative weighting function with $\int\limits_{\mathbb{R}} \theta(\tau) d\tau = 1$. In [18, 25] the following upper bound for the introduced error criterions has been found

$$\Delta^{\mathbf{R}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}),\ \Delta^{\mathbf{S}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}) \leq \sqrt{\Delta_0(\mathbf{z}, \tilde{\mathbf{z}}^{(m)})} \left(2\sqrt{\mathbf{E}\left\{ \|\tilde{\mathbf{z}}^{(m)}\|^2 \right\}} + \sqrt{\Delta_0(\mathbf{z}, \tilde{\mathbf{z}}^{(m)})} \right)$$

where $\Delta_0(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}) := \mathbf{E}\left\{ \left\| \mathbf{z} - \tilde{\mathbf{z}}^{(m)} \right\|^2 \right\}$. For the computation of the upper bound means and covariances of $\tilde{\mathbf{z}}^{(m)}$ and $\mathbf{z} - \tilde{\mathbf{z}}^{(m)}$ are needed, only. The above inequality suggests, that those basis elements $\mathbf{v}_1, \ldots, \mathbf{v}_n$ provide a small values of $\Delta^{\mathbf{R}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)})$ and $\Delta^{\mathbf{S}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)})$ for which the term $\Delta_0(\mathbf{z}, \tilde{\mathbf{z}}^{(m)})$ is small. This idea leads to the following suggestions for the selection of basis elements (see [18, 19, 25]).

i) Mean-square optimal projections:
Choose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ as the eigenvectors of the covariance matrix $\mathbf{\Gamma} := \mathbf{E}\{\mathbf{y}(t)\mathbf{y}^*(t)\}$ such that for the corresponding eigenvalues $\sigma_1^2, \ldots, \sigma_n^2$ it holds $\sigma_1^2 \geq \ldots \geq \sigma_n^2$.

ii) Modal decomposition:
Choose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ as the eigenvectors of $\mathbf{A}$ such that for the second-order moments of the transformed state variables $x_1, \ldots, x_n$ it holds $\mathbf{E}\{|x_1|^2\} \geq \ldots \geq \mathbf{E}\{|x_n|^2\}$.

4 Applications

For the purpose of illustration, this section deals with systems of linear ODEs which are the result of a semi-discretization of PDEs describing continuous mechanical vibration systems subjected to random external excitations. In this approach the PDE is discretized with respect to the spatial variables, only. The dimension of the resulting system of ODEs depends on the discretization and may become very large if complex domains in two or three dimensions have to be taken into account. Here, the method is described for a simple example with a one-dimensional spatial variable.

Consider the approximative computation of moment functions of a random function $u : \mathbb{R} \times [0,1] \times \Omega \to \mathbb{R}$ denoted by $u(t, x, \omega)$ which is a solution of the following boundary value problem for a PDE

$$\begin{aligned} \rho\, u_{tt} + 2\kappa\, u_t + \eta\, u_{xxxx} &= q(x)\xi(t,\omega) \\ u(t,0,\omega) &= u(t,1,\omega) = 0 \\ u_{xx}(t,0,\omega) &= u_{xx}(t,1,\omega) = 0 \end{aligned} \tag{13}$$

which arises in the description of transverse vibrations of a simply supported thin homogeneous beam of length 1 subject to a random load acting along the beams length (see Figure 1). Here, the subscripts t and x denote partial derivatives with respect to t and x, respectively, $u(t, x, \omega)$ denotes transverse displacement and the positive constants ρ, κ and η describe mass per unit length, viscous damping and bending stiffness, respectively (see Newland [7], p. 341 ff., Rao [10], p. 523 ff.)

The random load is described by the term $q(x)\xi(t, \omega)$ with a centered and stationary random process $\xi(t, \omega)$ and a non-random "modulation" function

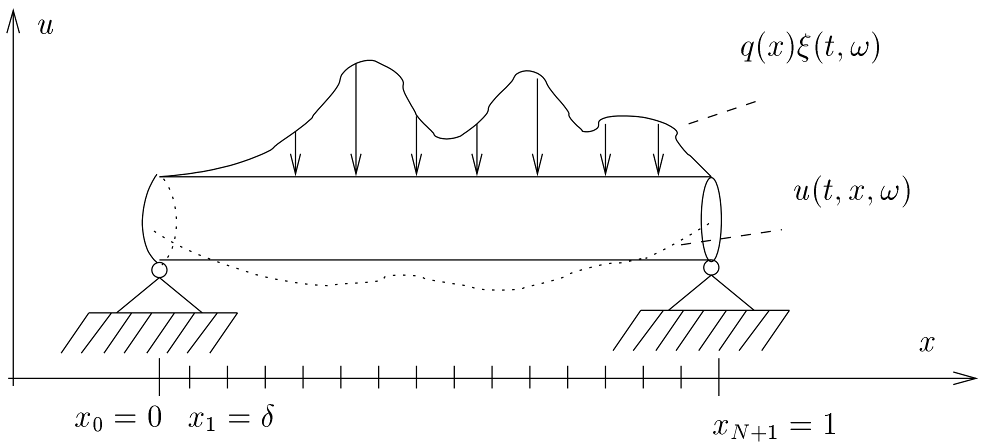

Figure 1: Transverse vibrations of a beam subject to a random load

$q(x)$. Similar problems as well the case of loads described by a random fields are considered in vom Scheidt [11],[12]. These investigations are based on Fourier method for solving PDEs. Further it is noted, that in [19] similar model reduction techniques are applied to equations arising from modelling axial vibrations of thin beams with one free end and a prescribed random displacement at the other end.

As in the context of systems of ODEs considered throughout this paper stationary solutions of (13) are of interest. In the present context stationary solutions $u(t,x,\omega)$ are defined by the property that for arbitrary $p \in \mathbb{N}$ and for all $x_1,\ldots,x_p \in [0,1]$ the vector functions $(u(t,x_1,\omega),\ldots,u(t,x_p,\omega))^\tau$ are stationary.

For the approximative computation of moment functions of $u(t,x,\omega)$ boundary value problem (13) is discretized with respect to the spatial variable x using finite difference method. In the following this procedure is briefly described. For more details see e.g. Großmann, Roos [3], p. 306 ff.

The transverse displacement $u(t,x,\omega)$ is only considered at grid points $x_i = i\delta$, $i = 0,\ldots,N+1$, $\delta = \frac{1}{N+1}$. Denoting by $u_i^N(t,\omega)$ the displacement at $x = x_i$ then it holds for $i = 0,\ldots,N+1$

$$u(t,x_i,\omega) = u_i^N(t,\omega), \quad u_t(t,x_i,\omega) = \dot{u}_i^N(t,\omega), \quad u_{tt}(t,x_i,\omega) = \ddot{u}_i^N(t,\omega).$$

For the derivatives of u with respect to x the finite difference approximations

$$u_{xxxx}(.,x_i,.) \approx \frac{1}{\delta^4}(u_{i-2}^N - 4u_{i-1}^N + 6u_i^N - 4u_{i+1}^N + u_{i+2}^N), \quad i = 2,\ldots,N-1,$$

are obtained. Taking into account the boundary conditions which imply

$$\begin{aligned} u_0^N = u_{N+1}^N &= 0 \\ u_{xxxx}(.,x_1,.) &\approx \frac{1}{\delta^4}(5u_1^N - 4u_2^N + u_3^N) \\ u_{xxxx}(.,x_N,.) &\approx \frac{1}{\delta^4}(u_{N-2}^N - 4u_{N-1}^N + 5u_N^N) \end{aligned}$$

it results an N-dimensional system of second-order ODE for $\mathbf{u}^N = (u_1^N, \ldots, u_N^N)^\tau$

$$\rho\,\ddot{\mathbf{u}}^N + 2\kappa\,\dot{\mathbf{u}}^N + \eta\,\mathbf{K}\mathbf{u}^N = \tilde{\mathbf{B}}\mathbf{h}$$

with the $N \times N$ matrix $\mathbf{K}$ and the $N \times 1$ matrix $\tilde{\mathbf{B}}$ defined by

$$\mathbf{K} = \frac{1}{\delta^4}\begin{pmatrix} 5 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 5 \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{B}} = \begin{pmatrix} q(x_1) \\ \vdots \\ q(x_N) \end{pmatrix}.$$

Further, the excitation function $\mathbf{h}$ is a scalar function given by $\mathbf{h}(t,\omega) = \xi(t,\omega)$. The corresponding first-order system reads as

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{h} \tag{14}$$

with $\mathbf{z} = \begin{pmatrix} \dot{\mathbf{u}}^N \\ \mathbf{u}^N \end{pmatrix}$ and a $2N \times 2N$ matrix $\mathbf{A}$ and a $2N \times 1$ matrix $\mathbf{B}$ defined by

$$\mathbf{A} = \begin{pmatrix} -\frac{2\kappa}{\rho}\mathbf{I}_N & -\frac{\eta}{\rho}\mathbf{K} \\ \mathbf{I}_N & \mathbf{O}_{N,N} \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \frac{1}{\rho}(q(x_1), \ldots, q(x_N), 0, \ldots, 0)^\tau.$$

The moment functions of the stationary solution $\mathbf{z}(t,\omega)$ can be used as approximations of the moment functions of $u(t,x,\omega)$ and $u_t(t,x,\omega)$ in the grid points $x_i = i\delta$, $i = 1, \ldots, N$. To this end for $\tau, \alpha \in \mathbb{R}$ and $x, x' \in [0,1]$ the moment functions

$$\begin{aligned} R_{uu}(\tau,x,x') &:= \mathbf{E}\left\{u(t,x)u(t+\tau,x')\right\} \qquad \text{and} \\ S_{uu}(\alpha,x,x') &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\mathrm{i}\alpha\tau} R_{uu}(\tau,x,x')d\tau \end{aligned}$$

are defined. Further, for $x = x_i = i\delta,\ x' = x_j = j\delta,\ i,j = 1,\dots,N$ the finite-dimensional approximations

$$\begin{aligned} R^N_{uu}(\tau,x,x') &:= R_{u^N_i u^N_j}(\tau) = R_{z_{N+i}z_{N+j}}(\tau) \\ S^N_{uu}(\alpha,x,x') &:= S_{u^N_i u^N_j}(\alpha) = S_{z_{N+i}z_{N+j}}(\alpha) \end{aligned}$$

are introduced. The functions R^N and S^N for values x between the grid points are obtained by interpolation. For the velocity u_t the moment functions $R_{u_t u_t}(\tau,x,x')$ and $S_{u_t u_t}(\alpha,x,x')$ are defined analogously. The corresponding finite-dimensional approximations for $x = i\delta,\ x' = j\delta,\ i,j = 1,\dots,N$ are

$$\begin{aligned} R^N_{u_t u_t}(\tau,x,x') &:= R_{\dot{u}^N_i \dot{u}^N_j}(\tau) = R_{z_i z_j}(\tau) \\ S^N_{u_t u_t}(\alpha,x,x') &:= S_{\dot{u}^N_i \dot{u}^N_j}(\alpha) = S_{z_i z_j}(\alpha). \end{aligned}$$

It is known that the goodness of the finite difference approximation depends on the discretization parameter $\delta > 0$ and therefore on the number of grid points. To ensure a quite accurate finite difference approximation the number N and consequently the dimension $n = 2N$ of the first-order system (1) has to be chosen sufficiently large. The result is a large-scale system (1) which has been the starting point of the described model reduction procedures.

For the subsequent numerical experiments the random function ξ is chosen as a stationary ε-correlated process ${}^{\varepsilon}f$ with continuous paths, a correlation length ε and an intensity μ. The modulation function is assumed to be $q(x) = 1 - x$. Table 1 gives the values of the parameters involved in the numerical example.

To avoid considerations of the dependence of the moment functions on the correlation length ε and the error due to the truncation of the expansions of moment functions in powers of ε the subsequent results refer to the a.s.-limit functions

$$\begin{aligned} {}^{o}\mathbf{z}(t,\omega) &:= \lim_{\varepsilon\downarrow 0} \frac{1}{\sqrt{\varepsilon}}\mathbf{z}(t,\omega), & {}^{o}\tilde{\mathbf{z}}^{(m)}(t,\omega) &:= \lim_{\varepsilon\downarrow 0} \frac{1}{\sqrt{\varepsilon}}\tilde{\mathbf{z}}^{(m)}(t,\omega) \\ {}^{o}u(t,x,\omega) &:= \lim_{\varepsilon\downarrow 0} \frac{1}{\sqrt{\varepsilon}}u(t,x,\omega). \end{aligned}$$

The first expansion terms of second-order moment functions for a given $\varepsilon = \varepsilon_0$ can be obtained by multiplying the corresponding quantities of the limit function by ε_0.

Figure 2 shows the finite-dimensional approximations of the correlation functions of ${}^{o}u$ and ${}^{o}u_t$, i.e. $R^N_{{}^{o}u\,{}^{o}u}(\tau,x,x')$ and $R^N_{{}^{o}u_t\,{}^{o}u_t}(\tau,x,x')$, as well as the corresponding spectral densities $S^N_{{}^{o}u\,{}^{o}u}(\alpha,x,x')$ and $S^N_{{}^{o}u_t\,{}^{o}u_t}(\alpha,x,x')$ for $x = x' \in (0,1)$. Further, Figure 3 shows model reduction approximations of the spectral density $S_{{}^{o}u_t\,{}^{o}u_t}(\alpha,x,x)$ which have been found from the corresponding spectral densities of ${}^{o}\tilde{z}^{(m)}_i, i = 1,\dots,N$. It can be seen that the approximations based on modal decomposition presented in the right column compare well with the spectral density plot of ${}^o u_t$ found from the $2N$-dimensional original system even for small dimensions $m \geq 8$. On the other hand, the approximations based on mean-square optimal projections show significant differences to the original system for $m \leq 32$.

parameter			value
material	mass per unit length	ρ	1
	viscous damping	κ	1
	bending stiffness	η	0.1
excitation	intensity	μ	10^4
discretization	number of internal grid points	N	25

Table 1: Parameter settings for the numerical example

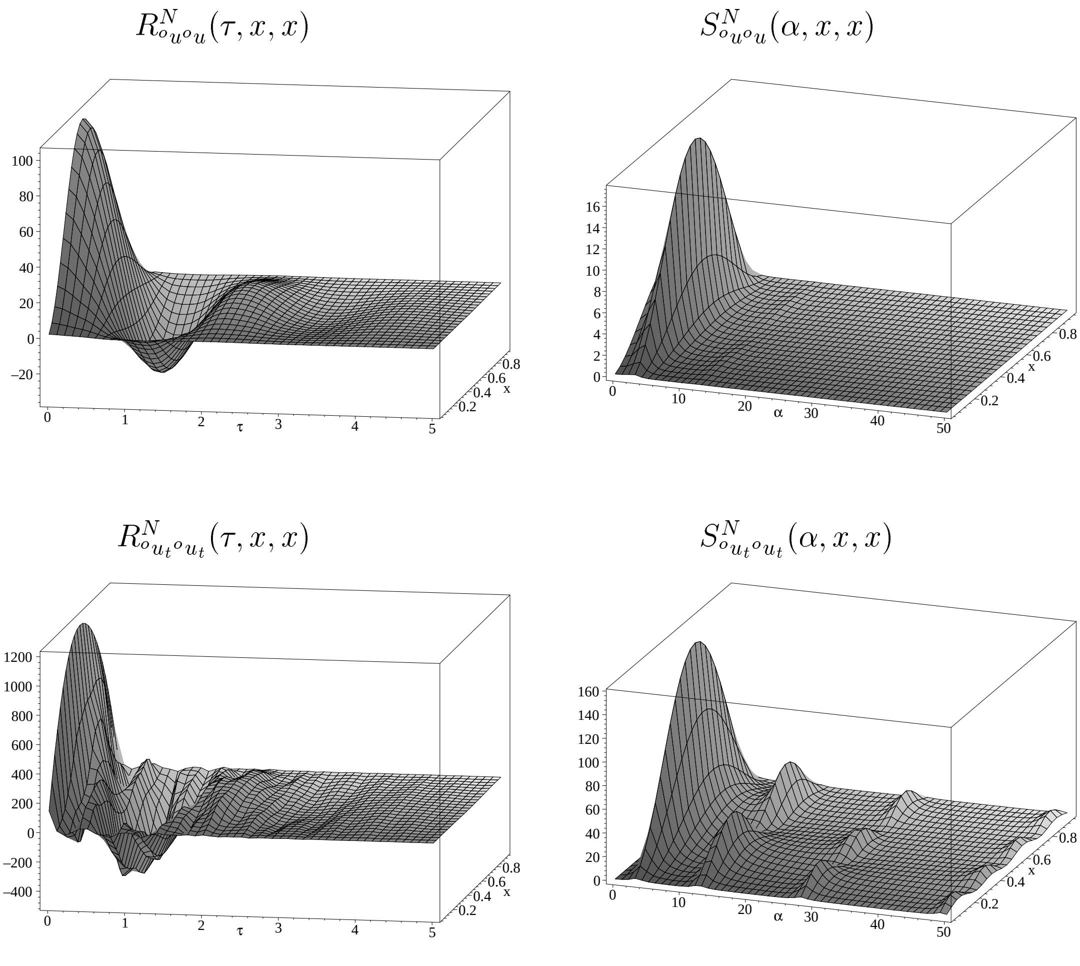

Figure 2: Finite-dimensional approximations of the moment functions of ${}^o u$ and ${}^o u_t$

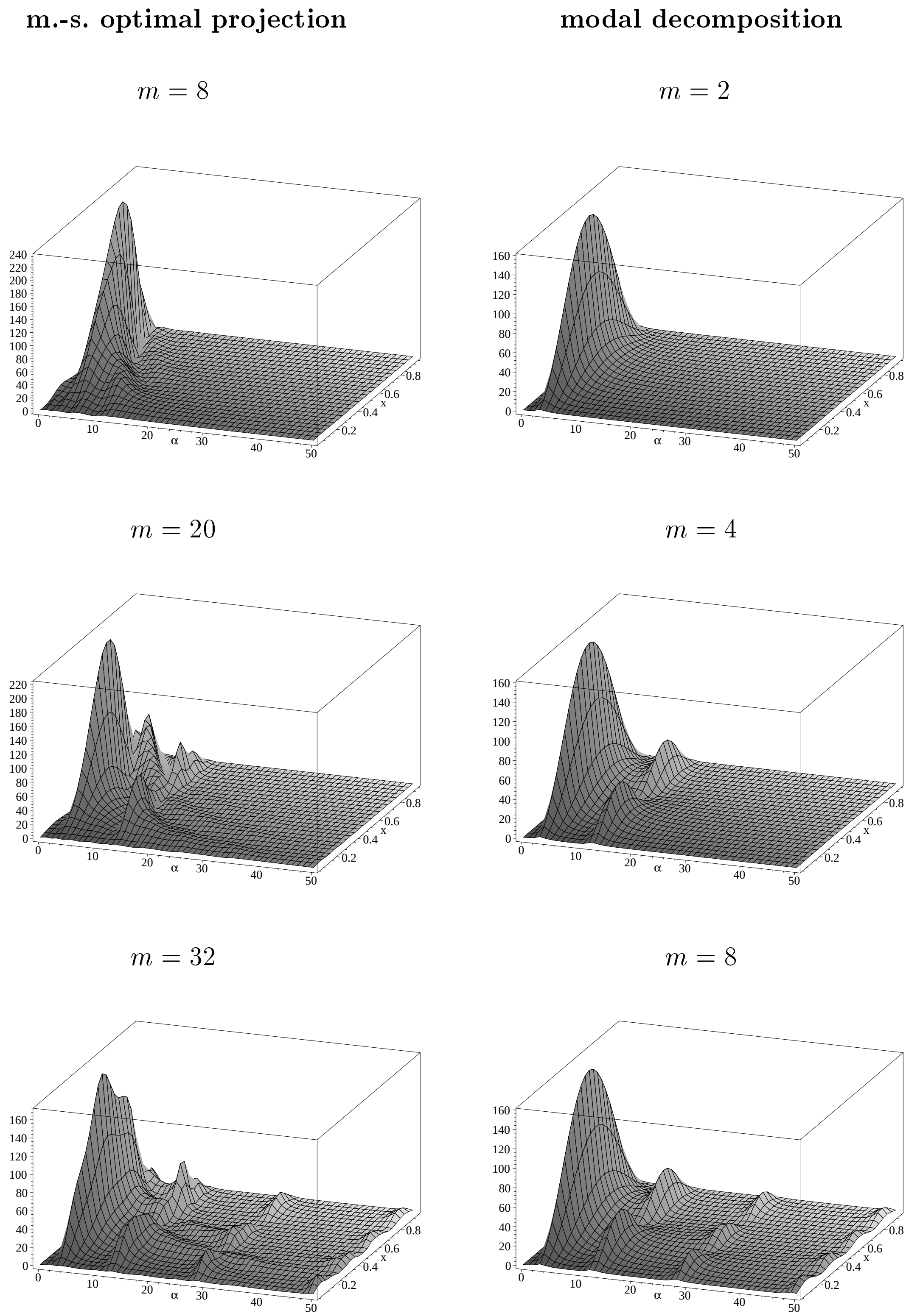

Figure 3: Approximations of the spectral density $S^N_{{}^o u_t {}^o u_t}(\alpha, x, x)$

Next, the modulation function is assumed to be $q(x) = \sin(k\pi x)$, $k \in \mathbb{N}$, which is an eigenfunction of the eigenvalue problem

$$\eta v''''(x) = \gamma v(x), \quad v(0) = v(1) = v''(0) = v''(1) = 0. \tag{15}$$

Here, the corresponding eigenvalue is $\gamma_k = \eta(k\pi)^4$. For this modulation function vom Scheidt [11, 12] provides an efficient analytical method for the computation of the moment functions of ${}^o u$ and ${}^o u_t$ which is based on Fourier method. Subsequently, the key results are briefly described. For the sake of shorter notation $\rho = 1$ is assumed.

Starting point is the assumption that the solution $u(t, x)$ of problem (13) can be expressed by an infinite series

$$u(t, x) = \sum_{j=1}^{\infty} c_j(t) w_j(x) \tag{16}$$

where $w_j(x) = \sqrt{2}\sin(j\pi x)$ are the eigenfunctions of (15) with $(w_i, w_j) = \delta_{ij}$, $i, j \in \mathbb{N}$ and c_j are some functions which are to be determined. Thereby $(.,.)$ denotes the scalar product of $\mathbf{L}_2([0,1])$. Substituting this series into PDE (13), using the representation $q(x) = \sum\limits_{j=1}^{\infty}(q, w_j)w_j(x)$ and equating the coefficients in front of w_j an infinite sequence of second-order ODE

$$\ddot{c}_j + 2\kappa\dot{c}_j + \gamma_j c_j = \xi(t, \omega)(q, w_j)$$

for the coefficient functions c_j, $j = 1, 2, \ldots$ is obtained. The functions c_j are determined as stationary solutions of the above equations. It is noted that these equations also arise in the mathematical modeling of the stationary response of a single degree of freedom oscillator with a random excitation force $\xi(t, \omega)(q, w_j)$, a linear spring and a linear damper described by the coefficients γ_j and κ, respectively. Since $(q, w_j) = \int\limits_0^1 \sin(k\pi x)\sqrt{2}\sin(j\pi x)dx = \sqrt{2}\delta_{jk}$ it holds $c_j(t, \omega) \equiv 0$ for $j \neq k$ while for $j = k$ the correlation function and spectral density of c_k and $\dot{c}_k$ can be deduced from the correponding first-order system

$$\frac{d}{dt}\begin{pmatrix}\dot{c}_k \\ c_k\end{pmatrix} = \begin{pmatrix} -2\kappa & -\gamma_k \\ 1 & 0 \end{pmatrix}\begin{pmatrix}\dot{c}_k \\ c_k\end{pmatrix} + \begin{pmatrix}\sqrt{2} \\ 0\end{pmatrix}\xi. \tag{17}$$

For $\xi = {}^\varepsilon f$ the expansion technique introduced in Section 2 can be applied. It follows

$$R_{c_k c_k}(\tau) = \varepsilon\mu\tilde{R}_k(\tau) + o(\varepsilon) \quad \text{and} \quad S_{c_k c_k}(\alpha) = \varepsilon\mu\tilde{S}_k(\alpha) + o(\varepsilon)$$

for $\tau \in \mathbb{R}$ and fixed values $\alpha \in \mathbb{R}$ where

$$\begin{aligned}
\tilde{R}_k(\tau) &= \frac{1}{4\gamma_k} e^{-\kappa|\tau|} \left[\frac{1}{\kappa} \cos(\kappa_k \tau) + \frac{1}{\kappa_k} \sin(\kappa_k |\tau|) \right] \qquad \text{with } \kappa_k = \sqrt{\gamma_k - \kappa^2} \\
\tilde{S}_k(\alpha) &= \frac{1}{2\pi} \frac{1}{(\gamma_k - \alpha^2)^2 + 4\kappa^2 \alpha^2}.
\end{aligned} \tag{18}$$

For the derivatives $\dot{c}_k$ it holds

$$R_{\dot{c}_k \dot{c}_k}(\tau) = -\varepsilon\mu \tilde{R}''_k(\tau) + o(\varepsilon) \quad \text{and} \quad S_{\dot{c}_k \dot{c}_k}(\alpha) = \varepsilon\mu\alpha^2 \tilde{S}(\alpha) + o(\varepsilon)$$

for $\tau \in \mathbb{R}$ and fixed α where

$$-\tilde{R}''_k(\tau) = \frac{1}{4} e^{-\kappa|\tau|} \left[\frac{1}{\kappa} \cos(\kappa_k \tau) - \frac{1}{\kappa_k} \sin(\kappa_k |\tau|) \right]$$

It can be seen that the Fourier series (16) contains only a single non-zero term (for $j = k$). The corresponding moment functions of the limit functions ${}^o u$ and ${}^o u_t$ are obtained as

$$\begin{aligned}
R_{{}^o u {}^o u}(\tau, x, x') &= \mu \tilde{R}_k(\tau) w_k(x) w_k(x') \\
R_{{}^o u_t {}^o u_t}(\tau, x, x') &= -\mu \tilde{R}''_k(\tau) w_k(x) w_k(x') \\
S_{{}^o u {}^o u}(\alpha, x, x') &= \mu \tilde{S}_k(\alpha) w_k(x) w_k(x') \\
S_{{}^o u_t {}^o u_t}(\alpha, x, x') &= \mu \alpha^2 \tilde{S}_k(\alpha) w_k(x) w_k(x').
\end{aligned}$$

Figure 4 shows the moment functions $R_{{}^o u {}^o u}(\tau, x, x')$ and $S_{{}^o u {}^o u}(\alpha, x, x')$ for $x = x'$ using $q(x) = \sin(2\pi x)$ and the parameters given in Table 1.

$R_{{}^o u {}^o u}(\tau, x, x)$ $\qquad$ $S_{{}^o u {}^o u}(\alpha, x, x)$

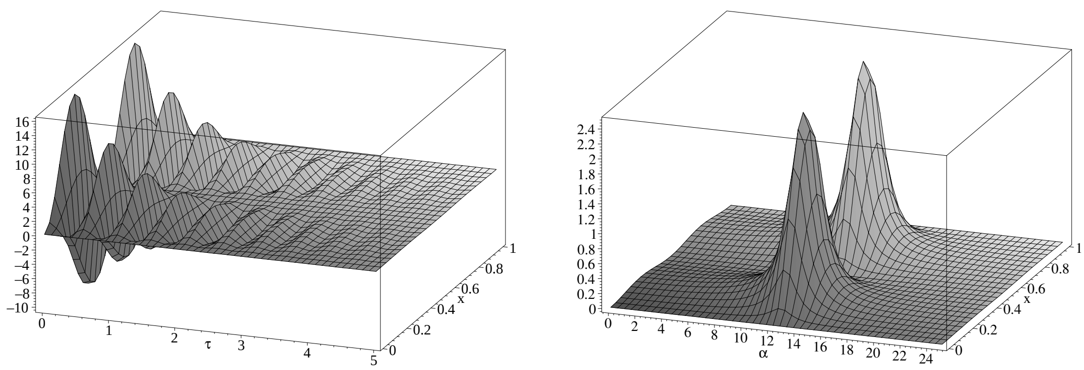

Figure 4: Moment functions of ${}^o u$ for $q(x) = \sin(2\pi x)$

The above analytical treatment of the problem with the help of Fourier method has shown that for $q(x) = \sin(k\pi x)$ the random dynamics of the infinite dimensional vibration system (13) can be described with the help of the stationary response of a single-degree-of-freedom oscillator or the stationary solution of two-dimensional system (17) of first-order ODEs. In view of this property one would expect that the finite-dimensional counterpart of problem (13) which results from the semi-discretization posesses a similar property.

Weiß shows in [23] that there exist reduced-order systems with a two dimensional state equation which are able to reproduce the stationary solution of the large-scale system (14) exactly. Starting point of the investigations in [23] is the eigenvalue decomposition of the matrix

$$\mathbf{A} = \begin{pmatrix} -2\kappa\mathbf{I}_N & -\eta\mathbf{K} \\ \mathbf{I}_N & \mathbf{O}_{N,N} \end{pmatrix}.$$

The submatrix $\eta\mathbf{K}$ is non-negative definite and symmetric, it possesses the non-negative eigenvalues γ_j^N and the pairwise orthogonal eigenvectors $\mathbf{w}_j^N$ which are found to be

$$\begin{aligned} \gamma_j^N &= 4\eta(1-\cos(j\pi\delta))^2 \\ \mathbf{w}_j^N &= (\sin(j\pi\delta), \sin(2j\pi\delta), \ldots, \sin(Nj\pi\delta)))^\tau, \quad j = 1, \ldots, N. \end{aligned}$$

A simple calculation shows that for $j = 1, \ldots, N$ and $\kappa < \gamma_1^N$

$$\lambda_j^\pm = -\kappa \pm \mathrm{i}\kappa_j^N, \quad \text{with } \kappa_j^N = \sqrt{\gamma_j^N - \kappa^2},$$

are the eigenvalues of $\mathbf{A}$ and

$$\mathbf{v}_j^\pm = \begin{pmatrix} \lambda_j^\pm \mathbf{w}_j^N \\ \mathbf{w}_j^N \end{pmatrix}$$

are the eigenvectors of $\mathbf{A}$. In view of the modal decomposition procedure presented in Section 3 the vectors

$$\mathring{\mathbf{v}}_j = \frac{1}{\|\mathbf{v}_j^+\|}\mathbf{v}_j^+ \quad \text{and} \quad \mathring{\mathbf{v}}_{N+j} = \frac{1}{\|\mathbf{v}_j^-\|}\mathbf{v}_j^-$$

and the matrices $\mathbf{V} = (\mathring{\mathbf{v}}_1, \ldots, \mathring{\mathbf{v}}_{2N})$ and $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1^+, \ldots, \lambda_N^+, \lambda_1^-, \ldots, \lambda_N^-)$ are defined. Then the original system (14) which is rewritten as

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}\xi, \qquad \mathbf{z} = \mathbf{I}_{2N}\mathbf{y},$$

with $\mathbf{B} = (q(\delta), \ldots, q(N\delta), 0, \ldots, 0)^\tau = \binom{\mathbf{w}_k^N}{\mathbf{0}_N}$ is transformed into

$$\dot{\mathbf{x}} = \mathbf{\Lambda}\mathbf{x} + \mathbf{D}\xi, \qquad \mathbf{z} = \mathbf{V}\mathbf{x},$$

with $\mathbf{D} = \mathbf{V}^{-1}\mathbf{B}$. In [23] it is shown that the $2N \times 1$ matrix $\mathbf{D}$ contains only two non-zero entries $D_{k,1} = d$ and $D_{N+k,1} = -d$ with some constant $d \in \mathbb{C}$.

This implies that the components $x_i(t,\omega)$ of the stationary solution of the state equation $\dot{\mathbf{x}} = \mathbf{\Lambda x} + \mathbf{D}\xi$ vanish for $i \neq k, N+k$. Neglecting of $2N-2$ state equations for these zero components leads to a truncated system (see (10)) with a two-dimensional state equation which coincides with the reduced-order system

$$\dot{\mathbf{y}}_2 = \begin{pmatrix} \lambda_k^+ & 0 \\ 0 & \lambda_k^- \end{pmatrix} \mathbf{y}_2 + \frac{\sqrt{2}\mathrm{i}}{2\kappa_k^N} \begin{pmatrix} d \\ -d \end{pmatrix} \xi, \qquad \tilde{\mathbf{z}}^{(2)} = (\overset{\circ}{\mathbf{v}}_k, \overset{\circ}{\mathbf{v}}_{N+k})\mathbf{y}_2. \tag{19}$$

For the output of the above system it holds $\tilde{\mathbf{z}}^{(2)} = \mathbf{z} = \mathcal{P}_2\mathbf{z}$, i.e. the projection $\mathcal{P}_2\mathbf{z}$ of $\mathbf{z}$ onto $\mathcal{S}_2 = \mathrm{span}(\overset{\circ}{\mathbf{v}}_k, \overset{\circ}{\mathbf{v}}_{N+k})$ is exact and its random dynamics can be obtained using the above system with only two state equations.

The identity $\tilde{\mathbf{z}}^{(2)} = (\overset{\circ}{\mathbf{v}}_k, \overset{\circ}{\mathbf{v}}_{N+k})\mathbf{y}_2$, the linear independence of $\overset{\circ}{\mathbf{v}}_k$ and $\overset{\circ}{\mathbf{v}}_{N+k}$ and $\mathrm{rank}\,\mathbf{R}_{\mathbf{y}_2\mathbf{y}_2}(0) = 2$ imply that it holds

$$\mathrm{rank}\,\mathbf{R}_{\mathbf{zz}}(0) = \mathrm{rank}\,[\mathbf{V}_2\mathbf{R}_{\mathbf{y}_2\mathbf{y}_2}(0)\mathbf{V}_2^*] = 2, \qquad \text{where } \mathbf{V}_2 = (\overset{\circ}{\mathbf{v}}_k, \overset{\circ}{\mathbf{v}}_{N+k}).$$

Hence, only two of the eigenvalues $\sigma_1^2, \ldots, \sigma_{2N}^2$ of the non-negative definite and symmetric covariance matrix $\mathbf{R}_{\mathbf{zz}}(0)$ are strictly positive, i.e. it holds $\sigma_1^2 \geq \sigma_2^2 > \sigma_3^2 = \ldots = \sigma_{2N}^2 = 0$. The application of the model reduction procedure based on mean-square optimal projections (see Section 3) using pairwise orthonormal eigenvectors $\check{\mathbf{v}}_1, \ldots, \check{\mathbf{v}}_{2N}$ of $\mathbf{R}_{\mathbf{zz}}(0)$ leads to a transformed original system which reads as

$$\dot{\mathbf{x}} = \mathbf{Cx} + \mathbf{D}\xi, \qquad \mathbf{z} = \mathbf{Vx}$$

with $\mathbf{V} = (\check{\mathbf{v}}_1, \ldots, \check{\mathbf{v}}_{2N})$, $\mathbf{C} = \mathbf{V}^{-1}\mathbf{AV}$ and $\mathbf{D} = \mathbf{V}^{-1}\mathbf{B}$. Since for the covariance matrix of $\mathbf{x}$ it holds $\mathbf{R}_{\mathbf{xx}}(0) = \mathbf{\Sigma} = \mathrm{diag}(\sigma_1^2, \sigma_2^2, 0, \ldots, 0)$ it follows $x_3(t,\omega) = \ldots = x_{2N}(t,\omega) = 0$ a.s..

As in the case of modal decomposition the truncated system (10) which contains only the first two state equations coincides with the reduced-order system which reads as

$$\dot{\mathbf{y}}_2 = \mathbf{C}_2\mathbf{y}_2 + \mathbf{D}_2\xi, \qquad \tilde{\mathbf{z}}^{(2)} = (\check{\mathbf{v}}_1, \check{\mathbf{v}}_2)\mathbf{y}_2.$$

Again, for the output it holds $\tilde{\mathbf{z}}^{(2)} = \mathbf{z} = \mathcal{P}_2\mathbf{z}$. i.e. there is no approximation error. Further, it can be seen that for the subspace $\mathcal{S}_2$ on which $\mathbf{z}$ is projected it holds

$$\mathcal{S}_2 = \mathrm{span}(\check{\mathbf{v}}_1, \check{\mathbf{v}}_2) = \mathrm{span}(\overset{\circ}{\mathbf{v}}_k, \overset{\circ}{\mathbf{v}}_{N+k}).$$

The computation of moment functions for the reduced-order system (19) leads to the following finite dimensional approximations of the moment functions of ${}^o u$ for $x = x_i = i\delta$, $x' = x_j = j\delta$, $i,j = 1,\ldots,N$

$$\begin{aligned} R^N_{{}^o u {}^o u}(\tau, x, x') &= R_{{}^o z_{N+i} {}^o z_{N+j}}(\tau) = \mu \tilde{R}_k^N(\tau) w_k(x) w_k(x') \\ S^N_{{}^o u {}^o u}(\alpha, x, x') &= S_{{}^o z_{N+i} {}^o z_{N+j}}(\alpha) = \mu \tilde{S}_k^N(\alpha) w_k(x) w_k(x') \end{aligned}$$

where

$$\tilde{R}_k^N(\tau) = \frac{1}{4\gamma_k^N} e^{-\kappa|\tau|} \left[\frac{1}{\kappa} \cos(\kappa_k^N \tau) + \frac{1}{\kappa_k} \sin(\kappa_k^N |\tau|) \right] \qquad \text{with } \kappa_k^N = \sqrt{\gamma_k^N - \kappa^2}$$
$$\tilde{S}_k^N(\alpha) = \frac{1}{2\pi} \frac{1}{(\gamma_k^N - \alpha^2)^2 + 4\kappa^2\alpha^2}. \tag{20}$$

Similar relations hold for the moment functions of the derivative ${}^o u_t$

Comparing Eqns. (20) with (18) it can be seen that the goodness of approximation of $R_{^o u ^o u}$ and $S_{^o u ^o u}$ by $R^N_{^o u ^o u}$ and $S^N_{^o u ^o u}$ depends on the approximation of the eigenvalues γ_k by γ_k^N and the quantities κ_k by κ_k^N. The following table gives some values of these quantities obtained for the parameter settings given in table 1. The plots of the finite dimensional approximations of the

k	γ_k	γ_k^N	κ_k	κ_k^N
1	9.741	9.717	2.957	2.952
2	155.854	154.344	12.444	12.383
3	789.014	771.904	28.072	27.765
4	2493.672	2398.269	49.927	48.962

Table 2: Approximation of γ_k and κ_k by its finite-dimensional counterparts

moment functions $R^N_{^o u ^o u}(\tau, x, x)$ and $S^N_{^o u ^o u}(\alpha, x, x)$ show no visible differences to the corresponding plots in Figure 4 and are therefore not presented.

Conclusion

Transverse vibrations of a thin homogeneous beam subjected to a random excitation modeled by a stationary ε–correlated random function were been investigated. Semi-discretization of the governing PDE leads to a (usually large-scale) system of ODEs with random inhomogeneous terms. Correlation function and spectral density of the stationary response of the continuous vibration system can be found from the corresponding moment functions of the stationary solution of this system of ODEs. Model reduction techniques are applied to find approximations of moment functions for the large-scale system by computing them for a suitable chosen low-dimensional system of ODEs.

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