Low-dimensional approximations for random axial vibrations of beams

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Abstract

The paper considers the computation of second-order moment functions of the solution of large-scale systems of linear random ODEs. Such systems arise from the semi-discretization of PDEs describing continuous vibration systems with random excitation. Since standard methods fail because of enormous computational problems model reduction techniques are applied. We find approximations of the desired moment functions of the large-scale system by computing them for a suitable low-dimensional system. Numerical results concerning axial vibrations of thin beams are presented.

1 Introduction

This paper investigates axial vibrations of a thin beam with a random excitation at one of its ends which is described by a stationary random process. Semi-discretization of the governing partial differential equation (PDE) with respect to the spatial variables generates a (usually large-scale) system of first-order linear ordinary differential equations (ODEs)

$$\dot{\mathbf{z}}(t) = \mathbf{A}\mathbf{z}(t) + \mathbf{B}\mathbf{h}(t,\omega), \tag{1}$$

which is studied in the first part of the paper. Here, $\mathbf{z}$ is an n-dimensional vector function and the inhomogeneous term contains the excitation function $\mathbf{h}$ which is assumed to be a centered and wide-sense stationary random vector process with values in $\mathbb{R}^r$, $r, n \in \mathbb{N}$. $\mathbf{A}$ and $\mathbf{B}$ denote an $n \times n$ and an $n \times r$ matrix, respectively. For the sake of greater generality the entries of the matrices $\mathbf{A}$ and $\mathbf{B}$ are supposed to be complex-valued, thus the values of $\mathbf{z}(t)$ are in $\mathbb{C}^n$. Moreover, it is assumed that $\mathbf{h}$ is defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ where $\mathcal{A}$ denotes a σ-algebra of subsets of Ω on which a probability measure $\mathbf{P}$ is defined. The expectation with respect to $\mathbf{P}$ is denoted by $\mathbf{E}\{.\}$

System (1) also arises as the state space formulation of multidegree-of-freedom vibration systems with damping and an external excitation which are usually described by systems of N real second-order ODEs

$$\mathbf{M}_2\ddot{\mathbf{y}}(t) + \mathbf{M}_1\dot{\mathbf{y}}(t) + \mathbf{M}_0\mathbf{y}(t) = \hat{\mathbf{B}}\mathbf{h}(t,\omega).$$

Here, $N \in \mathbb{N}$ denotes the number of degrees of freedom, $\mathbf{y} = (y_1, \ldots, y_N)^\tau$, is an N-dimensional vector function containing the response variables, $\mathbf{M}_0, \mathbf{M}_1, \mathbf{M}_2$ are some $N \times N$ matrices and $\hat{\mathbf{B}}$ is an $N \times r$ matrix. The transformation of the above system into a first-order system for the state vector function $\mathbf{z} = \binom{\dot{\mathbf{y}}}{\mathbf{y}}$ results in a system of the form (1) of dimension $n = 2N$.

The main tasks are the determination of so-called stationary solutions $\mathbf{z}$ and the computation of their distribution characteristics, in particular second-order moment functions. These questions arise in the investigation of the long-time behaviour of the response of systems described by Equation (1) to permanently acting random excitations.

If the dimension n becomes very large which is the typical case in the above mentioned semi-discretization of PDEs then procedures for the computation of moment functions of $\mathbf{z}$ become intractable because of enormous computational problems. To this end, in Section 3 model reduction techniques are investigated which provide approximations of the desired moment functions of the large-scale system by solving a suitable low-dimensional system.

In Section 4 the introduced approximation techniques are applied to the case where system (1) results from the semi-discretization of a boundary value problem for a PDE describing random axial vibrations of a thin beam. Some numerical results are presented.

2 Linear systems with stationary excitations

Consider the linear system (1) with a wide-sense stationary random excitation function $\mathbf{h}$. For the sake of shorter notation the term "wide-sense" is supressed throughout this paper. A random function $\mathbf{z}(t,\omega)$ satisfying (1) is called stationary solution of (1) if $\mathbf{z}$ and $\mathbf{h}$ are stationarily related, i.e. $\binom{\mathbf{z}}{\mathbf{h}}$ is a stationary process.

Lemma 1 (Bunke [2], Arnold, Wihstutz [1])
Assuming a centered stationary and a.s. continuous excitation function $\mathbf{h}$ *and a stable matrix* $\mathbf{A}$, *i.e. all eigenvalues of* $\mathbf{A}$ *possess negative real parts, then the linear first-order system (1) possesses the unique stationary solution*

$$\mathbf{z}(t,\omega) = \int_{-\infty}^{t} e^{\mathbf{A}(t-s)}\mathbf{B}\mathbf{h}(s,\omega)ds = \int_{0}^{\infty} e^{\mathbf{A}s}\mathbf{B}\mathbf{h}(t-s,\omega)ds. \qquad (2)$$

The representation (2) can be used for the probabilistic analysis of the stationary solution $\mathbf{z}(t,\omega)$ and the computation of its moment functions. Since $\mathbf{h}$ is assumed to be centered for the first-order moments it follows $\mathbf{E}\{\mathbf{z}(t)\} = 0$.

For the computation of second-order moments several procedures are known in the literature (see Newland [9], Preumont [10], Soong, Grigoriu [16], vom Scheidt, Fellenberg, Wöhrl [12]). A first group is based on time domain analysis. Here, several averages of the representation (2) are used. For the correlation function of $\mathbf{z}$ it follows

$$\begin{aligned}\mathbf{R}_{\mathbf{zz}}(\tau) &= \mathbf{E}\{\mathbf{z}(t)\mathbf{z}^*(t+\tau)\} \\ &= \int_0^\infty \int_0^\infty \mathrm{e}^{\mathbf{A}s_1}\mathbf{B}\mathbf{E}\{\mathbf{h}(t-s_1)\mathbf{h}^*(t+\tau-s_2)\}\mathbf{B}^*\mathrm{e}^{\mathbf{A}^*s_2}ds_1ds_2 \\ &= \int_0^\infty \int_0^\infty \mathrm{e}^{\mathbf{A}s_1}\mathbf{B}\mathbf{R}_{\mathbf{hh}}(\tau+s_1-s_2)\mathbf{B}^*\mathrm{e}^{\mathbf{A}^*s_2}ds_1ds_2.\end{aligned}$$

Other procedures are based on frequency domain analysis. Here, the spectral density of $\mathbf{z}$ which is defined by

$$\mathbf{S}_{\mathbf{zz}}(\alpha) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\mathrm{i}\alpha\tau}\mathbf{R}_{\mathbf{zz}}(\tau)\,d\tau$$

is related to the spectral density of $\mathbf{h}$.

Lemma 2 (Soong, Grigoriu [16], vom Scheidt, Fellenberg, Wöhrl [12])
Let the assumptions of Lemma 1 be fulfilled and assume that the excitation function $\mathbf{h}$ *possesses a spectral density denoted by* $\mathbf{S}_{\mathbf{hh}}(\alpha)$, $\alpha \in \mathbb{R}$. *Then, for the spectral density of* $\mathbf{z}$ *it holds for* $\alpha \in \mathbb{R}$

$$\mathbf{S}_{\mathbf{zz}}(\alpha) = \mathbf{H}(\alpha)\mathbf{S}_{\mathbf{hh}}(\alpha)\mathbf{H}^*(\alpha) \quad \textit{where} \quad \mathbf{H}(\alpha) = -(\mathrm{i}\alpha\mathbf{I}+\mathbf{A})^{-1}\mathbf{B}.$$

The matrix function $\mathbf{H}(\alpha)$ is known as the frequency response function or transition function.

3 Model reduction

For formulating the task of model reduction the first-order system (1) is rewritten in a slightly modified form known from control theory (see e.g. Knobloch, Kwakernaak [5], Zhou, Doyle [19])

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{h}, \qquad \mathbf{z} = \mathbf{T}\mathbf{y}, \tag{3}$$

with $\mathbf{T} = \mathbf{I}_n$, i.e. $\mathbf{z} \equiv \mathbf{y}$. Here, the first equation is called state equation while the second equation is the output equation. Further, $\mathbf{h}$, $\mathbf{y}$ and $\mathbf{z}$ are called input, state and output, respectively. An output $\mathbf{z}$ is called stationary output if $\mathbf{z}$ and $\mathbf{h}$ are stationarily related. The system (3) will be referred to as the original system. To find an approximation of the stationary output $\mathbf{z}$ one tries to find a system possessing a stable state equation with a dimension $m < n$, i.e.

$$\dot{\mathbf{y}}_m = \mathbf{A}_m \, \mathbf{y}_m + \mathbf{B}_m \, \mathbf{h}, \qquad \tilde{\mathbf{z}}^{(m)} = \mathbf{T}_m \, \mathbf{y}_m, \tag{4}$$

with an $m \times m$ matrix $\mathbf{A}_m$, an $m \times r$ matrix $\mathbf{B}_m$, an $n \times m$ matrix $\mathbf{T}_m$ and an m-dimensional vector function $\mathbf{y}_m : \mathbb{R} \times \Omega \to \mathbb{C}^m$. The above system is called reduced-order system to the original system (3) and its n-dimensional stationary output $\tilde{\mathbf{z}}^{(m)}$ is called the reduced-order approximation of $\mathbf{z}$.

There exists a great number of papers concerning the model reduction problem in the context of control theory where $\mathbf{h}$ is considered as input or control function and $\mathbf{z}$ is the output function which has to be controlled. In some papers the input function is allowed to be random - in most cases multi-dimensional white noise - see e.g. Hyland, Bernstein [4], Moore [7], Wilson [17].

In the present paper the construction of a suitable reduced-order system is based on the assumption that the random dynamics of the stationary output $\mathbf{z}$ can be expressed up to a certain degree of accuracy by projections of its values onto a m-dimensional subspace of $\mathbb{C}^n$. A similar approach is used in Sirovich, Knight, Rodriguez [15].

3.1 Reduced-order system

Consider the canonical basis $(\mathbf{e}_k)_{k=1,\ldots,n}$ of $\mathbb{C}^n$ and the representation of the state function $\mathbf{y}(t,\omega)$ with respect to this basis $\mathbf{y}(t,\omega) = \sum_{k=1}^{n} y_k(t,\omega)\mathbf{e}_k$. Now, let $(\mathbf{v}_k)_{k=1,\ldots,n}$ be a set of linear independent elements of $\mathbb{C}^n$ with $\|\mathbf{v}_k\| = 1$. Denote by $x_1, \ldots, x_n$ the new coordinates of $\mathbf{y} \in \mathbb{C}^n$ with respect to the basis $(\mathbf{v}_k)_{k=1,\ldots,n}$, i.e.

$$\mathbf{y}(t,\omega) = \sum_{k=1}^{n} x_k(t,\omega)\mathbf{v}_k. \tag{5}$$

Defining the $\mathbb{C}^n$-valued vector function $\mathbf{x}(t,\omega) = (x_1(t,\omega), \ldots, x_n(t,\omega))^\tau$ and the matrix $\mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ which is non-singular because of the linear independence of $\mathbf{v}_1, \ldots, \mathbf{v}_n$, the transformation (5) and its inverse transformation can be written as

$$\mathbf{y}(t,\omega) = \mathbf{V}\mathbf{x}(t,\omega) \qquad \text{and} \qquad \mathbf{x}(t,\omega) = \mathbf{V}^{-1}\mathbf{y}(t,\omega).$$

Next, sequences $(\mathcal{S}_m)_{m=1,\ldots,n}$ of linear subspaces of $\mathbb{C}^n$ are considered, where $\mathcal{S}_m$ is defined by

$$\mathcal{S}_m \;:=\; \mathrm{span}(\mathbf{v}_1,\ldots,\mathbf{v}_m), \quad m=1,\ldots,n.$$

To each sequence $(\mathcal{S}_m)_{m=1,\ldots,n}$ corresponds a sequence $(\mathcal{P}_m)_{m=1,\ldots,n}$ of projectors, where $\mathcal{P}_m : \mathbb{C}^n \to \mathcal{S}_m$ is defined by

$$\mathcal{P}_m\mathbf{y} \;:=\; \sum_{k=1}^{m} x_k\mathbf{v}_k, \quad m=1,\ldots,n, \tag{6}$$

as well as a sequence $(\mathcal{Q}_m)_{m=1,\ldots,n}$ of complementary projectors where $\mathcal{Q}_m : \mathbb{C}^n \to \mathrm{span}(\mathbf{v}_{m+1},\ldots,\mathbf{v}_n)$ is defined by

$$\mathcal{Q}_m\mathbf{y} \;:=\; \mathbf{y}-\mathcal{P}_m\mathbf{y}, \quad m=1,\ldots,n. \tag{7}$$

Defining for $m=1,\ldots,n-1$ the following subvectors of $\mathbf{x}$

$$\mathbf{x}_m := (x_1,\ldots,x_m)^\tau \quad\text{and}\quad \mathbf{x}_m^{\mathbf{c}} := (x_{m+1},\ldots,x_n)^\tau$$

and the submatrices of $\mathbf{V}$

$$\mathbf{V}_m := (\mathbf{v}_1,\ldots,\mathbf{v}_m) \quad\text{and}\quad \mathbf{V}_m^{\mathbf{c}} := (\mathbf{v}_{m+1},\ldots,\mathbf{v}_n)$$

the relations (6) and (7) for $m=1,\ldots,n-1$ can be written as

$$\mathcal{P}_m\mathbf{y}(t,\omega) = \mathbf{V}_m\,\mathbf{x}_m(t,\omega) \quad\text{and}\quad \mathcal{Q}_m\mathbf{y}(t,\omega) = \mathbf{V}_m^{\mathbf{c}}\,\mathbf{x}_m^{\mathbf{c}}(t,\omega).$$

A reduced-order system can be derived if an m-dimensional state equation for the m coordinate functions defining the projection $\mathcal{P}_m\mathbf{y}$ is considered. To this end, the original system (3) is written in terms of the transformed coordinates using the transformation $\mathbf{y}=\mathbf{V}\mathbf{x}$. Here, $\mathbf{x}=(x_1,\ldots,x_n)^\tau$ contains the coordinates of $\mathbf{y}$ with respect to the basis $(\mathbf{v}_k)_{k=1,\ldots,n}$. Then (3) reads as

$$\dot{\mathbf{x}} = \mathbf{C}\mathbf{x}+\mathbf{D}\mathbf{h}, \qquad \mathbf{z}=\mathbf{V}\mathbf{x}, \tag{8}$$

with $\mathbf{C} := \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ and $\mathbf{D} := \mathbf{V}^{-1}\mathbf{B}$ and is referred to as the transformed original system. Defining

$$\mathbf{C}_m := \{C_{ij}\}_{i,j=1,\ldots,m},\ \mathbf{C}_m^{\mathbf{c}} := \{C_{ij}\}_{\substack{i=1,\ldots,m\\ j=m+1,\ldots,n}},\ \mathbf{D}_m := \{D_{ij}\}_{\substack{i=1,\ldots,m\\ j=1,\ldots,r}}, \tag{9}$$

$$\text{i.e.} \qquad \mathbf{C} = \begin{pmatrix} \mathbf{C}_m & \mathbf{C}_m^{\mathbf{c}} \\ \cdots & \cdots \end{pmatrix} \quad\text{and}\quad \mathbf{D} = \begin{pmatrix} \mathbf{D}_m \\ \cdots \end{pmatrix},$$

for $m = 1, \ldots, n-1$, the truncated system using only the first m state equations of (8) reads as

$$\dot{\mathbf{x}}_m = \mathbf{C}_m \mathbf{x}_m + \mathbf{C}_m^{\mathrm{c}} \mathbf{x}_m^{\mathrm{c}} + \mathbf{D}_m \mathbf{h}, \qquad \mathcal{P}_m \mathbf{z} = \mathbf{V}_m \mathbf{x}_m. \tag{10}$$

The output of the above system is the projection of $\mathbf{z}$ onto the subspace $\mathcal{S}_m = \mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$. Provided the matrix $\mathbf{C}_m$ is stable, the state equations of (10) form a stable first-order system for $\mathbf{x}_m$ with the random excitation term $\mathbf{C}_m^{\mathrm{c}} \mathbf{x}_m^{\mathrm{c}} + \mathbf{D}_m \mathbf{h}$.

Since $\mathbf{z}$ and $\mathbf{h}$ and consequently $\mathbf{x}$ and $\mathbf{h}$ are centered and stationarily related it follows that the excitation term $\mathbf{C}_m^{\mathrm{c}} \mathbf{x}_m^{\mathrm{c}} + \mathbf{D}_m \mathbf{h}$ is stationary and centered. Then, the state equation of (10) possesses a centered stationary solution which can be represented as

$$\begin{aligned} \mathbf{x}_m(t,\omega) &= \int_0^\infty \mathrm{e}^{\mathbf{C}_m s} \left(\mathbf{D}_m \mathbf{h}(t-s,\omega) + \mathbf{C}_m^{\mathrm{c}} \mathbf{x}_m^{\mathrm{c}}(t-s,\omega) \right) ds \\ &= \mathbf{y}_m(t,\omega) + \hat{\mathbf{y}}_m(t,\omega) \end{aligned}$$

$$\text{with} \qquad \mathbf{y}_m(t,\omega) := \int_0^\infty \mathrm{e}^{\mathbf{C}_m s} \, \mathbf{D}_m \mathbf{h}(t-s,\omega) ds \tag{11}$$

$$\hat{\mathbf{y}}_m(t,\omega) := \int_0^\infty \mathrm{e}^{\mathbf{C}_m s} \, \mathbf{C}_m^{\mathrm{c}} \mathbf{x}_m^{\mathrm{c}}(t-s,\omega) ds. \tag{12}$$

The truncated system (10) possesses not the desired form (4) of a reduced-order system since its excitation term contains $\mathbf{C}_m^{\mathrm{c}} \mathbf{x}_m^{\mathrm{c}}$. The stationary output of system (10) and its moment functions cannot be determined exactly without the knowledge of $x_{m+1}, \ldots, x_n$ which requires the knowledge of the stationary solution of the large-scale state equation of the transformed original system (8).

An exception to this situation is the case of decoupled state equations of system (8). This case results if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are the eigenvectors of the diagonalizable matrix $\mathbf{A}$. Then, the matrix $\mathbf{C} := \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ is diagonal which implies $\mathbf{C}_m^{\mathrm{c}} = \mathbf{0}_{m,n-m}$ and $\mathbf{C}_m^{\mathrm{c}} \mathbf{x}_m^{\mathrm{c}} = \mathbf{0}_m$. In Subsection 3.2.2 this exceptional case is described more detailed.

To overcome the mentioned drawback of the non-vanishing term $\mathbf{C}_m^{\mathrm{c}} \mathbf{x}_m^{\mathrm{c}}$ a second approximation step is necessary. It consists in the approximation of the variables $x_{m+1}, \ldots, x_n$ by their zero means and generates an additional error. Then, $\tilde{\mathbf{z}}^{(m)} := \mathbf{V}_m \mathbf{y}_m$ instead of the projection $\mathcal{P}_m \mathbf{z} = \mathbf{V}_m \mathbf{x}_m$ is used as approximation of $\mathbf{z}$. Since the m-dimensional vector function $\mathbf{y}_m$ given in (11) is the stationary solution of $\dot{\mathbf{y}}_m = \mathbf{C}_m \mathbf{y}_m + \mathbf{D}_m \mathbf{h}$ if $\mathbf{C}_m$ is a stable matrix it

follows, that $\tilde{\mathbf{z}}^{(m)}$ is the stationary output of the system

$$\dot{\mathbf{y}}_m = \mathbf{C}_m \mathbf{y}_m + \mathbf{D}_m \mathbf{h}, \qquad \tilde{\mathbf{z}}^{(m)} = \mathbf{V}_m \mathbf{y}_m. \tag{13}$$

The above system may serve as the desired reduced-order system (4) if the settings $\mathbf{A}_m = \mathbf{C}_m$, $\mathbf{B}_m = \mathbf{D}_m$ and $\mathbf{T}_m = \mathbf{V}_m$ are used. If $\mathbf{C}_m$ is stable then system (13) possesses the same qualitative properties as the original system. Both systems are stable and possess stationary excitations. That means, procedures for the computation of moment functions of the original system can be applied to the reduced-order system (13), too.

The relation $\tilde{\mathbf{z}}^{(m)} = \mathbf{V}_m \mathbf{y}_m$ implies that the correlation function $\mathbf{R}_{\tilde{\mathbf{z}}^{(m)}\tilde{\mathbf{z}}^{(m)}}(\tau)$ and the spectral density $\mathbf{S}_{\tilde{\mathbf{z}}^{(m)}\tilde{\mathbf{z}}^{(m)}}(\alpha)$ which are used as approximations of the corresponding moment functions of $\mathbf{z}$ can be determined from the moment functions of $\mathbf{y}_m$ using

$$\mathbf{R}_{\tilde{\mathbf{z}}^{(m)}\tilde{\mathbf{z}}^{(m)}}(\tau) = \mathbf{V}_m \mathbf{R}_{\mathbf{y}_m\mathbf{y}_m}(\tau)\mathbf{V}_m^* \text{ and } \mathbf{S}_{\tilde{\mathbf{z}}^{(m)}\tilde{\mathbf{z}}^{(m)}}(\alpha) = \mathbf{V}_m \mathbf{S}_{\mathbf{y}_m\mathbf{y}_m}(\alpha)\mathbf{V}_m^*. \tag{14}$$

Now, the pathwise approximation error for the approximation of the stationary output $\mathbf{z}(t,\omega)$ of the large-scale original system (3) by the reduced-order approximation $\tilde{\mathbf{z}}^{(m)}(t,\omega)$ which is defined by $\hat{\mathbf{z}}^{(m)} := \hat{\mathbf{z}}^{(m)}(t,\omega) = \mathbf{z}(t,\omega) - \tilde{\mathbf{z}}^{(m)}(t,\omega)$ is considered. It holds

$$\begin{aligned} \hat{\mathbf{z}}^{(m)} &= \mathbf{z} - \tilde{\mathbf{z}}^{(m)} = (\mathcal{P}_m \mathbf{z} + \mathcal{Q}_m \mathbf{z}) - \tilde{\mathbf{z}}^{(m)} \\ &= (\mathbf{V}_m \mathbf{x}_m + \mathbf{V}_m^{\mathbf{c}} \mathbf{x}_m^{\mathbf{c}}) - \mathbf{V}_m \mathbf{y}_m = \mathbf{V}_m^{\mathbf{c}} \mathbf{x}_m^{\mathbf{c}} + \mathbf{V}_m(\mathbf{x}_m - \mathbf{y}_m) \\ &= \mathbf{V}_m^{\mathbf{c}} \mathbf{x}_m^{\mathbf{c}} + \mathbf{V}_m \hat{\mathbf{y}}_m = {}^1\hat{\mathbf{z}}^{(m)} + {}^2\hat{\mathbf{z}}^{(m)} \end{aligned} \tag{15}$$

with ${}^1\hat{\mathbf{z}}^{(m)} := \mathbf{V}_m^{\mathbf{c}} \mathbf{x}_m^{\mathbf{c}} = \mathcal{Q}_m \mathbf{z}$ and ${}^2\hat{\mathbf{z}}^{(m)} := \mathbf{V}_m \hat{\mathbf{y}}_m$.

Here, ${}^1\hat{\mathbf{z}}^{(m)}$ results from the first approximation step - the projection of $\mathbf{z}$ onto $\mathcal{S}_m$ - and is called projection error. The second approximation step - the replacement of $x_{m+1}, \ldots, x_n$ by their zero means - leads to the error term ${}^2\hat{\mathbf{z}}^{(m)}$ which is called defect.

From the relation $\hat{\mathbf{z}}^{(m)} = \mathbf{z} - \tilde{\mathbf{z}}^{(m)} = \mathbf{z} - \mathbf{V}_m\ \mathbf{y}_m$ it can be deduced that the pathwise approximation error is the output of the following augmented system with an $n+m$-dimensional state equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \mathbf{z} \\ \mathbf{y}_m \end{pmatrix} &= \begin{pmatrix} \mathbf{A} & \mathbf{O}_{n,m} \\ \mathbf{O}_{m,n} & \mathbf{C}_m \end{pmatrix} \begin{pmatrix} \mathbf{z} \\ \mathbf{y}_m \end{pmatrix} + \begin{pmatrix} \mathbf{B} \\ \mathbf{D}_m \end{pmatrix} \mathbf{h} \\ \hat{\mathbf{z}}^{(m)} &= (\mathbf{I}_n,\ -\mathbf{V}_m) \begin{pmatrix} \mathbf{z} \\ \mathbf{y}_m \end{pmatrix}. \end{aligned} \tag{16}$$

The system matrix of the state equation of the above system is stable since $\mathbf{A}$ and $\mathbf{C}_m$ are assumed to be stable. Consequently, it exists a centered stationary

output of (16) and the moment functions of the output $\hat{\mathbf{z}}^{(m)}$ can be determined using the same procedures as in case of the n-dimensional original system (3).

To assess the goodness of the approximation of the moment functions of $\mathbf{z}$ by those of the reduced-order approximations $\tilde{\mathbf{z}}^{(m)}$ the following error criterions are defined.

Definition 3
The quantity

$$\Delta_0(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}) := \mathbf{E}\left\{\left\|\mathbf{z} - \tilde{\mathbf{z}}^{(m)}\right\|^2\right\} = \mathbf{E}\left\{\left\|\hat{\mathbf{z}}^{(m)}\right\|^2\right\} = \sum_{i=1}^{n} \mathbf{E}\left\{|\hat{z}_i^{(m)}|^2\right\}$$

is called mean squared Euclidean distance *of* $\mathbf{z}$ *and* $\tilde{\mathbf{z}}^{(m)}$.

From the stationarity of $\mathbf{z}$ and $\tilde{\mathbf{z}}^{(m)}$ it follows that $\Delta_0(\mathbf{z}, \tilde{\mathbf{z}}^{(m)})$ does not depend on t. It can be seen that Δ_0 is related to the second-order moments of $\hat{z}_i^{(m)} = z_i - \tilde{z}_i^{(m)}$, $i = 1, \ldots, n$, i.e. to the values of the functions $R_{\hat{z}_i^{(m)}\hat{z}_i^{(m)}}(\tau)$ for $\tau = 0$, only. In the present case of model reduction approximations for systems of random differential equations not only these moments of the stationary solution but also (cross-) correlation functions $R_{z_i z_j}(\tau)$ for all $\tau \in \mathbb{R}$ and (cross-) spectral densities $S_{z_i z_j}(\alpha)$ for all $\alpha \in \mathbb{R}$ have to be approximated for $i, j = 1, \ldots, n$. Therefore, further error criterions with respect to the approximation of these moment functions are introduced. They are based on $\mathbf{L}_1$-distances of $R_{z_i z_j}(.)$ and $R_{\tilde{z}_i^{(m)}\tilde{z}_j^{(m)}}(.)$ as well as of $S_{z_i z_j}(.)$ and $S_{\tilde{z}_i^{(m)}\tilde{z}_j^{(m)}}(.)$ which are summed up for $i, j = 1, \ldots, n$.

Definition 4
Let $\mathcal{W} = \left\{\mathbf{\Psi} = \{\Psi_{ij}\}_{i,j=1}^{n} : \Psi_{ij} \in \mathbb{R}_+, i, j = 1, \ldots, n, \sum_{i,j=1}^{n} \Psi_{ij} = 1\right\}$ *be the set of all* $n \times n$ *matrices with non-negative entries whose sum is equal to one. The matrices* $\mathbf{\Psi} \in \mathcal{W}$ *are called* weighting matrices.
Further, let $\Theta = \left\{\theta(.) : \mathbb{R} \to \mathbb{R}_+, \int_{\mathbb{R}} \theta(\tau) d\tau = 1\right\}$ *be a set of non-negative scalar functions. The functions* $\theta \in \Theta$ *are called* weighting functions.

Definition 5
Let $\mathbf{\Psi} \in \mathcal{W}$ *and* $\theta \in \Theta$ *be a weighting matrix and a weighting function, respectively. The quantities*

$$\Delta^{\mathbf{R}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}) := \int_{\mathbb{R}} \sum_{i,j=1}^{n} \left|\mathbf{R}_{z_i z_j}(\tau) - \mathbf{R}_{\tilde{z}_i^{(m)}\tilde{z}_j^{(m)}}(\tau)\right| \Psi_{ij}\, \theta(\tau) d\tau$$

and

$$\Delta^{\mathbf{S}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}) := \int_{\mathbb{R}} \sum_{i,j=1}^{n} \left|\mathbf{S}_{z_i z_j}(\alpha) - \mathbf{S}_{\tilde{z}_i^{(m)}\tilde{z}_j^{(m)}}(\alpha)\right| \Psi_{ij} d\alpha$$

are called error criterions with respect to the correlation function and spectral density *for an approximation of* $\mathbf{z}$ *by* $\tilde{\mathbf{z}}^{(m)}$, *respectively.*

In [14, 18] the estimates for $m = 1, \ldots, n$

$$\Delta^{\mathbf{R},\mathbf{S}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}) \leq \overline{\Delta}_m := \sqrt{\Delta_0(\mathbf{z}, \tilde{\mathbf{z}}^{(m)})} \left(2\sqrt{\mathbf{E}\left\{ \|\tilde{\mathbf{z}}^{(m)}\|^2 \right\}} + \sqrt{\Delta_0(\mathbf{z}, \tilde{\mathbf{z}}^{(m)})} \right)$$

of the criterions $\Delta^{\mathbf{R}}$ and $\Delta^{\mathbf{S}}$ which are shortly denoted by $\Delta^{\mathbf{R},\mathbf{S}}$ by upper bounds $\overline{\Delta}_m$ are given. It can be seen, that the magnitude of the upper bounds depends essentially on the mean-squared Euclidean distance $\Delta_0(\mathbf{z}, \tilde{\mathbf{z}}^{(m)})$ If there is freedom in the selection of basis elements $\mathbf{v}_1, \ldots, \mathbf{v}_n$ one should prefer those elements which lead to minimal (or at least small) values of $\Delta_0(\mathbf{z}, \tilde{\mathbf{z}}^{(m)})$ and therefore provide small values of $\overline{\Delta}_m$. Results of this optimization procedure lead to special reduced-order systems and are discussed in the next subsection.

3.2 Special reduced-order systems

As announced in the preceding subsection now reduced-order systems are considered which provide small values of the upper bounds for the error criterions $\Delta^{\mathbf{R}}$ and $\Delta^{\mathbf{S}}$. To this end, the basis elements $\mathbf{v}_1, \ldots, \mathbf{v}_n$ should be chosen such that $\Delta_0\left(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}\right) = \mathbf{E}\left\{ \left\|\hat{\mathbf{z}}^{(m)}\right\|^2 \right\}$ is minimal or at least small. From the representation of the pathwise approximation error $\hat{\mathbf{z}}^{(m)}$ as the output of the augmented system (16) it can be seen that the basis elements $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are involved in the output equation as well as in the state equation since $\mathbf{C}_m$ and $\mathbf{D}_m$ are submatrices of $\mathbf{C} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}$ and $\mathbf{D} = \mathbf{V}^{-1}\mathbf{B}$, respectively. This implies a rather complicated dependence of the objective $\Delta_0\left(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}\right)$ on $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and a general solution of the mentioned minimization problem seems to be not available.

To find applicable guidelines for the selection of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ the decomposition (see Equation (15)) $\hat{\mathbf{z}}^{(m)} = {}^1\hat{\mathbf{z}}^{(m)} + {}^2\hat{\mathbf{z}}^{(m)}$ into the projection error ${}^1\hat{\mathbf{z}}^{(m)} = \mathbf{V}_m^{\mathbf{c}}\mathbf{x}_m^{\mathbf{c}} = \mathcal{Q}_m\mathbf{z}$ and the defect ${}^2\hat{\mathbf{z}}^{(m)} = \mathbf{V}_m\hat{\mathbf{y}}_m$ is considered. Now, the minimization of $\mathbf{E}\left\{ \left\|{}^j\hat{\mathbf{z}}^{(m)}\right\|^2 \right\}$, $j = 1, 2$, subject to $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is considered separately. The results are presented below.

3.2.1 Model reduction based on mean-square optimal projections

From [14], [18] it is known that

$$\mathbf{E}\left\{ \left\|{}^1\hat{\mathbf{z}}^{(m)}\right\|^2 \right\} = \mathbf{E}\left\{ \|\mathcal{Q}_m\mathbf{z}\|^2 \right\} = \Delta_0\left(\mathbf{z}, \mathcal{P}_m\mathbf{z}\right)$$

can be minimized if the basis elements $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are chosen as the eigenvectors of the covariance matrix $\mathbf{R}_{\mathbf{zz}}(0)$ of $\mathbf{z}$ which belong to the eigenvalues $\sigma_1^2 \geq \ldots \geq \sigma_n^2$ of $\mathbf{R}_{\mathbf{zz}}(0)$. The minimal value of the objective is $\sum_{k=m+1}^{n} \sigma_k^2$. The resulting projections are referred to as mean-square optimal projections. With the selected basis elements the reduced-order system can be derived by executing the following algorithm.

Algorithm 6

Step 1: Find the covariance matrix $\mathbf{R}_{\mathbf{zz}}(0)$ of the output of system (3).

Step 2: Determine the eigenvalues $\sigma_1^2 \geq \ldots \geq \sigma_n^2$ and corresponding orthonormal eigenvectors $\check{\mathbf{v}}_1, \ldots, \check{\mathbf{v}}_n$ of $\mathbf{R}_{\mathbf{zz}}(0)$.

Step 3: Choose $m \in \{1, \ldots, n-1\}$.

Step 4: Set $\mathbf{V}_m = (\check{\mathbf{v}}_1, \ldots, \check{\mathbf{v}}_m)$, $\mathbf{C}_m = \mathbf{V}_m^* \mathbf{A} \mathbf{V}_m$, $\mathbf{D}_m = \mathbf{V}_m^* \mathbf{B}$.

Step 5: If $\mathbf{C}_m$ is stable then Step 6 else Step 3.

Step 6: The reduced-order system is $\dot{\mathbf{y}}_m = \mathbf{C}_m \mathbf{y}_m + \mathbf{D}_m \mathbf{h}, \quad \tilde{\mathbf{z}}^{(m)} = \mathbf{V}_m \mathbf{y}_m$.

The moment functions of $\tilde{\mathbf{z}}^{(m)}$ can be derived from the corresponding moment functions of the m-dimensional stationary solution $\mathbf{y}_m$ of the state equation of the reduced-order system in Step 6 applying the transformation (14). For the error criterions with respect to the approximation of the moment functions the following estimate can be obtained.

Theorem 7 *([14])*
For the approximation of the moment functions of the stationary output $\mathbf{z}$ *of the original system (3) by the moment functions of the stationary output* $\tilde{\mathbf{z}}^{(m)}$ *of the reduced-order system derived in Algorithm 6 it holds*

$$\Delta^{\mathbf{R},\mathbf{S}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}) \leq \overline{\Delta}_m := \sqrt{\Delta_0(\mathbf{z}, \tilde{\mathbf{z}}^{(m)})} \left(2\sqrt{\mathbf{E}\left\{ \|\tilde{\mathbf{z}}^{(m)}\|^2 \right\}} + \sqrt{\Delta_0(\mathbf{z}, \tilde{\mathbf{z}}^{(m)})} \right)$$

where

$$\begin{aligned} \mathbf{E}\left\{ \left\|\tilde{\mathbf{z}}^{(m)}\right\|^2 \right\} &= \sum_{k=1}^{m} \mathbf{E}\left\{ |y_{m,k}|^2 \right\} = \mathbf{E}\left\{ \|\mathbf{y}_m\|^2 \right\} \\ \Delta_0(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}) &= \sum_{k=1}^{n} \sigma_k^2 + \mathbf{E}\left\{ \left\|\tilde{\mathbf{z}}^{(m)}\right\|^2 \right\} - 2\,\mathrm{Re}\left[\sum_{i=1}^{n} \sum_{k=1}^{m} \mathbf{E}\left\{ \overline{z_i}\, y_{m,k} \right\} V_{ik} \right]. \end{aligned}$$

Theorem 7 shows that the evaluation of the upper bounds $\overline{\Delta}_m$ requires the knowledge of the second-order moments $\mathbf{E}\{|y_{m,k}|^2\}$, $k = 1, \ldots, m$, of the eigenvalues σ_k^2, $k = m+1, \ldots, n$ of the covariance matrix $\mathbf{R}_{\mathbf{zz}}(0)$ and of the entries of the covariance matrix $\mathbf{E}\{\mathbf{y}_m \mathbf{z}^*\}$. While σ_k^2 and $\mathbf{E}\{|y_{m,k}|^2\}$ are by-products of the construction of the reduced-order system and the computation of moment functions for that system an additional effort is necessary for the computation of $\mathbf{E}\{\mathbf{y}_m \mathbf{z}^*\}$. As it has been pointed out in Subsection 3.1 the practical computation of $\mathbf{E}\{\mathbf{y}_m \mathbf{z}^*\}$ can be performed using the $n+m$-dimensional augmented system (16).

3.2.2 Model reduction based on modal decomposition

Now basis elements $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are searched such that to given m the mean squared norm $\mathbf{E}\left\{\left\|{}^2\hat{\mathbf{z}}^{(m)}\right\|^2\right\}$ of the defect ${}^2\hat{\mathbf{z}}^{(m)}$ is minimized. To this end, the eigenvalues $\lambda_1, \ldots, \lambda_n$ and the corresponding eigenvectors $\overset{\circ}{\mathbf{v}}_1, \ldots, \overset{\circ}{\mathbf{v}}_n$ with $\left\|\overset{\circ}{\mathbf{v}}_k\right\| = 1$, $k = 1, \ldots, n$, of the matrix $\mathbf{A}$ which has been assumed to be stable and diagonalizable are considered. The stability of $\mathbf{A}$ implies $\mathrm{Re}[\lambda_i] < 0$, $i = 1, \ldots, n$. In general, the eigenvectors $\overset{\circ}{\mathbf{v}}_1, \ldots, \overset{\circ}{\mathbf{v}}_n$ are not orthogonal since $\mathbf{A}$ needs not to be a Hermitian matrix.

For basis elements $\mathbf{v}_k = \overset{\circ}{\mathbf{v}}_k$, $k = 1, \ldots, n$, the matrices involved in the transformed original system (8)

$$\dot{\mathbf{x}} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{h}, \qquad \mathbf{z} = \mathbf{V}\mathbf{x}$$

are defined by $\mathbf{V} = \left(\overset{\circ}{\mathbf{v}}_1, \ldots, \overset{\circ}{\mathbf{v}}_n\right)$, $\mathbf{C} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V} = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n) =: \mathbf{\Lambda}$ and $\mathbf{D} = \mathbf{V}^{-1}\mathbf{B}$. The transformation $\mathbf{z} = \mathbf{V}\mathbf{x}$ is referred to as modal decomposition since the state equation of the above system consists of n decoupled equations

$$\dot{x}_i = \lambda_i x_i + \sum_{j=1}^{r} D_{ij} h_j, \quad i = 1, \ldots, n.$$

The matrix $\mathbf{C}_m = \{C_{ij}\}_{i,j=1,\ldots,m}$ which is the system matrix of the reduced-order system is a stable and diagonal matrix, it holds $\mathbf{C}_m = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_m)$. This implies, that for all $m = 1, \ldots, n$ the state equations of the reduced-order system (13) are stable and coincide with the first m equations of the transformed original system. The latter property is a consequence of the relation

$$\mathbf{C}_m^{\mathbf{c}} = \{C_{ij}\}_{\substack{i=1,\ldots,m \\ j=m+1,\ldots,n}} = \mathbf{O}_{m,n-m}.$$

Another consequence of this relation is that for the vector functions $\hat{\mathbf{y}}_m$ defined in (12) it holds $\hat{\mathbf{y}}_m = \hat{\mathbf{y}}_m(t,\omega) \equiv \mathbf{0}_m$, i.e. $\mathbf{x}_m = \mathbf{y}_m + \hat{\mathbf{y}}_m = \mathbf{y}_m$ and $\mathcal{P}_m \mathbf{z} =$

$\tilde{\mathbf{z}}^{(m)} = \mathbf{V}_m \mathbf{x}_m$. Further, for the defect ${}^2\hat{\mathbf{z}}^{(m)}$ it follows

$$ {}^2\hat{\mathbf{z}}^{(m)}(t,\omega) = \mathbf{V}_m \hat{\mathbf{y}}_m(t,\omega) \equiv \mathbf{0}_n. $$

It results, that for basis elements $\mathbf{v}_k = \overset{\circ}{\mathbf{v}}_k$, $k = 1,\ldots,n$, the value of the objective $\mathbf{E}\left\{\left\|{}^2\hat{\mathbf{z}}^{(m)}\right\|^2\right\}$ is equal to zero and therefore minimal.

The latter property holds for every permutation of indices of the sequence of basis elements $\overset{\circ}{\mathbf{v}}_1,\ldots,\overset{\circ}{\mathbf{v}}_n$ while the value of the mean-squared Euclidean distance

$$ \Delta_0(\mathbf{z},\tilde{\mathbf{z}}^{(m)}) = \mathbf{E}\left\{\left\|\hat{\mathbf{z}}^{(m)}\right\|^2\right\} = \mathbf{E}\left\{\left\|{}^1\hat{\mathbf{z}}^{(m)} + {}^2\hat{\mathbf{z}}^{(m)}\right\|^2\right\} = \mathbf{E}\left\{\left\|{}^1\hat{\mathbf{z}}^{(m)}\right\|^2\right\} $$

depends on the arrangement of the basis elements. It remains to find an arrangement which leads to a minimal or at least to a small value of $\Delta_0(\mathbf{z},\tilde{\mathbf{z}}^{(m)})$. A natural suggestion is to arrange $\overset{\circ}{\mathbf{v}}_1,\ldots,\overset{\circ}{\mathbf{v}}_n$ such that for the variances of the stationary solutions x_k of the decoupled state equations of the transformed original system it holds $\mathbf{E}\{|x_1|^2\} \geq \ldots \geq \mathbf{E}\{|x_n|^2\}$. This arrangement ensures that an m-dimensional reduced-order system contains those m state variables of the decoupled system $\dot{\mathbf{x}} = \Lambda\mathbf{x} + \mathbf{V}^{-1}\mathbf{B}\mathbf{h}$ which possess the largest variances.

The above suggestion can be supported by the following estimate of $\Delta_0(\mathbf{z},\tilde{\mathbf{z}}^{(m)})$. Using $\hat{\mathbf{y}}_m = \mathbf{0}_m$ and applying Cauchy-Schwarz inequality it follows

$$ \begin{aligned} \Delta_0(\mathbf{z},\tilde{\mathbf{z}}^{(m)}) &= \sum_{i=1}^{n} \mathbf{E}\left\{|\hat{z}_{m,i}|^2\right\} = \sum_{k,l=m+1}^{n} \mathbf{E}\left\{x_k\overline{x_l}\right\}(\overset{\circ}{\mathbf{v}}_k,\overset{\circ}{\mathbf{v}}_l) \\ &\leq \sum_{k,l=m+1}^{n} \sqrt{\mathbf{E}\{|x_k|^2\}\,\mathbf{E}\{|x_l|^2\}} = \left(\sum_{k=m+1}^{n} \sqrt{\mathbf{E}\{|x_k|^2\}}\right)^2. \end{aligned} $$

The following algorithm summarizes the above derivation of a reduced-order system.

Algorithm 8

Step 1: Find the eigenvalues $\lambda_1,\ldots,\lambda_n$ and the corresponding eigenvectors $\overset{\circ}{\mathbf{v}}_1,\ldots,\overset{\circ}{\mathbf{v}}_n$ of the matrix $\mathbf{A}$.

Step 2: Set $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$ and $\mathbf{V} = (\overset{\circ}{\mathbf{v}}_1,\ldots,\overset{\circ}{\mathbf{v}}_n)$.

Step 3: Determine the variances $\mathbf{E}\{|x_i|^2\}$, $i = 1,\ldots,n$, of the stationary solution of the system $\dot{\mathbf{x}} = \Lambda\mathbf{x} + \mathbf{V}^{-1}\mathbf{B}\mathbf{h}$.

Step 4: Rearrange the equations of the system in Step 3 and the corresponding eigenvalues $\lambda_1,\ldots,\lambda_n$ and eigenvectors $\overset{\circ}{\mathbf{v}}_1,\ldots,\overset{\circ}{\mathbf{v}}_n$ such that it holds

$$ \mathbf{E}\left\{|x_1|^2\right\} \geq \ldots \geq \mathbf{E}\left\{|x_n|^2\right\}. $$

Step 5: Choose $m \in \{1, \ldots, n-1\}$.

Step 6: Set $\mathbf{C}_m = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_m)$, $\mathbf{V}_m = \left(\overset{\circ}{\mathbf{v}}_1, \ldots, \overset{\circ}{\mathbf{v}}_m\right)$ and $\mathbf{D}_m = \{D_{ij}\}_{\substack{i=1,\ldots,m \\ j=1,\ldots,r}}$ where $\mathbf{D} = \mathbf{V}^{-1}\mathbf{B}$.

Step 7: The reduced-order system is $\dot{\mathbf{y}}_m = \mathbf{C}_m \mathbf{y}_m + \mathbf{D}_m \mathbf{h}, \quad \tilde{\mathbf{z}}^{(m)} = \mathbf{V}_m \mathbf{y}_m$.

For the error criterions with respect to the approximation of the moment functions of $\mathbf{z}$ the following estimates can be derived.

Theorem 9 *([14])*
For the approximation of the moment functions of the stationary output $\mathbf{z}$ of the original system (3) by the moment functions of the stationary output $\tilde{\mathbf{z}}^{(m)}$ of the reduced-order system derived in Algorithm 8 it holds

$$\Delta^{\mathbf{R},\mathbf{S}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}) \leq \overline{\Delta}_m := \sqrt{\Delta_0(\mathbf{z}, \tilde{\mathbf{z}}^{(m)})} \left(2\sqrt{\mathbf{E}\left\{ \|\tilde{\mathbf{z}}^{(m)}\|^2 \right\}} + \sqrt{\Delta_0(\mathbf{z}, \tilde{\mathbf{z}}^{(m)})} \right)$$

where

$$\begin{aligned} \mathbf{E}\left\{ \left\| \tilde{\mathbf{z}}^{(m)} \right\|^2 \right\} &= \sum_{i=1}^{n} \mathbf{E}\left\{ |\tilde{z}_i^{(m)}|^2 \right\} = \sum_{k,l=1}^{m} \mathbf{E}\left\{ x_k \overline{x_l} \right\} \left(\overset{\circ}{\mathbf{v}}_k, \overset{\circ}{\mathbf{v}}_l \right) \\ \Delta_0(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}) &= \sum_{i=m+1}^{n} \mathbf{E}\left\{ |\hat{z}_i^{(m)}|^2 \right\} = \sum_{k,l=m+1}^{n} \mathbf{E}\left\{ x_k \overline{x_l} \right\} \left(\overset{\circ}{\mathbf{v}}_k, \overset{\circ}{\mathbf{v}}_l \right), \end{aligned}$$

where $\mathbf{x} = (x_1, \ldots, x_n)^\tau$ denotes the stationary solution of the decoupled system derived in Step 3 of Algorithm 8.

4 Applications

For the purpose of illustration, this section deals with systems of linear ODEs which are the result of a semi-discretization of PDEs describing continuous mechanical vibration systems subjected to random external excitations. In this approach the PDE is discretized with respect to the spatial variables, only. The dimension of the resulting system of ODEs depends on the discretization and may become very large if complex domains in two or three dimensions have to be taken into account. Here, the method is described for a simple example with a one-dimensional spatial variable. This example deals with a homogeneous PDE with random boundary condition. Such problems arise from modelling axial vibrations of a beam which is randomly excited at one of

its ends. Similar investigations for transversal vibrations of a simply supported beam subjected to a random load acting along the beams length can be found in [13].

Consider the approximative computation of moment functions of a random function $u : \mathbb{R} \times [0,1] \times \Omega \to \mathbb{R}$ denoted by $u(t,x,\omega)$ which is a solution of the following boundary value problem for a PDE

$$\begin{aligned} \rho u_{tt} + \kappa_1 u_t - \eta u_{xx} - \kappa_2 u_{txx} &= 0 \\ u(t,0,\omega) &= \xi(t,\omega) \\ u_x(t,1,\omega) &= 0. \end{aligned} \tag{17}$$

Here, the subscripts t and x denote partial derivatives with respect to t and x, respectively. Further, ρ, κ_1, κ_2 and η are positive constants and $\xi(t,\omega)$ is assumed to be a twice continuously differentiable, centered and stationary random function.

As in the context of systems of ODEs considered throughout this paper stationary solutions of (17) are of interest. In the present context stationary solutions $u(t,x,\omega)$ are defined by the property that for arbitrary $p \in \mathbb{N}$ and for all $x_1, \ldots, x_p \in [0,1]$ the vector functions $(u(t,x_1,\omega), \ldots, u(t,x_p,\omega))^\tau$ are stationary.

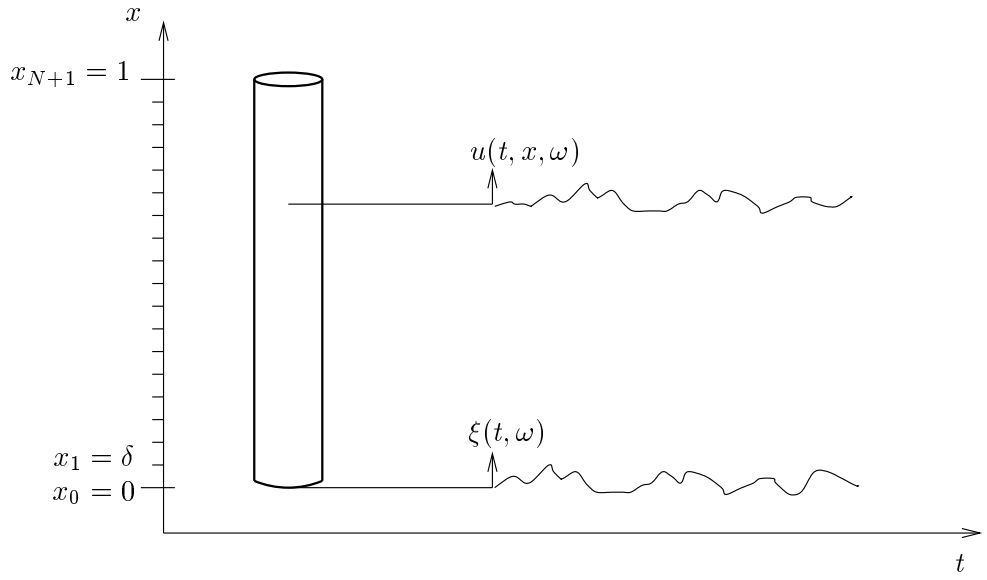


Figure 1: Axial vibrations of a beam with random excitation at the lower end

The above PDE arises from modeling axial (or longitudinal) vibrations of a thin homogeneous beam with one free end and a prescribed random displacement at the other end (see Figure 1). Here, u denotes the axial displacement, ρ the mass per unit length, η the material stiffness, κ_1 and κ_2 are viscous damping

coefficients (see Newland [8], p. 341 ff., Rao [11], p. 510 ff., and Manohar [6]). The moment functions of the stationary solutions are needed for the description of the long-term behaviour of those continuous vibration systems. This mechanical model can be used in the investigation of the transmission of the vertical vibration from the ground (as road or underground railway traffic) to a flexible building.

For the approximative computation of moment functions of $u(t,x,\omega)$ boundary value problem (17) is discretized with respect to the spatial variable x using finite difference method. In the following this procedure is briefly described. For more details see e.g. Großmann, Roos [3], p. 306 ff.

The axial displacement $u(t,x,\omega)$ is only considered at grid points $x_i = i\delta$, $i = 0,\ldots,N+1$, $\delta = \frac{1}{N+1}$. Denoting by $u_i^N(t,\omega)$ the displacement at $x = x_i$ then it holds for $i = 0,\ldots,N+1$

$$u(t,x_i,\omega) = u_i^N(t,\omega), \quad u_t(t,x_i,\omega) = \dot{u}_i^N(t,\omega), \quad u_{tt}(t,x_i,\omega) = \ddot{u}_i^N(t,\omega).$$

For the derivatives of u with respect to x the finite difference approximations

$$u_{xx}(.,x_i,.) \approx \frac{1}{\delta^2}\left(u_{i-1}^N - 2u_i^N + u_{i+1}^N\right), \; u_{txx}(.,x_i,.) \approx \frac{1}{\delta^2}\left(\dot{u}_{i-1}^N - 2\dot{u}_i^N + \dot{u}_{i+1}^N\right)$$

are used for $i = 2,\ldots,N-1$. The boundary condition $u(t,0,\omega) = u_0^N(t,\omega) = \xi(t,\omega)$ implies

$$u_{xx}(.,x_1,.) \approx \frac{1}{\delta^2}\left(\xi - 2u_1^N + u_2^N\right) \quad \text{and} \quad u_{txx}(.,x_1,.) \approx \frac{1}{\delta^2}\left(\dot{\xi} - 2\dot{u}_1^N + \dot{u}_2^N\right)$$

and from the boundary condition $u_x(t,1,\omega) = 0$ it can be deduced that it holds

$$\begin{aligned} u_{xx}(.,x_N,.) &\approx \frac{1}{\delta}\left(u_x(.,1,.) - u_x(.,x_N,.)\right) \approx \frac{1}{\delta}\left(0 - \frac{1}{\delta}\left(u_N^N - u_{N-1}^N\right)\right) \\ &= \frac{1}{\delta^2}\left(u_{N-1}^N - u_N^N\right) \end{aligned}$$

and analogously $u_{txx}(.,x_N,\omega) \approx \frac{1}{\delta^2}\left(\dot{u}_{N-1}^N - \dot{u}_N^N\right)$.

Forming the vector function $\mathbf{u}^N = \left(u_1^N,\ldots,u_N^N\right)^\tau$ it can be seen that the discretization of (17) results in the N-dimensional system of second-order ODEs for $\mathbf{u}^N$

$$\rho\ddot{\mathbf{u}}^N + \left(\kappa_1\mathbf{I}_N + \kappa_2\mathbf{K}\right)\dot{\mathbf{u}}^N + \eta\mathbf{K}\mathbf{u}^N = \widetilde{\mathbf{B}}\mathbf{h}$$

with the $N \times N$ matrix

$$\mathbf{K} = \frac{1}{\delta^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}.$$

Further, the $N \times 2$ matrix $\widetilde{\mathbf{B}}$ and the random vector function $\mathbf{h}(t,\omega)$ of dimension $r = 2$ defined by

$$\widetilde{\mathbf{B}} = \begin{pmatrix} \frac{\eta}{\delta^2} & \frac{\kappa_2}{\delta^2} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{h}(t,\omega) = \begin{pmatrix} \xi(t,\omega) \\ \dot{\xi}(t,\omega) \end{pmatrix}$$

are involved. Next, the above system of second-order ODEs can be transformed into the first-order system $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{h}$ for $\mathbf{z} = \begin{pmatrix} \dot{\mathbf{u}}^N \\ \mathbf{u}^N \end{pmatrix}$ with the $2N \times 2N$ matrix $\mathbf{A}$ and the $2N \times 2$ matrix $\mathbf{B}$ defined by

$$\mathbf{A} = \begin{pmatrix} -\frac{1}{\rho}(\kappa_1 \mathbf{I}_N + \kappa_2 \mathbf{K}) & -\frac{\eta}{\rho}\mathbf{K} \\ \mathbf{I}_N & \mathbf{O}_{N,N} \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} \frac{\eta}{\rho\delta^2} & \frac{\kappa_2}{\rho\delta^2} \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}.$$

Assuming the parameters $\rho, \eta, \kappa_1, \kappa_2$ are chosen such that $\mathbf{A}$ is a stable matrix and setting $n = 2N$ the standard form (1) of a random first-order system has been derived.

The moment functions of the stationary solution $\mathbf{z}(t,\omega)$ can be used as approximations of the moment functions of $u(t,x,\omega)$ and $u_t(t,x,\omega)$ in the grid points $x_i = i\delta$, $i = 1,\ldots,N$. To this end for $\tau, \alpha \in \mathbb{R}$ and $x, x' \in [0,1]$ the moment functions

$$\begin{aligned} R_{uu}(\tau,x,x') &:= \mathbf{E}\left\{u(t,x)u(t+\tau,x')\right\} \quad \text{and} \\ S_{uu}(\alpha,x,x') &:= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\mathrm{i}\alpha\tau} R_{uu}(\tau,x,x')d\tau \end{aligned}$$

are defined. Further, for $x = x_i = i\delta$, $x' = x_j = j\delta$, $i,j = 1,\ldots,N$ the finite-dimensional approximations

$$\begin{aligned} R^N_{uu}(\tau,x,x') &:= R_{u_i^N u_j^N}(\tau) = R_{z_{N+i}z_{N+j}}(\tau) \\ S^N_{uu}(\alpha,x,x') &:= S_{u_i^N u_j^N}(\alpha) = S_{z_{N+i}z_{N+j}}(\alpha) \end{aligned} \tag{18}$$

are introduced. The matrix-valued functions R^N and S^N for values x between the grid points are obtained by interpolation. For the velocity u_t the moment functions $R_{u_tu_t}(\tau,x,x')$ and $S_{u_tu_t}(\alpha,x,x')$ are defined analogously. The corresponding finite-dimensional approximations for $x = i\delta$, $x' = j\delta$, $i,j = 1,\ldots,N$ are

$$\begin{aligned} R^N_{u_tu_t}(\tau,x,x') &:= R_{\dot{u}_i^N \dot{u}_j^N}(\tau) = R_{z_iz_j}(\tau) \\ S^N_{u_tu_t}(\alpha,x,x') &:= S_{\dot{u}_i^N \dot{u}_j^N}(\alpha) = S_{z_iz_j}(\alpha). \end{aligned} \tag{19}$$

For $i = 0$, i.e. $x = x_0 = 0$, the relation $u(t, 0, \omega) = \xi(t, \omega)$ can be used while for $i = N + 1$, i.e. $x = x_{N+1} = 1$, the boundary condition $u_x(t, 1, \omega) = 0$ implies the approximation $u(t, 1, \omega) \approx u(t, x_N, \omega)$.

It is known that the goodness of the finite difference approximation depends on the discretization parameter $\delta > 0$ and therefore on the number of grid points. To ensure a quite accurate finite difference approximation the number N and consequently the dimension $n = 2N$ of the first-order system (1) has to be chosen sufficiently large. The result is a large-scale system (1) which has been the starting point of the described model reduction procedures.

For the subsequent numerical experiments the random boundary function $\xi(t, \omega)$ is chosen as a centered stationary random function with continuosly differentiable paths possessing a correlation function and a spectral density

$$\begin{aligned} R_{\xi\xi}(\tau) &= \sigma^2 \mathrm{e}^{-\gamma|\tau|} \left(\cos(p\tau) + \frac{\gamma}{p} \sin(p|\tau|) \right) \\ S_{\xi\xi}(\alpha) &= \frac{\sigma^2 \gamma \beta^2}{\pi} \frac{1}{\left(\alpha^2 - \beta^2\right)^2 + 4\gamma^2\alpha^2} \end{aligned}$$

where $\beta > \gamma > 0$ and $p = \sqrt{\beta^2 - \gamma^2}$. This selection ensures the mean square differentiability of ξ. The corresponding moment functions for the derivative $\dot{\xi}$ are

$$\begin{aligned} R_{\dot{\xi}\dot{\xi}}(\tau) &= -\frac{\partial^2}{\partial\tau^2} R_{\xi\xi}(\tau) = \sigma^2 \beta^2 \mathrm{e}^{-\gamma|\tau|} \left(\cos(p\tau) - \frac{\gamma}{p} \sin(p|\tau|) \right) \\ R_{\xi\dot{\xi}}(\tau) &= \frac{\partial}{\partial\tau} R_{\xi\xi}(\tau) = -\sigma^2 \frac{\beta^2}{p} \mathrm{e}^{-\gamma|\tau|} \sin(p\tau) \\ S_{\dot{\xi}\dot{\xi}}(\alpha) &= \alpha^2 S_{\xi\xi}(\alpha), \qquad S_{\xi\dot{\xi}}(\alpha) = \mathrm{i}\alpha S_{\xi\xi}(\alpha) \end{aligned}$$

For a special setting of the parameters which is given in Table 1 some results of computer simulations are presented.

The dimension N is chosen in a way that the computation of moment functions of the original system of dimension $n = 2N$ is still possible. This allows comparisons with model reduction approximations and the evaluation of the exact values of the error criterions.

Figure 2 shows the correlation function and the spectral density of the boundary function $\xi(t, \omega)$ together with finite-dimensional approximations of the correlation functions $R^N_{uu}(\tau, x, x')$ and $R^N_{u_t u_t}(\tau, x, x')$ as well as the corresponding spectral densities $S^N_{uu}(\alpha, x, x')$ and $S^N_{u_t u_t}(\alpha, x, x')$ for $x = x' \in (0, 1)$ which have been found from the corresponding moment functions of $z_1, \ldots, z_{2N}$ for values of $x = x_i$, $i = 1, \ldots, N$ (see (18),(19)), and linear interpolation between the grid points.

parameter			value
material	mass per unit length	ρ	1
	stiffness	η	1
	external viscous damping	κ_1	1
	internal viscous damping	κ_2	0.01
excitation		β	10
		γ	5
		σ^2	5
discretization	number of internal grid points	N	30

Table 1: Parameter settings for the numerical example

Further, model reduction approximations of the moment functions have been computed. For dimensions $m = 10, 14, 20$ Figure 3 shows the resulting spectral density plots for u. Here, the moment function approximations have been found from the corresponding moment functions of $\tilde{z}_i^{(m)}$, $i = N+1, \ldots, 2N$, of the reduced-order approximation of $\mathbf{z}$. The plots in the left column correspond to model reduction approximations based on mean-square optimal projections while the plots in the right column show the results of the procedure based on modal decomposition. It can be seen that in both cases the results for $m = 20$ compare well with the spectral density approximation of u found from the 60-dimensional original system shown in Figure 2. On the other hand, the model reduction approximations for $m = 10$ and $m = 14$ show more or less significant differences to the original system. Similar effects can be observed for the other second-order moment functions which are not presented.

After the visual comparison of the moment functions of $\mathbf{z}$ and $\tilde{\mathbf{z}}^{(m)}$ now the goodness of the approximation will be assessed by using the error criterions $\Delta^{\mathbf{R}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)})$ and $\Delta^{\mathbf{S}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)})$ defined in Definition 5. In order to normalize the values of the criterions the approximation $\tilde{\mathbf{z}}^{(m)}$ of $\mathbf{z}$ is compared with the approximation of $\mathbf{z}$ by its zero mean, i.e. by $\mathbf{0}_n$. To this end, the relative error criterions

$$\Delta_{\mathrm{rel}}^{\mathbf{R},\mathbf{S}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}) := \frac{\Delta^{\mathbf{R},\mathbf{S}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)})}{\Delta^{\mathbf{R},\mathbf{S}}(\mathbf{z}, \mathbf{0}_n)} \tag{20}$$

are introduced. A "good" approximation should be characterized by the property $\Delta^{\mathbf{R},\mathbf{S}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}) \ll \Delta^{\mathbf{R},\mathbf{S}}(\mathbf{z}, \mathbf{0}_n)$, i.e. by $\Delta_{\mathrm{rel}}^{\mathbf{R},\mathbf{S}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}) \ll 1$.

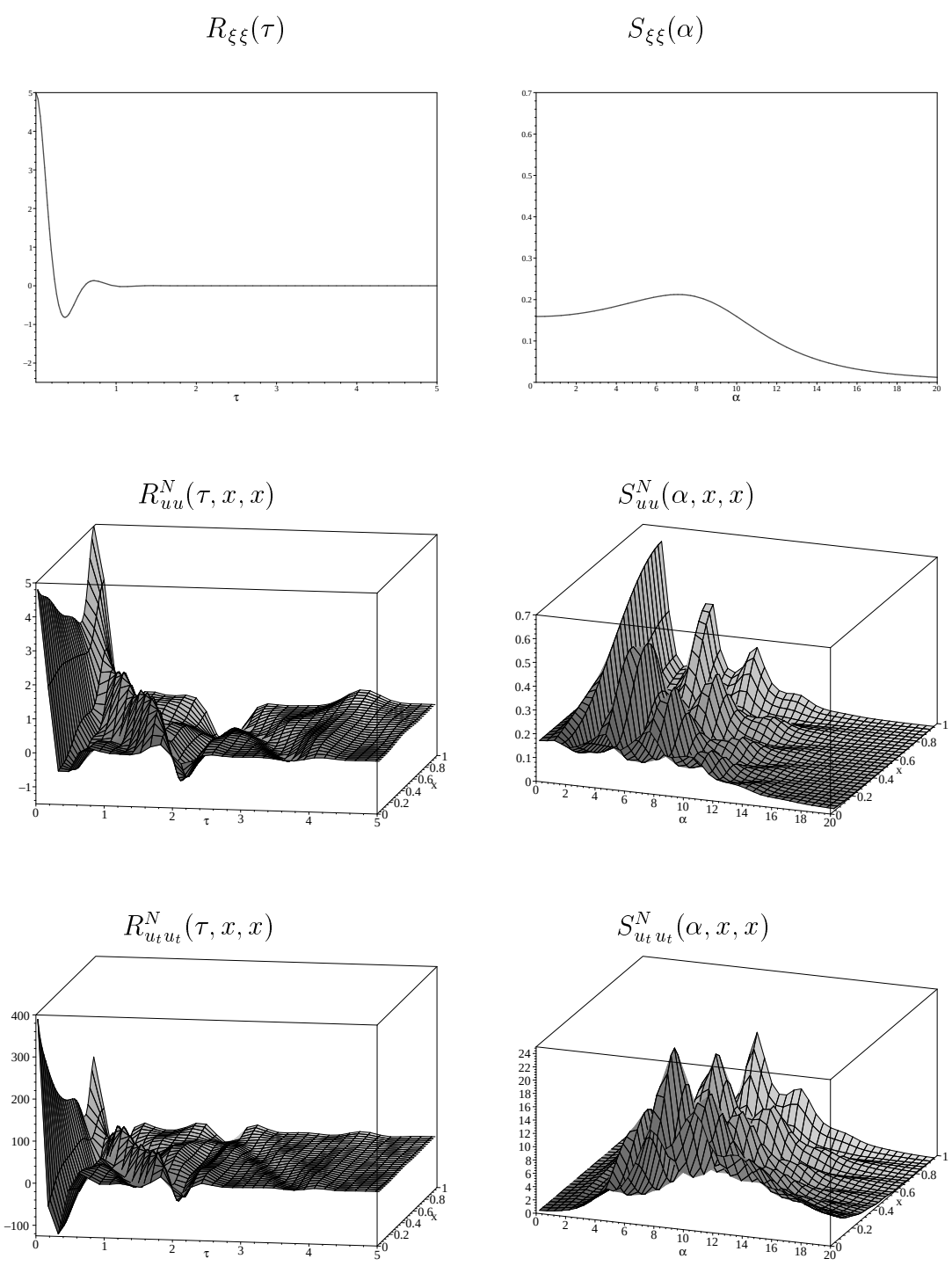
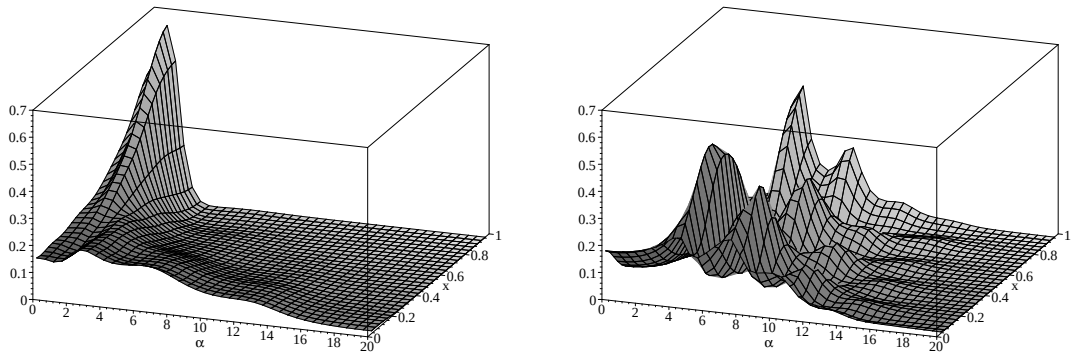


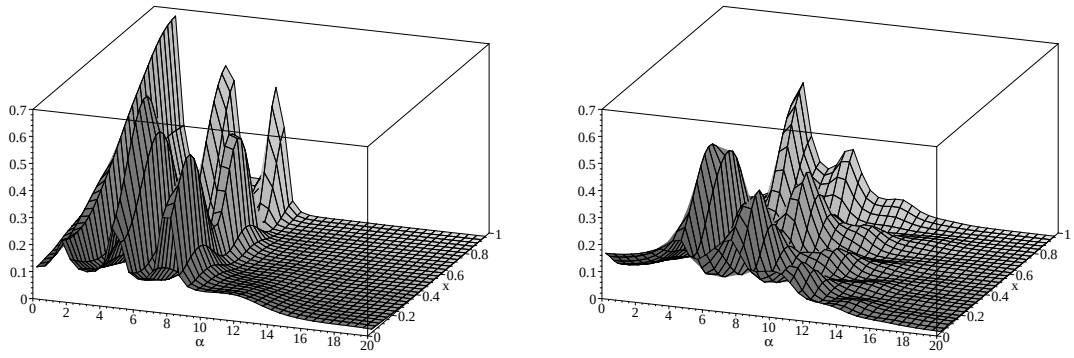
Figure 2: Finite-dimensional approximations of the moment functions of u and u_t

m.-s. optimal projection **modal decomposition**

$m = 10$ $m = 10$



$m = 14$ $m = 14$



$m = 20$ $m = 20$

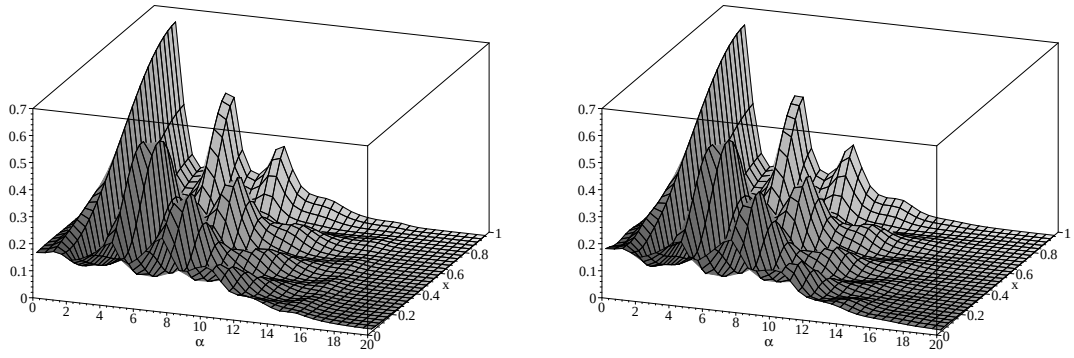


Figure 3: Model reduction approximations of the spectral density $S^N_{uu}(\alpha, x, x)$

In the present example only auto-correlation functions and auto-spectral densities, i.e. the diagonal entries of the corresponding matrix functions, are considered. Therefore, the error criterions are based on the weighting matrix $\mathbf{\Psi} = \frac{1}{n}\mathbf{I}_n$ (see Definition 4), i.e. it holds

$$\begin{aligned} \Delta^{\mathbf{R}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}) &= \int_{\mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} \left| R_{z_i z_i}(\tau) - R_{\tilde{z}_i^{(m)} \tilde{z}_i^{(m)}}(\tau) \right| \theta(\tau) d\tau \\ \text{and} \quad \Delta^{\mathbf{S}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}) &= \int_{\mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} \left| S_{z_i z_i}(\alpha) - S_{\tilde{z}_i^{(m)} \tilde{z}_i^{(m)}}(\alpha) \right| d\alpha. \end{aligned}$$

In [14], [18] it is proved, that in case of a weighting matrix $\mathbf{\Psi} = \frac{1}{n}\mathbf{I}_n$ the upper bounds $\overline{\Delta}_m$ for the error criterions $\Delta^{\mathbf{R}}$ and $\Delta^{\mathbf{S}}$ derived in Theorem 7 and 9 can be improved. It holds

$$\begin{aligned} \Delta^{\mathbf{R},\mathbf{S}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}) &\leq \frac{1}{n}\overline{\Delta}_m \\ \text{and} \quad \Delta^{\mathbf{R},\mathbf{S}}_{\text{rel}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}) &\leq \frac{\frac{1}{n}\overline{\Delta}_m}{\Delta^{\mathbf{R},\mathbf{S}}(\mathbf{z}, \mathbf{0}_n)}. \end{aligned}$$

Using the relative error criterions and their upper bounds it is possible to determine dimensions m which are necessary to guarantee a given level of accuracy of the approximation of the moment functions of $\mathbf{z}$ by those of $\tilde{\mathbf{z}}^{(m)}$. To a given level $c \in (0, 1)$ the integer number

$$m^{R,S}(c) := \min \left\{ m : 1 \leq m \leq n,\ \Delta^{\mathbf{R},\mathbf{S}}_{\text{rel}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}) \leq c \right\} \tag{21}$$

is called necessary dimension with respect to $\Delta^{\mathbf{R},\mathbf{S}}_{\text{rel}}$. Here, $\Delta^{\mathbf{R},\mathbf{S}}_{\text{rel}}(\mathbf{z}, \tilde{\mathbf{z}}^{(n)})$ is defined to be 0. The estimates of $\Delta^{\mathbf{R},\mathbf{S}}_{\text{rel}}$ imply the following estimates of the necessary dimension

$$m^{R,S}(c) \leq \overline{m}^{R,S}(c) := \min \left\{ m : 1 \leq m \leq n,\ \frac{\frac{1}{n}\overline{\Delta}_m}{\Delta^{\mathbf{R},\mathbf{S}}(\mathbf{z}, \mathbf{0}_n)} \leq c \right\} \tag{22}$$

where $\overline{\Delta}_n$ is defined to be 0.

The upper bounds for the relative criterions $\Delta^{\mathbf{R}}_{\text{rel}}$ and $\Delta^{\mathbf{S}}_{\text{rel}}$ contain

$$\begin{aligned} \Delta^{\mathbf{R}}(\mathbf{z}, \mathbf{0}_n) &= \int_{\mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} |R_{z_i z_i}(\tau)| \, \theta(\tau) d\tau \\ \text{and} \quad \Delta^{\mathbf{S}}(\mathbf{z}, \mathbf{0}_n) &= \int_{\mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} |S_{z_i z_i}(\alpha)| \, d\alpha = \frac{1}{n} \sum_{i=1}^{n} \sigma^2_{z_i}. \end{aligned}$$

That means, for the evaluation of the upper bounds and consequently of $\overline{m}^{R,S}(c)$ the knowledge of the moment functions of the stationary output of the original system is required. While in the present example n is chosen such that these moment functions can be determined they are not available in the general case in which the approximations $\Delta^{\mathbf{R},\mathbf{S}}(\mathbf{z},\mathbf{0}_n) \approx \Delta^{\mathbf{R},\mathbf{S}}(\tilde{\mathbf{z}}^{(m)},\mathbf{0}_n)$ can be used.

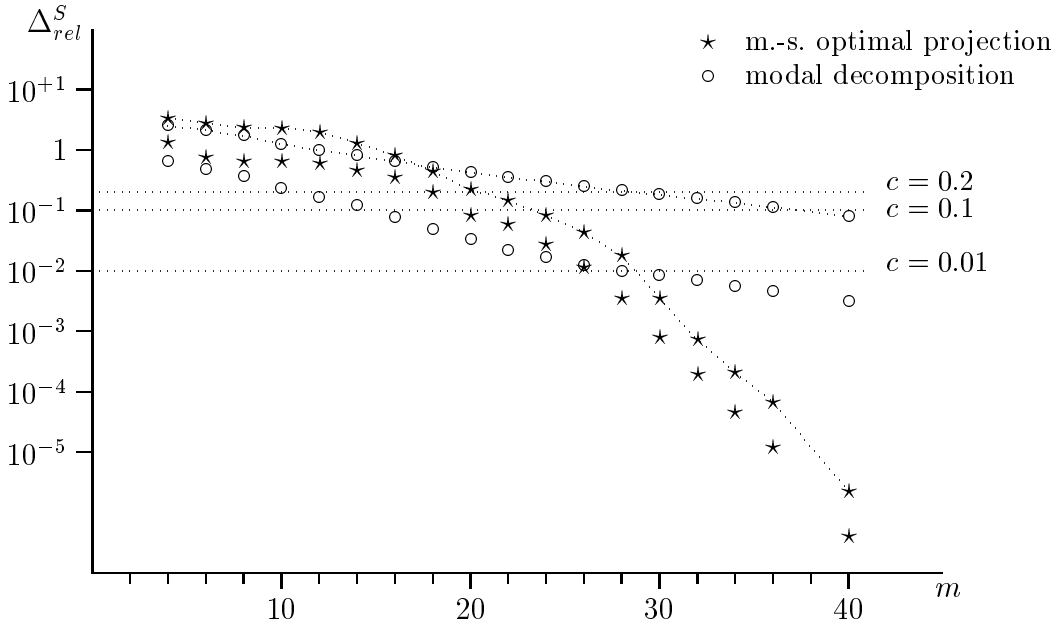


Figure 4: Error criterions $\Delta^{\mathbf{S}}_{rel}(\mathbf{z},\tilde{\mathbf{z}}^{(m)})$ and their estimates

Figure 4 shows some values of the relative error criterions $\Delta^{\mathbf{S}}_{\mathrm{rel}}(\mathbf{z},\tilde{\mathbf{z}}^{(m)})$ and the corresponding bounds $\frac{\frac{1}{n}\overline{\Delta}_m}{\Delta^{\mathbf{S}}(\mathbf{z},\mathbf{0}_n)}$ for several dimensions m using an ordinate with logarithmic scaling. It is noted, that $\Delta^{\mathbf{S}}_{\mathrm{rel}}$ depends on the spectral densities of all $n = 2N$ components of $\mathbf{z}$ and $\tilde{\mathbf{z}}^{(m)}$ while in Figure 3 only the spectral densities of $\tilde{z}_i^{(m)}$, $i = N+1,\ldots,2N$, are visualized. The symbol $\star$ marks values for the model reduction based on mean-square optimal projections while the corresponding values for the modal decomposition procedure are marked by the symbol $\circ$. The symbols for the upper bounds are connected by dotted lines.

It can be seen that the modal decomposition procedure leads to smaller values of $\Delta^{\mathbf{S}}_{\mathrm{rel}}(\mathbf{z},\tilde{\mathbf{z}}^{(m)})$ for $m \leq 24$ while for larger dimensions m the values resulting from mean-square optimal projections are smaller and decrease rapidely for increasing m.

Table 2 gives necessary dimensions $m^S(c)$ with respect to $\Delta^{\mathbf{S}}_{\mathrm{rel}}$ for some levels $c \in (0,1)$ together with the corresponding estimates $\overline{m}^S(c)$. Looking at the

		$c = 0.2$	$c = 0.1$	$c = 0.01$
$m^S(c)$	m.-s. optimal projection	18	20	27
	modal decomposition	12	16	28
$\overline{m}^S(c)$	m.-s. optimal projection	21	23	29
	modal decomposition	29	38	57

Table 2: Necessary dimensions $m^S(c)$ and estimates $\overline{m}^S(c)$

estimates $\overline{m}^S(c)$ which can be determined (at least approximately) without the knowledge of the moment functions of the original system, the following two observations are possible. First, there is only a slight overestimation of the actual values $m^S(c)$ in case of the mean-square optimal projection based procedure. Second, the estimates $\overline{m}^S(c)$ for the modal decomposition procedure are much larger than the actual values. That means, the good approximation properties of the modal decomposition procedure which can be observed for $m \leq 24$ cannot be predicted by the error estimates.

Finally, the error criterions $\Delta^{\mathbf{R}}_{\text{rel}}$ with respect to the correlation function are considered. For the evaluation of these criterions a weighting function θ (see Definition 4) has to be chosen. In the present example the three cases

$$\theta(\tau) = \theta_1(\tau) = \chi_{\{|\tau| \leq \tau^*\}} \frac{1}{2\tau^*} \quad \text{with } \tau^* = 6$$

$$\theta(\tau) = \theta_2(\tau) = d_1 \mathrm{e}^{-d_2|\tau|} \quad \text{with } d_1, d_2 > 0,\ d_2 \text{ such that } \mathrm{e}^{-6d_2} = 10^{-2} \text{ and } \int_{-\infty}^{\infty} \theta_2(\tau) d\tau = 1$$

$$\theta(\tau) = \theta_3(\tau) = \delta(\tau) \quad \text{with Dirac's delta functional } \delta(.)$$

are considered.

The results for the uniform weighting function $\theta_1(\tau)$ and the exponential weighting function $\theta_2(\tau)$ are similar to the results for $\Delta^{\mathbf{S}}$ presented in Figure 4 and Table 2. An interesting effect can be observed in case $\theta(\tau) = \theta_3(\tau)$ where $\Delta^{\mathbf{R}}_{\text{rel}}$ depends only on variances of $\mathbf{z}$ and $\tilde{\mathbf{z}}^{(m)}$, it holds

$$\Delta^{\mathbf{R}}_{\text{rel}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)}) = \frac{\Delta^{\mathbf{R}}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)})}{\Delta^{\mathbf{R}}(\mathbf{z}, \mathbf{0}_n)} = \frac{\sum_{i=1}^{n} \left| \sigma^2_{z_i} - \sigma^2_{\tilde{z}_i^{(m)}} \right|}{\sum_{i=1}^{n} \sigma^2_{z_i}}.$$

Here, the model reduction based on mean-square optimal projection leads to significantly smaller values of $\Delta^{\mathbf{R}}_{\text{rel}}$ than the modal decomposition procedure as it can be seen in Figure 5.

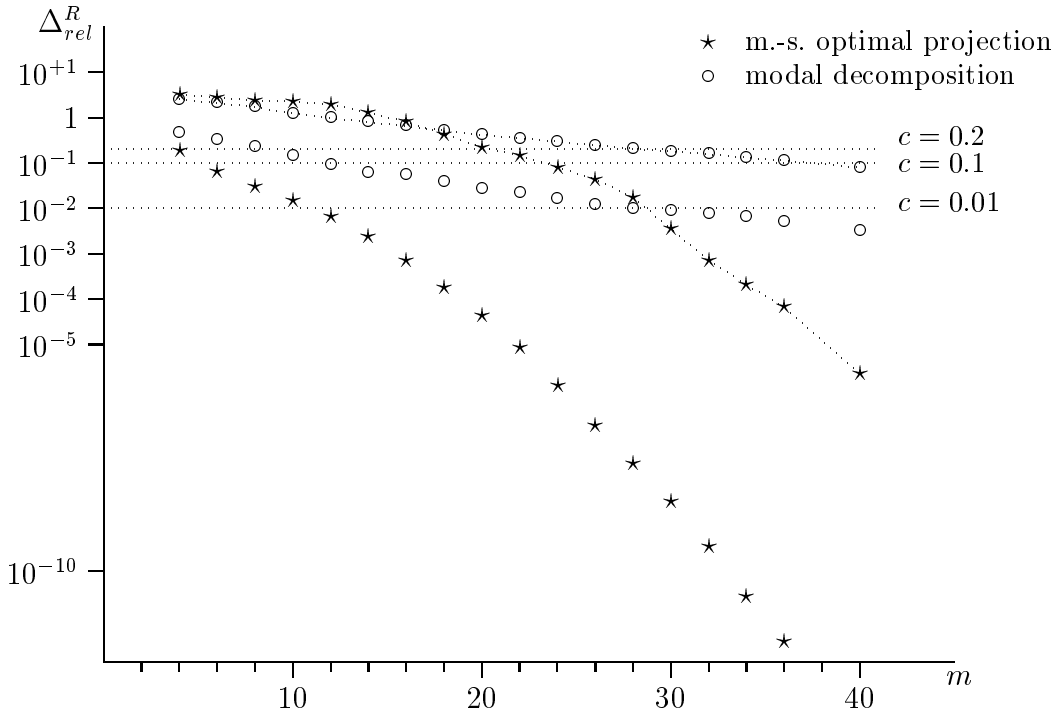


Figure 5: Error criterions $\Delta^{\mathbf{R}}_{rel}(\mathbf{z}, \tilde{\mathbf{z}}^{(m)})$ and their estimates for the weighting function $\theta(\tau) = \delta(\tau)$

Numerical experiments with finer as well as coarser discretizations of the problem (17), e.g. with $N = 20, 25, 40, 50$ internal discretization points, leads to similar results. In particular, the necessary dimensions $m^{R,S}(c)$ seem to be invariant with respect to a (sufficiently fine) discretization.

This supports the conjecture that the random dynamics of the considered infinite dimensional vibration system described by the PDE (17) can be expressed approximately by using only a few properly chosen first-order ODE.

Conclusion

Axial vibrations of a thin homogeneous beam subjected to a random excitation modeled by a stationary random process were been investigated. Semi-discretization of the governing PDE leads to a (usually large-scale) system of ODEs with random inhomogeneous terms. Correlation function and spectral density of the stationary response of the continuous vibration system can be found from the corresponding moment functions of the stationary solution of this system of ODEs. Model reduction techniques are applied to find approximations of moment functions for the large-scale system by computing them

for a suitable chosen low-dimensional system of ODEs.

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