Compactness and Generalized Approximation Spaces

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Abstract. We show that generalized approximation spaces can be used to describe the relatively compact sets of Banach spaces. This leads to compactness and convergence criteria in the approximation spaces themselves. If these spaces can be described with the help of moduli of smoothness, then the criteria can be formulated in terms of the moduli. As applications we give a generalization of Bernstein's theorem about existence of elements with prescribed best approximation errors, compactness criteria for operators, a criterion for compactness in Sobolev type spaces, and a generalization of Simon's compactness criterion for subsets of L^p-spaces of Banach-space-valued functions.

1. Introduction

Many classical Banach spaces, like Hölder-Zygmund spaces, Besov spaces on an interval, certain Sobolev spaces, and Lorentz sequence spaces can be viewed as classical approximation spaces. These spaces are defined as follows ([16]):

Let X be a Banach space and let X_n ($n \in \mathbb{N} := \{1, 2, \ldots\}$) be finite-dimensional linear subspaces of X with $X_1 \subseteq X_2 \subseteq X_3 \subseteq \ldots$. Set $X_0 := \{0\}$ and

$$E_n^X(f) := \inf\{\|f - P_n\|_X : P_n \in X_n\} \qquad (\text{particularly, } E_0^X(f) = \|f\|_X).$$

For $0 < s < \infty$ and $1 \leq q \leq \infty$ the classical approximation space X_q^s is defined by

$$X_q^s = \{f \in X : \|f\|_{X_q^s} = \left\| \left\{ (n+1)^{s-(1/q)} E_n^X(f) \right\}_{n=0}^{\infty} \right\|_q < \infty\}, \tag{1.1}$$

where $\|\,.\,\|_q$ denotes the l^q-norm. Sometimes we also write $X_q^s(\{X_n\})$, but usually the sequence $\{X_n\}$ is viewed as fixed so that it is not necessary to include it in the notations. (This means that we use the same $\{X_n\}$ in all what follows.) For example, the weighted Besov space

$$B_q^s(L^p_{\rho,\tau}) := \{f \in L^p_{\rho,\tau} : \|f\|_{B_q^s(L^p_{\rho,\tau})} = \|f\|_{L^p_{\rho,\tau}} + \|t^{-s-(1/q)}\, \Omega^r_\varphi(f,t)_{p,\rho,\tau}\|_{L^q(0,1)} < \infty\},$$

where $r \in \mathbb{N} \setminus (0, s]$, $1 \leq p \leq \infty$, $\rho, \tau \left\{ \begin{array}{ll} > -1/p\,, & p < \infty \\ \geq 0\,, & p = \infty \end{array} \right\}$ and

$L^p_{\rho,\tau} := \{f : f v^{\rho,\tau} \in L^p(-1,1)\}$ $(\|f\|_{L^p_{\rho,\tau}} := \|f v^{\rho,\tau}\|_{L^p})$, $v^{\rho,\tau}(x) = (1-x)^\rho (1+x)^\tau$,

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is (in the sense of equivalent norms) equal to $(L^p_{\rho,\tau})^s_q(\{\Pi_n\})$, $\Pi_n = \text{span}\{x^k\}_{k=0}^{n-1}$ ([7], Theo.3.1). Here we denote by $\Omega^r_\varphi(f,t)_{p,\rho,\tau}$ the so called weighted (main part) modulus of smoothness of order r which is defined by

$$\Omega^r_\varphi(f,t)_{p,\rho,\tau} := \sup_{0<h\leq t} \|v^{\rho,\tau}\Delta^r_{h\varphi}f\|_{L^p(-1+2r^2h^2,1-2r^2h^2)}\,,\ (\Delta^r_{h\varphi}f)(x) := (\Delta^r_{h\sqrt{1-x^2}}f)(x),$$

where $(\Delta^1_\delta f)(x) := f\left(x+\frac{\delta}{2}\right) - f\left(x-\frac{\delta}{2}\right)$ and $\Delta^r_\delta f := \Delta^1_\delta(\Delta^{r-1}_\delta f)$ $(r>1)$.

The theory of the classical approximation spaces X^s_q can be found in [16] (even for quasi-Banach spaces X, non-linear X_n , and $0 < q \leq \infty$). But recently there also appeared applications in which more general approximation spaces are needed (see e.g. [11], [14], [2]). More precisely, instead of the sequence $\{(n+1)^s\}$ in (1.1) other sequences are considered in the definition of approximation spaces. This suggests the following definition of generalized approximation spaces (see [13] and [1]):

Definition 1.1 *Let $\mathcal{B} = \{b_n\}_{n=0}^\infty$ be a sequence of nonnegative real numbers with $b_0 > 0$ and let $1 \leq q \leq \infty$. The approximation space with respect to $\mathcal{B}$ and q is defined by*

$$X^{\mathcal{B}}_q = \left\{ f \in X : \|f\|_{X,\mathcal{B},q} = \left\| \left\{ b_n(n+1)^{-1/q} E^X_n(f) \right\}_{n=0}^\infty \right\|_q < \infty \right\}. \tag{1.2}$$

We include the factor $(n+1)^{-1/q}$ into the definition (1.2) to have an easy connection between the sequence $\mathcal{B}$ and the parameter s in the case of the classical approximation space X^s_q which corresponds to $b_n = (n+1)^s$. But sometimes it is easier to work with the numbers

$$\widetilde{b}_n := b_n(n+1)^{-1/q} \tag{1.3}$$

which we will use likewise in the sequel.

Of course, it is of interest to know how compactness in $X^{\mathcal{B}}_q$ can be characterized. In the present paper we will prove compactness criteria for subsets of such spaces. The basis is a nice compactness criterion for subsets of X which points out the close connection between compactness in X and boundedness in $X^{\mathcal{B}}_q$. This criterion goes back to the following well-known result:

Theorem 1.2 ([18], Section 2.5.1) *Let $\bigcup X_n$ be dense in X. Then, for every $M \subseteq X$, the following assertions are equivalent.*

(i) M is relatively compact in X (i.e. every sequence of elements of M contains a subsequence which converges in X).

(ii) M is bounded in X and $\lim_{n\to\infty} E^X_n(f) = 0$ uniformly in $f \in M$.

(For expressions $A_n(f)$ which depend on n and f, we write $\lim_{n\to\infty} A_n(f) = 0$ uniformly in $f \in M$ iff $\lim_{n\to\infty} \sup_{f\in M} |A_n(f)| = 0$.)

Remark 1.3 *Later (in Sections 6 and 7) we will use the following fact: The implication "(i)⇒(ii)" of Theorem 1.2 is also true without the assumption* $\dim X_n < \infty$. *To make this clear, we give the proof of the theorem.*

Proof of Theorem 1.2. (i)⇒(ii): Let $\varepsilon > 0$. By Hausdorff's theorem, we can find a finite ε-net $\{f_1, \dots, f_m\}$ for M. Now we choose some n_0 such that $\max_{i=1,\dots m} E_n^X(f_i) < \varepsilon$ for all $n \geq n_0$. Then it follows, for all $f \in M$ and for $i = i(f)$ with $\|f - f_i\|_X < \varepsilon$,

$$E_n^X(f) \leq E_n^X(f - f_i) + E_n^X(f_i) \leq \|f - f_i\|_X + E_n^X(f_i) < 2\varepsilon\,, \quad n \geq n_0\,.$$

(ii)⇒(i): Let $\varepsilon > 0$ and choose some $n = n(\varepsilon)$ such that $\sup_{f\in M} E_n^X(f) < \varepsilon/2$. Then it is clear that $M_n := X_n \cap \{f \in X : \operatorname{dist}(f, M) < \varepsilon/2\}$ is an $(\varepsilon/2)$-net for M. M_n is a bounded subset of the finite-dimensional space X_n. Hence, M_n is relatively compact in X and it follows the existence of a finite $(\varepsilon/2)$-net M_ε for M_n (Hausdorff's theorem). Since M_n is an $(\varepsilon/2)$-net for M, we conclude that M_ε is a finite ε-net for M. Thus, M is relative compact in X (again by Hausdorff's theorem). ∎

The paper is organized as follows: In Section 2 we give some properties of generalized approximation spaces. In Section 3 we characterize the relatively compact subsets of X with the help of the spaces $X_q^{\mathcal{B}}$ (where q is fixed). In Section 4 we use this characterization to prove criteria for compactness and convergence in approximation spaces. Section 5 is devoted to norm equivalences and compactness criteria in terms of moduli of smoothness. In Section 6 we give generalizations of the theory and in Section 7 we present some applications which show that the compactness criteria are powerful tools.

2. Properties of generalized approximation spaces

Take the notations and assumptions of Section 1. In all what follows we suppose

$$\left\| \left\{ \widetilde{b}_n \right\}_{n=0}^{\infty} \right\|_q = \infty\,, \tag{2.1}$$

since otherwise $X_q^{\mathcal{B}} = X$ in the sense of equivalent norms.

In the following theorem we give a list of known results from [13] (see also [1]). We also give the proofs, since [13] and [1] are probably not available for the reader.

Theorem 2.1 *Let (2.1) be satisfied. Then the following assertions hold true.*

(i) $X_q^{\mathcal{B}}$ *is a Banach space.*

(ii) The embedding $X_q^{\mathcal{B}} \subseteq X$ *is compact.*

(iii) If $\mathcal{C} = \{a_n b_n\}$, *where* $\{a_n\}_{n=0}^{\infty}$ *is a sequence of nonnegative real numbers with* $a_0 > 0$ *and* $\lim_{n\to\infty} a_n = \infty$, *then the embedding* $X_q^{\mathcal{C}} \subseteq X_q^{\mathcal{B}}$ *is compact.*

(iv) *If $P_n \in X_n$ is a best approximation for $f \in X_q^{\mathcal{B}}$ in the norm of X, i.e. $\|f-P_n\|_X = E_n^X(f)$, then it is also a best approximation in the norm of $X_q^{\mathcal{B}}$, i.e. $\|f-P_n\|_{X,\mathcal{B},q} = \inf_{P\in X_n}\|f-P\|_{X,\mathcal{B},q}$.*

(v) *If $q<\infty$ then $\bigcup_{n=0}^{\infty} X_n$ is dense in $X_q^{\mathcal{B}}$. If $q=\infty$ then the closure of $\bigcup X_n$ in $X_\infty^{\mathcal{B}}$ is equal to the subspace*

$$X_{\infty,0}^{\mathcal{B}} = \{f\in X : \lim_{n\to\infty} b_n E_n^X(f) = 0\}\ . \tag{2.2}$$

(vi) *For arbitrary sequences $\mathcal{B}_1 = \{b_n^{(1)}\}_{n=0}^{\infty}$ and $\mathcal{B}_2 = \{b_n^{(2)}\}_{n=0}^{\infty}$ of nonnegative real numbers with $b_0^{(1)}, b_0^{(2)} > 0$, we have*

$$(X_q^{\mathcal{B}_1})_q^{\mathcal{B}_2} = (X_q^{\mathcal{B}_2})_q^{\mathcal{B}_1} = X_q^{\mathcal{C}}$$

(in the sense of equivalent norms), where $\mathcal{C} = \{c_n\}_{n=0}^{\infty}$ with

$$c_n = b_n^{(1)} \left\| \left\{\widetilde{b}_m^{(2)}\right\}_{m=0}^{n}\right\|_q + b_n^{(2)} \left\| \left\{\widetilde{b}_m^{(1)}\right\}_{m=0}^{n}\right\|_q\ .$$

Remark 2.2 *If we take $\mathcal{B}_1 = \mathcal{B}$ and $\mathcal{B}_2 = \{(n+1)^{-1/q}\}$, then the sequence $\mathcal{C}$ in (vi) has positive entries. On the other hand, (2.1) is not satisfied for $\mathcal{B}_2$, i.e. $(X_q^{\mathcal{B}_1})_q^{\mathcal{B}_2} = X_q^{\mathcal{B}}$. Thus, $X_q^{\mathcal{B}} = X_q^{\mathcal{C}}$ in the sense of equivalent norms. So we see that, without loss of generality, the approximation spaces may be defined only for sequences $\mathcal{B}$ of positive numbers. In case $q=\infty$ we even have $\max_{m\le n} b_m \le c_n \le 2\max_{m\le n} b_m$ which shows that the approximation space does not change if we replace b_n by $\max_{0\le m\le n} b_m$. So we do not lose generality if we define $X_\infty^{\mathcal{B}}$ only for increasing sequences $\mathcal{B}$ of positive numbers.*

Proof of Theorem 2.1. In all what follows, we denote by $\{P_n(f)\}$ a sequence of best approximations of f, i.e. $P_n(f)\in X_n$ and $\|f-P_n(f)\|_X = E_n^X(f)$.

(i) Obviously, $X_q^{\mathcal{B}}$ is a normed space (this follows from $E_n^X(f+g)\le E_n^X(f)+E_n^X(g)$ and $E_n^X(\nu f) = |\nu| E_n^X(f)$). To show that $X_q^{\mathcal{B}}$ is complete, we first remark that

(2.3) The closure in X of a set $\mathcal{M}\subseteq X_q^{\mathcal{B}}$ which is bounded by $M<\infty$ in $X_q^{\mathcal{B}}$ (i.e. $\|f\|_{X,\mathcal{B},q}\le M$ for all $f\in\mathcal{M}$) is again bounded by M in $X_q^{\mathcal{B}}$.

This follows from the relation

$$\lim_{m\to\infty} E_n^X(f_m) = E_n^X(f) \text{ for } f_m\in\mathcal{M} \text{ with } X\text{-}\lim_{m\to\infty} f_m = f \tag{2.4}$$

(use $|E_n^X(f)-E_n^X(g)|\le E_n^X(f-g)\le\|f-g\|_X$) which we apply for every k to the left hand side of $\|\{\widetilde{b}_n E_n^X(f_m)\}_{n=0}^k\|_q \le M$. Now, let $\{f_m\}_{m=1}^{\infty}$ be a Cauchy sequence in $X_q^{\mathcal{B}}$. Since X is complete, there exists some $f\in X$ with $\lim_{m\to\infty}\|f_m-f\|_X = 0$. From (2.3) it follows $f\in X_q^{\mathcal{B}}$. Let $\varepsilon>0$. Then there exists an $m_0\in\mathbb{N}$ such that $\|f_l-f_m\|_{X,\mathcal{B},q}\le\varepsilon$ for all $l,m\ge m_0$. Since $\mathcal{M} := \{f_l-f_m : l,m\ge m_0\}$ is bounded by ε in $X_q^{\mathcal{B}}$ and $f-f_m\in \operatorname{clos}_X\mathcal{M}$ for all $m\ge m_0$, (2.3) yields $\|f-f_m\|_{X,\mathcal{B},q}\le\varepsilon$ for all $m\ge m_0$ which shows that f_m converges to f in the norm of $X_q^{\mathcal{B}}$.

(ii) Let $\mathcal{M}$ be a bounded subset of $X_q^{\mathcal{B}}$. Then $\mathcal{M}$ is bounded in X and the estimate

$$E_m^X(f)\left\|\left\{\widetilde{b}_n\right\}_{n=0}^m\right\|_q \le \left\|\left\{\widetilde{b}_n E_n^X(f)\right\}_{n=0}^m\right\|_q \le \|f\|_{X,\mathcal{B},q} \tag{2.5}$$

shows that $\lim_{n\to\infty} E_n^X(f) = 0$ uniformly in $f \in \mathcal{M}$. By Theorem 1.2 (applied with $\operatorname{clos}_X \bigcup X_n$ instead of X), this implies the relative compactness of $\mathcal{M}$ in X.

(iii) Let $\mathcal{M} \subseteq X_q^{\mathcal{C}}$ ($\mathcal{C} = \{c_n\}$, $c_n = a_n b_n$) with $\|f\|_{X,\mathcal{C},q} \le M < \infty$ for all $f \in \mathcal{M}$. In view of (ii), every infinite subset of $\mathcal{M}$ contains a sequence $\{f_m\}_{m=1}^\infty$ which converges in X, i.e. there exists an $f \in X$ such that $\lim_{m\to\infty} \|f - f_m\|_X = 0$. From (2.3) it follows $f \in X_q^{\mathcal{C}}$. Let $\varepsilon > 0$. Choose an N with $a_n\,\varepsilon > 2(\|f\|_{X,\mathcal{C},q} + M)$ for all $n > N$ and an m_0 such that

$$\|f - f_m\|_X \le \frac{\varepsilon}{2}\left\|\left\{\widetilde{b}_n\right\}_{n=0}^N\right\|_q^{-1} \quad \text{for all} \quad m \ge m_0\ .$$

Then, for all $m \ge m_0$,

$$\begin{aligned}
\|f - f_m\|_{X,\mathcal{B},q} &\le \left\|\left\{\widetilde{b}_n E_n^X(f - f_m)\right\}_{n=0}^N\right\|_q + \left\|\left\{\widetilde{b}_n E_n^X(f - f_m)\right\}_{n=N+1}^\infty\right\|_q \\
&\le \left\|\left\{\widetilde{b}_n \|f - f_m\|_X\right\}_{n=0}^N\right\|_q + \left\|\{a_n^{-1}\widetilde{c}_n E_n^X(f - f_m)\}_{n=N+1}^\infty\right\|_q \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2(\|f\|_{X,\mathcal{C},q} + M)}\left\|\{\widetilde{c}_n E_n^X(f - f_m)\}_{n=N+1}^\infty\right\|_q \le \varepsilon\ ,
\end{aligned}$$

which shows that f_m converges to f in the norm of $X_q^{\mathcal{B}}$.

(iv) To prove (iv) we need that, for every $f \in X$,

$$E_n^X(f - P_m(f)) = E_{\max\{n,m\}}^X(f)\ , \quad n, m = 0, 1, \dots\ , \tag{2.6}$$

which can be shown by the estimate

$$\begin{aligned}
E_{\max\{n,m\}}^X(f) &\le \|f - P_m(f) - P_n(f - P_m(f))\|_X = E_n^X(f - P_m(f)) \\
&\le \|f - P_m(f) - [P_{\max\{n,m\}}(f) - P_m(f)]\,\|_X = E_{\max\{n,m\}}^X(f).
\end{aligned} \tag{2.7}$$

Now we get, for all $P \in X_m$,

$$\begin{aligned}
\|f - P\|_{X,\mathcal{B},q} &= \left\|\left\{\widetilde{b}_n E_n^X(f - P)\right\}_{n=0}^\infty\right\|_q \ge \left\|\left\{\widetilde{b}_n E_{\max\{n,m\}}^X(f - P)\right\}_{n=0}^\infty\right\|_q \\
&= \left\|\left\{\widetilde{b}_n E_{\max\{n,m\}}^X(f)\right\}_{n=0}^\infty\right\|_q = \|f - P_m(f)\|_{X,\mathcal{B},q}\ ,
\end{aligned}$$

where we used $E_n^X(f - P) = E_n^X(f)$ for $n \ge m$ (since (2.7) is in case $n \ge m$ also valid for arbitrary $P_m \in X_m$).

(v) Let $f \in X_q^{\mathcal{B}}$ ($q < \infty$) or $f \in X_{\infty,0}^{\mathcal{B}}$ ($q = \infty$). From (2.1) and (2.5) it follows

$$\lim_{n\to\infty} E_n^X(f) = 0 \quad \text{for} \quad f \in X_q^{\mathcal{B}}\ . \tag{2.8}$$

Thus, there exist $m_1 < m_2 < m_3 < \dots$ such that

$$E^X_{m_j}(f) \leq j^{-1} \left\| \left\{ \widetilde{b}_n \right\}_{n=0}^{j-1} \right\|_q^{-1}, \quad j = 1, 2, \dots .$$

For $P_j \in X_{m_j}$ with $E^X_{m_j}(f) = \|f - P_j\|_X$ we get, in view of (2.6),

$$\begin{aligned} \|f - P_j\|_{X,\mathcal{B},q} &\leq \left\| \left\{ \widetilde{b}_n \|f - P_j\|_X \right\}_{n=0}^{j-1} \right\|_q + \left\| \left\{ \widetilde{b}_n E^X_{\max\{n,m_j\}}(f) \right\}_{n=j}^{\infty} \right\|_q \\ &\leq j^{-1} + \left\| \left\{ \widetilde{b}_n E^X_n(f) \right\}_{n=j}^{\infty} \right\|_q . \end{aligned}$$

The right hand side of this estimate tends to zero for $j \to \infty$, since $f \in X^{\mathcal{B}}_q$, $q < \infty$, and $f \in X^{\mathcal{B}}_{\infty,0}$, $q = \infty$. So we have proved that $\bigcup X_n$ is dense in $X^{\mathcal{B}}_q$ $(q < \infty)$ and $X^{\mathcal{B}}_{\infty,0}$ $(q = \infty)$, respectively. It remains to show that $X^{\mathcal{B}}_{\infty,0}$ is a closed subspace of $X^{\mathcal{B}}_{\infty}$. For this aim, let $f_m \in X^{\mathcal{B}}_{\infty,0}$ and $f \in X^{\mathcal{B}}_{\infty}$ such that $\lim_{m\to\infty} \|f - f_m\|_{X,\mathcal{B},\infty} = 0$. For $\varepsilon > 0$ we choose an $m(\varepsilon)$ such that $\sup_{n\geq 0} b_n E^X_n(f - f_{m(\varepsilon)}) < \varepsilon/2$ and an $n(\varepsilon)$ such that $b_n E^X_n(f_{m(\varepsilon)}) < \varepsilon/2$ for all $n \geq n(\varepsilon)$. Then $b_n E^X_n(f) \leq b_n E^X_n(f - f_{m(\varepsilon)}) + b_n E^X_n(f_{m(\varepsilon)}) < \varepsilon$ for all $n \geq n(\varepsilon)$, i.e. $f \in X^{\mathcal{B}}_{\infty,0}$.

(vi) Let $f \in X^{\mathcal{B}_1}_q$. From (iv) and (2.6) it follows

$$E^{X^{\mathcal{B}_1}_q}_n(f) = \|f - P_n(f)\|_{X,\mathcal{B}_1,q} = \left\| \left\{ \widetilde{b}^{(1)}_m E^X_{\max\{n,m\}}(f) \right\}_{m=0}^{\infty} \right\|_q . \tag{2.9}$$

Consequently (with the notation $\{a_n\}_{n=n_1}^{n_2} := \{a'_n\}_{n=0}^{\infty}$, where $a'_n = a_n$ if $n \in [n_1, n_2]$ and $a'_n = 0$ if $n \notin [n_1, n_2]$),

$$\begin{aligned} \|f\|_{X^{\mathcal{B}_1}_q,\mathcal{B}_2,q} &= \left\| \left\{ \widetilde{b}^{(2)}_n \left\| \left\{ \widetilde{b}^{(1)}_m E^X_{\max\{n,m\}}(f) \right\}_{m=0}^{\infty} \right\|_q \right\}_{n=0}^{\infty} \right\|_q \\ &= \left\| \left\{ \left\| \widetilde{b}^{(2)}_n E^X_n(f) \left\{ \widetilde{b}^{(1)}_m \right\}_{m=0}^{n} + \widetilde{b}^{(2)}_n \left\{ \widetilde{b}^{(1)}_m E^X_m(f) \right\}_{m=n+1}^{\infty} \right\|_q \right\}_{n=0}^{\infty} \right\|_q \\ &\sim \left\| \left\{ \widetilde{b}^{(2)}_n E^X_n(f) \left\| \left\{ \widetilde{b}^{(1)}_m \right\}_{m=0}^{n} \right\|_q + \widetilde{b}^{(2)}_n \left\| \left\{ \widetilde{b}^{(1)}_m E^X_m(f) \right\}_{m=n+1}^{\infty} \right\|_q \right\}_{n=0}^{\infty} \right\|_q \\ &\sim \left\| \left\{ \widetilde{b}^{(2)}_n E^X_n(f) \left\| \left\{ \widetilde{b}^{(1)}_m \right\}_{m=0}^{n} \right\|_q \right\}_{n=0}^{\infty} \right\|_q + \left\| \left\{ \widetilde{b}^{(2)}_n \left\| \left\{ \widetilde{b}^{(1)}_m E^X_m(f) \right\}_{m=n+1}^{\infty} \right\|_q \right\}_{n=0}^{\infty} \right\|_q \\ &= \left\| \left\{ \widetilde{b}^{(2)}_n E^X_n(f) \left\| \left\{ \widetilde{b}^{(1)}_m \right\}_{m=0}^{n} \right\|_q \right\}_{n=0}^{\infty} \right\|_q + \left\| \left\{ \widetilde{b}^{(1)}_m E^X_m(f) \left\| \left\{ \widetilde{b}^{(2)}_n \right\}_{n=0}^{m-1} \right\|_q \right\}_{m=1}^{\infty} \right\|_q \\ &\sim \left\| \left\{ \widetilde{b}^{(2)}_n E^X_n(f) \left\| \left\{ \widetilde{b}^{(1)}_m \right\}_{m=0}^{n} \right\|_q \right\}_{n=0}^{\infty} \right\|_q + \left\| \left\{ \widetilde{b}^{(1)}_m E^X_m(f) \left\| \left\{ \widetilde{b}^{(2)}_n \right\}_{n=0}^{m} \right\|_q \right\}_{m=0}^{\infty} \right\|_q \\ &\sim \left\| \left\{ E^X_n(f) \left(\widetilde{b}^{(2)}_n \left\| \left\{ \widetilde{b}^{(1)}_m \right\}_{m=0}^{n} \right\|_q + \widetilde{b}^{(1)}_n \left\| \left\{ \widetilde{b}^{(2)}_m \right\}_{m=0}^{n} \right\|_q \right) \right\}_{n=0}^{\infty} \right\|_q , \end{aligned}$$

where $A \sim B$ means that $c_1 B \leq A \leq c_2 B$ with constants $c_1, c_2 > 0$. ∎

3. Compactness in Banach spaces

Let $\bigcup X_n$ be dense in X. Obviously, Theorem 1.2 can be rewritten in the following form: $M \subseteq X$ is relatively compact in X if and only if M is bounded in $X_\infty^{\mathcal{B}}$ for some sequence $\mathcal{B} = \{b_n\}_{n=0}^\infty$ of positive numbers b_n with $\lim_{n\to\infty} b_n = \infty$. The following result is a generalization of this fact.

Theorem 3.1 *Let $\bigcup X_n$ be dense in X and let $q \in [1,\infty]$ be fixed. For every set $M \subseteq X$, the following assertions are equivalent.*

(i) M is a relatively compact subset of X.

(ii) There exists a sequence $\mathcal{B} = \{b_n\}_{n=0}^\infty$ of nonnegative numbers b_n with $b_0 > 0$ and $\|\{\widetilde{b}_n\}_{n=0}^\infty\|_q = \infty$ such that M is a bounded subset of $X_q^{\mathcal{B}}$.

Moreover, if M is relatively compact, then it is possible to find a sequence $\mathcal{B}$ of positive numbers such that (ii) holds true. In case $q = \infty$ one even can replace "$b_0 > 0$" in (ii) by "$0 < b_0 < b_1 < b_2 < \ldots$".

Proof. (ii)⇒(i): This follows immediately from Theorem 2.1,(ii).

(i)⇒(ii): Let M be relatively compact in X. From Theorem 1.2 it follows the existence of numbers $a_0 > a_1 > a_2 > \ldots > 0$ with $\lim_{n\to\infty} a_n = 0$ such that

$$E_n^X(f) \le a_n \quad \text{for all } f \in M \text{ and all } n = 0,1,2,\ldots .$$

In case $q = \infty$ this implies assertion (ii) (set $b_n = a_n^{-1}$). In case $q < \infty$ we choose a subsequence $\{a_{n_k}\}_{k=1}^\infty$ such that $a_{n_k} \le 2^{-k}$ for all $k \in \mathbb{N}$. If we set

$$b_n = (n+1)^{1/q}\,\widetilde{b}_n \quad \text{with} \quad \widetilde{b}_n = \begin{cases} 1\,, & n = n_1, n_2, \ldots \\ (2^n a_n)^{-1}, & n \notin \{n_1, n_2, \ldots\}\,, \end{cases}$$

then we have $\|\{\widetilde{b}_n\}\|_q^q \ge \sum_{k=1}^\infty \widetilde{b}_{n_k}^q = \sum_{k=1}^\infty 1 = \infty$ and

$$\begin{aligned} \|f\|_{X,\mathcal{B},q}^q &= \left\|\left\{\widetilde{b}_n E_n^X(f)\right\}\right\|_q^q \le \left\|\left\{\widetilde{b}_n a_n\right\}\right\|_q^q = \sum_{n\notin\{n_1,n_2,\ldots\}} \widetilde{b}_n^q a_n^q + \sum_{k=1}^\infty \widetilde{b}_{n_k}^q a_{n_k}^q \\ &\le \sum_{n\notin\{n_1,n_2,\ldots\}} 2^{-nq} + \sum_{k=1}^\infty 2^{-kq} = c < \infty \quad \text{for all } f \in M\,, \end{aligned}$$

i.e. M is bounded in $X_q^{\mathcal{B}}$. ∎

4. Compactness and convergence in approximation spaces

In this section we will show that Theorem 3.1 leads to compactness criteria in generalized approximation spaces and, as a corollary, also to a nice criterion for convergence in such spaces. The idea is the following: Apply Theorem 3.1 with $X_q^{\mathcal{B}}$ instead of X and use assertion (vi) of Theorem 2.1. We must pay attention to

the density-assumption of Theorem 3.1 which is, in view of Theorem 2.1,(v), only satisfied for $q < \infty$, so that we have to consider the space $X^{\mathcal{B}}_{\infty,0}$ (see (2.2)) in the case $q = \infty$. To have a uniform notation, we set

$$X^{\mathcal{B}}_{q,0} := \begin{cases} X^{\mathcal{B}}_q & \text{if} \quad 1 \leq q < \infty\,, \\ X^{\mathcal{B}}_{\infty,0} \text{ (with the norm of } X^{\mathcal{B}}_{\infty}) & \text{if} \quad q = \infty\,. \end{cases}$$

Theorem 4.1 *Let $\mathcal{B}$ satisfy (2.1) and let M be a subset of X. The following assertions are equivalent.*

(i) M is a relatively compact subset of $X^{\mathcal{B}}_{q,0}$.

(ii) There exists a sequence $\{a_n\}_{n=0}^{\infty}$ with $a_n \geq 0$, $a_0 > 0$, and $\lim_{n\to\infty} a_n = \infty$ such that M is a bounded subset of $X_q^{\{b_n a_n\}}$.

Moreover, in (ii) one can replace "$a_n \geq 0$, $a_0 > 0$" by "$0 < a_0 < a_1 < a_2 < \ldots$".

Proof. The implication "(ii)⇒(i)" follows from Theorem 2.1,(iii) and the estimate

$$b_n E_n^X(f) = \frac{a_n b_n E_n^X(f)}{a_n} \leq \frac{\|f\|_{X,\{a_n b_n\},\infty}}{a_n}\,, \quad n \geq n_0\,, \tag{4.1}$$

which shows that $X_{\infty}^{\{b_n a_n\}}$ is a subset of $X^{\mathcal{B}}_{\infty,0}$ in case $q = \infty$. It remains to prove that (i) implies (ii) with $0 < a_0 < a_1 < \ldots < a_n \to \infty$. So, let M be relatively compact in $X^{\mathcal{B}}_{q,0}$. In view of Theorem 3.1 and Theorem 2.1,(v), there exists a sequence $\mathcal{C} = \{c_n\}_{n=0}^{\infty}$ (which is strictly increasing in case $q = \infty$) with $c_n > 0$ and $\|\{\widetilde{c}_n\}\|_q = \infty$ such that M is bounded in $(X^{\mathcal{B}}_{q,0})^{\mathcal{C}}_q$. Obviously, $(X^{\mathcal{B}}_{q,0})^{\mathcal{C}}_q$ is a subspace of $(X^{\mathcal{B}}_q)^{\mathcal{C}}_q$ (with the same norm) and, consequently, M is bounded in $(X^{\mathcal{B}}_q)^{\mathcal{C}}_q$. Now we have

$$(X^{\mathcal{B}}_q)^{\mathcal{C}}_q = X^{\mathcal{D}}_q \quad \text{with} \quad d_n = b_n \left\|\{\widetilde{c}_m\}_{m=0}^n\right\|_q + c_n \left\|\left\{\widetilde{b}_m\right\}_{m=0}^n\right\|_q$$

(Theorem 2.1,(vi)). Set $a_n := \|\{\widetilde{c}_m\}_{m=0}^n\|_q$. Then $0 < a_0 < a_1 < \ldots < a_n \to \infty$ and $a_n b_n \leq d_n$ which implies that $X^{\mathcal{D}}_q$ is continuously embedded into $X_q^{\{a_n b_n\}}$. Thus, M is bounded in $X_q^{\{a_n b_n\}}$. ■

Remark 4.2 *If M is bounded in $X_{\infty}^{\{a_n b_n\}}$ with a sequence $\{a_n\}$ of positive numbers which converges to infinity, then*

$$\sqrt{a_n}\, b_n E_n^X(f) = \frac{a_n b_n E_n^X(f)}{\sqrt{a_n}} \leq \frac{c}{\sqrt{a_n}} \quad \textit{for all } f \in M \textit{ and } n = 0, 1, \ldots$$

which shows that M is also bounded in $X_{\infty,0}^{\{\sqrt{a_n}\, b_n\}}$. Consequently, Theorem 4.1 remains true if we replace $X_q^{\{a_n b_n\}}$ by $X_{q,0}^{\{a_n b_n\}}$.

Sometimes the following variation of Theorem 4.1 is useful. Although this result was already proved by Acosta and Almira in [3], we will give another proof which better fits into the concept of the present paper. (Later, in Theorem 6.14, we will give a generalization of Acosta's and Almira's proof.)

Theorem 4.3 ([3], Theorems 3.1 and 3.2) *Let $\mathcal{B}$ satisfy (2.1) and let M be a subset of X. The following assertions are equivalent.*

(i) M is a relatively compact subset of $X^{\mathcal{B}}_{q,0}$.

(ii) M is bounded in X and $\lim_{m\to\infty} \left\| \left\{ \widetilde{b}_n E^X_n(f) \right\}_{n=m}^{\infty} \right\|_q = 0$ uniformly in $f \in M$.

Proof. (i)⇒(ii): If M is relatively compact in $X^{\mathcal{B}}_{q,0}$, then M is bounded in X and, in view of Theorem 4.1,

$$\left\| \left\{ a_n \widetilde{b}_n E^X_n(f) \right\}_{n=0}^{\infty} \right\|_q \le c \quad \text{for all } \ f \in M$$

with some constant $c < \infty$ and some increasing sequence $\{a_n\}$ of positive numbers which converges to infinity. This implies

$$\left\| \left\{ \widetilde{b}_n E^X_n(f) \right\}_{n=m}^{\infty} \right\|_q \le \frac{1}{a_m} \left\| \left\{ a_n \widetilde{b}_n E^X_n(f) \right\}_{n=m}^{\infty} \right\|_q \le \frac{c}{a_m}, \quad f \in M, \ \ m \in \mathbb{N}$$

and (ii) follows.

(ii)⇒(i): If (ii) is satisfied, then we can find natural numbers $n_1 < n_2 < \dots$ with

$$\sup_{f\in M} \left\| \left\{ \widetilde{b}_n E^X_n(f) \right\}_{n=n_k}^{\infty} \right\|_q \le 2^{-2k}, \quad k = 1, 2, \dots .$$

Now we set $n_0 := 0$ and define $\{a_n\}_{n=0}^{\infty}$ by $a_n := 2^k$ if $n \in [n_k, n_{k+1})$. It follows

$$\left\| \left\{ a_n \widetilde{b}_n E^X_n(f) \right\}_{n=n_1}^{\infty} \right\|_q = \left\| \left\{ \left\| \left\{ a_n \widetilde{b}_n E^X_n(f) \right\}_{n=n_k}^{n_{k+1}-1} \right\|_q \right\}_{k=1}^{\infty} \right\|_q \le \left\| \{2^{-k}\}_{k=1}^{\infty} \right\|_q = c$$

for all $f \in M$. This implies, together with $\left\| \left\{ a_n \widetilde{b}_n E^X_n(f) \right\}_{n=0}^{n_1-1} \right\|_q \le \text{const} \, \|f\|_X$ and the boundedness of M in X, that M is bounded in $X_q^{\{a_n b_n\}}$. Since, obviously, $\lim_{n\to\infty} a_n = \infty$, we may apply Theorem 4.1 to conclude that M is relatively compact in $X^{\mathcal{B}}_{q,0}$. ∎

Corollary 4.4 ([3], Cor. 3.1) *Let $\mathcal{B}$ satisfy (2.1) and let M be a subset of X.*

(i) If $q < \infty$ and $\sum_{n=0}^{\infty} \left[\widetilde{b}_n \sup_{f\in M} E^X_n(f) \right]^q < \infty$, then M is relatively compact in $X^{\mathcal{B}}_q$.

(ii) If $q = \infty$, then M is relatively compact in $X^{\mathcal{B}}_{\infty,0}$ if and only if M is bounded in X and $\lim_{n\to\infty} b_n \sup_{f\in M} E^X_n(f) = 0$.

Proof. In case $q = \infty$, (ii) is only a reformulation of Theorem 4.3, since the assertions $\sup_{n\geq n_0} b_n E_n^X(f) \leq \varepsilon$ for all $f \in M$ and $b_n \sup_{f\in M} E_n^X(f) \leq \varepsilon$ for all $n \geq n_0$ are equivalent. If $q < \infty$, then $\sum_{n=0}^{\infty} \left[\widetilde{b}_n \sup_{f\in M} E_n^X(f)\right]^q < \infty$ implies

$$\widetilde{b}_0 \sup_{f\in M} E_0^X(f) < \infty \quad \text{and} \quad \lim_{m\to\infty} \left\| \left\{ \widetilde{b}_n \sup_{f\in M} E_n^X(f) \right\}_{n=m}^{\infty} \right\|_q = 0 \ .$$

The first assertion means that M is bounded in X (since $E_0^X(f) = \|f\|_X$) and the second one implies that $\lim_{m\to\infty} \left\| \left\{ \widetilde{b}_n E_n^X(f) \right\}_{n=m}^{\infty} \right\|_q = 0$ uniformly in $f \in M$. So we see that (ii) in Theorem 4.3 is satisfied, i.e. M is relatively compact in $X_q^{\mathcal{B}}$. ■

From Theorem 4.3 it also follows a nice characterization of convergence in $X_{q,0}^{\mathcal{B}}$.

Proposition 4.5 (cf. [3], Theorems 2.1 and 3.1) *Let $\mathcal{B}$ satisfy (2.1) and let $\{f_k\}$ be a sequence in $X_{q,0}^{\mathcal{B}}$. The following assertions are equivalent.*

(i) $\{f_k\}$ converges in $X_{q,0}^{\mathcal{B}}$.

(ii) $\{f_k\}$ is convergent in X and relatively compact in $X_{q,0}^{\mathcal{B}}$.

In other words: If $\{f_k\}$ is a relatively compact sequence in $X_{q,0}^{\mathcal{B}}$ which converges to $f \in X$ in the norm of X, then $f \in X_{q,0}^{\mathcal{B}}$ and $\lim f_k = f$ in the norm of $X_q^{\mathcal{B}}$.

Proof. The implication "(i)⇒(ii)" is obvious. Let $\{f_k\}$ be relatively compact in $X_{q,0}^{\mathcal{B}}$ and convergent in X with $X\text{-}\lim f_k = f$. Since $\{f_k\}$ contains a subsequence which converges in $X_{q,0}^{\mathcal{B}}$ (obviously with the same limit as in X), we obtain $f \in X_{q,0}^{\mathcal{B}}$. It follows that $\{f - f_k\}$ is a relatively compact subset of $X_{q,0}^{\mathcal{B}}$. Consequently, for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$\left\| \left\{ \widetilde{b}_n E_n^X(f - f_k) \right\}_{n=n_0}^{\infty} \right\|_q < \frac{\varepsilon}{2} \quad \text{for all } k \,. \tag{4.2}$$

Moreover, we can find a $k_0 = k_0(\varepsilon, n_0)$ such that

$$\left\| \left\{ \widetilde{b}_n E_n^X(f - f_k) \right\}_{n=0}^{n_0-1} \right\|_q \leq \|f - f_k\|_X \left\| \left\{ \widetilde{b}_n \right\}_{n=0}^{n_0-1} \right\|_q < \frac{\varepsilon}{2} \quad \text{for all } k \geq k_0 \,. \tag{4.3}$$

(4.2) and (4.3) together give $\|f - f_k\|_{X,\mathcal{B},q} = \left\| \left\{ \widetilde{b}_n E_n^X(f - f_k) \right\}_{n=0}^{\infty} \right\|_q < \varepsilon$ for all $k \geq k_0$, i.e. f_k converges to f in $X_{q,0}^{\mathcal{B}}$. ■

5. Characterization of compactness with the help of moduli of smoothness

Let us suppose that there exists a fixed $t_0 > 0$ and a function

$$\omega : X \times (0, t_0] \to [0, \infty) \quad \text{(we call it modulus of smoothness)}$$

such that, with some constants $C > 0$, $t_1 \in (0, t_0]$, $n_0 \in \mathbb{N}$ $(n_0 \geq t_0^{-1})$ and $r > 0$, the following assumptions are satisfied:

(A1) $\omega(f, \,.\,) : (0, t_0] \to [0, \infty)$ is an increasing function for all $f \in X$.

(A2) $\omega(f, t) \leq C\|f\|_X$ for all $f \in X$ and all $t \in (0, t_0]$.

(A3) $E_n^X(f) \leq C \sum_{k=0}^{\infty} \omega(f, n^{-1}2^{-k})$ for all $n \geq n_0$ and all $f \in X$.

(A4) $\omega(f, t) \leq C\, t^r \sum_{0 \leq n \leq 1/t} (n+1)^{r-1} E_{n+1}^X(f)$ for all $t \in (0, t_1]$ and all $f \in X$.

((A3) and (A4) are called weak Jackson and Stechkin inequality, respectively.) We remark that in many cases it is possible to define moduli of smoothness with these properties which are related to the smoothness properties of $f \in X$. For example, the Ditzian-Totik modulus of smoothness $\Omega_\varphi^r(f, t)_{p,\rho,\tau}$ (see Section 1) satisfies (A1)$-$(A4) with $X = L^p_{\rho,\tau}$ $(1 \leq p \leq \infty)$ and $X_n = \Pi_n$ (cf. [6], Lemma 8.1.2 and Theorem 8.2.1).

Remark 5.1 *Sometimes one only has weaker connections between $\omega(f, t)$ and $E_n^X(f)$. For example, if $X = L^p(-1, 1)$ with $0 < p < 1$ (this is only a quasi-Banach space) and $X_n = \Pi_n$, then the assumptions (A1)-(A3) (even with $E_n^X(f) \leq C\,\omega(f, n^{-1})$) are satisfied for the Ditzian-Totik modulus of smoothness $\omega(f, t) = \omega_\varphi^r(f, t)_p$ (see [4]), but instead of (A4) we only have*

$$\omega(f, t) \leq C\, t^r \left(\sum_{0 \leq n \leq 1/t} (n+1)^{rp-1} E_{n+1}^X(f)^p \right)^{1/p} \tag{5.1}$$

(see [5]). Such situations will be not treated here. Nevertheless, we think that the following considerations can be generalized to the case of quasi-Banach spaces X, approximation spaces $X_q^{\mathcal{B}}$ with $0 < q \leq \infty$, and moduli which satisfy (A1)$-$(A3) and (5.1) (for some $p \in (0, 1]$). At least for the classical approximation spaces X_q^s ($0 < s < \infty$, $0 < q \leq \infty$, X: quasi-Banach space) and for a modulus which satisfies (A1), (A2), (5.1) (for some $p \in (0, 1]$), and $E_n^X(f) \leq C\,\omega(f, n^{-1})$ this seems to be possible, since S. Li ([12]) already showed that X_q^s can be defined equivalently with the help of ω in this case.

For a certain class of sequences $\mathcal{B}$, the space $X_q^{\mathcal{B}}$ can be described with the help of ω (cf. [13], Theorem 2.65, or, in case $b_n = (n+1)^\gamma$ with $0 < \gamma < r$, [7]). Here we restrict ourselves to sequences $\mathcal{B} = \{b_n\}_{n=0}^\infty$ of the following form:

$$b_n = G(n+1) \text{ with a function } G : [1,\infty) \to (0,\infty)$$

which satisfies the conditions (G1), (G2), and (G3) below.

(G1) There exists a constant γ with $0 < \gamma < r$ such that $t^{-\gamma}G(t)$ is a decreasing function on $[1,\infty)$.

(G2) $G(n+1)G(m) \leq \text{const}\, G(nm+1)$ for all $n, m \in \mathbb{N}$.

(G3) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{G(2^k)} < \infty.$

For example, $G(t) = t^\gamma \ln^{-\delta}(t+1)$ with $0 < \gamma < r$ and $\delta \geq 0$ satisfies the above conditions.

Lemma 5.2 *Let $b_n = G(n+1)$, where $G : [1,\infty) \to (0,\infty)$ satisfies (G1) ($\gamma < r$ is not necessary here). If $\{E_n\}_{n=1}^\infty$ is a decreasing sequence of nonnegative real numbers, then, for all $i_0 \in \mathbb{N} \cup \{0\}$,*

$$2^{-\gamma-(1/q)} \left\|\{b_{2^i}E_{2^i}\}_{i=i_0+1}^\infty\right\|_q \leq \left\|\left\{\widetilde{b}_n E_n\right\}_{n=2^{i_0}}^\infty\right\|_q \leq 2^\gamma \left\|\{b_{2^i}E_{2^i}\}_{i=i_0}^\infty\right\|_q$$

(with the γ from (G1)).

Proof. First we remark that (G1) implies

$$\begin{aligned} 2^\gamma b_{2^i} &= (2^{i+1}+2)^\gamma \frac{G(2^i+1)}{(2^i+1)^\gamma} \geq (n+1)^\gamma \frac{G(n+1)}{(n+1)^\gamma} = b_n \\ &\geq \left(2^i + \frac{1}{2}\right)^\gamma \frac{G(2^{i+1}+1)}{(2^{i+1}+1)^\gamma} = 2^{-\gamma} b_{2^{i+1}} \quad \text{for } 2^i \leq n \leq 2^{i+1}. \end{aligned}$$

Consequently, for all $i \in \mathbb{N} \cup \{0\}$ and all $n \in [2^i, 2^{i+1}]$,

$$b_n E_n \leq 2^\gamma b_{2^i} E_{2^i} \quad \text{and} \quad b_n E_n \geq 2^{-\gamma} b_{2^{i+1}} E_{2^{i+1}}. \tag{5.2}$$

From the first inequality it follows

$$\begin{aligned} \left\|\{\widetilde{b}_n E_n\}_{n=2^{i_0}}^\infty\right\|_q &= \left\|\left\{\|\{\widetilde{b}_n E_n\}_{n=2^i}^{2^{i+1}-1}\|_q\right\}_{i=i_0}^\infty\right\|_q \\ &\leq 2^\gamma \left\|\left\{b_{2^i} E_{2^i} \|\{(n+1)^{-1/q}\}_{n=2^i}^{2^{i+1}-1}\|_q\right\}_{i=i_0}^\infty\right\|_q \\ &\leq 2^\gamma \left\|\left\{b_{2^i} E_{2^i} 2^{-i/q} \|\{1\}_{n=2^i}^{2^{i+1}-1}\|_q\right\}_{i=i_0}^\infty\right\|_q = 2^\gamma \left\|\{b_{2^i}E_{2^i}\}_{i=i_0}^\infty\right\|_q, \end{aligned}$$

while the second one yields

$$\begin{aligned}\left\|\{\widetilde{b}_n E_n\}_{n=2^{i_0}}^{\infty}\right\|_q &= \left\|\left\{\|\{\widetilde{b}_n E_n\}_{n=2^{i-1}}^{2^i-1}\|_q\right\}_{i=i_0+1}^{\infty}\right\|_q \\ &\geq 2^{-\gamma}\left\|\left\{b_{2^i} E_{2^i}\|\{(n+1)^{-1/q}\}_{n=2^{i-1}}^{2^i-1}\|_q\right\}_{i=i_0+1}^{\infty}\right\|_q \\ &\geq 2^{-\gamma}\left\|\left\{b_{2^i} E_{2^i} 2^{-i/q}\|\{1\}_{n=2^{i-1}}^{2^i-1}\|_q\right\}_{i=i_0+1}^{\infty}\right\|_q = 2^{-\gamma-(1/q)}\left\|\{b_{2^i} E_{2^i}\}_{i=i_0+1}^{\infty}\right\|_q .\end{aligned}$$

■

Remark 5.3 *From (5.2) (with $E_n = 1$) it follows $b_n \geq 2^{-\gamma} b_{2^{i+1}} \geq 2^{-2\gamma} b_{2^{i+2}-1} = 2^{-2\gamma} G(2^{i+2})$ for all $n \in [2^i, 2^{i+1}]$. This shows that $b_n = G(n+1)$ converges to infinity if (G1) and (G3) are satisfied. Particularly, condition (2.1) is fulfilled in this case.*

In the following theorem we give the connection between the weighted l^q-norms of $\{E_n^X(f)\}$ and $\{\omega(f, n^{-1})\}$.

Theorem 5.4 *Let assumptions (A1)–(A4) be fulfilled and let $b_n = G(n+1)$ with a function $G : [1, \infty) \to (0, \infty)$.*

(i) If (G1), (G2), and (G3) are satisfied, then, for all $f \in X$ and all $i_0 \in \mathbb{N} \cup \{0\}$ with $2^{i_0} \geq n_0$ (n_0 from (A3)),

$$\left\|\left\{\widetilde{b}_n E_n^X(f)\right\}_{n=2^{i_0+1}}^{\infty}\right\|_q \leq c\left\|\left\{\widetilde{b}_n\, \omega(f, n^{-1})\right\}_{n=2^{i_0}}^{\infty}\right\|_q ,$$

where c is independent of f and i_0.

(ii) If (G1) holds, then, with $n_1 = \min\{2^i : i \in \mathbb{N} \cup \{0\},\ 2^{-i} \leq t_1\}$ (t_1 from (A4)),

$$\left\|\left\{\widetilde{b}_n\, \omega(f, n^{-1})\right\}_{n=n_1}^{\infty}\right\|_q \leq c\left\|\left\{\widetilde{b}_n E_n^X(f)\right\}_{n=1}^{\infty}\right\|_q$$

for all $f \in X$, where c is independent of f.

Particularly, if (G1)–(G3) are satisfied, then

$$\|f\|_{X,\mathcal{B},q} \sim \|f\|_X + \left\|\left\{\widetilde{b}_n\, \omega(f, n^{-1})\right\}_{n \geq 1/t_0}\right\|_q \quad \text{for all } f \in X .$$

Proof. (i) Let (G1)–(G3) be satisfied and let $f \in X$. In view of (A3) and Lemma 5.2 we have, for i_0 with $2^{i_0} \geq n_0$,

$$\begin{aligned}\left\|\left\{\widetilde{b}_n E_n^X(f)\right\}_{n=2^{i_0+1}}^{\infty}\right\|_q &\leq c\left\|\left\{\widetilde{b}_n \sum_{k=0}^{\infty} \omega(f, n^{-1}2^{-k})\right\}_{n=2^{i_0+1}}^{\infty}\right\|_q \\ &\leq c\sum_{k=0}^{\infty}\left\|\left\{\widetilde{b}_n\, \omega(f, n^{-1}2^{-k})\right\}_{n=2^{i_0+1}}^{\infty}\right\|_q \leq c\sum_{k=0}^{\infty}\left\|\left\{b_{2^i}\, \omega(f, 2^{-i-k})\right\}_{i=i_0+1}^{\infty}\right\|_q .\end{aligned}$$

Assumption (G2) yields $b_{2^i} G(2^k) = G(2^i+1)G(2^k) \le c\, G(2^{i+k}+1) = c\, b_{2^{i+k}}$. Together with (G3) and Lemma 5.2 it follows

$$\begin{aligned}
\left\| \left\{ \widetilde{b}_n E_n^X(f) \right\}_{n=2^{i_0+1}}^{\infty} \right\|_q &\le c \sum_{k=0}^{\infty} \frac{1}{G(2^k)} \left\| \left\{ b_{2^{i+k}}\, \omega(f, 2^{-i-k}) \right\}_{i=i_0+1}^{\infty} \right\|_q \\
&\le c \sum_{k=0}^{\infty} \frac{1}{G(2^k)} \left\| \left\{ b_{2^i}\, \omega(f, 2^{-i}) \right\}_{i=i_0+1}^{\infty} \right\|_q \le c \left\| \left\{ b_{2^i}\, \omega(f, 2^{-i}) \right\}_{i=i_0+1}^{\infty} \right\|_q \\
&\le c \left\| \left\{ \widetilde{b}_n\, \omega(f, n^{-1}) \right\}_{n=2^{i_0}}^{\infty} \right\|_q .
\end{aligned}$$

(ii) Let (G1) hold true and let $n_1 = 2^{i_1} \ge t_1^{-1}$. In view of Lemma 5.2 and (A4),

$$\begin{aligned}
&\left\| \left\{ \widetilde{b}_n\, \omega(f, n^{-1}) \right\}_{n=n_1}^{\infty} \right\|_q \le c \left\| \left\{ b_{2^i}\, \omega(f, 2^{-i}) \right\}_{i=i_1}^{\infty} \right\|_q \\
&\le c \left\| \left\{ b_{2^i}\, 2^{-ir} \sum_{n=0}^{2^i} (n+1)^{r-1} E_{n+1}^X(f) \right\}_{i=i_1}^{\infty} \right\|_q \le c \left\| \left\{ b_{2^i}\, 2^{-ir} \sum_{n=1}^{2^i+1} n^{r-1} E_n^X(f) \right\}_{i=0}^{\infty} \right\|_q \\
&\le c \left\| \left\{ b_{2^i}\, 2^{-ir} \sum_{k=0}^{i} 2^{kr} E_{2^k}^X(f) \right\}_{i=0}^{\infty} \right\|_q \quad \text{for all } \ f \in X .
\end{aligned}$$

(In the last estimate we applied Lemma 5.2 with $q = 1$ and $i_0 = 0$ to the sequence $\{E_n\}$ defined by $E_n = E_n^X(f)$ if $n \le 2^i + 1$ and $E_n = 0$ if $n > 2^i + 1$.) Now, using (G1), we may estimate

$$b_{2^i} = \frac{G(2^i+1)}{(2^i+1)^\gamma} (2^i+1)^\gamma \le c\, \frac{G(2^k+1)}{(2^k+1)^\gamma} (2^i+1)^\gamma \le c\, 2^{i\gamma}\, b_{2^k}\, 2^{-k\gamma} \quad \text{for } \ i \ge k .$$

Together with the Hölder inequality we obtain

$$\begin{aligned}
\left\| \left\{ \widetilde{b}_n\, \omega(f, n^{-1}) \right\}_{n=n_1}^{\infty} \right\|_q &\le c \left\| \left\{ 2^{-i(r-\gamma)} \sum_{k=0}^{i} 2^{k(r-\gamma)} b_{2^k} E_{2^k}^X(f) \right\}_{i=0}^{\infty} \right\|_q \\
&= c \left\| \left\{ 2^{-i(r-\gamma)} \sum_{k=0}^{i} 2^{k(r-\gamma)(1-(1/q))} \cdot 2^{k(r-\gamma)/q} b_{2^k} E_{2^k}^X(f) \right\}_{i=0}^{\infty} \right\|_q \\
&\le c \left\| \left\{ 2^{-i(r-\gamma)} \left[\sum_{k=0}^{i} 2^{k(r-\gamma)} \right]^{1-(1/q)} \left\| \left\{ 2^{k(r-\gamma)/q} b_{2^k} E_{2^k}^X(f) \right\}_{k=0}^{i} \right\|_q \right\}_{i=0}^{\infty} \right\|_q \\
&\le c \left\| \left\{ 2^{-i(r-\gamma)/q} \left\| \left\{ 2^{k(r-\gamma)/q} b_{2^k} E_{2^k}^X(f) \right\}_{k=0}^{i} \right\|_q \right\}_{i=0}^{\infty} \right\|_q \\
&= c \left\| \left\{ b_{2^k} E_{2^k}^X(f)\, 2^{k(r-\gamma)/q} \left\| \left\{ 2^{-i(r-\gamma)/q} \right\}_{i=k}^{\infty} \right\|_q \right\}_{k=0}^{\infty} \right\|_q \le c \left\| \left\{ b_{2^k} E_{2^k}^X(f) \right\}_{k=0}^{\infty} \right\|_q .
\end{aligned}$$

In view of Lemma 5.2, the last term can be estimated by $c \left\| \left\{ \widetilde{b}_n E_n^X(f) \right\}_{n=1}^{\infty} \right\|_q$. ∎

Remark 5.5 *In the above proof we have not used that the subspaces X_n are finite-dimensional. Thus, also in case* $\dim X_n = \infty$ *($n \geq N_0$), Theorem 5.4 gives an equivalent characterization for the norm of the space $X_q^{\mathcal{B}}$ (which remains a Banach space; see Theorem 6.1).*

Now we are able to prove the following main result of this section. An alternative proof, which is due to Acosta and Almira ([3]), can be found in Section 6 (see Example 6.16).

Theorem 5.6 *Let assumptions (A1)–(A4) be fulfilled and let $b_n = G(n+1)$ with a function $G : [1, \infty) \to (0, \infty)$ satisfying (G1)–(G3). Then, for every $M \subseteq X$, the following assertions are equivalent.*

(i) M is a relatively compact subset of $X_{q,0}^{\mathcal{B}}$.

(ii) M is bounded in X and there exist positive numbers a_n ($n \geq t_0^{-1}$) with $\lim_{n\to\infty} a_n = \infty$ such that $\sup_{f\in M} \left\| \left\{ \widetilde{b}_n a_n \, \omega(f, n^{-1}) \right\}_{n \geq 1/t_0} \right\|_q < \infty$.

(iii) M is bounded in X and $\lim_{m\to\infty} \left\| \left\{ \widetilde{b}_n \, \omega(f, n^{-1}) \right\}_{n=m}^{\infty} \right\|_q = 0$ uniformly in $f \in M$.

Proof. (ii)⇒(iii): From $\left\| \left\{ \widetilde{b}_n a_n \, \omega(f, n^{-1}) \right\}_{n \geq 1/t_0} \right\|_q \leq c$, $f \in M$, it follows

$$\left\| \left\{ \widetilde{b}_n \, \omega(f, n^{-1}) \right\}_{n=m}^{\infty} \right\|_q \leq \frac{\left\| \left\{ \widetilde{b}_n a_n \, \omega(f, n^{-1}) \right\}_{n=m}^{\infty} \right\|_q}{\inf_{n\geq m} a_n} \leq \frac{c}{\inf_{n\geq m} a_n}, \ f \in M, \, m \geq t_0^{-1}$$

which shows that (ii) implies (iii).

(iii)⇒(i): From (iii) and Theorem 5.4 it follows

$$\sup_{f\in M} \left\| \left\{ \widetilde{b}_n E_n^X(f) \right\}_{n=2^{i_0}}^{\infty} \right\|_q \leq c \sup_{f\in M} \left\| \left\{ \widetilde{b}_n \, \omega(f, n^{-1}) \right\}_{n=2^{i_0-1}}^{\infty} \right\|_q < \varepsilon$$

if $i_0 = i_0(\varepsilon)$ is large enough. Together with the boundedness of M in X this implies, in view of Theorem 4.3, the relative compactness of M in $X_{q,0}^{\mathcal{B}}$.

(i)⇒(ii): From (i) it follows the existence of numbers $0 < a_0 < a_1 < \ldots$ such that

$$\lim_{n\to\infty} a_n = \infty \quad \text{and} \quad S := \sup_{f\in M} \|f\|_{X,\{b_n a_n\},q} < \infty$$

(see Theorem 4.1). Choose $\delta \in (0, r - \gamma)$ (r from (A4) and γ from (G1)) and set $c_n := (n+1)^{\delta} \min_{m=0,\ldots,n} (m+1)^{-\delta} a_m$. Then we get

$$\sup_{f\in M} \|f\|_{X,\{b_n c_n\},q} \leq S \quad \text{and} \quad \lim_{n\to\infty} c_n = \infty \,, \tag{5.3}$$

since $c_n \leq a_n$, $(n+1)^{\delta} \min\limits_{\sqrt{n}\leq m\leq n} (m+1)^{-\delta} a_m \geq a_{[\sqrt{n}]}$, $(n+1)^{\delta} \min\limits_{m\leq\sqrt{n}} (m+1)^{-\delta} a_m \geq c\, n^{\delta/2}$. (By $[A]$ we denote the integer part of A.) Moreover,

$$\frac{b_n c_n}{(n+1)^{\gamma+\delta}} = \frac{G(n+1)}{(n+1)^{\gamma}} \min_{m=0,\ldots,n} \frac{a_m}{(m+1)^{\delta}} \text{ is a decreasing sequence},$$

which shows that $b_n c_n = g(n+1)$, where $g(t) := t^{\gamma+\delta}[t]^{-\gamma-\delta} b_{[t-1]} c_{[t-1]}$ satisfies (G1) with $\gamma+\delta$ instead of γ. Now it follows from Theorem 5.4 and (5.3) that M is bounded in X and

$$\sup_{f\in M} \left\| \left\{ \widetilde{b}_n c_n\, \omega(f, n^{-1}) \right\}_{n\geq 1/t_0} \right\|_q \leq c \sup_{f\in M} \|f\|_{X,\{b_n c_n\},q} < \infty\ ,$$

i.e. assertion (ii) holds true. ■

Remark 5.7 *In assertion (ii) of Theorem 5.6 one can replace "positive numbers a_n" by "positive numbers a_n with $a_n < a_{n+1}$". Indeed, if (ii) holds true with a_n, then also with $a_n^* = (1-2^{-n-1}) \min_{m\geq n} a_n$ (obviously, the minimum exists, since $\lim_{n\to\infty} a_n = \infty$) and $\{a_n^*\}$ is a strictly increasing sequence converging to infinity.*

Remark 5.8 *If even the strong Jackson inequality*

$$E_n^X(f) \leq C\, \omega(f, n^{-1}), \quad f \in X,\ n \geq n_0\ , \tag{5.4}$$

holds true instead of (A3), then all assertions of Theorem 5.4 hold true if G satisfies only (G1). So, in this case, Theorem 5.6 remains true if we replace conditions (G2) and (G3) by (2.1). If $q=\infty$, then (G1) is satisfied for $b_n = G = 1$, but (2.1) is not satisfied in this case. Using Theorem 3.1 instead of Theorems 4.1,4.3 in the proof of Theorem 5.6, we see that, in case $q=\infty$, Theorem 5.6 and Remark 5.7 remain true for $b_n = 1$, if (5.4) holds true instead of (A3) and if $\bigcup X_n$ is dense in X (so that $X^{\mathcal{B}}_{\infty,0} = X$). The following theorem shows that this can be generalized also to the case $q < \infty$.

Theorem 5.9 *Let $\bigcup X_n$ be dense in X, let $q \in [1,\infty]$ be fixed, and let $M \subseteq X$. If (A1), (A2), (A4), (5.4) are satisfied, then the following assertions are equivalent.*

(i) M is a relatively compact subset of X.

(ii) M is bounded in X and there exists a sequence $\mathcal{B} = \{b_n\}_{n=0}^{\infty}$ with $b_n \geq 0$, $b_0 > 0$, and $\|\{\widetilde{b}_n\}_{n=0}^{\infty}\|_q = \infty$ such that $\sup\limits_{f\in M} \left\| \left\{ \widetilde{b}_n\, \omega(f, n^{-1}) \right\}_{n\geq 1/t_0} \right\|_q < \infty$.

Moreover, if M is relatively compact, then it is possible to find a sequence $\mathcal{B}$ of positive numbers such that (ii) holds true. In case $q=\infty$ one even can replace "$b_n \geq 0$, $b_0 > 0$" in (ii) by "$0 < b_0 < b_1 < b_2 < \ldots$".

Proof. (ii)⇒(i): This follows immediately from (5.4) and Theorem 3.1.

(i)⇒(ii): For $q = \infty$ this is already proved (see Remark 5.8) and for $q < \infty$ one can proceed in the same way as in the proof of Theorem 3.1. (One only has to replace $E_n^X(f)$ by $\omega(f, n^{-1})$.)

■

Often it is useful to replace weighted l^q-norms of $\{\omega(f, n^{-1})\}$ by weighted L^q-norms of $\omega(f, t)$. The connection between these two kinds of norms is given in the following lemma.

Lemma 5.10 *Let (A1) be fulfilled and let $b_n = G(n+1)$ with a function $G : [1, \infty) \to (0, \infty)$ satisfying (G1) and, additionally,*

$$G(n) \le \operatorname{const} G(n+1) \quad \text{for all } n = 1, 2, \dots \,. \tag{5.5}$$

Then, for all $f \in X$ and all $i_0 \in \mathbb{N}$ with $2^{1-i_0} \le t_0$,

$$\left\| \left\{ \widetilde{b}_n\, \omega(f, n^{-1}) \right\}_{n=2^{i_0}}^{\infty} \right\|_q \le c \left\| t^{-1/q} G(t^{-1})\, \omega(f,t) \right\|_{L^q(0, 2^{1-i_0})} \quad \text{and}$$

$$\left\| t^{-1/q} G(t^{-1})\, \omega(f,t) \right\|_{L^q(0, 2^{-i_0})} \le c \left\| \left\{ \widetilde{b}_n\, \omega(f, n^{-1}) \right\}_{n=2^{i_0-1}}^{\infty} \right\|_q ,$$

where c does not depend on f and i_0.

Proof. First we remark that, in view of (A1), (G1) and (5.5), for all $i \ge i_0$,

$$b_{2^i}\, \omega(f, 2^{-i}) = (2^i+1)^\gamma \frac{G(2^i+1)}{(2^i+1)^\gamma}\, \omega(f, 2^{-i}) \le c\, G(t^{-1})\, \omega(f,t), \quad 2^{-i} \le t \le 2^{1-i},$$

$$b_{2^i}\, \omega(f, 2^{-i}) \ge c\, 2^{\gamma i} \frac{G(2^i)}{2^{\gamma i}}\, \omega(f, 2^{-i}) \ge c\, G(t^{-1})\, \omega(f,t), \quad 2^{-i-1} \le t \le 2^{-i}.$$

Together with Lemma 5.2, the first inequality implies

$$\begin{aligned}
\left\| \left\{ \widetilde{b}_n\, \omega(f, n^{-1}) \right\}_{n=2^{i_0}}^{\infty} \right\|_q &\le 2^\gamma \left\| \left\{ b_{2^i}\, \omega(f, 2^{-i}) \right\}_{i=i_0}^{\infty} \right\|_q \\
&= 2^\gamma \left\| \left\{ b_{2^i}\, \omega(f, 2^{-i})\, 2^{i/q} \|1\|_{L^q(2^{-i}, 2^{1-i})} \right\}_{i=i_0}^{\infty} \right\|_q \\
&\le c \left\| \left\{ \|t^{-1/q} G(t^{-1})\, \omega(f,t)\|_{L^q(2^{-i}, 2^{1-i})} \right\}_{i=i_0}^{\infty} \right\|_q \\
&= c \left\| t^{-1/q} G(t^{-1})\, \omega(f,t) \right\|_{L^q(0, 2^{1-i_0})} ,
\end{aligned}$$

while the second one yields

$$\begin{aligned}
\left\| \left\{ \widetilde{b}_n\, \omega(f, n^{-1}) \right\}_{n=2^{i_0-1}}^{\infty} \right\|_q &\ge 2^{-\gamma-(1/q)} \left\| \left\{ b_{2^i}\, \omega(f, 2^{-i}) \right\}_{i=i_0}^{\infty} \right\|_q \\
&= 2^{-\gamma} \left\| \left\{ b_{2^i}\, \omega(f, 2^{-i})\, 2^{i/q} \|1\|_{L^q(2^{-i-1}, 2^{-i})} \right\}_{i=i_0}^{\infty} \right\|_q \\
&\ge c \left\| \left\{ \|t^{-1/q} G(t^{-1})\, \omega(f,t)\|_{L^q(2^{-i-1}, 2^{-i})} \right\}_{i=i_0}^{\infty} \right\|_q \\
&= c \left\| t^{-1/q} G(t^{-1})\, \omega(f,t) \right\|_{L^q(0, 2^{-i_0})} .
\end{aligned}$$

■

Since (G2) implies (5.5) (apply (G2) with $n = 1$), we get the following theorems as immediate consequence of Lemma 5.10 and Theorem 5.6 (together with Remark 5.8 for the case $q = \infty$, $b_n = G = 1$).

Theorem 5.11 *Let assumptions (A1)−(A4) be fulfilled and let $b_n = G(n+1)$ with a function $G : [1, \infty) \to (0, \infty)$ satisfying (G1)−(G3). Then, for every $M \subseteq X$, the following assertions are equivalent.*

(i) M is a relatively compact subset of $X_{q,0}^{\mathcal{B}}$.

(ii) M is bounded in X and $\lim_{\delta \downarrow 0} \left\| t^{-1/q} G(t^{-1})\, \omega(f,t) \right\|_{L^q(0,\delta)} = 0$ uniformly in $f \in M$.

Theorem 5.12 *Let $\bigcup X_n$ be dense in X. If (A1), (A2), (A4), and (5.4) are satisfied, then, for every $M \subseteq X$, the following assertions are equivalent.*

(i) M is a relatively compact subset of X.

(ii) M is bounded in X and $\lim_{\delta \downarrow 0} \|\omega(f,t)\|_{L^\infty(0,\delta)} = 0$ uniformly in $f \in M$.

Obviously, the second assertion of (ii) means $\lim_{t \downarrow 0} \omega(f,t) = 0$ uniformly in $f \in M$ (use, for example, $\|\omega(f,t)\|_{L^\infty(0,T)} \geq \omega(f, T/2)$). Since, for all fixed $f \in M$, $M = \{f\}$ is relatively compact in X, we get $\lim_{t \downarrow 0} \omega(f,t) = 0$ for all $f \in X$ if $\bigcup X_n$ is dense in X and (A1), (A2), (A4), (5.4) are satisfied. On the other hand, if (5.4) is satisfied, then it is clear that $\bigcup X_n$ is dense in X if $\lim_{t \downarrow 0} \omega(f,t) = 0$ for all $f \in X$. Thus, the following corollary holds true.

Corollary 5.13 *Let (A1), (A2), (A4), and (5.4) be satisfied. Then $\bigcup X_n$ is dense in X if and only if $\lim_{t \downarrow 0} \omega(f,t) = 0$ for all $f \in X$.*

At the end of this section we shortly discuss the following question: What happens if $\gamma > r$ instead of $\gamma < r$ in assumption (G1)? In the classical case $G(t) = t^\gamma$ (but also in more general cases; see [13], Corollary 2.78), one often has a norm characterization of the following kind:

$$\text{If } \gamma > r \text{ then } \quad \|f\|_{X,\mathcal{B},q} \sim \|f\|_X + \|D_r f\|_{Y,\{(n+1)^{-r} b_n\},q} \ ,$$

where Y is a certain Banach space, $D_r : X_q^{\mathcal{B}} \to Y$ is a linear operator, and $Y_q^{\mathcal{B}}$ is defined with respect to a sequence $Y_1 \subseteq Y_2 \subseteq \ldots$ of finite-dimensional subspaces of Y, e.g. $Y_n = D_r(X_n)$. (Usually, X and Y are weighted L^p-spaces on an interval, r is a natural number, and $D_r = \left(\frac{d}{dx}\right)^r$, where $\frac{d}{dx}$ is the operator of generalized differentiation.) In this case, one can easily show that

(5.6) M is relatively compact in $X_q^{\mathcal{B}}$ if and only if M is relatively compact in X and $D_r(M)$ is relatively compact in $Y_q^{\{(n+1)^{-r} b_n\}}$.

So we see that the problem of relative compactness in $X_q^{\mathcal{B}}$ leads to the question of characterizing relative compactness in the space $Y_q^{\{(n+1)^{-r} b_n\}}$, where the sequence

$\{(n+1)^{-r}b_n\}$ has the "reduced order" $\gamma - r$ instead of γ. If, for example, $\gamma > 1$ is a non-integer and if (5.6) holds true with the integer part r of γ, then the relative compactness in $X_q^{\mathcal{B}}$ can be described with the help of a first order modulus of smoothness in Y (i.e. a modulus which satisfies (A1)–(A4) with Y and 1 instead of X and r).

6. Generalizations

6.1. The case $\dim \mathbf{X_n} = \infty$ ($\mathbf{n \geq n_0}$)

First we will generalize the results of the preceding sections to the case of linear subspaces $X_1 \subseteq X_2 \subseteq \ldots$ which are not necessarily finite-dimensional. So, take all notations and assumptions of Section 1, but omit the condition $\dim X_n < \infty$.

Theorem 6.1 (cf. [13], Section 2.4) *Let (2.1) be satisfied. Then the assertions (i), (iv), (v), and (vi) of Theorem 2.1 remain true.*

Proof. The proofs of (i) and (iv) remain the same with one exception: In (2.7) write $\inf_{P\in X_n} \|\ldots - P\|_X$ instead of $\|\ldots - P_n(f - P_m(f))\|_X$ in the first line and $\inf_{P\in X_{\max\{n,m\}}} \|\ldots - [P - P_m(f)]\|_X$ instead of $\|\ldots - [P_{\max\{n,m\}}(f) - P_m(f)]\|_X$ in the second line. In the proofs of (v) and (vi) we used relation (2.6) for the best approximations $P_m(f)$. But now the existence of best approximations $P_m(f)$ is not ensured. Nevertheless, for each sequence $\{\varepsilon_m\}$ of positive numbers, we can choose $P_m(f, \varepsilon_m) \in X_m$ such that

$$\|f - P_m(f, \varepsilon_m)\|_X \leq E_m^X(f) + \varepsilon_m \, .$$

With these P_m one can easily prove

$$(6.1) \quad E^X_{\max\{n,m\}}(f) \leq E_n^X(f - P_m(f, \varepsilon_m)) \leq E^X_{\max\{n,m\}}(f) + \begin{cases} 0\,, & n \geq m \\ \varepsilon_m\,, & n < m \end{cases}$$

instead of (2.6). Now, in the proof of (v), we only have to take $P_j = P_{m_j}(f, \varepsilon_j)$, where $\varepsilon_j \leq \left(j \, \|\{\widetilde{b}_n\}_{n=0}^{m_j}\|_q\right)^{-1}$. To prove (vi) we need relation (2.9) which is also true in our more general setting, as the following considerations show: From (6.1) it follows, for all $P_m \in X_m$,

$$\begin{aligned} E_m^{X_q^{\mathcal{B}}}(f) &\leq \|f - P_m(f, \varepsilon)\|_{X,\mathcal{B},q} \leq \left\|\{\widetilde{b}_n E^X_{\max\{n,m\}}(f)\}_{n=0}^{\infty}\right\|_q + \varepsilon \left\|\{\widetilde{b}_n\}_{n=0}^{m}\right\|_q \\ &= \left\|\{\widetilde{b}_n E^X_{\max\{n,m\}}(f - P_m)\}_{n=0}^{\infty}\right\|_q + \varepsilon \left\|\{\widetilde{b}_n\}_{n=0}^{m}\right\|_q \\ &\leq \left\|\{\widetilde{b}_n E_n^X(f - P_m)\}_{n=0}^{\infty}\right\|_q + \varepsilon \left\|\{\widetilde{b}_n\}_{n=0}^{m}\right\|_q = \|f - P_m\|_{X,\mathcal{B},q} + \varepsilon \left\|\{\widetilde{b}_n\}_{n=0}^{m}\right\|_q . \end{aligned}$$

Take the infimum over all $P_m \in X_m$ and let ε tend to zero. Then it follows

$$E_m^{X_q^{\mathcal{B}}}(f) = \left\|\{\widetilde{b}_n E^X_{\max\{n,m\}}(f)\}_{n=0}^{\infty}\right\|_q \, .$$

Now the proof of (vi) remains exactly the same as in Theorem 2.1. ∎

Remark 6.2 *Let* $\dim X_n = \infty$ $(n \geq n_0)$. *Then assertions (ii) and (iii) of Theorem 2.1 are false. More precisely,* $X_q^{\mathcal{B}}$ *cannot be compactly imbedded into* X *or* $X_q^{\mathcal{C}}$ *(for all choices of* $\mathcal{B}$ *and* $\mathcal{C}$*), since this would imply the compact embeddings*

$$(X_n\,,\,\|\,.\,\|_{X,\mathcal{B},q}) \subseteq (X_n\,,\,\|\,.\,\|_X) \quad or \quad (X_n\,,\,\|\,.\,\|_{X,\mathcal{B},q}) \subseteq (X_n\,,\,\|\,.\,\|_{X,\mathcal{C},q})\,.$$

But on X_n *the norms* $\|\,.\,\|_{X,\mathcal{B},q}$, $\|\,.\,\|_X$, *and* $\|\,.\,\|_{X,\mathcal{C},q}$ *are equivalent (obviously,* $b_0\|P_n\|_X \leq \|P_n\|_{X,\mathcal{B},q} \leq \|\{\widetilde{b}_m\}_{m=0}^n\|_q\,\|P_n\|_X$, $P_n \in X_n$*). So it would follow that boundedness and relative compactness with respect to* $\|\,.\,\|_X$ *are equivalent in* X_n, *i.e.* $\dim X_n < \infty$ *in contradiction to our assumption.*

In view of Remark 6.2, it seems to be impossible to generalize Theorems 3.1 and 4.1, since we used assertions (ii) and (iii) of Theorem 2.1 to prove them. But nevertheless, we will see that generalizations are possible. First we consider Theorem 3.1. If $\dim X_n = \infty$ $(n \geq n_0)$ then it is clear that not every bounded subset M of $X_q^{\mathcal{B}}$ is relatively compact in X (see Remark 6.2). But if we know that M is bounded in $X_q^{\mathcal{B}}$, then it is much easier to check the relative compactness in X than without this knowledge. Indeed, the following Lemma shows that the relative compactness in X of a bounded subset of $X_q^{\mathcal{B}}$ can be checked as easier as faster the numbers $\|\{\widetilde{b}_m\}_{m=0}^n\|_q$ converge to infinity.

Lemma 6.3 *Let* $\mathcal{B}$ *satisfy (2.1) and set* $c_n := \|\{\widetilde{b}_m\}_{m=0}^n\|_q$. *A bounded subset* M *of* $X_q^{\mathcal{B}}$ *is relatively compact in* X *if and only if for every sequence* $\{f_m\} \subseteq M$ *there exist numbers* $a_1, a_2, \ldots \geq 0$ *with* $\lim_{n\to\infty} a_n = 0$ *such that, for all fixed* $n \in \mathbb{N}$, *the following criterion is fulfilled: Every subsequence of* $\{f_m\}$ *contains a subsequence* $\{f_m\}_{m\in\mathbb{N}_0}$ *(*$\mathbb{N}_0$ *an infinite subset of* $\mathbb{N}$*) such that*

$$\|f_m - f_k\|_X \leq a_n\left[E_n^X(f_m)c_n + E_n^X(f_k)c_n + 1\right] + \varepsilon \quad \text{for } m,k \in \mathbb{N}_0 \setminus [0,m_0), \tag{6.2}$$

where $\varepsilon > 0$ *can be arbitrary small and* $m_0 = m_0(\varepsilon)$ *may depend on* ε.

Proof. If (6.2) is true for all fixed n, then we choose natural $n_0 < n_1 < \ldots$ and infinite $\mathbb{N}_i \subseteq \mathbb{N}$ with $\mathbb{N}_0 \supseteq \mathbb{N}_1 \supseteq \ldots$ such that $a_{n_i}(2\sup_{f\in M}\|f\|_{X,\mathcal{B},q}+1) \leq 2^{-i}$ and

$$\|f_m - f_k\|_X \leq a_{n_i}\left[E_{n_i}^X(f_m)c_{n_i} + E_{n_i}^X(f_k)c_{n_i} + 1\right] + 2^{-i} \quad \text{for all } m,k \in \mathbb{N}_i$$

$(i = 0, 1, \ldots)$. Since $E_n^X(f)c_n \leq \|f\|_{X,\mathcal{B},q}$ for all n (cf. (2.5)), it follows

$$\|f_m - f_k\|_X \leq 2^{1-i} \quad \text{for all } m,k \in \mathbb{N}_i \text{ and all } i = 0,1,\ldots\,. \tag{6.3}$$

If we choose $m_0 < m_1 < \ldots$ with $m_i \in \mathbb{N}_i$, then we have $m_k \in \mathbb{N}_i$ for all $k \geq i$ and (6.3) implies $\|f_{m_i} - f_{m_k}\|_X \leq 2^{1-i}$ for $k \geq i$. This shows that $\{f_{m_k}\}_{k=0}^\infty$ is convergent in X. Thus, M is relatively compact in X. The other direction is trivial (set $a_n = 0$ for all n). ∎

If $\{f_m^{(1)}\}, \{f_m^{(2)}\}, \ldots$ are sequences with $\|f_m - f_m^{(n)}\|_X \leq a_n\left[E_n^X(f_m)c_n + 1\right]$ for all n, m, then it is clear that the left hand side of (6.2) can be replaced by $\|f_m^{(n)} - f_k^{(n)}\|_X$. So we arrive at the following condition on M which is weaker than relative compactness in X and which is equal to relative compactness in X if M is bounded in $X_q^{\mathcal{B}}$.

Definition 6.4 *Let $\mathcal{B}$ satisfy (2.1) and set $c_n := \|\{\widetilde{b}_m\}_{m=0}^n\|_q$. We say that $M \subseteq X$ satisfies the $(\mathcal{B},q)$-condition in X, if for every sequence $\{f_m\}_{m=1}^\infty \subseteq M$ there exist numbers $a_1, a_2, a_3, \ldots \geq 0$ and sequences $\{f_m^{(1)}\}_{m=1}^\infty, \{f_m^{(2)}\}_{m=1}^\infty, \ldots \subseteq X$ such that the following assertions hold true:*

(i) $\lim_{n\to\infty} a_n = 0$.

(ii) $\|f_m - f_m^{(n)}\|_X \leq a_n \left[E_n^X(f_m)c_n + 1\right]$ *for all* $n, m \in \mathbb{N}$.

(iii) For all fixed $n \in \mathbb{N}$, the following criterion is fulfilled:
Every subsequence of $\{f_m^{(n)}\}_{m=1}^\infty$ contains a subsequence $\{f_m^{(n)}\}_{m\in\mathbb{N}_0}$ ($\mathbb{N}_0$ an infinite subset of $\mathbb{N}$) such that

$$\|f_m^{(n)} - f_k^{(n)}\|_X \leq a_n\big[E_n^X(f_m)c_n + E_n^X(f_k)c_n + 1\big] + \varepsilon \text{ for } m,k \in \mathbb{N}_0 \setminus [0,m_0),$$

where $\varepsilon > 0$ can be arbitrary small and $m_0 = m_0(\varepsilon)$ may depend on ε.

Remark 6.5 *If $\dim X_n < \infty$ for all n, then the $(\mathcal{B},q)$-condition is automatically fulfilled for all bounded $M \subseteq X$ and all $\mathcal{B}$ satisfying (2.1). Indeed, for every f_m we choose best approximations $f_m^{(n)} \in X_n$, i.e. $E_n^X(f_m) = \|f_m - f_m^{(n)}\|_X$. Then, (i) and (ii) are satisfied with $a_n = c_n^{-1}$ and (iii) follows from the relative compactness of the sequences $\{f_m^{(n)}\}_{m=1}^\infty$. (The bounded sequence $\{f_m^{(n)}\}_{m=1}^\infty \subseteq X_n$ is relatively compact, since $\dim X_n < \infty$.)*

Remark 6.6 *Let $\mathcal{B}$ satisfy (2.1). One can use the following criterion to check the $(\mathcal{B},q)$-condition: If there exist uniformly bounded linear projections P_n from X onto X_n, then the $(\mathcal{B},q)$-condition is satisfied if all images $P_n(M)$ are relatively compact in X.*

Proof. Let $P_n(M)$ be relatively compact for all n and let $\{f_m\} \subseteq M$. Set $f_m^{(n)} := P_n f_m$ and $a_n = c_n^{-1}(1 + \|P_n\|_{X\to X})$ (c_n from Definition 6.4). Then (i) and (ii) in Definition 6.4 are obviously satisfied. Moreover, every subsequence of $\{f_m^{(n)}\}_{m=1}^\infty \subseteq P_n(M)$ contains a convergent subsequence, which implies assertion (iii) of Definition 6.4. ■

The following theorem is a generalization of Theorem 3.1.

Theorem 6.7 *Let $\bigcup X_n$ be dense in X and let $q \in [1,\infty]$ be fixed. For every set $M \subseteq X$, the following assertions are equivalent.*

(i) M is a relatively compact subset of X.

(ii) There exists some $\mathcal{B} = \{b_n\}_{n=0}^\infty$ with $b_n \geq 0$, $b_0 > 0$, and $\|\{\widetilde{b}_n\}\|_q = \infty$ such that M is a bounded subset of $X_q^{\mathcal{B}}$ which satisfies the $(\mathcal{B},q)$-condition in X.

Moreover, if M is relatively compact, then it is possible to find a sequence $\mathcal{B}$ of positive numbers such that (ii) holds true. In case $q = \infty$ one even can replace "$b_n \geq 0$, $b_0 > 0$" in (ii) by "$0 < b_0 < b_1 < b_2 < \ldots$".

Proof. (i)⇒(ii): If M is relatively compact in X, then the $(\mathcal{B},q)$-condition is satisfied for all $\mathcal{B}$ (set $a_n = 0$ and $f_m^{(n)} = f_m$). The remaining assertions follow in the same way as in the proof of Theorem 3.1 (since the implication "(i)⇒(ii)" of Theorem 1.2 remains true; see Remark 1.3).

(ii)⇒(i): This implication is already proved (see Lemma 6.3 and the considerations before Definition 6.4). ■

Now we are able to generalize Theorem 4.1:

Theorem 6.8 *Let $\mathcal{B}$ satisfy (2.1) and let M be a subset of X. The following assertions are equivalent.*

(i) M is a relatively compact subset of $X_{q,0}^{\mathcal{B}}$.

(ii) There exists a sequence $\{\alpha_n\}_{n=0}^{\infty}$ with $\alpha_n \geq 0$, $\alpha_0 > 0$, and $\lim_{n\to\infty} \alpha_n = \infty$ such that M is a bounded subset of $X_q^{\{b_n\alpha_n\}}$ which satisfies the $(\{b_n\alpha_n\},q)$-condition in X.

Moreover, in (ii) one can replace "$\alpha_n \geq 0$, $\alpha_0 > 0$" by "$0 < \alpha_0 < \alpha_1 < \alpha_2 < \ldots$".

Proof. If (i) is satisfied, then M fulfills the $(\{b_n\alpha_n\},q)$-condition, since M is relatively compact in X. The remaining assertions of (ii) follow in the same way as in the proof of Theorem 4.1. If (ii) is satisfied, then M is a subset of $X_{q,0}^{\mathcal{B}}$ (see the proof of Theorem 4.1) which is relatively compact in X (cf. Theorem 6.7). Now, the relative compactness of M in $X_q^{\mathcal{B}}$ follows as in the proof of Theorem 2.1,(iii). ■

In the same way as Theorem 4.3 one can prove the following

Theorem 6.9 *Let $\mathcal{B}$ satisfy (2.1) and let M be a subset of X. The following assertions are equivalent.*

(i) M is a relatively compact subset of $X_{q,0}^{\mathcal{B}}$.

(ii) M is a bounded subset of X which satisfies the $(\mathcal{B},q)$-condition in X and which has the property $\lim_{m\to\infty} \left\| \left\{ \widetilde{b}_n E_n^X(f) \right\}_{n=m}^{\infty} \right\|_q = 0$ uniformly in $f \in M$.

(Remark that the $(\mathcal{B},q)$-condition is stronger than the $(\{b_n\alpha_n\},q)$-condition of Theorem 6.8,(ii).)

Obviously, the $(\mathcal{B},q)$-condition in assertion (ii) of Theorem 6.9 can be omitted if we replace "M is a bounded subset of X" by "M is a relatively compact subset of X". In the next subsection we will show that, in this form, Theorem 6.9 holds true in a much more general framework. For this aim, we will proceed in another way as in Section 4: First we prove a convergence criterion which generalizes Proposition 4.5 and then we will conclude compactness criteria.

6.2. A general approach via convergence

Let X be a normed space and let $\mathcal{E} : X \to \{\{E_n\}_{n=1}^{\infty} : E_n \geq 0\}$, $f \to \{E_n(f)\}$ be a mapping such that, for all $n \in \mathbb{N}$, the following assumptions are satisfied:

(E1) $E_n(\,.\,)$ is a semi-norm on X.

(E2) There exists a constant $C_n > 0$ such that $E_n(f) \leq C_n \|f\|_X$ for all $f \in X$.

For $n = 0$ we set $E_0(f) := \|f\|_X$.

Remark 6.10 *It is clear that a semi-norm E_n which satisfies (E2) is a continuous map from X into $\mathbb{R}$ (use $|E_n(f) - E_n(g)| \leq E_n(f-g) \leq C_n \|f-g\|_X$). The counterdirection is also true: If the continuous map $E_n : X \to \mathbb{R}$ has the properties of a semi-norm, then (E2) is satisfied. Indeed, if we assume that (E2) is not satisfied, then we can find $f_k \in X \setminus \{0\}$ such that $E_n(f_k) \geq k\|f_k\|_X$ $(k = 1, 2, \ldots)$. This implies $\lim_{k\to\infty} E_n(g_k) = \infty$ for $g_k = f_k/\|f_k\|_X$ and, consequently, $\lim_{k\to\infty} g_k/E_n(g_k) = 0$ in X. From the continuity of E_n it follows $\lim_{k\to\infty} E_n(g_k/E_n(g_k)) = E_n(0) = 0$, in contradiction to $E_n(g_k/E_n(g_k)) = 1$.*

For all sequences $\mathcal{B} = \{b_n\}_{n=0}^{\infty}$ of nonnegative real numbers with $b_0 > 0$ and all $q \in [1, \infty]$ we can define the normed space

$$X_q^{\mathcal{B}} = X_q^{\mathcal{B}}(\mathcal{E}) := \left\{ f \in X : \|f\|_{X,\mathcal{B},q} = \left\| \left\{ \widetilde{b}_n E_n(f) \right\}_{n=0}^{\infty} \right\|_q < \infty \right\}$$

($\widetilde{b}_n$ is defined in (1.3)). In case $q = \infty$ we investigate the linear subspace

$$X_{\infty,0}^{\mathcal{B}} = \{ f \in X : \lim_{n\to\infty} b_n E_n(f) = 0 \}$$

of $X_{\infty}^{\mathcal{B}}$. To have a uniform notation, we set

$$X_{q,0}^{\mathcal{B}} := \begin{cases} X_q^{\mathcal{B}} & \text{if} \quad 1 \leq q < \infty\,, \\ X_{\infty,0}^{\mathcal{B}} \text{ (with the norm of } X_{\infty}^{\mathcal{B}}) & \text{if} \quad q = \infty\,. \end{cases}$$

Proposition 6.11 *$X_{\infty,0}^{\mathcal{B}}$ is a closed subspace of $X_{\infty}^{\mathcal{B}}$.*

Proof. See the end of the proof of assertion (v) of Theorem 2.1. ∎

Lemma 6.12 *Let $n_0 \in \mathbb{N} \cup \{0\}$ and $M \in [0, \infty)$. If $\mathcal{M} \subseteq X$ has the property*

$$\left\| \left\{ \widetilde{b}_n E_n(f) \right\}_{n=n_0}^{\infty} \right\|_q \leq M \quad \textit{for all } f \in \mathcal{M}\,, \tag{6.4}$$

then the closure of $\mathcal{M}$ in X has the same property (i.e. (6.4) holds true for all $f \in \operatorname{clos}_X \mathcal{M}$).

Proof. This can be proved in the same way as (2.3). ∎

Corollary 6.13 *If X is complete, then $X_q^{\mathcal{B}}$ is a Banach space.*

Proof. Since (2.3) is also true in our more general setting (Lemma 6.12 with $n_0 = 0$), the proof is the same as that of assertion (i) of Theorem 2.1. ∎

Now we are able to prove the following characterization of convergence in $X_{q,0}^{\mathcal{B}}$.

Theorem 6.14 *Let $\{f_k\} \subseteq X_{q,0}^{\mathcal{B}}$. The following assertions are equivalent.*

(i) $\{f_k\}$ converges in $X_{q,0}^{\mathcal{B}}$.

(ii) $\{f_k\}$ is convergent in X and $\lim_{m\to\infty} \left\| \left\{ \widetilde{b}_n E_n(f_k) \right\}_{n=m}^{\infty} \right\|_q = 0$ uniformly in k.

(Especially, if (ii) holds, then the X-limit of $\{f_k\}$ is equal to the $X_{q,0}^{\mathcal{B}}$-limit of $\{f_k\}$.)

Proof. (i)⇒(ii): Let $\lim_{k\to\infty} f_k = f$ in $X_{q,0}^{\mathcal{B}}$. Then f is also the X-limit of $\{f_k\}$, since $X_{q,0}^{\mathcal{B}}$ is continuously embedded into X. Moreover, for every $\varepsilon > 0$ there exists some $k_0 = k_0(\varepsilon)$ such that

$$\left\| \left\{ \widetilde{b}_n E_n(f - f_k) \right\}_{n=0}^{\infty} \right\|_q < \frac{\varepsilon}{2} \quad \text{for all } \ k > k_0 \ .$$

Since f and $f_1, \ldots, f_{k_0}$ belong to $X_{q,0}^{\mathcal{B}}$, we have

$$\left\| \left\{ \widetilde{b}_n E_n(f) \right\}_{n=n_0}^{\infty} \right\|_q < \frac{\varepsilon}{2} \quad \text{and} \quad \max_{k=1,\ldots,k_0} \left\| \left\{ \widetilde{b}_n E_n(f_k) \right\}_{n=n_0}^{\infty} \right\|_q < \varepsilon \tag{6.5}$$

for some $n_0 = n_0(\varepsilon)$. Consequently, $\|\{\widetilde{b}_n E_n(f_k)\}_{n=n_0}^{\infty}\|_q < \varepsilon$ for all k, since this is contained in (6.5) for $k \leq k_0$ and for $k > k_0$ we may estimate

$$\left\| \left\{ \widetilde{b}_n E_n(f_k) \right\}_{n=n_0}^{\infty} \right\|_q \leq \left\| \left\{ \widetilde{b}_n E_n(f_k - f) \right\}_{n=n_0}^{\infty} \right\|_q + \left\| \left\{ \widetilde{b}_n E_n(f) \right\}_{n=n_0}^{\infty} \right\|_q < \varepsilon \, .$$

(ii)⇒(i): Let f be the limit of $\{f_k\}$ in X and choose some $n_0 = n_0(\varepsilon)$ such that $\|\{\widetilde{b}_n E_n(f_k)\}_{n=n_0}^{\infty}\|_q \leq \varepsilon/4$ for all k. From Lemma 6.12, applied with $\mathcal{M} = \{f_k\}$, it follows $\|\{\widetilde{b}_n E_n(f)\}_{n=n_0}^{\infty}\|_q \leq \varepsilon/4$ (especially, $f \in X_{q,0}^{\mathcal{B}}$), which implies

$$\left\| \left\{ \widetilde{b}_n E_n(f - f_k) \right\}_{n\geq n_0} \right\|_q \leq \left\| \left\{ \widetilde{b}_n E_n(f) \right\}_{n\geq n_0} \right\|_q + \left\| \left\{ \widetilde{b}_n E_n(f_k) \right\}_{n\geq n_0} \right\|_q \leq \frac{\varepsilon}{2}$$

for all k. Moreover, $E_n(f - f_k) \leq (\max_{n<n_0} C_n) \, \|f - f_k\|_X \leq (\varepsilon/2) \, \|\{\widetilde{b}_n\}_{n=0}^{n_0-1}\|_q^{-1}$ for all $k \geq k_0 = k_0(\varepsilon)$ and all $n < n_0$. Consequently,

$$\left\| \left\{ \widetilde{b}_n E_n(f - f_k) \right\}_{n=0}^{\infty} \right\|_q \leq \left\| \left\{ \widetilde{b}_n E_n(f - f_k) \right\}_{n<n_0} \right\|_q + \left\| \left\{ \widetilde{b}_n E_n(f - f_k) \right\}_{n\geq n_0} \right\|_q \leq \varepsilon$$

for all $k \geq k_0$, i.e. f_k converges to f in the norm of $X_q^{\mathcal{B}}$. ∎

Theorem 6.14 leads to the following compactness criterion.

Theorem 6.15 *Let M be a subset of X. The following assertions are equivalent.*

(i) M is a relatively compact subset of $X_{q,0}^{\mathcal{B}}$.

(ii) M is relatively compact in X and $\lim_{m\to\infty} \|\{\widetilde{b}_n E_n(f)\}_{n=m}^{\infty}\|_q = 0$ uniformly in $f \in M$.

Proof. (ii)⇒(i): Obviously, (ii) implies that M is a subset of $X_{q,0}^{\mathcal{B}}$. Now, (i) follows immediately from (ii) and Theorem 6.14.

(i)⇒(ii): Let M be a relatively compact subset of $X_{q,0}^{\mathcal{B}}$. Then M is also relatively compact in X, since $X_{q,0}^{\mathcal{B}}$ is continuously embedded into X. Let us assume that the second assertion of (ii) is not true. Then there exists a constant $\varepsilon_0 > 0$ and a sequence $\{f_m\}_{m=1}^{\infty} \subseteq M$ such that

$$\left\| \left\{ \widetilde{b}_n E_n(f_m) \right\}_{n=m}^{\infty} \right\|_q \geq \varepsilon_0 \quad \text{for all } m \in \mathbb{N}\,. \tag{6.6}$$

But $\{f_m\}$ contains a subsequence $\{f_{m_j}\}_{j=1}^{\infty}$ which converges in $X_{q,0}^{\mathcal{B}}$. In view of Theorem 6.14, this implies the existence of some $n_0 \in \mathbb{N}$ such that

$$\left\| \left\{ \widetilde{b}_n E_n(f_{m_j}) \right\}_{n=n_0}^{\infty} \right\|_q < \varepsilon_0 \quad \text{for all } j \in \mathbb{N}\,,$$

which is in contradiction with (6.6). ■

Example 6.16 *Let X be a Banach space and let $\omega : X \times (0,1] \to [0,\infty)$ satisfy the assumptions (A1)−(A4) of Section 5 with $t_0 = 1$, where $E_n^X(f)$ are the best approximation errors w.r.t. linear subspaces $\{0\} = X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$ of X ($\dim X_n < \infty$ is not supposed). If the additional conditions*

(A5) $\omega(\lambda f, t) = |\lambda|\,\omega(f,t)$ for all $f \in X$, $t \in (0,1]$, and all scalars λ,

(A6) $\omega(f+g,t) \leq \omega(f,t) + \omega(g,t)$ for all $f, g \in X$ and all $t \in (0,1]$

are satisfied (usually, a modulus of smoothness has these properties), then

$$E_n(f) = \omega(f, n^{-1}) \quad (n \in \mathbb{N})$$

fulfills (E1) and (E2). Moreover, if $b_n = G(n+1)$ satisfy the assumptions (G1)−(G3) of Section 5, then the space $X_{q,0}^{\mathcal{B}}(\mathcal{E})$ is equal to the approximation space $X_{q,0}^{\mathcal{B}}(\{X_n\})$ in the sense of equivalent norms (see Theorem 5.4 and Remark 5.5; for the proof of $X_{\infty,0}^{\mathcal{B}}(\{X_n\}) \subseteq X_{\infty,0}^{\mathcal{B}}(\mathcal{E})$ use that the implications "(i) ⇒ (ii) ⇒ (iii)" of Theorem 5.6 are also true without the assumption $\dim X_n < \infty$). Consequently, Theorem 6.15 gives a compactness criterion for subsets of $X_{q,0}^{\mathcal{B}}(\{X_n\})$ in terms of the modulus ω. This generalizes the equivalence "(i)⇔(iii)" of Theorem 5.6 (in (iii) we have to replace "M is bounded in X" by "M is relatively compact in X") to the case $\dim X_n = \infty$ ($n \geq N_0$).

7. Applications

7.1. Existence of elements with prescribed best approximation errors

Let X be a Banach space and let $X_1 \subset X_2 \subset \ldots$ be non-trivial (i.e. $X_1 \neq \{0\}$) closed linear subspaces of X. ($X_i \subset X_{i+1}$ means that X_i is a proper subset of X_{i+1}.) If $\dim X_n < \infty$ for all n, then it is well-known that, for every decreasing sequence $\{\varepsilon_n\} \subseteq [0, \infty)$ with $\lim_{n\to\infty} \varepsilon_n = 0$, there exists an $f \in X$ such that

$$E_n(f) = \varepsilon_n \text{ for all } n = 0, 1, \ldots, \text{ where } E_n(f) := \inf_{P_n \in X_n} \|f - P_n\|_X \text{ and } X_0 := \{0\}$$

(see [18], Section 2.5). In the present subsection we will give an easy proof of this fact, in which we use Theorem 6.7. Moreover, if X is a Hilbert space, then we will see that this proof also works if $\dim X_n = \infty$ ($n \geq n_0$).

Although the proof of the following lemma is standard, we will give it for the sake of completeness.

Lemma 7.1 *Let $f \in X$ such that, for some $m \in \mathbb{N}$ and some $\varepsilon \geq 0$, $E_m(f) \leq \varepsilon$. If P_m is an element of X_m such that*

P_m is a best approximation of f (i.e. $\|f - P_m\|_X = E_m(f)$) if $E_{m-1}(f) > E_m(f)$, $P_m \notin X_{m-1}$ if $E_{m-1}(f) = E_m(f)$,

then there exists some $\lambda \in \mathbb{R}$ such that $E_{m-1}(f + \lambda P_m) = \varepsilon$.

Proof. $E_{m-1}(f + \lambda P_m)$ is a continuous function in λ, since

$$\begin{aligned} |E_{m-1}(f + \lambda P_m) - E_{m-1}(f + \mu P_m)| &\leq E_{m-1}((f + \lambda P_m) - (f + \mu P_m)) \\ &= |\lambda - \mu|\, E_{m-1}(P_m)\,. \end{aligned}$$

Moreover, $\lim_{\lambda\to\infty} E_{m-1}(f + \lambda P_m) = \infty$, since

$$E_{m-1}(f + \lambda P_m) + E_{m-1}(-f) \geq E_{m-1}(\lambda P_m) = |\lambda|\, E_{m-1}(P_m) \text{ and } E_{m-1}(P_m) > 0.$$

(If $E_{m-1}(f) > E_m(f)$, then $P_m \notin X_{m-1}$, since otherwise $E_{m-1}(f) \leq \|f - P_m\|_X = E_m(f)$.) Now we have

$$E_{m-1}(f + \lambda P_m) \begin{cases} \leq \|f + \lambda P_m\|_X \leq \varepsilon \text{ for } \lambda = -1 & \text{if } E_{m-1}(f) > E_m(f) \\ = E_{m-1}(f) = E_m(f) \leq \varepsilon \text{ for } \lambda = 0 & \text{if } E_{m-1}(f) = E_m(f). \end{cases}$$

Consequently, there must exist some λ_0 such that $E_{m-1}(f + \lambda_0 P_m) = \varepsilon$. ■

Theorem 7.2 *Let $\varepsilon_0 \geq \varepsilon_1 \geq \varepsilon_2 \geq \ldots$ such that $\lim_{n\to\infty} \varepsilon_n = 0$ and let one of the following additional conditions (a) or (b) be satisfied:*

(a) All subspaces X_n are finite-dimensional.

(b) X is a Hilbert space.

Then there exists some $f \in X$ such that $E_n(f) = \varepsilon_n$ for all $n = 0, 1, 2, \ldots$.

Proof. Choose $Q_n \in X_n \setminus X_{n-1}$ with $\|Q_n\|_X = 1$ $(n = 1, 2, \ldots)$. If (a) is satisfied and X is no Hilbert space, then we set $P_n := Q_n$. Otherwise, let $g := \sum_{k=1}^{\infty} 2^{-k}(Q_k - \mathcal{P}_{k-1}Q_k)$, where $\mathcal{P}_n : X \to X_n$ denotes the orthogonal projection onto X_n (obviously, this series is absolutely convergent in X), and set $P_n := \mathcal{P}_n g$. Also in this case we have $P_n \notin X_{n-1}$, since $P_n = \mathcal{P}_n g = \sum_{k=1}^{n} 2^{-k}(Q_k - \mathcal{P}_{k-1}Q_k) \in X_{n-1}$ would imply $Q_n \in X_{n-1}$. Now, let $n \in \mathbb{N}$ be fixed. From Lemma 7.1, applied with $f = 0$ and $m = n$, it follows

$$E_{n-1}(\lambda_n P_n) = \varepsilon_{n-1} \quad \text{for some } \lambda_n \in \mathbb{R}.$$

If $n = 1$, then we set $f_1 := \lambda_1 P_1$. Otherwise, we proceed as follows: If $E_{n-2}(\lambda_n P_n) > E_{n-1}(\lambda_n P_n)$, then let $p_{n-1} \in X_{n-1}$ be a best approximation of P_n. If $E_{n-2}(\lambda_n P_n) = E_{n-1}(\lambda_n P_n)$, then we set $p_{n-1} := P_{n-1}$. In case (b), the best approximation of P_n is given by

$$\mathcal{P}_{n-1}P_n = \mathcal{P}_{n-1}\mathcal{P}_n g = \mathcal{P}_{n-1}g = P_{n-1} \ .$$

So we see that Lemma 7.1, applied with $f = \lambda_n P_n$, $m = n - 1$, and $\lambda_n\, p_{n-1}$ (if $E_{n-2}(\lambda_n P_n) > E_{n-1}(\lambda_n P_n)$) or p_{n-1} (if $E_{n-2}(\lambda_n P_n) = E_{n-1}(\lambda_n P_n)$) instead of P_m , yields

$$E_{n-2}(\lambda_n P_n + \lambda_{n-1} p_{n-1}) = \varepsilon_{n-2} \text{ for some } \lambda_{n-1} \in \mathbb{R}, \text{ where } p_{n-1} = P_{n-1} \text{ in case (b).}$$

If $n = 2$, then we set $f_2 := \lambda_2 P_2 + \lambda_1 p_1$. Otherwise we find, in the same way as above, some $\lambda_{n-2} \in \mathbb{R}$ and some $p_{n-2} \in X_{n-2}$ such that

$$E_{n-3}(\lambda_n P_n + \lambda_{n-1} p_{n-1} + \lambda_{n-2} p_{n-2}) = \varepsilon_{n-3} \ , \quad \text{where } p_{n-2} = P_{n-2} \text{ in case (b).}$$

Repeating this procedure, we finally get $\lambda_n, \ldots, \lambda_1 \in \mathbb{R}$ and $p_n \in X_n, \ldots, p_1 \in X_1$ $(p_n = P_n)$ such that, for $m = 0, 1, \ldots, n-1$,

$$E_m(\lambda_n p_n + \ldots + \lambda_{m+1} p_{m+1}) = \varepsilon_m \ , \quad \text{where } p_i = P_i \text{ in case (b).}$$

Set $f_n = \lambda_n p_n + \ldots + \lambda_1 p_1$. Then we have $E_m(f_n) = E_m(\lambda_n p_n + \ldots + \lambda_{m+1} p_{m+1})$ $(m < n)$. Thus, f_n satisfies

$$E_m(f_n) = \begin{cases} \varepsilon_m & \text{if } m < n \,, \\ 0 & \text{if } m \geq n \,. \end{cases} \tag{7.1}$$

If $\varepsilon_m = 0$ for $m \geq n_0$, then $f = f_{n_0}$ fulfills the assertion of the theorem. Otherwise we have $\varepsilon_m > 0$ for all m and (7.1) shows that the sequence $\{f_n\}_{n=1}^{\infty}$ is bounded in $X_\infty^{\mathcal{B}}$, where $\mathcal{B} = \{\varepsilon_m^{-1}\}_{m=0}^{\infty}$. Moreover, in case (b), the image $\mathcal{P}_m(\{f_n\}_{n=1}^{\infty})$ is relatively compact in X, since it is a bounded subset of the finite-dimensional subspace $\operatorname{span}\{P_1, \ldots, P_m\}$. Indeed,

$$\mathcal{P}_m f_n = \lambda_n \mathcal{P}_m \mathcal{P}_n g + \lambda_{n-1} \mathcal{P}_m \mathcal{P}_{n-1} g + \ldots + \lambda_1 \mathcal{P}_m \mathcal{P}_1 g = \sum_{k=1}^{n} \lambda_k \mathcal{P}_{\min\{k,m\}} g \ .$$

In view of Remarks 6.5 and 6.6, the $(\mathcal{B},\infty)$-condition is satisfied for $M=\{f_n\}_{n=1}^{\infty}$ in both cases (a) and (b). Consequently, $\{f_n\}_{n=1}^{\infty}$ is relatively compact in X (see Theorem 6.7). Thus, there exists a convergent subsequence $\{f_{n_i}\}_{i=1}^{\infty}$. The limit f satisfies $E_m(f)=\varepsilon_m$ for all $m\in\mathbb{N}\cup\{0\}$. This follows from (7.1), applied with $n=n_i$, by taking the limit $i\to\infty$ (see also (2.4)). ■

7.2. Compactness of operators

Let X and Y be Banach spaces and let $\{0\}=X_0\subseteq X_1\subseteq X_2\subseteq\ldots$ and $\{0\}=Y_0\subseteq Y_1\subseteq Y_2\subseteq\ldots$ be finite-dimensional linear subspaces of X and Y, respectively. In all what follows, we will take these subspaces as underlying spaces in the definition of the approximation spaces $X_q^{\mathcal{B}}$ and $Y_q^{\mathcal{B}}$, respectively. Moreover, by $\mathcal{L}(X,Y)$ ($\mathcal{K}(X,Y)$) we denote the set of all bounded (compact) linear operators from X into Y.

Theorem 7.3 *Let $\bigcup Y_n$ be dense in Y and let $q\in[1,\infty]$ be fixed. Then $A\in\mathcal{K}(X,Y)$ if and only if $A\in\mathcal{L}(X,Y_q^{\mathcal{B}})$ for some sequence $\mathcal{B}=\{b_n\}_{n=0}^{\infty}\subseteq(0,\infty)$ with $\|\{\widetilde{b}_n\}_{n=0}^{\infty}\|_q=\infty$. If $q=\infty$, then "$\|\{\widetilde{b}_n\}_{n=0}^{\infty}\|_q=\infty$" can be replaced by "$b_0<b_1<b_2<\ldots$ and $\lim_{n\to\infty}b_n=\infty$".*

Proof. If $A\in\mathcal{L}(X,Y_q^{\mathcal{B}})$ with $\|\{\widetilde{b}_n\}_{n=0}^{\infty}\|_q=\infty$, then $A\in\mathcal{K}(X,Y)$, since $Y_q^{\mathcal{B}}$ is compactly embedded into Y (see assertion (ii) of Theorem 2.1). If $A\in\mathcal{K}(X,Y)$, then the image $A(B_1)$ of the closed unit ball $B_1=\{f\in X:\|f\|_X\leq 1\}$ is relatively compact in Y. In view of Theorem 3.1, it follows the existence of some $\mathcal{B}=\{b_n\}\subseteq(0,\infty)$ with $\|\{\widetilde{b}_n\}_{n=0}^{\infty}\|_q=\infty$ such that $A(B_1)$ is a bounded subset of $Y_q^{\mathcal{B}}$. This implies $A\in\mathcal{L}(X,Y_q^{\mathcal{B}})$. (Use that $Af/\|f\|_X=A(f/\|f\|_X)$ belongs to $A(B_1)$ for all $0\neq f\in X$.) ■

Remark 7.4 *Theorem 7.3 generalizes the following well-known fact: If there exist uniformly bounded linear projections P_n from X onto X_n and if $\bigcup X_n$ is dense in X, then $A\in\mathcal{K}(X,X)$ if and only if $P_nA\rightrightarrows A$ (convergence in operator norm). Indeed, from $\sup_n\|P_n\|_{X\to X}<\infty$ it follows $E_n^X(Af)\sim\|(A-P_nA)f\|_X$ for all n and f. Consequently, $A\in\mathcal{K}(X,X)$ if and only if there exist $b_n>0$ with $\lim_{n\to\infty}b_n=\infty$ such that $b_n\|(A-P_nA)f\|_X\leq\text{const}\,\|f\|_X$ for all $f\in X$ and all n (Theorem 7.3 with $q=\infty$).*

Corollary 7.5 *If $K\in\mathcal{K}(X,Y)$ and $K(X_n)\subseteq Y_n$ for all n, then $K\in\mathcal{K}(X_q^{\mathcal{B}},Y_q^{\mathcal{B}})$ for all $\mathcal{B}=\{b_n\}\subseteq(0,\infty)$ and all $q\in[1,\infty]$.*

Proof. If $\|\{\widetilde{b}_n\}\|_q<\infty$, then $X_q^{\mathcal{B}}=X$ and $Y_q^{\mathcal{B}}=Y$ in the sense of equivalent norms and the assertion is clear. Now, let $\|\{\widetilde{b}_n\}\|_q=\infty$. Without loss of generality we may assume that $\bigcup Y_n$ is dense in Y. (Otherwise we replace X and Y by the closure of $\bigcup X_n$ and $\bigcup Y_n$ in X and Y, respectively. In view of (2.8), this does not change $X_q^{\mathcal{B}}$ and $Y_q^{\mathcal{B}}$.) Obviously, $A\in\mathcal{L}(X,Y)$ and $A(X_n)\subseteq Y_n$ imply $A\in\mathcal{L}(X_q^{\mathcal{B}},Y_q^{\mathcal{B}})$. (Use that $E_n^Y(Af)\leq\|A(f-f_n)\|_Y\leq\text{const}\,\|f-f_n\|_X$ for all $f_n\in X_n$.) It follows that $K\in\mathcal{K}(X,Y)$, i.e. $K\in\mathcal{L}(X,Y_q^{\mathcal{A}})$ for some

$\mathcal{A} = \{a_n\}$ with $\|\{\widetilde{a}_n\}\|_q = \infty$, implies $K \in \mathcal{L}(X_q^{\mathcal{B}}, (Y_q^{\mathcal{A}})_q^{\mathcal{B}}) = \mathcal{L}(X_q^{\mathcal{B}}, Y_q^{\mathcal{C}})$, where $c_n = a_n\|\{\widetilde{b}_m\}_{m=0}^n\|_q + b_n\|\{\widetilde{a}_m\}_{m=0}^n\|_q$ (Theorem 2.1,(vi)). We have $\lim_{n\to\infty} c_n/b_n = \infty$ and, consequently, $Y_q^{\mathcal{C}}$ is compactly embedded into $Y_q^{\mathcal{B}}$ (Theorem 2.1,(iii)). Consequently, $K \in \mathcal{K}(X_q^{\mathcal{B}}, Y_q^{\mathcal{B}})$. ■

Theorem 7.6 *Let $q \in [1,\infty]$ be fixed and let $\mathcal{B} = \{b_n\}_{n=0}^\infty$ be a sequence of positive numbers with $\|\{\widetilde{b}_n\}\|_q = \infty$. Then $A \in \mathcal{K}(X, Y_{q,0}^{\mathcal{B}})$ if and only if $A \in \mathcal{L}(X, Y_q^{\{a_n b_n\}})$ for some sequence $\{a_n\}_{n=0}^\infty \subseteq (0,\infty)$ with $\lim_{n\to\infty} a_n = \infty$. The same holds true if we replace "$\{a_n\}_{n=0}^\infty \subseteq (0,\infty)$" by "$0 < a_0 < a_1 < a_2 < \ldots$".*

Proof. The assertions follow from Theorem 4.1 in the same way as in the proof of Theorem 7.3 (replace Y, $\mathcal{B}$, and $\|\{\widetilde{b}_n\}_{n=0}^\infty\|_q = \infty$ by $Y_{q,0}^{\mathcal{B}}$, $\{b_n a_n\}$, and $\lim_{n\to\infty} a_n = \infty$ (or $a_n < a_{n+1} \to \infty$)). ■

7.3. Convergence and compactness in Sobolev type spaces

Many Sobolev type spaces H^s ($s > 0$), based on a Hilbert space H (usually, $H = L^2$), are defined by the scalar product $\langle f, g\rangle_s = \sum_{n=1}^\infty n^{2s} c_n(f)\,\overline{c_n(g)}$, where $c_n(f)$ are the Fourier coefficients of $f \in H$ w.r.t. some orthonormal basis $\{e_n\}_{n=1}^\infty$ of H. The following proposition shows that such spaces are special types of classical approximation spaces.

Proposition 7.7 ([13], Proposition 2.22) *Let H be a separable Hilbert space with scalar product $\langle\,.\,,\,.\,\rangle$ and let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis of H. Then, for arbitrary $\mathcal{B} = \{b_n\}_{n=0}^\infty \subseteq (0,\infty)$, the approximation space $H_2^{\mathcal{B}}$ w.r.t. the subspaces*

$$H_n := \operatorname{span}\{e_1, e_2, \ldots, e_n\} \qquad (H_0 := \{0\})$$

is also a separable Hilbert space, where the scalar product is given by

$$\langle f, g\rangle_{\mathcal{B}} = \sum_{n=1}^\infty c_n(f)\,\overline{c_n(g)}\,\left\|\left\{\widetilde{b}_k\right\}_{k=0}^{n-1}\right\|_2^2 \qquad \left(\widetilde{b}_k = (k+1)^{-1/2} b_k\right) .$$

($c_n(f) = \langle f, e_n\rangle$ denote the Fourier coefficients of f w.r.t. the basis $\{e_n\}$.) The weighted basis elements $e_n^{\mathcal{B}} := \|\{\widetilde{b}_k\}_{k=0}^{n-1}\|_2^{-1} e_n$ form an orthonormal basis of $H_2^{\mathcal{B}}$ and the Fourier coefficients of $f \in H_2^{\mathcal{B}}$ w.r.t. $\{e_n^{\mathcal{B}}\}$ are $c_n^{\mathcal{B}}(f) = \|\{\widetilde{b}_k\}_{k=0}^{n-1}\|_2\, c_n(f)$. Especially, the Sobolev space H^s ($s > 0$) defined above is equal to the classical approximation space H_2^s in the sense of equivalent norms (since $\|\{\widetilde{b}_k\}_{k=0}^{n-1}\|_2^2 \sim n^{2s}$ in this case).

Proof. It is well-known that $E_n^H(f) = \|\{c_k(f)\}_{k=n+1}^\infty\|_2$ for all n. Consequently,

$$\|f\|_{H,\mathcal{B},2}^2 = \sum_{n=0}^\infty \sum_{k=n+1}^\infty \frac{b_n^2\,|c_k(f)|^2}{(n+1)} = \sum_{k=1}^\infty \sum_{n=0}^{k-1} \frac{b_n^2\,|c_k(f)|^2}{(n+1)} = \langle f, f\rangle_{\mathcal{B}} .$$

Clearly, $H_2^{\mathcal{B}}$ endowed with $\langle\,.\,,\,.\,\rangle_{\mathcal{B}}$ is a Hilbert space and $\left\{e_n^{\mathcal{B}}\right\}$ is an orthonormal system in $H_2^{\mathcal{B}}$. Thus, in view of Theorem 2.1,(v) (or the equivalence $\|\,.\,\|_H \sim$

$\|\,.\,\|_{H,\mathcal{B}_2}$ if $\|\{\widetilde{b}_n\}\|_2 < \infty$), $\left\{e_n^{\mathcal{B}}\right\}$ is an orthonormal basis of $H_2^{\mathcal{B}}$. The formula for $c_n^{\mathcal{B}}(f)$ is obvious. ■

Proposition 7.7 shows that the spaces $H_2^{\mathcal{B}}$ can be viewed as generalizations of the above Sobolev type spaces H^s. Now we want to characterize convergence and compactness in $H_2^{\mathcal{B}}$ with the help of Fourier coefficients.

Theorem 7.8 *Take the notations and assumptions of Proposition 7.7. For all $M \subseteq H$, the following assertions are equivalent.*

(i) M is a relatively compact subset of $H_2^{\mathcal{B}}$.

(ii) M is bounded in H and $\lim_{m\to\infty} \sum_{n=m}^{\infty} |c_n(f)|^2 \, \|\{\widetilde{b}_k\}_{k=0}^{n-1}\|_2^2 = 0$ uniformly in $f \in M$.

Moreover, a sequence $\{f_k\} \subseteq H_2^{\mathcal{B}}$ is convergent in $H_2^{\mathcal{B}}$ if and only if it is convergent in H and relatively compact in $H_2^{\mathcal{B}}$.

Proof. In view of Proposition 7.7, we have

$$E_m^{H_2^{\mathcal{B}}}(f)^2 = \sum_{n=m+1}^{\infty} |c_n^{\mathcal{B}}(f)|^2 = \sum_{n=m+1}^{\infty} |c_n(f)|^2 \left\|\{\widetilde{b}_k\}_{k=0}^{n-1}\right\|_2^2 .$$

Now the assertions follow from Theorem 1.2 and Proposition 4.5. ■

Remark 7.9 *Theorem 7.8 also contains the case $\|\{\widetilde{b}_n\}\|_2 < \infty$ in which $H_2^{\mathcal{B}} = H$ (in the sense of equivalent norms) and $\|\{\widetilde{b}_k\}_{k=0}^{n-1}\|_2 \sim 1$.*

7.4. Compactness in $\mathbf{L^p(R^d, B)}$

Let B be a Banach space and let $1 \leq p < \infty$. We will show that Theorem 6.7 can be used to prove a compactness criterion in $L^p(\mathbb{R}^d, B)$ which generalizes both Kolmogoroff's criterion for compactness in $L^p(\mathbb{R}^d)$ ([19], p.201) and Simon's criterion for compactness in $L^p((0,T),B)$ ([17], Theorem 3.1). Let us first recall the definition and some properties of the space $L^p(\mathbb{R}^d, B)$ (cf. [8]). The Banach space $L^p(\mathbb{R}^d, B)$ consists of all (equivalence classes of) measurable B-valued functions $f : \mathbb{R}^d \to B$, for which the norm

$$\|f\|_{L^p(\mathbb{R}^d,B)} = \|f\|_p := \left(\int_{\mathbb{R}^d} \|f(\mathbf{x})\|_B^p \, d\mathbf{x}\right)^{1/p}$$

is finite. (Remark that the measurability of f implies measurability of $\|f(\,.\,)\|_B$.) Here measurability of f means that $f(\mathbf{x}) = \lim_{n\to\infty} f_n(\mathbf{x})$ (in B) for almost all $\mathbf{x} \in \mathbb{R}^d$, where $f_n : \mathbb{R}^d \to B$ are finitely-valued functions, i.e. picewise constant functions w.r.t. decompositions of $\mathbb{R}^d$ into finitely many Lebesgue-measurable subsets. We need the following facts (see [10], Theorems 3.7.4,3.7.6, and [8], III.3.8):

(A) A measurable function $f : \mathbb{R}^d \to B$ belongs to $L^1(\mathbb{R}^d, B)$ if and only if the Bochner integral $\int_{\mathbb{R}^d} f(\mathbf{x})\, d\mathbf{x} \in B$ exists. In this case we have

$$\left\| \int_{\mathbb{R}^d} f(\mathbf{x})\, d\mathbf{x} \right\|_B \leq \int_{\mathbb{R}^d} \|f(\mathbf{x})\|_B \, d\mathbf{x}\,.$$

(B) The set of all finitely-valued $f : \mathbb{R}^d \to B$ with $\operatorname{meas}\{\mathbf{x} : f(\mathbf{x}) \neq 0\} < \infty$ (such functions are called simple) is dense in $L^p(\mathbb{R}^d, B)$.

Simple functions are nothing else than finite sums of functions of the type $\chi_A\, b$, where $A \subseteq \mathbb{R}^d$ is a Lebesgue-measurable set with $\operatorname{meas} A < \infty$, χ_A is the characteristic function of A, and b is an element of B. It is well-known that the set $C_0(\mathbb{R}^d)$ of all real-valued continuous functions on $\mathbb{R}^d$ with compact support is dense in $L^p(\mathbb{R}^d, \mathbb{R})$. Thus, χ_A is the $L^p(\mathbb{R}^d, \mathbb{R})$-limit of certain $f_m \in C_0(\mathbb{R}^d)$ and it follows $\chi_A\, b = \lim_{m\to\infty} f_m b$ in $L^p(\mathbb{R}^d, B)$, where $f_m b$ are continuous and compactly supported functions from $\mathbb{R}^d$ into B. So we obtain, as a consequence of (B),

(C) The set of all continuous and compactly supported functions $f : \mathbb{R}^d \to B$ is dense in $L^p(\mathbb{R}^d, B)$.

Of course, the Bochner integral and the L^p-space of B-valued functions are also defined for measurable subsets $M \subseteq \mathbb{R}^d$. One only needs to extend functions f on M to the whole $\mathbb{R}^d$ by $f(\mathbf{x}) := 0$ outside M.

For $\mathbf{x} = (x_1, \ldots, x_d)$ and $\mathbf{y} = (y_1, \ldots, y_d)$ we write $\mathbf{x} < \mathbf{y}$ if $x_j < y_j$ for all j. In this case we set

$$[\mathbf{x}, \mathbf{y}) := [x_1, y_1) \times \ldots \times [x_d, y_d) \qquad \text{(such sets are called intervals in } \mathbb{R}^d\text{)}.$$

For every $n \in \mathbb{N}$, $[-n, n)^d$ can be decomposed into the disjoint subintervals

$$I^n_{\mathbf{j}} := [2^{-n}\mathbf{j}, 2^{-n}(\mathbf{j} + \mathbf{1})),\ \ \mathbf{j} = (j_1, \ldots, j_d) \in J_n = \{-2^n n, -2^n n + 1, \ldots, 2^n n - 1\}^d,$$

where $\mathbf{1} := (1, 1, \ldots, 1)$. Now we define the following linear subspaces X_n $(n \in \mathbb{N})$ of $X = L^p(\mathbb{R}^d, B)$:

$$X_n := \{f \in X : f \equiv 0 \text{ outside } [-n, n)^d \text{ and } f \equiv b_{\mathbf{j}} \in B \text{ on } I^n_{\mathbf{j}}\ (\mathbf{j} \in J_n)\}.$$

Obviously, the operator $P_n : X \to X_n$ defined by

$$P_n f \equiv 0 \text{ outside } [-n, n)^d \quad \text{and} \quad P_n f \equiv 2^{nd} \int_{I^n_{\mathbf{j}}} f(\mathbf{x})\, d\mathbf{x} \text{ on } I^n_{\mathbf{j}}\ (\mathbf{j} \in J_n)$$

is a bounded linear projection onto X_n with $\|P_n\| = 1$. ($\|P_n\| \geq 1$ is clear and the estimate

$$\|P_n f\|^p_X \leq \sum_{\mathbf{j}} 2^{ndp} |I^n_{\mathbf{j}}| \left(\int_{I^n_{\mathbf{j}}} \|f(\mathbf{x})\|_B \, d\mathbf{x} \right)^p \leq \sum_{\mathbf{j}} 2^{ndp} |I^n_{\mathbf{j}}|^p \|f\|^p_{L^p(I^n_{\mathbf{j}}, B)}$$

shows that $\|P_n\| \leq 1$.)

Lemma 7.10 *Let $f \in L^p(\mathbb{R}^d, B)$ and set $\omega(f,\delta) := \sup_{0<|\mathbf{h}|\leq\delta} \|f(\,.\,+\mathbf{h}) - f(\,.\,)\|_p$, where $|\mathbf{h}| := \max\{|h_1|,\ldots,|h_d|\}$. Then we have $\lim_{n\to\infty} P_n f = f$ in $L^p(\mathbb{R}^d, B)$ and*

$$\|f - P_n f\|_p^p \leq \int_{|\mathbf{x}|\geq n} \|f(\mathbf{x})\|_B^p \, d\mathbf{x} + \left(2^d \omega(f, 2^{-n})\right)^p \quad \text{for all } n \in \mathbb{N}\,.$$

Especially, $\bigcup X_n$ is dense in $X = L^p(\mathbb{R}^d, B)$.

Proof. Using the substitution $\mathbf{y} = \mathbf{x} + \mathbf{h}$ we obtain, for all $\mathbf{j} \in J_n$,

$$\begin{aligned}
\int_{I_{\mathbf{j}}^n} \|(f - P_n f)(\mathbf{x})\|_B^p \, d\mathbf{x} &= \int_{I_{\mathbf{j}}^n} \left\| \int_{I_{\mathbf{j}}^n} 2^{nd}[f(\mathbf{x}) - f(\mathbf{y})]\, d\mathbf{y} \right\|_B^p d\mathbf{x} \\
&\leq \int_{I_{\mathbf{j}}^n} \left(\int_{I_{\mathbf{j}}^n} 2^{nd} \|f(\mathbf{x}) - f(\mathbf{y})\|_B \, d\mathbf{y} \right)^p d\mathbf{x} \\
&\leq \int_{I_{\mathbf{j}}^n} \left(\int_{|\mathbf{h}|\leq 2^{-n}} 2^{nd} \|f(\mathbf{x}) - f(\mathbf{x}+\mathbf{h})\|_B \, d\mathbf{h} \right)^p d\mathbf{x}\,.
\end{aligned}$$

(The integrals are well-defined, since $f(\mathbf{x}) - f(\mathbf{y})$ and $f(\mathbf{x}) - f(\mathbf{x}+\mathbf{h})$ are measurable functions from $\mathbb{R}^d \times \mathbb{R}^d$ into B.) Together with the generalized Minkowski inequality

$$\left[\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} g(\mathbf{h}, \mathbf{x})\, d\mathbf{h} \right)^p d\mathbf{x} \right]^{1/p} \leq \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} g(\mathbf{h}, \mathbf{x})^p \, d\mathbf{x} \right]^{1/p} d\mathbf{h}$$

($g : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ measurable; cf. [9], p.148, or [15], Theorem V.6.2) we get

$$\begin{aligned}
\|f - P_n f\|_X^p &= \int_{|\mathbf{x}|\geq n} \|f(\mathbf{x})\|_B^p \, d\mathbf{x} + \sum_{\mathbf{j}\in J_n} \int_{I_{\mathbf{j}}^n} \|(f - P_n f)(\mathbf{x})\|_B^p \, d\mathbf{x} \\
&\leq \int_{|\mathbf{x}|\geq n} \|f(\mathbf{x})\|_B^p \, d\mathbf{x} + \int_{\mathbb{R}^d} \left(\int_{|\mathbf{h}|\leq 2^{-n}} 2^{nd} \|f(\mathbf{x}) - f(\mathbf{x}+\mathbf{h})\|_B \, d\mathbf{h} \right)^p d\mathbf{x} \\
&\leq \int_{|\mathbf{x}|\geq n} \|f(\mathbf{x})\|_B^p \, d\mathbf{x} + \left(\int_{|\mathbf{h}|\leq 2^{-n}} 2^{nd} \left[\int_{\mathbb{R}^d} \|f(\mathbf{x}) - f(\mathbf{x}+\mathbf{h})\|_B^p \, d\mathbf{x} \right]^{1/p} d\mathbf{h} \right)^p \\
&\leq \int_{|\mathbf{x}|\geq n} \|f(\mathbf{x})\|_B^p \, d\mathbf{x} + \left(2^d \omega(f, 2^{-n})\right)^p\,.
\end{aligned}$$

If f is a continuous and compactly supported function, then it is clear that the right hand side of this estimate converges to zero for $n \to \infty$. Now the strong convergence $P_n \to I$ follows from (C) and the Banach-Steinhaus theorem. ■

Now we are able to prove the main theorem of the present subsection:

Theorem 7.11 *A bounded set $\mathcal{M} \subseteq L^p(\mathbb{R}^d, B)$ is relatively compact in $L^p(\mathbb{R}^d, B)$ if and only if the following three conditions are satisfied:*

(i) $\lim_{k\to\infty} \int_{|\mathbf{x}|\geq k} \|f(\mathbf{x})\|_B^p \, d\mathbf{x} = 0$ *uniformly in* $f \in \mathcal{M}$.

(ii) *The set* $\left\{ \int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{x})\, d\mathbf{x} : f \in \mathcal{M} \right\} \subseteq B$ *is relatively compact in* B *for all* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ *with* $\mathbf{a} < \mathbf{b}$.

(iii) $\lim_{\delta \downarrow 0} \omega(f, \delta) = 0$ *uniformly in* $f \in \mathcal{M}$.

Remark 7.12 *It is clear that (iii) is equivalent to*

$$\lim_{|\mathbf{h}|\to 0} \|f(\,.\,+\mathbf{h}) - f(\,.\,)\|_p = 0 \text{ uniformly in } f \in \mathcal{M}.$$

Now, using (i), it is easy to prove that (iii) can be replaced by

(iii)' *For every compact* $K \subseteq \mathbb{R}^d$ *we have* $\lim_{|\mathbf{h}|\to 0} \|f(\,.\,+\mathbf{h}) - f(\,.\,)\|_{L^p(K,B)} = 0$ *uniformly in* $f \in \mathcal{M}$.

Proof of Theorem 7.11. Let $\mathcal{M}$ be relatively compact. The union of all $Y_n := L^p([-n,n]^d, B)$ (considered as subspaces of $L^p(\mathbb{R}^d, B)$) is dense in $X = L^p(\mathbb{R}^d, B)$ (see (C)). Consequently, by Theorem 1.2 (together with Remark 1.3), $\lim_{n\to\infty} \inf_{f_n \in Y_n} \|f - f_n\|_X = \lim_{n\to\infty} \|f\|_{L^p(\mathbb{R}^d \setminus [-n,n]^d, B)} = 0$ uniformly in $f \in \mathcal{M}$. Thus, (i) is satisfied. From (A) and Hölder's inequality it follows that the linear operator $f \in X \to \int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{x})\, d\mathbf{x} \in B$ is bounded. This shows that (ii) holds true. From Lemma 7.10 it follows that the implication "(i)⇒(ii)" of Theorem 1.2 (see Remark 1.3) can be applied with the subspaces X_n (which are obviously nested). Consequently, $E_n(f) := \inf_{f_n \in X_n} \|f - f_n\|_X$, i.e. $\|f - P_n f\|_X$ ($\leq 2E_n(f)$ since $\|P_n\| = 1$), converges to zero uniformly in $f \in \mathcal{M}$. Let $0 < |\mathbf{h}| \leq 1$. By $\chi_{n,\mathbf{h}}$ we denote the characteristic function of the set

$$N_{n,\mathbf{h}} := \bigcup_{\mathbf{j}\in J_n} \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x} - \mathbf{y}| \leq |\mathbf{h}| \text{ for some } \mathbf{y} \in \partial I_{\mathbf{j}}^n\}\,.$$

It is clear that $(P_n f)(\mathbf{x}+\mathbf{h}) = (P_n f)(\mathbf{x})$ for all $\mathbf{x} \notin N_{n,\mathbf{h}}$ and it follows

$$\begin{aligned}\|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})\|_B \leq \|f(\mathbf{x}+\mathbf{h}) - (P_n f)(\mathbf{x}+\mathbf{h})\|_B + \|f(\mathbf{x}) - (P_n f)(\mathbf{x})\|_B \\ + \|(P_n f)(\mathbf{x}+\mathbf{h}) - (P_n f)(\mathbf{x})\|_B \, \chi_{n,\mathbf{h}}(\mathbf{x}).\end{aligned}$$

Now we obtain, for all $n \in \mathbb{N}$ and all $\mathbf{h} \in \mathbb{R}^d$ with $0 < |\mathbf{h}| \leq 1$,

$$\|f(\,.\,+\mathbf{h}) - f(\,.\,)\|_p \leq 4E_n(f) + \left(\int_{N_{n,\mathbf{h}}} \|(P_n f)(\mathbf{x}+\mathbf{h}) - (P_n f)(\mathbf{x})\|_B^p \, d\mathbf{x} \right)^{1/p}.$$

For $\varepsilon > 0$ we choose n_0 such that $4E_{n_0}(f) < \varepsilon/2$ for all $f \in \mathcal{M}$. For all $f_0 = \sum_{\mathbf{j}\in J_0} \chi_{\mathbf{j}}\, b_{\mathbf{j}} \in X_{n_0}$ ($J_0 := J_{n_0}$; $\chi_{\mathbf{j}}$: characteristic function of $I_{\mathbf{j}}^{n_0}$) we have $\sup\{\|f_0(x)\|_B : x \in \mathbb{R}^d\} = \max\{\|b_{\mathbf{j}}\|_B : \mathbf{j} \in J_0\} \leq (\sum_{\mathbf{j}\in J_0} \|b_{\mathbf{j}}\|_B^p)^{1/p} = 2^{n_0 d/p} \|f_0\|_p$. Thus, $\|(P_{n_0} f)(\mathbf{x})\|_B \leq 2^{n_0 d/p} \|f\|_p \leq C(n_0)$ for all $f \in \mathcal{M}$. Now we use that the Lebesgue-measure of the set $N_{n_0,\mathbf{h}}$ can be estimated by $c(n_0)|\mathbf{h}|$, and it follows

$$\|f(\,.\,+\mathbf{h}) - f(\,.\,)\|_p \leq \frac{\varepsilon}{2} + [(2C(n_0))^p \operatorname{meas} N_{n_0,\mathbf{h}}]^{1/p} \leq \frac{\varepsilon}{2} + 2C(n_0)\,(c(n_0)|\mathbf{h}|)^{1/p}$$

for all $f \in \mathcal{M}$. If $|\mathbf{h}|$ is small enough, then the right hand side is smaller than ε. Thus, (iii) is proved. Now, let (i)−(iii) be satisfied. In view of Theorem 6.7 (with $q = \infty$), Remark 6.6 and Lemma 7.10, it remains to show that $P_n(\mathcal{M})$ is relatively compact in B for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be fixed and let $\{P_n f_k\}_{k=0}^{\infty} \subseteq P_n(\mathcal{M})$. From (ii) it follows the existence of a subsequence $\{f_{k_i}\}_{i=1}^{\infty}$ such that, for all $\mathbf{j} \in J_n$, $\int_{I_{\mathbf{j}}^n} f_{k_i}(\mathbf{x})d\mathbf{x}$ converges to some $b_{\mathbf{j}} \in B$ for $i \to \infty$. Obviously, this implies that $\{P_n f_{k_i}\}_{i=1}^{\infty}$ is convergent in X.

■

Remark 7.13 *If $\mathcal{M} \subseteq L^p(\mathbb{R}^d, B)$ satisfies the conditions (i)−(iii) of Theorem 7.11, then $\mathcal{M}$ is bounded in $L^p(\mathbb{R}^d, B)$. This means that it is not necessary to suppose the boundedness of $\mathcal{M}$ in Theorem 7.11.*

Proof. If (i)−(iii) are satisfied, then we can find some n_0 such that $\|f - P_{n_0} f\|_p \leq 1$ for all $f \in \mathcal{M}$ (see Lemma 7.10). From (ii) it follows that the values $P_{n_0} f_{|I_{\mathbf{j}}^{n_0}} = 2^{n_0 d} \int_{I_{\mathbf{j}}^{n_0}} f(\mathbf{x})\, d\mathbf{x}$ are uniformly bounded in $f \in \mathcal{M}$ and $\mathbf{j} \in J_{n_0}$. This implies $C := \sup\{\|P_{n_0} f\|_p : f \in \mathcal{M}\} < \infty$ and, consequently, $\|f\|_p \leq C + 1$ for all $f \in \mathcal{M}$.

■

Remark 7.14 *Let $M \subseteq \mathbb{R}^d$ be a measurable set. Then $L^p(M, B)$ can be viewed as a subspace of $L^p(\mathbb{R}^d, B)$ if functions on M are extended by zero onto the whole $\mathbb{R}^d$. Thus, Theorem 7.11 can also be used to characterize compactness in $L^p(M, B)$.*

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