Generalized Approximation Spaces and Applications

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In the paper we generalize the theory of classical approximation spaces to a much wider class of spaces which are defined with the help of best approximation errors. We also give some applications. For example, we show that generalized approximation spaces can be used to find natural (in some sense) domains of definition of unbounded operators.

Key Words: Approximation spaces; Besov spaces; Embedding theorems; Extension of unbounded operators; Representation theorem

1. INTRODUCTION

One of the main tasks in approximation theory is the following problem: How good can complicated functions of a large space X be approximated by more simple functions (as polynomials, splines or rational functions) which belong to a certain subset $A \subset X$. For example, we may take $X = L^p_{2\pi}$, the space of all 2π-periodic L^p-functions on $\mathbb{R}$, and $A = \mathcal{T}$, the set of all trigonometric polynomials. Then, for any $f \in L^p_{2\pi}$, we may define the sequence of errors of best approximation $\{E^*_n(f)\}_{n\in\mathbb{N}}$, where

* Partially supported by Junta de Andalucía, Grupo de Investigación Aproximación y Métodos Numéricos, código FQM 0178

$E_n^*(f) = \inf_{T\in\mathcal{T}_n} \|f-T\|_{L_p(0,2\pi)}$ and $\mathcal{T}_n$ is the set of all trigonometric polynomials of degree less than n. It is natural to hope that some rule of the kind "as smoother the function as smaller the best approximation errors" holds true and in fact it is one of the basic aims in approximation theory to find out the connection between smoothness and best approximation errors.

For approximation by trigonometric polynomials it were Jackson and Bernstein who solved this problem proving the so called direct and inverse theorems. The inverse was more difficult and did not have an algebraic counterpart until the work of Timan, Nikolskii, Dzyadyk and others.

From direct and inverse theorems it follows that certain classical function spaces can be viewed as special approximation spaces and in our opinion this is one of the best mathematical expression of equivalences between the degree of smoothness of functions and the behaviour of their best approximation errors.

The aim of this paper is twofold: the exposition of this philosophy and the generalization of the classical concept of approximation spaces (where we will prove several elementary facts on these newer spaces without the use of moduli of smoothness or K-functionals, which are natural tools for the classical theory). As applications, we will use generalized approximation spaces to give proofs of a few classical and new theorems.

2. SOME CLASSICAL FUNCTION SPACES

For the sake of simplicity we only consider spaces of functions defined on $[-1,1]$ or spaces of 2π-periodic functions defined on $\mathbb{R}$.

2.1. Hölder spaces

Let $k \in \{0,1,2,\dots\}$ and $\alpha \in (0,1]$. Hölder spaces on $\mathbb{R}$ (periodic case) or $[-1,1]$ are defined by

$$C_{2\pi}^{k,\alpha} := \{f \in C_{2\pi}^k : f^{(k)} \in \operatorname{Lip}\alpha\} \quad \text{and}$$
$$C^{k,\alpha}[-1,1] := \{f \in C^k[-1,1] : f^{(k)} \in \operatorname{Lip}\alpha\},$$

where $C_{2\pi}^k$ denotes the space of all 2π-periodic functions with continuous k-th derivative and $f^{(k)} \in \operatorname{Lip}\alpha$ means that $f^{(k)}$ is Hölder continuous with exponent α. With the norm

$$\|f\|_{C^{k,\alpha}} := \sum_{\nu=0}^{k} \sup_x |f^{(\nu)}(x)| + \sup_{x\neq y} \frac{|f^{(k)}(x)-f^{(k)}(y)|}{|x-y|^\alpha}$$

$C_{2\pi}^{k,\alpha}$ and $C^{k,\alpha}[-1,1]$ become Banach spaces.

2.2. Besov spaces of 2π-periodic functions

Let $f \in L^p_{2\pi}$ $(1 \le p \le \infty)$ and $r \in \mathbb{N}$. The classical L^p-modulus of smoothness of order r is defined by $\omega_r(f,t)_p := \sup\limits_{0<h\le t} \|\Delta^r_h f\|_{L^p(0,2\pi)}$, where

$$(\Delta^r_h f)(x) := \sum_{k=0}^{r} (-1)^k \begin{pmatrix} r \\ k \end{pmatrix} f\left(x + \left(\frac{r}{2} - k\right) h\right) .$$

One possibility to define the classical Besov spaces $B^s_q(L^p_{2\pi})$ $(0 < s < \infty$, $1 \le q \le \infty)$ is the following: Choose some natural number $r > s$ (the definition will not depend on this choice in the sense of equivalent norms) and denote by $B^s_q(L^p_{2\pi})$ the set of all $f \in L^p_{2\pi}$ for which

$$\|f\|_{B^s_q(L^p_{2\pi})} = \|f\|_{L^p(0,2\pi)} + \|t^{-s-(1/q)}\omega_r(f,t)_p\|_{L^q(0,1)} < \infty .$$

It is well-known that $B^s_q(L^p_{2\pi})$ is a Banach space. (One can also define $B^s_q(L^p_{2\pi})$ for $0 < p < 1$ and/or $0 < q < 1$. But in these cases we only get quasi-Banach spaces.)

Remark 2.1 *If $\alpha \ne 1$ then $C^{k,\alpha}_{2\pi} = B^{k+\alpha}_\infty(L^\infty_{2\pi})$. If $\alpha = 1$ then the Hölder space $C^{k,1}_{2\pi}$ is different from the so-called Zygmund space $B^{k+1}_\infty(L^\infty_{2\pi})$. The spaces belonging to the scale $B^s_\infty(L^\infty_{2\pi})$ are called Hölder-Zygmund spaces.*

2.3. Weighted Besov spaces on $[-1,1]$

Let $f \in L^p_{\rho,\tau}$ $(1 \le p \le \infty)$, where $\rho, \tau \left\{ \begin{array}{ll} > -1/p & , \ p < \infty \\ \ge 0 & , \ p = \infty \end{array} \right\}$ and

$$L^p_{\rho,\tau} := \{f : fv^{\rho,\tau} \in L^p\} \ (\|f\|_{L^p_{\rho,\tau}} := \|fv^{\rho,\tau}\|_{L^p}), \ \ v^{\rho,\tau}(x) = (1-x)^\rho(1+x)^\tau$$

$(L^p = L^p(-1,1))$. With the help of the r-th difference $(\Delta^r_{h\varphi} f)(x) := (\Delta^r_{h\sqrt{1-x^2}} f)(x)$ $(r \in \mathbb{N})$ one can define the weighted (main part) φ-modulus of smoothness of order r:

$$\Omega^r_\varphi(f,t)_{p,\rho,\tau} := \sup_{0<h\le t} \|v^{\rho,\tau} \Delta^r_{h\varphi} f\|_{L^p(-1+2r^2h^2, 1-2r^2h^2)}$$

(see [8]). Analogously to the periodic case one may introduce the weighted Besov spaces $B^s_q(L^p_{\rho,\tau})$ $(0 < s < \infty$, $1 \le q \le \infty)$ as follows: Choose some natural number $r > s$ (the definition will not depend on this choice in the sense of equivalent norms) and denote by $B^s_q(L^p_{\rho,\tau})$ the set of all $f \in L^p_{\rho,\tau}$ for which

$$\|f\|_{B^s_q(L^p_{\rho,\tau})} = \|f\|_{L^p_{\rho,\tau}} + \|t^{-s-(1/q)}\,\Omega^r_\varphi(f,t)_{p,\rho,\tau}\|_{L^q(0,1)} < \infty .$$

It is well-known that $B^s_q(L^p_{\rho,\tau})$ is a Banach space. (For the quasi-Banach space setting, i.e. $0<p<1$ and/or $0<q<1$, we refer to [10].)

We mention that Remark 2.1 is not true if we replace $C^{k,\alpha}_{2\pi}$ and $B^{k+\alpha}_\infty(L^\infty_{2\pi})$ by $C^{k,\alpha}[-1,1]$ and $B^{k+\alpha}_\infty(L^\infty_{0,0})$. Nevertheless, also $C^{k,\alpha}[-1,1]$ with $\alpha\neq 1$ belongs to a certain class of Besov spaces. But these spaces have to be defined with the help of classical moduli of smoothness instead of the non-classical $\Omega^r_\varphi(f,t)_{\infty,0,0}$.

2.4. Besov norms and best approximation errors

Take the notations and assumptions of Subsections 2.2 and 2.3 (for the Banach space setting, i.e. $1\le p,q\le\infty$). Moreover, for $f\in L^p_{2\pi}$ and $f\in L^p_{\rho,\tau}$, respectively, set

$$E^*_n(f)_p := \inf_{T\in\mathcal{T}_n}\|f-T\|_{L^p(0,2\pi)} \quad\text{and}\quad E^{\rho,\tau}_n(f)_p := \inf_{P\in\mathcal{P}_n}\|f-P\|_{L^p_{\rho,\tau}},$$

where $\mathcal{T}_n$ and $\mathcal{P}_n$ denote the sets of all polynomials of degree less than n (trigonometric and algebraic, respectively). For $n=0$ this means $E^*_0(f)_p=\|f\|_{L^p(0,2\pi)}$ and $E^{\rho,\tau}_0(f)_p=\|f\|_{L^p_{\rho,\tau}}$. As announced in the introduction, the Besov spaces can be characterized with the help of best approximation errors:

Theorem 2.2 ([6], 7.Theo.9.2 and [7], Theo.3.1)

$$B^s_q(L^p_{2\pi}) = \{f\in L^p_{2\pi} : \|f\|^{\mathrm{trig}}_{B^s_q} = \|\{(n+1)^{s-(1/q)}E^*_n(f)_p\}^\infty_{n=0}\|_{l^q}<\infty\},$$
$$B^s_q(L^p_{\rho,\tau}) = \{f\in L^p_{\rho,\tau} : \|f\|^{\mathrm{alg}}_{B^s_q} = \|\{(n+1)^{s-(1/q)}E^{\rho,\tau}_n(f)_p\}^\infty_{n=0}\|_{l^q}<\infty\}$$

(in the sense of equivalent norms).

We mention that also classical (non-weighted) Besov spaces on $[-1,1]$ (defined with classical moduli of smoothness), especially $C^{k,\alpha}[-1,1]$ with $\alpha\neq 1$, can be defined equivalently with the help of best approximation errors. But then we have to consider, for example, approximation by polynomial splines with equally spaced knots (cf. [6], 12.Theo.2.4).

Remark 2.3 *Recently, S. Li proved that the second identity in Theorem 2.2 also holds true if $1\le p\le\infty$ and $0<q<1$. If $0<p<1$ and $0<q\le\infty$, then the Besov space $B^s_q(L^p_{\rho,\tau})$ must be defined with another modulus of smoothness $\omega^r_\varphi(f,t)_{p,\rho,\tau}$ (see [8]) instead of $\Omega^r_\varphi(f,t)_{p,\rho,\tau}$ to ensure that the second identity in Theorem 2.2 remains true. (cf. [10])*

Theorem 2.2 says more than the classical counterparts of the type

$$E_n(f)\le \mathrm{const}\, n^{-s},\ n\in\mathbb{N} \iff \omega(f,t)\le \mathrm{const}\, t^s,\ t\in(0,1]. \qquad (2.1)$$

Indeed, (2.1) corresponds to the case $q = \infty$ of Theorem 2.2. So we see that connections between the behaviours of moduli of smoothness and best approximation errors can be written as equivalences between Besov norms and norms of so-called approximation spaces. Moreover, the generalized approximation spaces, which we will study in the following sections, can be viewed as generalizations of classical function spaces and it is surprising how many properties of classical spaces can be proved in the framework of generalized approximation spaces.

3. GENERALIZED APPROXIMATION SPACES

3.1. Definition

Classical approximation spaces are defined as follows ([13]): Let X be a quasi-normed space and let $\{A_n\}$ be a sequence of subsets of X such that

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots, \tag{3.1}$$

$$\lambda A_n \subseteq A_n \text{ for all scalars } \lambda \text{ and all } n \in \mathbb{N}, \tag{3.2}$$

$$A_n + A_m \subseteq A_{n+m} \text{ for all } n, m \in \mathbb{N}. \tag{3.3}$$

(Later we will replace (3.3) by the more general condition (3.6).) Set $A_0 := \{0\}$ and

$$E_n^X(f) := \inf\{\|f - f_n\|_X : f_n \in A_n\} \quad \text{(particularly, } E_0^X(f) = \|f\|_X\text{)}.$$

For $0 < s < \infty$ and $0 < q \le \infty$ the classical approximation space $A_q^s(X)$ is defined by

$$A_q^s(X) = \{f \in X : \|f\|_{A_q^s} = \left\| \left\{ (n+1)^{s-(1/q)} E_n^X(f) \right\}_{n=0}^{\infty} \right\|_q < \infty\}, \tag{3.4}$$

where $\|\,.\,\|_q$ denotes the l^q-norm. Sometimes we also write $A_q^s(X, \{A_n\})$, but usually, the sequence $\{A_n\}$ is viewed as fixed so that it is not necessary to include it in the notations. (This means that we use the same $\{A_n\}$ in all what follows.)

The theory of the classical approximation spaces $A_q^s(X)$ can be found in [13]. We only mention that $A_q^s(X)$ is a quasi-normed space, which is complete if X is complete. Moreover, if X is a normed space, all A_n are linear subspaces, and $q \ge 1$, then $A_q^s(X)$ is a normed space.

Remark 3.1 *We recall that a quasi-norm on a linear space X is defined in the same way as a norm, but the triangle equation is replaced by*

$$\|f + g\|_X \le c_X(\|f\|_X + \|g\|_X)\,,$$

where $c_X \geq 1$ *is a certain constant. A linear space* X*, endowed with a quasi-norm, is called quasi-normed space. If* $(X, \|\,.\,\|_X)$ *is a quasi-normed space, then there exists an equivalent quasi-norm* $|||\,.\,|||_X$ *such that*

$$|||f+g|||_X^p \leq |||f|||_X^p + |||g|||_X^p\,,\ \ f,g \in X,\ \ \textit{where}\ \ p = (1+\log_2 c_X)^{-1}. \quad (3.5)$$

This shows that the quasi-norm $\|\,.\,\|_X$ *generates a metrisable topology on* X*. The corresponding metric is given by* $d(f,g) = |||f-g|||_X^p$*. With respect to this metric, things like Cauchy sequence, completeness, and boundedness are well-defined. The same things are also defined w.r.t.* $\|\,.\,\|_X$*, but fortunately this does not give other definitions (at least for the above properties). A complete quasi-normed space is called quasi-Banach space. Many results, which are true in normed spaces (Banach spaces), remain also true in quasi-normed spaces (quasi-Banach spaces). For example, a linear operator* $A : X \to Y$ *(*X, Y*: quasi-normed spaces) is continuous if and only if it is bounded, i.e.* $\|Af\|_Y \leq \mathrm{const}\,\|f\|_X$ *for all* $f \in X$*. Moreover, all quasi-norms on a finite-dimensional linear space* X *are equivalent. One can prove these facts in the same way as in normed spaces. (For the proof of the last assertion we may assume* $X = \mathbb{K}^n$*,* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$*, and* $\|\,.\,\| = |||\,.\,|||$*. Then it is easy to prove* $\|x\| \leq \mathrm{const}\,|x|$ *and the evaluation* $|x| \leq \mathrm{const}\,\|x\|$ *can be done by using the continuity of* $\|\,.\,\| : (\mathbb{K}^n, \|\,.\,\|) \to \mathbb{R}$ *(cf. Lemma 3.9 with* $n = 0$*), which implies the continuity of* $\|\,.\,\| : (\mathbb{K}^n, |\,.\,|) \to \mathbb{R}$ *(since* $I : (\mathbb{K}^n, |\,.\,|) \to (\mathbb{K}^n, \|\,.\,\|)$ *is continuous) and, hence, the existence of* $\min_{x:|x|=1} \|x\|$*.)*

The Besov spaces $B_q^s(L_{2\pi}^p)$ and $B_q^s(L_{\rho,\tau}^p)$ are particular cases of approximation spaces $A_q^s(X)$ as we have already seen in Subsection 2.4. Also other classical spaces, for example Lorentz sequence spaces, fit into the concept of classical approximation spaces. But recently there also appeared applications in which more general approximation spaces are needed (see e.g. [9], [12], [3], [2]). More precisely, instead of the sequence $\{(n+1)^{s-(1/q)}\}$ $(s > 0)$ in (3.4), more general sequences are considered in the definition of approximation spaces. This means that the weighted l^q-norm $\|\{(n+1)^{s-(1/q)}\,.\,\}\|_q$ is replaced by another quasi-norm on a certain sequence space. Instead of weighted l^q-spaces one can also use other sequence spaces S to define approximation spaces $A(X,S)$. To ensure that $\|\,.\,\|_{A(X,S)}$ remains a quasi-norm, we have to give certain assumptions on S. If even $A_n + A_n \subseteq A_n$ instead of $A_n + A_n \subseteq A_{2n}$ (see (3.3)), then the assumptions on S are weaker than in the general case. For this reason we replace (3.3) by the more general condition

$$\begin{aligned} &A_n + A_n \subseteq A_{K(n)}\,,\ n \in \mathbb{N}_0 := \{0,1,2,\ldots\}\,, \\ &\text{with certain numbers } n \leq K(n) \leq K(n+1) \text{ and } K(0) := 0, \end{aligned} \quad (3.6)$$

which allows us to give the conditions on S in dependence on the numbers $K(n)$. Thus, in all what follows $\{A_n\}_{n=0}^{\infty}$ is a fixed sequence of subsets of X with $A_0 = \{0\}$, such that (3.1), (3.2) and (3.6) are satisfied.

Definition 3.2 *Let S be a real linear space of sequences $\{a_n\}_{n=0}^{\infty} \subseteq \mathbb{R}$ (with elementwise defined operations), equipped with a quasi-norm $\|\,.\,\|_S$. S is called admissible sequence space if the following assumptions are satisfied:*

(A1) All finite sequences $\{a_n\}_{n=0}^{N}$ belong to S.

(A2) If $0 \le a_n \le b_n$ for all $n = 0, 1, 2, \ldots$ and $\{b_n\}_{n=0}^{\infty} \in S$, then $\{a_n\}_{n=0}^{\infty} \in S$ and $\|\{a_n\}\|_S \le \|\{b_n\}\|_S$.

(A3) If $a_0 \ge a_1 \ge a_2 \ge \ldots \ge 0$ and $\{a_{K(n)}\}_{n=0}^{\infty} \in S$, then $\{a_n\} \in S$ and

$$\|\{a_n\}_{n=0}^{\infty}\|_S \le C_S \, \|\{a_{K(n)}\}_{n=0}^{\infty}\|_S \,,$$

where C_S is a constant, which only depends on S and $\{K(n)\}_{n=0}^{\infty}$.

(In (A1) we used the notation

$$\{a_n\}_{n=n_0}^{n_1} := \{a'_n\}_{n=0}^{\infty} \,, \quad \text{where } a'_n := \begin{cases} a_n \,, & n \in [n_0, n_1] \,, \\ 0 \,, & \text{otherwise} \,, \end{cases}$$

which will be also useful in the sequel.)

Remark 3.3 *The assumption $K(n) \le K(n+1)$ in (3.6) is not really a restriction. If it is not satisfied, then $A_n + A_n \subseteq A_{\widetilde{K}(n)}$ with $\widetilde{K}(n) := \min_{m \ge n} K(m)$ (since $A_n + A_n \subseteq A_m + A_m$ for $m \ge n$) and $\{\widetilde{K}(n)\}$ is increasing. Moreover, all admissible S satisfy (A3) with $\widetilde{K}(n)$ instead of $K(n)$, since $\widetilde{K}(n) \le K(n)$. In most of the theorems, which we will prove in the present paper, the assumption $K(n) \le K(n+1)$ is not needed. To be more precise: We only need it in some applications (Sections 4 and 5) and for the admissibility assertion of the reiteration theorem (Theorem 3.35).*

Now we are able to define generalized approximation spaces $A(X,S)$. For weighted l^q-spaces S and linear subspaces A_n of a Banach space X, the theory of such spaces was developed in [11]. The general case was first treated in [1].

Definition 3.4 *Let X be a quasi-normed space and let A_n $(n \in \mathbb{N}_0)$ be subsets of X with $A_0 = \{0\}$, such that (3.1), (3.2), and (3.6) are satisfied. Moreover, let S be an admissible sequence space. Then the space*

$$A(X,S) := \left\{ f \in X : \{E_n^X(f)\}_{n=0}^{\infty} \in S \right\}, \quad \text{endowed with}$$
$$\|f\|_{X,S} := \left\| \{E_n^X(f)\}_{n=0}^{\infty} \right\|_S \,,$$

is called (generalized) approximation space.

Remark 3.5 *If $K(n) < K(n+1)$ for all n and if S is defined by the quasi-norm*

$$\|\{a_n\}_{n=0}^{\infty}\|_S = \|\{b_n a_n\}_{n=0}^{\infty}\|_q \quad (\text{with fixed } b_n > 0 \text{ and } 0 < q \le \infty) \tag{3.7}$$

(we write $S = l^q(\{b_n\})$ in this case), then S is admissible if there exists some constant $C > 0$ such that, for all $n \in \mathbb{N}_0$,

$$\begin{aligned} &\max\{b_{K(n)}, b_{K(n)+1}, \dots, b_{K(n+1)-1}\} \le C\, b_n \quad \text{and} \\ &[K(n+1) - K(n)]^{1/q} \le C\,. \end{aligned} \tag{3.8}$$

(But admissibility of $l^q(\{b_n\})$ does not necessarily imply (3.8). A necessary and sufficient condition will be proved later in Corollary 4.14.) Particularly, if $K(n) = Kn$ with some $K \in \mathbb{N}$ and $b_n = (n+1)^{s-(1/q)}$ with some $s > 0$ (classical weights), then (3.8) is satisfied and (3.7) defines an admissible sequence space. In this case we also write $A_q^s(X)$ (or $A_q^s(X, \{A_n\})$) instead of $A(X, S)$.

Proof. It is clear that S is a quasi-Banach space (since l^q is a quasi-Banach space for all $0 < q \le \infty$) satisfying (A1) and (A2) if its quasi-norm is defined by (3.7). Moreover, if (3.8) is satisfied then, for all $a_0 \ge a_1 \ge a_2 \ge \dots \ge 0$,

$$\begin{aligned} &\|\{b_n a_n\}_{n=0}^{\infty}\|_q \\ &= \left\| \left\{ \|\{b_n a_n\}_{n=K(m)}^{K(m+1)-1}\|_q \right\}_{m=0}^{\infty} \right\|_q \le \left\| \left\{ a_{K(m)} \|\{b_n\}_{n=K(m)}^{K(m+1)-1}\|_q \right\}_{m=0}^{\infty} \right\|_q \\ &\le C \left\| \left\{ b_m a_{K(m)} \|\{1\}_{n=K(m)}^{K(m+1)-1}\|_q \right\}_{m=0}^{\infty} \right\|_q \le C^2\, \|\{b_m a_{K(m)}\}_{m=0}^{\infty}\|_q\,, \end{aligned}$$

which shows that (A3) is also satisfied. ▮

Example 3.6 *Let $1 \le p, q \le \infty$ and $s > 0$ be fixed. Then we have $B_q^s(L_{2\pi}^p) = A_q^s(L_{2\pi}^p, \{\mathcal{T}_n\})$ and $B_q^s(L_{\rho,\tau}^p) = A_q^s(L_{\rho,\tau}^p, \{\mathcal{P}_n\})$ in the sense of equivalent norms (see Theorem 2.2).*

3.2. Quasi-norm properties of $\|\,.\,\|_{\mathbf{X},\mathbf{S}}$. Completeness of $\mathbf{A(X,S)}$

In all what follows, $A(X, S)$ is an approximation space.

Lemma 3.7 *Let $f, g \in X$ and $n \in \mathbb{N}_0$. Then we have $E_n^X(\lambda f) = |\lambda| E_n^X(f)$ for all scalars λ and $E_{K(n)}^X(f+g) \le c_X[E_n^X(f) + E_n^X(g)]$, where c_X is the constant from the quasi-triangle equation in X.*

Proof. $E_n^X(\lambda f) = |\lambda| E_n^X(f)$ follows easily from $\lambda A_n = A_n$. Moreover,

$$E_{K(n)}^X(f+g) \le \|f+g-f_n-g_n\|_X \le c_X(\|f-f_n\|_X + \|g-g_n\|_X)$$

for all $f_n, g_n \in A_n$, which proves $E_{K(n)}^X(f+g) \le c_X[E_n^X(f) + E_n^X(g)]$. ▮

Proposition 3.8 *$A(X,S)$ is a quasi-normed space, which is continuously embedded into X. If X and S are normed spaces and all A_n are linear subspaces of X, then $A(X,S)$ is even a normed space.*

Proof. In view of Lemma 3.7, $f \in A(X,S)$ implies $\lambda f \in A(X,S)$ and $\|\lambda f\|_{X,S} = |\lambda| \|f\|_{X,S}$ for all scalars λ. If $f,g \in A(X,S)$, then the estimate $E_{K(n)}^X(f+g) \le c_X(E_n^X(f) + E_n^X(g))$ (see Lemma 3.7) and property (A2) show that $\{E_{K(n)}(f+g)\}$ belongs to S and that

$$\|\{E_{K(n)}^X(f+g)\}\|_S \le c_X \|\{E_n^X(f)\} + \{E_n^X(g)\}\|_S \le c_X c_S(\|f\|_{X,S} + \|g\|_{X,S}).$$

In view of (A3), we obtain $\{E_n^X(f+g)\} \in S$, i.e. $f+g \in A(X,S)$, and

$$\|f+g\|_{X,S} \le C_S \|\{E_{K(n)}^X(f+g)\}\|_S \le C_S c_X c_S\,(\|f\|_{X,S} + \|g\|_{X,S})\ .$$

Moreover, $\|f\|_X = [\|\{1\}_{n=0}^0\|_S]^{-1} \|\{E_n^X(f)\}_{n=0}^0\|_S \le [\|\{1\}_{n=0}^0\|_S]^{-1} \|f\|_{X,S}$ for all $f \in A(X,S)$. Consequently, $A(X,S)$ is a quasi-normed space, which is continuously embedded into X. If X and S are normed spaces and all A_n are linear subspaces of X, then we have $C_S = c_S = c_X = 1$, i.e. $A(X,S)$ is a normed space in this case. ▮

In all what follows, by c we denote positive constants which can have different values at different places. Moreover, by $c \ne c(n,f,\ldots)$ we will indicate that c is independent of $n, f, \ldots$. If A and B are two expressions which depend on certain variables $v_1, \ldots, v_k$, then the notation "$A \sim B$ for all $v_1,\ldots,v_k$" means that there exists a positive constant $c \ne c(v_1,\ldots,v_k)$ such that $c^{-1}A \le B \le cA$ for all $v_1,\ldots,v_k$.

Lemma 3.9 *For every n, the error $\mathcal{E}_n^X(f) := \inf\{|||f-f_n|||_X : f_n \in A_n\}$ (see Remark 3.1) defines a continuous function $\mathcal{E}_n^X(\,.\,) : X \to \mathbb{R}$. Moreover, $E_n^X(f)$ and $\mathcal{E}_n^X(f)$ are equivalent, i.e.*

$$E_n^X(f) \sim \mathcal{E}_n^X(f) \quad \text{for all } f \in X \text{ and all } n \in \mathbb{N}_0\,.$$

Particularly (in view of (A2)), $A(X,S) = A((X, |||\,.\,|||_X), S)$ in the sense of equivalent quasi-norms.

Proof. Let $\varepsilon > 0$ and choose $g_\varepsilon \in A_n$ such that $|||g-g_\varepsilon|||_X^p \le \mathcal{E}_n^X(g)^p + \varepsilon$ (p from Remark 3.1). It follows

$$\mathcal{E}_n^X(f)^p - \mathcal{E}_n^X(g)^p \le |||f-g_\varepsilon|||_X^p - |||g-g_\varepsilon|||_X^p + \varepsilon \le |||f-g|||_X^p + \varepsilon\,.$$

Thus, $\mathcal{E}_n^X(f)^p - \mathcal{E}_n^X(g)^p \le |||f-g|||_X^p$ for all $f,g \in X$, which implies $|\mathcal{E}_n^X(f)^p - \mathcal{E}_n^X(g)^p| \le |||f-g|||_X^p$ for all $f,g\in X$. This shows that $\mathcal{E}_n^X(\,.\,)$ is continuous on X. The second assertion follows easily from the equivalence of the quasi-norms $\|\,.\,\|_X$ and $|||\,.\,|||_X$. ∎

Lemma 3.10 *Let $n_0 \in \mathbb{N}_0$ and $M \in [0,\infty)$. If $\mathcal{M} \subseteq X$ has the property*

$$\lim_{k\to\infty} \left\| \left\{ E_n^X(f) \right\}_{n=n_0}^k \right\|_S \le M \quad \textit{for all } f \in \mathcal{M}\,, \tag{3.9}$$

then the closure of $\mathcal{M}$ in X has the same property with exception of some constant $C_{X,S} \ge 1$ which only depends on X and S. This means that (3.9) implies

$$\lim_{k\to\infty} \left\| \left\{ E_n^X(f) \right\}_{n=n_0}^k \right\|_S \le C_{X,S} M \quad \textit{for all } f \in \operatorname{clos}\mathcal{M}\,.$$

(Remark that, in view of (A2), $\|\{E_n^X(f)\}_{n=n_0}^{n_0}\|_S \le \|\{E_n^X(f)\}_{n=n_0}^{n_0+1}\|_S \le \|\{E_n^X(f)\}_{n=n_0}^{n_0+2}\|_S \le \dots$, which implies that the limit in (3.9) is equal to the supremum over all $k \ge n_0$.)

Proof. For every $k \ge n_0$, the linear mapping $(a_n)_{n=n_0}^k \in \mathbb{R}^{k-n_0+1} \to \{a_n\}_{n=n_0}^k \in S$ is continuous, since $\|(a_n)_{n=n_0}^k\|_{\mathbb{R}^{k-n_0+1}}$ and $\|\{a_n\}_{n=n_0}^k\|_S$ are equivalent quasi-norms on $\mathbb{R}^{k-n_0+1}$ (see Remark 3.1). This implies that $(a_n)_{n=n_0}^k \in \mathbb{R}^{k-n_0+1} \to |||\{a_n\}_{n=n_0}^k|||_S \in \mathbb{R}$ ($|||\,.\,|||_S$ from Remark 3.1) is a continuous mapping, since $|||\,.\,|||_S$ is a continuous function on S (see Lemma 3.9 with $n=0$). In view of Lemma 3.9, we obtain that the mapping $f \in X \to |||\{\mathcal{E}_n^X(f)\}_{n=n_0}^k|||_S \in \mathbb{R}$ is continuous for all $k \ge n_0$. Now, the proof of the lemma is easy: Let $f \in \operatorname{clos}\mathcal{M}$ and choose $f_m \in \mathcal{M}$ such that $f = \lim_{m\to\infty} f_m$ in X. Then we may take the limit $m\to\infty$ in $|||\{\mathcal{E}_n^X(f_m)\}_{n=n_0}^k|||_S \le c\|\{\mathcal{E}_n^X(f_m)\}_{n=n_0}^k\|_S \le CM$, which gives $|||\{\mathcal{E}_n^X(f)\}_{n=n_0}^k|||_S \le CM$ for all $k \ge n_0$. From the equivalences $|||\,.\,|||_S \sim \|\,.\,\|_S$ and $\mathcal{E}_n^X(f) \sim E_n^X(f)$ it follows $\|\{E_n^X(f)\}_{n=n_0}^k\|_S \le C_{X,S}M$ for all $k \ge n_0$. ∎

Lemma 3.11 *If $\{a_n\} \subseteq [0,\infty)$ is a decreasing sequence with the properties*

$$\lim_{n\to\infty} a_n = 0 \quad \textit{and} \quad \lim_{l\to\infty}\lim_{k\to\infty} \|\{a_{K(n)}\}_{n=l}^k\|_S = 0\,,$$

then $\lim\limits_{l\to\infty}\lim\limits_{k\to\infty} \|\{a_n\}_{n=l}^k\|_S = 0$. (We remark that $\lim\limits_{k\to\infty}\|\{a_n\}_{n=l}^k\|_S = \sup\limits_{k\ge l}\|\{a_n\}_{n=l}^k\|_S$ and, hence, $\lim\limits_{l\to\infty}\lim\limits_{k\to\infty}\|\{a_n\}_{n=l}^k\|_S = \inf\limits_{l\ge 0}\sup\limits_{k\ge l}\|\{a_n\}_{n=l}^k\|_S$.)

Proof. Let C_S and c_S be the constants from (A3) and from the quasi-triangle equation in S, respectively. For every $\varepsilon > 0$, there exists some

$l_0 \in \mathbb{N}$ such that

$$\lim_{k\to\infty} \|\{a_{K(n)}\}_{n=l_0}^{k}\|_S \le (2c_S C_S)^{-1}\varepsilon\,.$$

Moreover, there is an $n_0 \ge K(l_0-1)$ such that

$$a_{n_0}\|\{1\}_{n=0}^{l_0-1}\|_S \le (2c_S C_S)^{-1}\varepsilon\,.$$

Together with (A3) we obtain

$$\begin{aligned}\lim_{k\to\infty} \|\{a_n\}_{n=n_0}^{k}\|_S &\le \lim_{k\to\infty} \|\{a_{\max\{n,n_0\}}\}_{n=0}^{k}\|_S \\ &\le C_S \lim_{k\to\infty} \|\{a_{\max\{K(n),n_0\}}\}_{n:K(n)\le k}\|_S \\ &\le C_S \lim_{k\to\infty} \|a_{n_0}\{1\}_{n=0}^{l_0-1} + \{a_{\max\{K(n),n_0\}}\}_{n=l_0}^{k}\|_S \\ &\le C_S c_S\left[a_{n_0}\|\{1\}_{n=0}^{l_0-1}\|_S + \lim_{k\to\infty} \|\{a_{K(n)}\}_{n=l_0}^{k}\|_S\right] \le \varepsilon,\end{aligned}$$

which shows that $\inf_{l\ge 0}\lim_{k\to\infty} \|\{a_n\}_{n=l}^{k}\|_S = 0$. ∎

Theorem 3.12 (Completeness of A(X,S))
Suppose that S has the property

$$\begin{aligned}\|\{a_n\}_{n=0}^{\infty}\|_S \le C \lim\nolimits_{k\to\infty} \|\{a_n\}_{n=0}^{k}\|_S \\ \text{for all } \{a_n\} \in S \text{ with } a_0 \ge a_1 \ge \ldots \ge 0\,,\end{aligned} \tag{3.10}$$

where $C > 0$ is a constant depending only on S. If X is complete and if one of the following conditions (a), (b) or (c) is satisfied, then $A(X,S)$ is also complete.

(a) X is a Banach space, all A_n are linear subspaces, $\|\{a_n\}\|_S = \|\{|a_n|\}\|_S$ for all $\{a_n\} \in S$, and every Cauchy sequence $\{\mathbf{a}_m\}$ in S, which consists of decreasing sequences $\mathbf{a}_m = \{a_{mn}\}_{n=0}^{\infty} \subseteq [0,\infty)$, is convergent in S.

(b) A decreasing sequence $\{a_n\} \subseteq [0,\infty)$ belongs to S if and only if the limit $\lim_{k\to\infty} \|\{a_n\}_{n=0}^{k}\|_S$ ($= \sup_k \|\{a_n\}_{n=0}^{k}\|_S$) is finite.

(c) A decreasing sequence $\{a_n\} \subseteq [0,\infty)$ belongs to S if and only if

$$\lim\nolimits_{l\to\infty} \lim\nolimits_{k\to\infty} \|\{a_n\}_{n=l}^{k}\|_S = 0\,. \tag{3.11}$$

Moreover, $\lim_{k\to\infty} \|\{1\}_{n=0}^{k}\|_S = \infty$.

Proof. Let $\{f_m\}_{m=1}^\infty$ be a Cauchy sequence in $A(X,S)$ and let $\varepsilon > 0$. Then there exists an m_0 such that $\mathcal{M} := \{f_l - f_m : l, m \geq m_0\}$ is bounded by $\varepsilon / C_{X,S}$ in $A(X,S)$ ($C_{X,S}$ from Lemma 3.10). Moreover, $\{f_m\}$ is also a Cauchy sequence in X, since $A(X,S)$ is continuously embedded into X. If f is the X-limit of $\{f_m\}$, then $f - f_m$ belongs to the closure of $\mathcal{M}$ in X for all $m \geq m_0$ and Lemma 3.10 shows that

$$\lim_{k\to\infty} \|\{E_n^X(f - f_m)\}_{n=0}^k\|_S \leq \varepsilon \quad \text{for all } m \geq m_0 . \tag{3.12}$$

It remains to show that f belongs to $A(X,S)$, since in this case (3.12) and (3.10) mean that $\{f_m\}$ converges to f in $A(X,S)$. If (a) is satisfied then, in view of Lemma 3.7, $E_n^X(g+h) \leq E_n^X(g) + E_n^X(h)$, which implies

$$\begin{aligned} \|\{E_n^X(f_m) - E_n^X(f_l)\}_{n=0}^\infty\|_S &= \|\{|E_n^X(f_m) - E_n^X(f_l)|\}_{n=0}^\infty\|_S \\ &\leq \|\{E_n^X(f_m - f_l)\}_{n=0}^\infty\|_S \leq \varepsilon \end{aligned}$$

for all $m, l \geq m_0$, i.e. $\{\{E_n^X(f_m)\}_{n=0}^\infty\}_{m=1}^\infty$ is a Cauchy sequence in S. Thus, there exists some $\{E_n\} \in S$, such that

$$\lim_{m\to\infty} \|\{E_n^X(f_m) - E_n\}_{n=0}^\infty\|_S = \lim_{m\to\infty} \|\{|E_n^X(f_m) - E_n|\}_{n=0}^\infty\|_S = 0 .$$

This implies that, for every fixed $n \in \mathbb{N}_0$, $\lim_{m\to\infty} |E_n^X(f_m) - E_n| = (\|\{1\}_{j=n}^n\|_S)^{-1} \lim_{m\to\infty} \|\{|E_j^X(f_m) - E_j|\}_{j=n}^n\|_S = 0$. Consequently, $E_n = \lim_{m\to\infty} E_n^X(f_m) = E_n^X(f)$ in view of Lemma 3.9. Thus, $f \in A(X,S)$. If (b) is satisfied, then

$$A(X,S) = \{g \in X : \lim_{k\to\infty} \|\{E_n^X(g)\}_{n=0}^k\|_S < \infty\}$$

and $f \in A(X,S)$ follows from (3.12) (since $f = (f - f_{m_0}) + f_{m_0}$ and $f - f_{m_0}, f_{m_0} \in A(X,S)$). If (c) is satisfied, then

$$A(X,S) = \{g \in X : \lim_{l\to\infty} \lim_{k\to\infty} \|\{E_n^X(g)\}_{n=l}^k\|_S = 0\}$$

and we must show that $\lim_{l\to\infty} \lim_{k\to\infty} \|\{E_n^X(f)\}_{n=l}^k\|_S = 0$. For this aim we remark that $f_{m_0} \in A(X,S)$ implies

$$\lim_{k\to\infty} \|\{E_n^X(f_{m_0})\}_{n=l}^k\|_S \leq \varepsilon \ \text{ for all } l \geq l_0 .$$

Together with (3.12) and Lemma 3.7 it follows

$$\begin{aligned} &\lim_{k\to\infty} \|\{E_{K(n)}^X(f)\}_{n=l}^k\|_S \leq c_X \lim_{k\to\infty} \|\{E_n^X(f - f_{m_0}) + E_n^X(f_{m_0})\}_{n=l}^k\|_S \\ &\quad \leq c_S c_X \lim_{k\to\infty} (\|\{E_n^X(f - f_{m_0})\}_{n=0}^k\|_S + \|\{E_n^X(f_{m_0})\}_{n=l}^k\|_S) \leq 2 c_S c_X \, \varepsilon \end{aligned}$$

for all $l \geq l_0$, which shows that

$$\lim_{l\to\infty} \lim_{k\to\infty} \|\{E^X_{K(n)}(f)\}^k_{n=l}\|_S = 0\,. \tag{3.13}$$

If $g \in X$ satisfies $E^X_n(g) \geq \text{const} > 0$, then $\lim_{k\to\infty} \|\{E^X_n(g)\}^k_{n=0}\|_S \geq \text{const} \lim_{k\to\infty} \|\{1\}^k_{n=0}\|_S = \infty$. So we have proved that the assumption $\lim_{k\to\infty} \|\{1\}^k_{n=0}\|_S = \infty$ implies

$$\lim_{n\to\infty} E^X_n(g) = 0 \text{ for all } g \in X \text{ with } \lim_{k\to\infty} \|\{E^X_n(g)\}^k_{n=0}\|_S < \infty.$$

Lemma 3.10, applied with $\mathcal{M} = \{f_m\}$, yields $\lim_{k\to\infty} \|\{E^X_n(f)\}^k_{n=0}\|_S < \infty$. So we conclude $\lim_{n\to\infty} E^X_n(f) = 0$. Together with (3.13) and Lemma 3.11 it follows $\lim_{l\to\infty} \lim_{k\to\infty} \|\{E^X_n(f)\}^k_{n=l}\|_S = 0$, i.e. $f \in A(X, S)$. ▌

Remark 3.13 *If S is complete and if $\|\{a_n\}\|_S = \|\{|a_n|\}\|_S$ for all $\{a_n\} \in S$, then the conditions (3.10) and (c) from Theorem 3.12 are satisfied if*

$$\lim_{l\to\infty} \|\{a_n\}^\infty_{n=l}\|_S = 0 \quad \textit{for all } \{a_n\} \in S \textit{ with } a_0 \geq a_1 \geq \ldots \geq 0 \tag{3.14}$$

and $\lim_{k\to\infty} \|\{1\}^k_{n=0}\|_S = \infty$. If (3.14) is satisfied for all $\{a_n\} \in S$, then the condition $\|\{a_n\}\|_S = \|\{|a_n|\}\|_S$ can be omitted in the previous sentence and instead of (3.10) we even have

$$\|\{a_n\}^\infty_{n=0}\|_S \sim \limsup_{k\to\infty} \|\{a_n\}^k_{n=0}\|_S \sim \liminf_{k\to\infty} \|\{a_n\}^k_{n=0}\|_S \tag{3.15}$$

for all $\{a_n\} \in S$ (not only for $\{a_n\} \in S$ with $a_n \geq a_{n+1} \geq 0$) in this case.

Proof. Let $\{a_n\} \in S$ with $a_0 \geq a_1 \geq \ldots \geq 0$. Then, for all $k \geq l \geq 0$,

$$\big| |||\{a_n\}^\infty_{n=l}|||^q_S - |||\{a_n\}^k_{n=l}|||^q_S \big| \leq |||\{a_n\}^\infty_{n=k+1}|||^q_S \leq c\|\{a_n\}^\infty_{n=k+1}\|^q_S\,,$$

where $q = q(S) \leq 1$ is a certain positive number and $|||\,.\,|||_S$ is the equivalent quasi-norm from Remark 3.1. Together with (3.14) it follows

$$|||\{a_n\}^\infty_{n=l}|||_S = \lim_{k\to\infty} |||\{a_n\}^k_{n=l}|||_S \text{ for all } l \in \mathbb{N}_0\,.$$

Especially, (3.10) is satisfied, since $|||\,.\,|||_S \sim \|\,.\,\|_S$. If (3.14) is satisfied for all $\{a_n\} \in S$, then the assumption $a_0 \geq a_1 \geq \ldots \geq 0$ can be omitted in the above considerations and we obtain (3.15). Now, let $\{a_n\}$ be a decreasing sequence of nonnegative numbers. We have to show that $\{a_n\} \in S$ if and only if (3.11) is fulfilled. If $\{a_n\} \in S$, then (3.11) follows from (3.14), since $\|\{a_n\}^k_{n=l}\|_S \leq \|\{a_n\}^\infty_{n=l}\|_S$ for all $k \geq l$. For the proof of the counterdirection we remark that the second limit in (3.11) is equal to the supremum over all $k \geq l$. Consequently, if (3.11) is satisfied, then there exists some $l_0 = l_0(\varepsilon)$ such that

$$\|\{a_n\}_{n=0}^{k} - \{a_n\}_{n=0}^{l}\|_S = \|\{a_n\}_{n=l+1}^{k}\|_S \leq \|\{a_n\}_{n=l}^{k}\|_S < \varepsilon \text{ for } k > l \geq l_0,$$

i.e. $\{\{a_n\}_{n=0}^{k}\}_{k=0}^{\infty}$ is a Cauchy sequence in S and, consequently, convergent in S. The limit $\{c_n\}$ is equal to $\{a_n\}$, since $|c_{n_0} - a_{n_0}|$ can be estimated by

$$\|\{1\}_{n=n_0}^{n_0}\|_S^{-1}\|\{|c_n - a_n|\}_{n=n_0}^{n_0}\|_S \leq \|\{1\}_{n=n_0}^{n_0}\|_S^{-1}\|\{c_n\} - \{a_n\}_{n=0}^{k}\|_S$$

for all $k \geq n_0$. In the last estimate we used $\|\{\,.\,\}\|_S = \|\{|\,.\,|\}\|_S$. If (3.14) is satisfied for all $\{a_n\} \in S$, then $\{c_n\} = \{a_n\}$ can be proved without this assumption in the following way: The estimate

$$\|\{c_n - a_n\}_{n=0}^{k}\|_S \leq c_S \left(\|\{c_n\} - \{a_n\}_{n=0}^{k}\|_S + \|\{c_n\}_{n=k+1}^{\infty}\|_S\right)$$

shows that $\lim_{k\to\infty} \|\{c_n - a_n\}_{n=0}^{k}\|_S = 0$ which yields $\{c_n\} = \{a_n\}$ in view of (3.15). ▌

Remark 3.14 *If (3.10) is satisfied and if*

$$\lim_{k\to\infty} \|\{1\}_{n=0}^{k}\|_S < \infty\,, \tag{3.16}$$

then $\|\,.\,\|_X$ is a quasi-norm in $A(X,S)$, which is equivalent to $\|\,.\,\|_{X,S}$. Indeed, for all $f \in A(X,S)$ we have $\|f\|_X \leq \text{const}\,\|f\|_{X,S}$, since $A(X,S)$ is continuously imbedded into X, and

$$\|f\|_{X,S} \leq C \lim_{k\to\infty} \|\{E_n^X(f)\}_{n=0}^{k}\|_S \leq C\|f\|_X \lim_{k\to\infty} \|\{1\}_{n=0}^{k}\|_S = \text{const}\,\|f\|_X,$$

since $\lim_{k\to\infty} \|\{1\}_{n=0}^{k}\|_S < \infty$. Consequently, $A(X,S)$ can be viewed as a subspace of X (with the same quasi-norm) in this case. If one is interested in a criterion, which ensures the completeness of such subspaces of X, one can use the following fact:

If (3.16) is satisfied, then the conditions "X is a Banach space" and "all A_n are linear subspaces" can be omitted in assumption (a) of Theorem 3.12.

Proof. In view of Lemma 3.9, we may assume $\|\,.\,\|_X = |||\,.\,|||_X$ without loss of generality. In the proof of Theorem 3.12, case (a), the above two conditions are only needed to show that $\{\{E_n^X(f_m)\}_{n=0}^{\infty}\}_{m=1}^{\infty}$ is a Cauchy sequence in S. But now this can be done as follows: In the proof of Lemma 3.9 we have seen that $|E_n^X(f)^p - E_n^X(g)^p| \leq \|f-g\|_X^p$ for some $p \in (0,1]$. Since $x^{1/p}$ is a Lipschitz continuous function on $[0, M^p]$, we obtain

$$\begin{aligned} |E_n^X(f) - E_n^X(g)| &= |(E_n^X(f)^p)^{1/p} - (E_n^X(g)^p)^{1/p}| \\ &\leq c(M,p)|E_n^X(f)^p - E_n^X(g)^p| \leq c(M,p)\|f-g\|_X^p \end{aligned}$$

for all $f, g \in X$ with $\|f\|_X, \|g\|_X \le M$. Since the Cauchy sequence $\{f_m\}$ is bounded in X, we conclude

$$\begin{aligned}\|\{E_n^X(f_m) - E_n^X(f_l)\}_{n=0}^\infty\|_S &\le C \lim_{k\to\infty} \|\{|E_n^X(f_m) - E_n^X(f_l)|\}_{n=0}^k\|_S \\ &\le c\,\|f_m - f_l\|_X^p \lim_{k\to\infty} \|\{1\}_{n=0}^k\|_S = \text{const}\,\|f_m - f_l\|_X^p\,,\end{aligned}$$

which shows that $\{\{E_n^X(f_m)\}_{n=0}^\infty\}_{m=1}^\infty$ is a Cauchy sequence in S. ▮

3.3. Density of $\bigcup \mathbf{A_n}$ in $\mathbf{A(X,S)}$

Obviously, $\bigcup_{n=0}^\infty A_n$ is a linear subspace of every approximation space $A(X,S)$. Often it is of great importance to know whether $\bigcup A_n$ is dense in $A(X,S)$ or, if it is not dense, to know the closure of $\bigcup A_n$ in $A(X,S)$. In the present subsection we will tackle these questions.

Lemma 3.15 *Let S be an admissible sequence space. If $\{a_n\}_{n=0}^\infty \subseteq [0,\infty)$ is a decreasing sequence with the properties*

$$\lim_{n\to\infty} a_n = 0\,,\quad \{a_{K(n)}\}_{n=0}^\infty \in S\,,\quad \lim_{l\to\infty} \|\{a_{K(n)}\}_{n=l}^\infty\|_S = 0\,,$$

then $\{a_n\} \in S$ and $\lim_{l\to\infty} \|\{a_n\}_{n=l}^\infty\|_S = 0$.

Proof. $\{a_n\} \in S$ follows from (A3) and $\lim_{l\to\infty} \|\{a_n\}_{n=l}^\infty\|_S = 0$ can be shown in the same way as in the proof of Lemma 3.11. (We only have to replace $\lim_{k\to\infty} \|\{\,\cdot\,\}_{n=\dots}^k\|_S$ by $\|\{\,\cdot\,\}_{n=\dots}^\infty\|_S$.) ▮

Lemma 3.16 *Let $A(X,S)$ be an approximation space and let $f \in A(X,S)$. Then we have*

$$\begin{aligned}C_S^{-1}\|\{E_{\max\{K(n),m\}}^X(f)\}_{m=0}^\infty\|_S &\le E_n^{A(X,S)}(f) \\ &\le c_S C_S \|\{E_{\max\{n,m\}}^X(f)\}_{m=0}^\infty\|_S\end{aligned} \tag{3.17}$$

for all $n \in \mathbb{N}_0$, where C_S and c_S are the constants from (A3) and the quasi-triangle equation in S, respectively.

Proof. Let $f_n \in A_n$ such that $\|f - f_n\|_X \le E_n^X(f) + \varepsilon$. If $m \ge n$ then $g_m - f_n \in A_m + A_m \subseteq A_{K(m)}$ for all $g_m \in A_m$. Consequently,

$$E_{K(m)}^X(f - f_n) \le \inf_{g_m \in A_m} \|f - f_n - (g_m - f_n)\|_X = E_m^X(f) \quad \text{if } m \ge n.$$

If $m \le n$ then $E_{K(m)}^X(f - f_n) \le \|f - f_n\|_X \le E_n^X(f) + \varepsilon$. Together with (A2) it follows $\{E_{K(m)}^X(f - f_n)\}_{m=0}^\infty \in S$ and

$$\begin{aligned}\|\{E_{K(m)}^X(f - f_n)\}_{m=0}^\infty\|_S &\le \|\{E_{\max\{m,n\}}^X(f)\}_{m=0}^\infty + \{\varepsilon\}_{m=0}^n\|_S \\ &\le c_S\Big(\|\{E_{\max\{m,n\}}^X(f)\}_{m=0}^\infty\|_S + \varepsilon\|\{1\}_{m=0}^n\|_S\Big).\end{aligned}$$

In view of (A3), we conclude

$$\begin{aligned} E_n^{A(X,S)}(f) &\le \|\{E_m^X(f-f_n)\}_{m=0}^\infty\|_S \\ &\le c_S C_S \left(\|\{E_{\max\{m,n\}}^X(f)\}_{m=0}^\infty\|_S + \varepsilon\|\{1\}_{m=0}^n\|_S\right), \end{aligned}$$

which yields the second inequality of (3.17) for $\varepsilon \to 0$. For all $f_n \in A_n$ and $g_m \in A_m$ we have $f_n + g_m \in A_{\max\{K(n),K(m)\}}$. Consequently,

$$E_{\max\{K(n),K(m)\}}^X(f) \le \inf_{g_m \in A_m} \|f-f_n-g_m\|_X = E_m^X(f-f_n) \text{ for all } f_n \in A_n.$$

Together with (A2) it follows $\{E_{\max\{K(n),K(m)\}}^X(f)\}_{m=0}^\infty \in S$ and

$$\|\{E_{\max\{K(n),K(m)\}}^X(f)\}_{m=0}^\infty\|_S \le \inf_{f_n \in A_n} \|\{E_m^X(f-f_n)\}_{m=0}^\infty\|_S = E_n^{A(X,S)}(f).$$

In view of (A3) we obtain $\|\{E_{\max\{K(n),m\}}^X(f)\}_{m=0}^\infty\|_S \le C_S E_n^{A(X,S)}(f)$. ∎

Theorem 3.17 *Let $A(X,S)$ be an approximation space. Then the space*

$$S_0 := \left\{\{a_n\} \in S : \lim_{l\to\infty} \|\{a_n\}_{n=l}^\infty\|_S = 0\right\}, \quad \textit{endowed with } \|\,.\,\|_S,$$

is an admissible sequence space. If $\{1\}_{n=0}^\infty \notin S$, then $A(X,S_0)$ is equal to the closure of $\bigcup_{n=0}^\infty A_n$ in $A(X,S)$. Moreover, if X and S_0 are complete, then $A(X,S_0)$ is also complete if $\lim_{k\to\infty} \|\{1\}_{n=0}^k\|_S = \infty$.

Proof. Obviously, S_0 is a linear subspace of S. Moreover, (A1) and (A2) are satisfied in S_0. This follows immediately from the corresponding properties of S. If $\{1\} \in S$ and $\lim_{l\to\infty} \|\{1\}_{n=l}^\infty\|_S > 0$ then all bounded $\{a_n\} \subseteq [0,\infty)$ belong to S (since $a_n \le \mathrm{const}\cdot 1$; cf. (A2)) and

$$c \inf_{n\ge l} a_n \le \|\{1\}_{n=l}^\infty\|_S \inf_{n\ge l} a_n \le \|\{a_n\}_{n=l}^\infty\|_S \le \|\{1\}\|_S \sup_{n\ge l} a_n .$$

This shows that, in this case, a convergent sequence $\{a_n\} \subseteq [0,\infty)$ belongs to S_0 if and only if $\lim_{n\to\infty} a_n = 0$. Consequently, (A3) is satisfied in S_0 if $\{1\} \in S$ and $\lim_{l\to\infty} \|\{1\}_{n=l}^\infty\|_S > 0$. If $\{1\} \in S$ and $\lim_{l\to\infty} \|\{1\}_{n=l}^\infty\|_S = 0$, then all bounded $\{a_n\} \subseteq [0,\infty)$ belong to S_0, since $a_n \le c$ implies, in view of (A2), $\{a_n\} \in S$ and

$$\lim_{l\to\infty} \|\{a_n\}_{n=l}^\infty\|_S \le c \lim_{l\to\infty} \|\{1\}_{n=l}^\infty\|_S = 0 .$$

This shows that S_0 has the property (A3) also in this case. Now we assume that $\{1\} \notin S$. If $a_0 \ge a_1 \ge \ldots \ge 0$ and $\{a_{K(n)}\}_{n=0}^\infty \in S_0$, then we have

$\lim_{n\to\infty} a_n = 0$, since

$$\{a_n\} \in S,\ \{a_n\} \subseteq [0,\infty) \text{ and } \{1\} \notin S \text{ imply } \inf_{n\geq 0} a_n = 0. \tag{3.18}$$

(Otherwise we would have $1 \leq c\,a_n$, which implies $\{1\} \in S$ in view of (A2).) From Lemma 3.15 it follows $\{a_n\} \in S_0$, i.e. (A3) is satisfied in S_0. Now we show that $A(X,S_0)$ is equal to the closure clos $\bigcup A_n$ of $\bigcup A_n$ in $A(X,S)$. Let $f \in A(X,S_0)$. From (3.18) it follows $\lim_{n\to\infty} E_n^X(f) = 0$. Thus, there exist $n_1 < n_2 < n_3 < \ldots$ such that

$$E_{n_j}^X(f) \leq j^{-1} \|\{1\}_{m=0}^{j-1}\|_S^{-1}, \quad j = 1,2,\ldots .$$

In view of (3.17), we obtain

$$\begin{aligned} E_{n_j}^{A(X,S)}(f) &\leq c_S C_S \|E_{n_j}^X(f)\{1\}_{m=0}^{j-1} + \{E_{\max\{n_j,m\}}^X(f)\}_{m=j}^{\infty}\|_S \\ &\leq c_S^2 C_S \left(E_{n_j}^X(f)\|\{1\}_{m=0}^{j-1}\|_S + \|\{E_m^X(f)\}_{m=j}^{\infty}\|_S\right) \\ &\leq c_S^2 C_S \left(j^{-1} + \|\{E_m^X(f)\}_{m=j}^{\infty}\|_S\right) . \end{aligned}$$

The right hand side of this estimate tends to zero for $j \to \infty$, since $f \in A(X,S_0)$. So we have proved that $\bigcup_{n=0}^{\infty} A_n$ is dense in $A(X,S_0)$, i.e. $A(X,S_0) \subseteq$ clos $\bigcup A_n$. On the other hand, if $f \in$ clos $\bigcup A_n$, then we have $\lim_{n\to\infty} E_n^{A(X,S)}(f) = 0$, which implies, in view of Lemma 3.16,

$$\lim_{n\to\infty} \|\{E_m^X(f)\}_{m=K(n)}^{\infty}\|_S \leq \lim_{n\to\infty} \|\{E_{\max\{K(n),m\}}^X(f)\}_{m=0}^{\infty}\|_S = 0$$

This means $\lim_{l\to\infty} \|\{E_m^X(f)\}_{m=l}^{\infty}\|_S = \inf_{l\geq 0} \|\{E_m^X(f)\}_{m=l}^{\infty}\|_S = 0$, i.e. $f \in A(X,S_0)$. Thus, $A(X,S_0) \supseteq$ clos $\bigcup A_n$. It remains to show that completeness of X and S_0 imply completeness of $A(X,S_0)$ if $\lim_{k\to\infty} \|\{1\}_{n=0}^{k}\|_S = \infty$. But this is an immediate consequence of Remark 3.13 and Theorem 3.12. ▮

Remark 3.18 *If $\|\{a_n\}\|_S = \|\{|a_n|\}\|_S$ (or at least $\sup_{l\geq 0} \|\{a_n\}_{n=l}^{\infty}\|_S \leq \|\{a_n\}_{n=0}^{\infty}\|_S$) for all $\{a_n\} \in S$, then S_0 is a closed subspace of S.*

Proof. Let $\{a_n\}$ belong to the closure of S_0 in S and let $\{a_{n,m}\}_{n=0}^{\infty} \in S_0$ such that $\lim_{m\to\infty} \|\{a_n - a_{n,m}\}_{n=0}^{\infty}\|_S = 0$. Choose some $m(\varepsilon)$ such that $\|\{a_n - a_{n,m(\varepsilon)}\}_{n=0}^{\infty}\|_S < c_S^{-1}\varepsilon$. Then

$$\begin{aligned} \|\{a_n\}_{n=l}^{\infty}\|_S &\leq c_S \left(\|\{a_n - a_{n,m(\varepsilon)}\}_{n=l}^{\infty}\|_S + \|\{a_{n,m(\varepsilon)}\}_{n=l}^{\infty}\|_S\right) \\ &\leq c_S \left(\|\{a_n - a_{n,m(\varepsilon)}\}_{n=0}^{\infty}\|_S + \|\{a_{n,m(\varepsilon)}\}_{n=l}^{\infty}\|_S\right) < \varepsilon \end{aligned}$$

for all sufficiently large $l \geq l(\varepsilon)$, since $\{a_{n,m(\varepsilon)}\}_{n=0}^{\infty} \in S_0$. ▮

Remark 3.19 *If $\{1\}_{n=0}^{\infty} \in S$, then $A(X,S) = X$ in the sense of equivalent quasi-norms, since $E_n^X(f) \le \|f\|_X = \|f\|_X \cdot 1$ implies $\{E_n^X(f)\} \in S$ and $\|\{E_n^X(f)\}\|_S \le \|\{1\}\|_S \|f\|_X$ (cf. (A2)). This shows that the assertion $\operatorname{clos} \bigcup A_n = A(X,S_0)$ of Theorem 3.17 may be false in this case. (For example, consider the case $S_0 = S$ and $\operatorname{clos} \bigcup A_n \ne X$.)*

Corollary 3.20 *If $\{1\}_{n=0}^{\infty} \notin S$ and if (3.14) is satisfied, then $\bigcup A_n$ is dense in $A(X,S)$.*

Proof. (3.14) implies $A(X,S) = A(X,S_0)$, which yields the assertion in view of Theorem 3.17. ▮

Example 3.21 *Let $S = l^q(\{b_n\})$ and suppose that the conditions of Remark 3.5 are satisfied. Moreover, assume that $\{1\}$ does not belong to S, i.e. $\|\{b_n\}_{n=0}^{\infty}\|_q = \infty$. If $q < \infty$, then $\bigcup A_n$ is dense in $A(X, l^q(\{b_n\}))$. If $q = \infty$, then the closure of $\bigcup A_n$ in $A(X, l^\infty(\{b_n\}))$ is equal to the subspace*

$$A(X, l_0^\infty(\{b_n\})) = \{f \in X : \lim_{n\to\infty} b_n E_n^X(f) = 0\},$$

since $S_0 = l_0^\infty(\{b_n\}) = \{\{a_n\} : \lim_{n\to\infty} b_n a_n = 0\}$ in this case. This shows that, in case $q = \infty$, the classes $A_q^s(X)$ and even $A(X, l^q(\{b_n\}))$ are, in general, not sufficient to characterize the closure of $\bigcup A_n$ in $A(X, l^\infty(\{b_n\}))$ as approximation space belonging to such a class.

3.4. Convergence and compactness in $\mathbf{A(X,S_0)}$

Let $A(X,S)$ be an approximation space and let S_0 be the sequence space defined in Theorem 3.17.

Theorem 3.22 (Characterization of convergence in $\mathbf{A(X,S_0)}$) *Let f_m ($m \in \mathbb{N}$) and f be elements of $A(X,S_0)$. If $\lim_{k\to\infty} \|\{1\}_{n=0}^{k}\|_S = \infty$, then the following assertions are equivalent.*

- *(i) $f_m \to f$ in $A(X,S_0)$.*
- *(ii) $f_m \to f$ in X and $\lim_{l\to\infty} \|\{E_n^X(f_m)\}_{n=l}^{\infty}\|_S = 0$ uniformly in m.*

(The second assertion of (ii) means that, for every $\varepsilon > 0$, there exists an $l_0 = l_0(\varepsilon)$ such that $\|\{E_n^X(f_m)\}_{n=l_0}^{\infty}\|_S < \varepsilon$ for all m.)

Proof. (i)⇒(ii): Let $\lim_{m\to\infty} f_m = f$ in $A(X,S_0)$. Then f is also the X-limit of $\{f_m\}$, since $A(X,S_0)$ is continuously embedded into X. Moreover, for every $\varepsilon > 0$, there exists some $m_0 = m_0(\varepsilon)$ such that

$$\|\{E_n^X(f - f_m)\}_{n=0}^{\infty}\|_S < (4c_S^2 c_X C_S)^{-1}\varepsilon \quad \text{for all } m > m_0,$$

where c_S, c_X and C_S are the constants from the quasi-triangle equations and (A3), respectively. f and $f_1, \dots, f_{m_0}$ belong to $A(X, S_0)$. Hence,

$$\|\{E_n^X(f)\}_{n=l_0}^\infty\|_S < (4c_S^2 c_X C_S)^{-1}\varepsilon\,, \quad \max_{m\le m_0} \|\{E_n^X(f_m)\}_{n=l_0}^\infty\|_S < \varepsilon \quad (3.19)$$

for some $l_0 = l_0(\varepsilon) \in \mathbb{N}$. For $m \le m_0$ the second assertion of (ii) is contained in (3.19). For $m > m_0$ we may estimate

$$\begin{aligned}\|\{E_{K(n)}^X(f_m)\}_{n=l_0}^\infty\|_S &\le c_X c_S(\|\{E_n^X(f_m - f)\}_{n=l_0}^\infty\|_S + \|\{E_n^X(f)\}_{n=l_0}^\infty\|_S) \\ &< (2c_S C_S)^{-1}\varepsilon\,, \qquad (3.20)\end{aligned}$$

where we used Lemma 3.7 and (A2). Moreover, the estimate

$$\begin{aligned}E_k^X(f_m)\,\|\{1\}_{n=0}^k\|_S &= \|\{E_k^X(f_m)\}_{n=0}^k\|_S \\ &\le \|\{E_n^X(f_m)\}_{n=0}^k\|_S \le \|f_m\|_{X,S} \le c\end{aligned} \qquad (3.21)$$

shows that $\lim_{n\to\infty} E_n^X(f_m) = 0$ uniformly in m. Thus, there exists some $n_0 = n_0(\varepsilon) \ge K(l_0 - 1)$ such that

$$E_{n_0}^X(f_m)\,\|\{1\}_{n=0}^{l_0-1}\|_S \le (2c_S C_S)^{-1}\varepsilon \quad \text{for all } m\,. \qquad (3.22)$$

As in the proof of Lemma 3.11 (last estimate with $\|\{E_{\dots}^X(f_m)\}_{n=\dots}^\infty\|_S$ instead of $\lim_{k\to\infty}\|\{a_{\dots}\}_{n=\dots}^k\|_S$) we see that (3.20) and (3.22) imply

$$\|\{E_n^X(f_m)\}_{n=n_0}^\infty\|_S < \varepsilon \quad \text{for all } m > m_0\,.$$

(ii)$\Rightarrow$(i): Let $\varepsilon > 0$. From (ii) and $f \in A(X, S_0)$ it follows the existence of some $l_0 = l_0(\varepsilon)$ such that

$$\|\{E_n^X(f)\}_{n=l_0}^\infty\|_S < [4c_S^2 C_S c_X]^{-1}\varepsilon \text{ and } \|\{E_n^X(f_m)\}_{n=l_0}^\infty\|_S < [4c_S^2 C_S c_X]^{-1}\varepsilon$$

for all m. This implies, in view of Lemma 3.7 and (A2),

$$\begin{aligned}\|\{E_{K(n)}^X(f - f_m)\}_{n=l_0}^\infty\|_S &\le c_X c_S\left(\|\{E_n^X(f)\}_{n=l_0}^\infty\|_S + \|\{E_n^X(f_m)\}_{n=l_0}^\infty\|_S\right) \\ &< (2c_S C_S)^{-1}\varepsilon \quad \text{for all } m\,. \qquad (3.23)\end{aligned}$$

Using $\|\{E_n^X(f_m)\}_{n=0}^\infty\|_S \le c_S\left(\|f_m\|_X\|\{1\}_{n=0}^{k-1}\|_S + \|\{E_n^X(f_m)\}_{n=k}^\infty\|_S\right)$, we see that $\{f_m\}$ is bounded in $A(X, S)$. Consequently, $\{f - f_m\}$ is also bounded in $A(X, S)$. Thus, the considerations after (3.20) can be repeated with $f - f_m$ instead of f_m and so we conclude from (3.23) that

$$\|\{E_n^X(f - f_m)\}_{n=n_0}^\infty\|_S < \varepsilon \quad \text{for all } m\,,$$

where $n_0 = n_0(\varepsilon)$ does not depend on m. Moreover,

$$\|\{E_n^X(f-f_m)\}_{n=0}^{n_0-1}\|_S \le \|f-f_m\|_X \|\{1\}_{n=0}^{n_0-1}\|_S < \varepsilon \text{ for } m \ge m_0 = m_0(\varepsilon).$$

The last two estimates show that $\|f-f_m\|_{X,S} < 2c_S\varepsilon$ for all $m \ge m_0$, i.e. f_m converges to f in $A(X,S_0)$. ▮

Remark 3.23 *If (3.10) is satisfied, then it is sufficient to suppose $f \in A(X,S)$ in Theorem 3.22 (i.e. (ii) and $f \in A(X,S)$ imply $f \in A(X,S_0)$ in this case).*

Proof. (A2) and (3.10) imply that

$$\lim_{k\to\infty} \|\{E_n^X(f)\}_{n=n_0}^k\|_S \le \|\{E_n^X(f)\}_{n=n_0}^\infty\|_S \le C \lim_{k\to\infty} \|\{E_n^X(f)\}_{n=n_0}^k\|_S$$

for all $f \in A(X,S)$. Now the assertion follows from Lemma 3.10, applied with $\mathcal{M} = \{f_m\}$ and $n_0 = n_0(\varepsilon)$ such that $\|\{E_n^X(f_m)\}_{n=n_0}^\infty\|_S \le (C_{X,S}C)^{-1}\varepsilon$ for all m. ▮

The following theorem gives a criterion for convergence in $A(X,S_0)$ in which the knowledge of the X-limit is not needed.

Theorem 3.24 *Let S_0 be complete and suppose that $\lim_{k\to\infty} \|\{1\}_{n=0}^k\|_S = \infty$. Then, for $\{f_m\} \subseteq X$, the following assertions are equivalent.*

(i) $\{f_m\}$ is convergent in $A(X,S_0)$ (especially, $\{f_m\} \subseteq A(X,S_0)$).

(ii) $\{f_m\} \subseteq A(X,S_0)$, $\{f_m\}$ converges in X, and $\lim_{l\to\infty} \|\{E_n^X(f_m)\}_{n=l}^\infty\|_S = 0$ uniformly in m.

(iii) $\{f_m\}$ is convergent in X and $\lim_{l\to\infty} \sup_{k\ge l} \|\{E_n^X(f_m)\}_{n=l}^k\|_S = 0$ uniformly in m.

Proof. In view of Theorem 3.22 it remains to show that (iii) implies that $\{f_m\}$ is a subset of $A(X,S_0)$ with the properties

$$\begin{aligned} &\lim\nolimits_{l\to\infty} \|\{E_n^X(f_m)\}_{n=l}^\infty\|_S = 0 \text{ uniformly in } m \quad \text{and} \\ &f := X\text{-}\lim\nolimits_{m\to\infty} f_m \in A(X,S_0)\,. \end{aligned} \tag{3.24}$$

From Remark 3.13 (applied with S_0 instead of S) it follows that all decreasing $\{a_n\} \subseteq [0,\infty)$ with $\lim_{l\to\infty}\lim_{k\to\infty} \|\{a_n\}_{n=l}^k\|_S = 0$ belong to S_0 and that, for all $\{a_n\} \in S_0$, $\|\{a_n\}\|_S \le C \liminf_{k\to\infty} \|\{a_n\}_{n=0}^k\|_S$, where C only depends on S. So we obtain the following fact:

$$f \in X \text{ and } \lim_{l\to\infty}\lim_{k\to\infty} \|\{E_n^X(f)\}_{n=l}^k\|_S \; (= \lim_{l\to\infty}\sup_{k\ge l} \|\{E_n^X(f_m)\}_{n=l}^k\|_S) = 0$$

$$\text{imply } f \in A(X,S_0) \text{ and } \|\{E_n^X(f)\}_{n=l}^\infty\|_S \le C \lim_{k\to\infty} \|\{E_n^X(f)\}_{n=l}^k\|_S. \tag{3.25}$$

Now it is clear that we can conclude from (iii) that $\{f_m\}$ is a subset of $A(X, S_0)$, which satisfies the first assertion of (3.24). In view of (3.25) it remains to prove that (iii) implies

$$\lim_{l\to\infty}\lim_{k\to\infty}\|\{E_n^X(f)\}_{n=l}^k\|_S = 0 \quad \text{for} \quad f := X\text{-}\lim_{m\to\infty} f_m\,.$$

But this is an easy consequence of Lemma 3.10, applied with $\mathcal{M} = \{f_m\}$ and $M = C_{X,S}^{-1}\varepsilon$. (Use that $\lim_{l\to\infty}\lim_{k\to\infty}\|\{E_n^X(f)\}_{n=l}^k\|_S$ is equal to $\inf_{l\geq 0}\lim_{k\to\infty}\|\{E_n^X(f)\}_{n=l}^k\|_S$.) ▮

Remark 3.25 *If $\|\{a_n\}\|_S = \|\{|a_n|\}\|_S$ for all $\{a_n\} \in S$, then we may assume completeness of S instead of completeness of S_0, since S_0 is a closed subspace of S in this case (see Remark 3.18).*

Theorem 3.26 (Compactness criterion in $\mathbf{A(X, S_0)}$)
Let S_0 be complete and suppose that $\lim_{k\to\infty}\|\{1\}_{n=0}^k\|_S = \infty$. Then, for $\mathcal{M} \subseteq X$, the following assertions are equivalent.

(i) $\mathcal{M}$ is a relatively compact subset of $A(X, S_0)$.

(ii) $\mathcal{M}$ is a subset of $A(X, S_0)$ which is relatively compact in X and $\lim_{l\to\infty}\|\{E_n^X(f)\}_{n=l}^\infty\|_S = 0$ uniformly in $f \in \mathcal{M}$.

(iii) $\mathcal{M}$ is relatively compact in X and $\lim_{l\to\infty}\sup_{k\geq l}\|\{E_n^X(f)\}_{n=l}^k\|_S = 0$ uniformly in $f \in \mathcal{M}$.

The implications "(i)⇒(ii)⇒(iii)" are also true if S_0 is not complete.

Proof. The implication "(ii)⇒(iii)" is obvious. Moreover, (iii) implies (i). This follows from the implication "(iii)⇒(i)" of Theorem 3.24. It remains to prove that (i) implies (ii). Obviously, a relatively compact subset $\mathcal{M}$ of $A(X, S_0)$ is also relatively compact in X, since $A(X, S_0)$ is continuously embedded into X. To prove the last assertion of (ii), let us assume that $\lim_{l\to\infty}\|\{E_n^X(f)\}_{n=l}^\infty\|_S = 0$ does not hold uniformly in $f \in \mathcal{M}$, i.e.

$$C := \lim_{l\to\infty}\sup_{f\in\mathcal{M}}\|\{E_n^X(f)\}_{n=l}^\infty\|_S = \inf_{l\geq 0}\sup_{f\in\mathcal{M}}\|\{E_n^X(f)\}_{n=l}^\infty\|_S > 0\,.$$

Then there exists a sequence $\{f_l\}_{l=0}^\infty \subseteq \mathcal{M}$ such that

$$\|\{E_n^X(f_l)\}_{n=l}^\infty\|_S \geq C/2 \quad \text{for all } l \in \mathbb{N}_0\,. \tag{3.26}$$

But $\{f_l\}$ possesses a subsequence $\{f_{l_j}\}_{j=0}^\infty$ which converges in $A(X, S_0)$. In view of Theorem 3.22, this implies

$$\begin{aligned}\lim\nolimits_{k\to\infty}\|\{E_n^X(f_{l_k})\}_{n=l_k}^\infty\|_S &\leq \lim\nolimits_{k\to\infty}\|\{E_n^X(f_{l_k})\}_{n=k}^\infty\|_S \\ &\leq \lim\nolimits_{k\to\infty}\sup\nolimits_{j\geq 0}\|\{E_n^X(f_{l_j})\}_{n=k}^\infty\|_S = 0\end{aligned}$$

which is in contradiction with (3.26). ▮

Corollary 3.27 *Let $\{f_m\} \subseteq A(X, S_0)$. If $\lim_{k\to\infty} \|\{1\}_{n=0}^k\|_S = \infty$, then the following assertions are equivalent.*

(i) $\{f_m\}$ converges in $A(X, S_0)$.

(ii) $\{f_m\}$ is convergent in X and relatively compact in $A(X, S_0)$.

Proof. (i)⇒(ii) is obvious. (ii)⇒(i) follows from Theorems 3.26 and 3.22. (Remark that the relative compactness in $A(X, S_0)$ implies that the X-limit of f_m belongs to $A(X, S_0)$.) ▮

Example 3.28 *Let $S = l^q(\{b_n\})$ and suppose that the conditions of Remark 3.5 are satisfied. Set $l_0^q(\{b_n\}) := S_0$ (i.e. $l_0^q(\{b_n\}) = l^q(\{b_n\})$ for $q < \infty$ and $l_0^\infty(\{b_n\}) = \{\{a_n\} : \lim_{n\to\infty} b_n a_n = 0\}$; see Example 3.21). If $\|\{b_n\}\|_q = \infty$ then, for $\mathcal{M} \subseteq X$, the following assertions are equivalent.*

(i) $\mathcal{M}$ is a relatively compact subset of $A(X, l_0^q(\{b_n\}))$.

(ii) $\mathcal{M}$ is relatively compact in X and $\lim_{l\to\infty} \|\{b_n E_n^X(f)\}_{n=l}^\infty\|_q = 0$ uniformly in $f \in \mathcal{M}$.

Moreover, a sequence $\{f_m\} \subseteq A(X, l_0^q(\{b_n\}))$ converges in $A(X, l_0^q(\{b_n\}))$ if and only if it is convergent in X and relatively compact in $A(X, l_0^q(\{b_n\}))$.

3.5. Embedding theorems

In all what follows $A(X, S)$, $A(X, S_1)$ and $A(X, S_2)$ are approximation spaces (w.r.t. the same $\{A_n\}$). We will study embeddings of the type $A(X, S) \hookrightarrow X$ and $A(X, S_1) \hookrightarrow A(X, S_2)$. The notation $X_1 \hookrightarrow X_2$ means that X_1 is continuously embedded into X_2, i.e. $X_1 \subseteq X_2$ and the embedding operator $I : X_1 \to X_2$ is continuous. We say that the embedding $X_1 \hookrightarrow X_2$ is compact, if $I : X_1 \to X_2$ is a compact operator.

Proposition 3.29 *If there is a constant $C = C(S_1, S_2) > 0$ such that $\{a_n\} \in S_1$ and $a_0 \geq a_1 \geq \ldots \geq 0$ imply $\{a_n\} \in S_2$ and $\|\{a_n\}\|_{S_2} \leq C\|\{a_n\}\|_{S_1}$, then $A(X, S_1)$ is continuously embedded into $A(X, S_2)$.*

Proof. This follows from the definition of the approximation spaces. ▮

Example 3.30 *Let $0 < q \leq \infty$ and suppose that $0 < K(n+1) - K(n) \leq C^q$ with some constant C. Moreover, let $\mathcal{B}_1 = \{b_n^{(1)}\}_{n=0}^\infty$ and $\mathcal{B}_2 = \{b_n^{(1)}\}_{n=0}^\infty$ be sequences of positive numbers satisfying (3.8). If $b_n^{(2)} \leq \mathrm{const}\, b_n^{(1)}$ then $A(X, l^q(\mathcal{B}_1)) \hookrightarrow A(X, l^q(\mathcal{B}_2))$.*

Example 3.31 *Let $0 < q \le \infty$ and suppose that $K(n) = K\,n$ with some constant $K \in \mathbb{N}$. If $\{b_n\} \subseteq (0, \infty)$ satisfies*

$$\max\{b_n, \ldots, b_{2n}\} \le \text{const} \min\{b_n, \ldots, b_{2n}\} \quad \textit{for all } n \in \mathbb{N}_0 ,$$

then one can show that (3.8) is fulfilled and that

$$\|\{a_n b_n\}_{n=1}^{\infty}\|_q \sim \|\{2^{i/q} a_{2^i} b_{2^i}\}_{i=0}^{\infty}\|_q \quad \textit{for all } a_1 \ge a_2 \ge \ldots \ge 0 \tag{3.27}$$

(see [3], proof of Lemma 5.1). If $0 < r < q$ then it follows, in view of Jensen's inequality,

$$\begin{aligned} \|\{a_n b_n\}_{n=1}^{\infty}\|_q &\sim \|\{2^{i/q} a_{2^i} b_{2^i}\}_{i=0}^{\infty}\|_q \\ &\le \|\{2^{i/q} a_{2^i} b_{2^i}\}_{i=0}^{\infty}\|_r \sim \|\{a_n (n+1)^{\frac{1}{q}-\frac{1}{r}} b_n\}_{n=1}^{\infty}\|_r . \end{aligned}$$

(In the last equivalence we used (3.27) with r and $(n+1)^{\frac{1}{q}-\frac{1}{r}} b_n$ instead of q and b_n .) Together with Proposition 3.29 it follows

$$A(X, l^r(\{(n+1)^{\frac{1}{q}-\frac{1}{r}} b_n\})) \hookrightarrow A(X, l^q(\{b_n\})) \quad \textit{for } 0 < r < q \le \infty . \tag{3.28}$$

Especially, we obtain the well-known embedding $A^s_r(X) \hookrightarrow A^s_q(X)$. We remark that, in certain non-classical cases, (3.28) can be improved. Moreover, embedding theorems of the type $A(X, l^q(\{b_n\})) \hookrightarrow A(X, l^r(\{c_n\}))$ $(q > r)$ can be obtained with the help of the Hölder inequality. For example, $A^s_q(X) \hookrightarrow A^{s-\varepsilon}_r(X)$ for all $\varepsilon > 0$. The general situation will be treated later in Theorem 4.12 and Corollary 5.5.

We already know that $A(X, S)$ is continuously embedded into X (see Proposition 3.8). Now we will show that, under certain conditions, this embedding is even compact if all A_n are boundedly compact, especially, if all A_n are finite-dimensional linear subspaces of X. (A subset $\mathcal{M}$ of X is said to be boundedly compact if every bounded subset of $\mathcal{M}$ is relatively compact in X.)

Theorem 3.32 *Suppose that X is complete and that all A_n are boundedly compact. If $\lim_{k\to\infty} \|\{1\}_{n=0}^{k}\|_S = \infty$, then the embedding $A(X, S) \hookrightarrow X$ is compact.*

Proof. Let $\mathcal{M}$ be a bounded subset of $A(X, S)$. The estimate (3.21) (applied with $f \in \mathcal{M}$ instead of f_m) shows that

$$\lim_{n\to\infty} E_n^X(f) = 0 \;\text{ uniformly in } f \in \mathcal{M}. \tag{3.29}$$

If X is a Banach space and all A_n are finite-dimensional subspaces, then, by a well-known theorem of Timan ([15, Section 2.5.1]), (3.29) implies the relative compactness of $\mathcal{M}$ in X. Fortunately, Timan's proof can be generalized to our situation as follows: Let $\varepsilon > 0$ and set $\delta := (2c_X)^{-1}\varepsilon$, where c_X is the constant from the quasi-triangle equation in X. In view of (3.29), we can choose some $n = n(\varepsilon)$ such that $E_n^X(f) < \delta$ for all $f \in \mathcal{M}$. This means that every open ball $B_\delta(f)$ (in X) with middle point $f \in \mathcal{M}$ and radius δ contains an element of A_n. This implies that the set

$$\mathcal{M}_n := B_\delta(\mathcal{M}) \cap A_n \qquad (\, B_\delta(\mathcal{M}) := \bigcup_{f\in\mathcal{M}} B_\delta(f)\,)$$

is a δ-net for $\mathcal{M}$. Moreover, $\mathcal{M}_n$ is a bounded subset of A_n, since boundedness of $\mathcal{M}$ (in X) implies boundedness of $B_\delta(\mathcal{M})$. Hence, $\mathcal{M}_n$ is relatively compact in X and it follows the existence of a finite δ-net $\mathcal{M}_\varepsilon$ for $\mathcal{M}_n$. Since $\mathcal{M}_n$ is a δ-net for $\mathcal{M}$, we conclude that $\mathcal{M}_\varepsilon$ is a finite ε-net for $\mathcal{M}$. Thus, $\mathcal{M}$ is relatively compact in X. ▌

Theorem 3.33 *Suppose that X is complete and that all A_n are boundedly compact. Let S_1 and S_2 be admissible sequence spaces with the following properties.*

(i) $\lim_{k\to\infty} \|\{1\}_{n=0}^k\|_{S_1} = \infty$.

(ii) S_2 is complete and $\|\{a_n\}\|_{S_2} = \|\{|a_n|\}\|_{S_2}$ for all $\{a_n\} \in S_2$.

(iii) There exists a sequence $\{c_l\}_{l=0}^\infty \subseteq [0,\infty)$ with $\lim_{l\to\infty} c_l = 0$ such that $\{a_n\} \in S_1$ and $a_0 \geq a_1 \geq \ldots \geq 0$ imply $\{a_n\} \in S_2$ and

$$\|\{a_n\}_{n=l}^\infty\|_{S_2} \leq c_l \, \|\{a_n\}_{n=0}^\infty\|_{S_1} \quad \textit{for all } l \in \mathbb{N}_0\,.$$

Then the embedding $A(X,S_1) \hookrightarrow A(X,S_2)$ is compact.

Proof. If $f \in A(X,S_1)$ then, in view of (iii), $\lim_{l\to\infty} \|\{E_n^X(f)\}_{n=l}^\infty\|_{S_2} = 0$. Together with Proposition 3.29 we see that $A(X,S_1)$ is continuously embedded into $A(X,(S_2)_0)$. ($(S_2)_0$ is defined in Theorem 3.17.) Moreover, every bounded subset $\mathcal{M}$ of $A(X,S_1)$ is relatively compact in X (see Theorem 3.32). If $\lim_{k\to\infty} \|\{1\}_{n=0}^k\|_{S_2} = \infty$ then we may apply Theorem 3.26 to show that $\mathcal{M}$ is even relatively compact in $A(X,(S_2)_0)$. (Remark that the completeness of $(S_2)_0$ follows from assumption (ii) and Remark 3.25.) Thus, in this case it remains to show that

$$\lim_{l\to\infty} \|\{E_n^X(f)\}_{n=l}^\infty\|_{S_2} = 0 \;\text{ uniformly in } f \in \mathcal{M}.$$

But this follows easily from assumption (iii). If $\lim_{k\to\infty} \|\{1\}_{n=0}^k\|_{S_2} < \infty$ then $\|\,.\,\|_X$ is equivalent to $\|\,.\,\|_{X,S_2}$ on $A(X,(S_2)_0)$. This follows from Remark 3.14 (applied with $S = (S_2)_0$), since (3.10) holds true in $(S_2)_0$ (see Remark 3.13 and note that we do not need the assumption $\lim_{k\to\infty} \|\{1\}_{n=0}^k\|_S = \infty$ in the corresponding part of the proof of this remark). ▌

Example 3.34 *Let $0 < q \leq \infty$ and suppose that $0 < K(n+1) - K(n) \leq C^q$ with some constant C. Moreover, let $\mathcal{B} = \{b_n\}_{n=0}^\infty$, $\mathcal{B}_1 = \{b_n^{(1)}\}_{n=0}^\infty$ and $\mathcal{B}_2 = \{b_n^{(2)}\}_{n=0}^\infty$ be sequences of positive numbers satisfying (3.8). If X is complete and all A_n are boundedly compact, then the following assertions hold true.*

(i) If $\|\{b_n\}_{n=0}^\infty\|_q = \infty$, then $A(X, l^q(\mathcal{B})) \hookrightarrow X$ compact.

(ii) If $\lim_{n\to\infty} b_n^{(2)}/b_n^{(1)} = 0$ and $\|\{b_n^{(1)}\}_{n=0}^\infty\|_q = \infty$, then the embedding $A(X, l^q(\mathcal{B}_1)) \hookrightarrow A(X, l^q(\mathcal{B}_2))$ is compact.

Especially, for $s > t > 0$ the embeddings $A_q^s(X) \hookrightarrow A_q^t(X) \hookrightarrow X$ are compact if X is complete, all A_n are boundedly compact, and $K(n) = Kn$ with some $K \in \mathbb{N}$.

3.6. The reiteration theorem

Let X be a quasi-normed space and let $\{0\} = A_0 \subseteq A_1 \subseteq \ldots \subseteq X$ satisfy (3.2) and (3.6). If S_1 and S_2 are admissible sequence spaces, then the iterated space $A(A(X, S_1), S_2)$ is well-defined, since also $A(X, S_1)$ is a quasi-normed space which contains all A_n. We will show that $A(A(X, S_1), S_2) = A(X, S)$ for some admissible $S = S(S_1, S_2)$.

Theorem 3.35 *Let S_1 and S_2 be admissible sequence spaces and define*

$$S = S(S_1, S_2) := \left\{\{a_n\} \in S_1 : \left\{\|\{a_{\max\{m,n\}}\}_{m=0}^\infty\|_{S_1}\right\}_{n=0}^\infty \in S_2\right\} ,$$
$$\|\{a_n\}\|_S := \left\|\left\{\|\{a_{\max\{m,n\}}\}_{m=0}^\infty\|_{S_1}\right\}_{n=0}^\infty\right\|_{S_2} .$$

Then S is an admissible sequence space and $A(A(X, S_1), S_2) = A(X, S)$ in the sense of equivalent quasi-norms.

Proof. $f \in X$ belongs to $A(A(X, S_1), S_2)$ if and only if

$$f \in A(X, S_1) \quad \text{and} \quad \{E_n^{A(X,S_1)}(f)\}_{n=0}^\infty \in S_2 ,$$

where the S_2-norm of the last sequence defines the $A(A(X, S_1), S_2)$-norm of f. In view of (3.17) and (A2), the following assertions hold true:

(i) $f \in A(A(X, S_1), S_2)$ implies $\left\{\|\{E_{\max\{K(n),m\}}^X(f)\}_{m=0}^\infty\|_{S_1}\right\}_{n=0}^\infty \in S_2$ and

$$\left\|\left\{\|\{E_{\max\{K(n),m\}}^X(f)\}_{m=0}^\infty\|_{S_1}\right\}_{n=0}^\infty\right\|_{S_2} \leq C_{S_1} \|f\|_{A(A(X,S_1),S_2)} .$$

(ii) $f \in A(X, S_1)$ and $\left\{\|\{E_{\max\{n,m\}}^X(f)\}_{m=0}^\infty\|_{S_1}\right\}_{n=0}^\infty \in S_2$ imply $f \in A(A(X, S_1), S_2)$ and

$$\|f\|_{A(A(X,S_1),S_2)} \leq c_{S_1} C_{S_1} \left\|\left\{\|\{E_{\max\{n,m\}}^X(f)\}_{m=0}^\infty\|_{S_1}\right\}_{n=0}^\infty\right\|_{S_2} .$$

Together with (A3) it follows that $A(A(X,S_1),S_2)=A(X,S)$ and

$$(C_{S_1}C_{S_2})^{-1}\|f\|_{X,S}\le\|f\|_{A(A(X,S_1),S_2)}\le c_{S_1}C_{S_1}\|f\|_{X,S}\,,\quad f\in A(X,S)\,.$$

It remains to prove that S is an admissible sequence space. Obviously, S is a linear subspace of S_1 which satisfies (A1) and (A2). The quasi-norm properties of $\|\,.\,\|_S$ are also obvious. Since $\{K(n)\}$ is increasing, we have

$$K(\max\{m,n\})=\max\{K(m),K(n)\}\,.$$

This shows that $\{a_{K(n)}\}\in S$ (i.e. $\big\{\|\{a_{K(\max\{m,n\})}\}_{m=0}^\infty\|_{S_1}\big\}_{n=0}^\infty\in S_2$) and $a_0\ge a_1\ge\ldots\ge 0$ imply $\big\{\|\{a_{\max\{m,K(n)\}}\}_{m=0}^\infty\|_{S_1}\big\}_{n=0}^\infty\in S_2$ and

$$\big\|\big\{\|\{a_{\max\{m,K(n)\}}\}_{m=0}^\infty\|_{S_1}\big\}_{n=0}^\infty\big\|_{S_2}\le C_{S_1}\|\{a_{K(n)}\}\|_S\,,$$

since $\|\{a_{\max\{m,K(n)\}}\}_{m=0}^\infty\|_{S_1}\le C_{S_1}\|\{a_{\max\{K(m),K(n)\}}\}_{m=0}^\infty\|_{S_1}$ ((A3) in S_1). Again with (A3) (now in S_2) if follows $\big\{\|\{a_{\max\{m,n\}}\}_{m=0}^\infty\|_{S_1}\big\}_{n=0}^\infty\in S_2$ (i.e. $\{a_n\}\in S$) and

$$\|\{a_n\}\|_S=\big\|\big\{\|\{a_{\max\{m,n\}}\}_{m=0}^\infty\|_{S_1}\big\}_{n=0}^\infty\big\|_{S_2}\le C_{S_1}C_{S_2}\|\{a_{K(n)}\}\|_S\,.$$

Thus, (A3) is satisfied in S. ▌

Example 3.36 *Let $0<q\le\infty$ and suppose that $0<K(n+1)-K(n)\le C^q$ with some constant C. Moreover, let $\mathcal{B}_1=\{b_n^{(1)}\}_{n=0}^\infty$ and $\mathcal{B}_2=\{b_n^{(2)}\}_{n=0}^\infty$ be sequences of positive numbers satisfying (3.8). For $S=S(l^q(\mathcal{B}_1),l^q(\mathcal{B}_2))$ and arbitrary sequences $\{a_n\}\subset\mathbb{R}$ we have*

$$\begin{aligned}
\|\{a_n\}\|_S &= \left\|\left\{b_n^{(2)}\left\|\left\{b_m^{(1)}a_{\max\{n,m\}}\right\}_{m=0}^\infty\right\|_q\right\}_{n=0}^\infty\right\|_q\\
&\sim \left\|\left\{b_n^{(2)}|a_n|\left\|\left\{b_m^{(1)}\right\}_{m=0}^n\right\|_q+b_n^{(2)}\left\|\left\{b_m^{(1)}a_m\right\}_{m=n+1}^\infty\right\|_q\right\}_{n=0}^\infty\right\|_q\\
&\sim \left\|\left\{b_n^{(2)}|a_n|\left\|\left\{b_m^{(1)}\right\}_{m=0}^n\right\|_q\right\}_{n=0}^\infty\right\|_q+\left\|\left\{b_n^{(2)}\left\|\left\{b_m^{(1)}a_m\right\}_{m=n+1}^\infty\right\|_q\right\}_{n=0}^\infty\right\|_q\\
&= \left\|\left\{b_n^{(2)}|a_n|\left\|\left\{b_m^{(1)}\right\}_{m=0}^n\right\|_q\right\}_{n=0}^\infty\right\|_q+\left\|\left\{b_m^{(1)}|a_m|\left\|\left\{b_n^{(2)}\right\}_{n=0}^{m-1}\right\|_q\right\}_{m=1}^\infty\right\|_q\\
&\sim \left\|\left\{b_n^{(2)}|a_n|\left\|\left\{b_m^{(1)}\right\}_{m=0}^n\right\|_q\right\}_{n=0}^\infty\right\|_q+\left\|\left\{b_m^{(1)}|a_m|\left\|\left\{b_n^{(2)}\right\}_{n=0}^{m}\right\|_q\right\}_{m=0}^\infty\right\|_q\\
&\sim \left\|\left\{a_n\left(b_n^{(2)}\left\|\left\{b_m^{(1)}\right\}_{m=0}^n\right\|_q+b_n^{(1)}\left\|\left\{b_m^{(2)}\right\}_{m=0}^n\right\|_q\right)\right\}_{n=0}^\infty\right\|_q\,,
\end{aligned}$$

which shows that $S = l^q(\mathcal{B})$ (in the sense of equivalent quasi-norms), where

$$\mathcal{B} = \{b_n\}_{n=0}^{\infty} \quad \text{with} \quad b_n = b_n^{(1)} \|\{b_m^{(2)}\}_{m=0}^n\|_q + b_n^{(2)} \|\{b_m^{(1)}\}_{m=0}^n\|_q \,.$$

Consequently,

$$A(A(X, l^q(\mathcal{B}_1)), l^q(\mathcal{B}_2)) = A(A(X, l^q(\mathcal{B}_2)), l^q(\mathcal{B}_1)) = A(X, l^q(\mathcal{B}))$$

in the sense of equivalent quasi-norms. (Remark that $l^q(\mathcal{B})$ is admissible, since, in view of Theorem 3.35, S is admissible.) Under more restrictive assumptions on $\mathcal{B}_1$, $\mathcal{B}_2$, and $K(n)$ one can prove that

$$A(A(X, l^q(\mathcal{B}_1)), l^r(\mathcal{B}_2)) = A(X, l^r(\{b_n^{(1)} b_n^{(2)} (n+1)^{1/q}\}))\,,$$

also if $r > 0$ is different from $q > 0$ (cf. [4]). Especially, $A_r^{s_2}(A_q^{s_1}(X)) = A_r^{s_1+s_2}(X)$ for all $s_1, s_2 > 0$, which is the well-known classical reiteration theorem.

3.7. Homogenity of approximation spaces

Let X be a homogeneous quasi-normed space, i.e. the elements of X are (classes of) 2π-periodic measurable functions on $\mathbb{R}$ and the following two properties hold true:

$$f \in X \text{ implies } f_\zeta \in X \text{ and } \|f\|_X = \|f_\zeta\|_X \text{ for all } \zeta \in \mathbb{R}, \\ \text{where } f_\zeta(x) := f(x - \zeta). \tag{3.30}$$

$$\lim_{\zeta \to \zeta_0} \|f_\zeta - f_{\zeta_0}\|_X = 0 \text{ for all } f \in X \text{ and all } \zeta_0 \in \mathbb{R}. \tag{3.31}$$

We will show that, under certain conditions, also $A(X, S_0)$ is homogeneous. (S_0 is defined in Theorem 3.17.)

Remark 3.37 *Our definition of homogenity is not standard. Usually, X has to be a Banach space which is continuously embedded into $L^1_{2\pi}$ and which satisfies (3.30) and (3.31). Here we use a more general definition, since norm properties of $\|\,.\,\|_{X,S}$ and completeness of $A(X, S_0)$ do not follow immediately from the corresponding properties of $(X, \|\,.\,\|_X)$, but can be checked easily by Proposition 3.8 and Theorems 3.12 and 3.17.*

Homogeneous Banach spaces are very useful for the study of trigonometric approximation. For example, it is well-known that trigonometric polynomials are dense in all homogeneous Banach spaces.

Theorem 3.38 *Let $A(X, S)$ be an approximation space based on a homogeneous quasi-normed space X and suppose that $\lim_{k\to\infty} \|\{1\}_{n=0}^k\|_S = \infty$. If $(A_n)_\zeta \subseteq A_n$ for all $n \in \mathbb{N}_0$ and all $\zeta \in \mathbb{R}$, where $(A_n)_\zeta := \{f_\zeta : f \in A_n\}$, then $A(X, S_0)$ is homogeneous.*

Proof. Let $f \in X$ and $\zeta \in \mathbb{R}$. For all $f_n \in A_n$ we have

$$E_n^X(f_\zeta) \le \|f_\zeta - (f_n)_\zeta\|_X = \|f - f_n\|_X \,.$$

Taking the infimum over all $f_n \in A_n$ it follows $E_n^X(f_\zeta) \le E_n^X(f)$, which also implies $E_n^X(f) = E_n^X((f_\zeta)_{-\zeta}) \le E_n^X(f_\zeta)$. Thus,

$$E_n^X(f_\zeta) = E_n^X(f) \quad \text{for all} \;\; n \in \mathbb{N}_0 \,.$$

This shows that $f \in A(X, S_0)$ implies $f_\zeta \in A(X, S_0)$ and $\|f_\zeta\|_{X,S} = \|f\|_{X,S}$. It remains to prove that (3.31) is satisfied in $A(X, S_0)$ or, equivalently, that $\lim_{m\to\infty} \|f - f_{\zeta_m}\|_{X,S} = 0$ for all $f \in A(X, S_0)$ and all $\{\zeta_m\} \subset \mathbb{R}$ with $\lim_{m\to\infty} \zeta_m = 0$. From the homogenity of X it follows that

$$\lim_{m\to\infty} \zeta_m = 0 \;\text{ implies }\; \lim_{m\to\infty} \|f - f_{\zeta_m}\|_X = 0 \;\text{ for all } f \in X.$$

If $f \in A(X, S_0)$ then $\lim_{l\to\infty} \|\{E_n^X(f_{\zeta_m})\}_{n=l}^\infty\|_S = \lim_{l\to\infty} \|\{E_n^X(f)\}_{n=l}^\infty\|_S = 0$ uniformly in m. Theorem 3.22 now shows that $\lim_{m\to\infty} \|f - f_{\zeta_m}\|_{X,S} = 0$. ▌

4. APPLICATIONS

4.1. Existence of elements with prescribed best approximation errors

In this subsection we will show that the following well-known Theorem of Bernstein can be proved easily with the help of approximation spaces:

Theorem 4.1 ([15], Section 2.5) *Let X be a Banach space and let $\{0\} = A_0 \subset A_1 \subset A_2 \subset \dots$ be finite-dimensional linear subspaces of X. ($A_i \subset A_{i+1}$ means that A_i is a proper subset of A_{i+1}.) Then, for every decreasing sequence $\{E_n\}_{n=0}^\infty \subset \mathbb{R}$ with $\lim_{n\to\infty} E_n = 0$, there exists some $f \in X$ with $E_n^X(f) = E_n$ for all $n \in \mathbb{N}_0$.*

Proof. For any $g \in X$ and any $E \ge E_{n+1}^X(g)$ one can find some $\lambda \in \mathbb{R}$ such that $E_n^X(g + \lambda g_n) = E$, where $g_n \in A_{n+1}$ is a best approximation of g (i.e. $E_{n+1}^X(g) = \|g - g_n\|_X$) in case $E_{n+1}^X(g) < E_n^X(g)$ and g_n is an arbitrary element of $A_{n+1} \setminus A_n$ if $E_{n+1}^X(g) = E_n^X(g)$. This follows from the continuity of the function $\phi(\lambda) := E_n^X(g + \lambda g_n)$, which tends to infinity for $\lambda \to \pm\infty$. Starting with $g = 0$ and $E = E_n$ one can use this fact $n + 1$ times to prove that there exists an element $f_n = \lambda_n g_n + \dots + \lambda_0 g_0 \in A_{n+1}$ with

$$E_m^X(f_n) = E_m^X(\lambda_n g_n + \dots + \lambda_m g_m) = E_m \,, \;\; m = 0, \dots, n \,. \qquad (4.1)$$

This can be done for every $n \in \mathbb{N}_0$ and so we get a sequence $f_0, f_1, f_2, \dots$ of elements $f_n \in A_{n+1}$ with the above property. Until now we followed the same ideas as in the usual standard proof. But in a second step we have to show that $\{f_n\}$ possesses a convergent subsequence and this can be done easily with the help of approximation space theory: Without loss of generality we may assume $E_n > 0$ for all n. (Otherwise we have $E_n = 0$ for all $n > n_0$ and $f = f_{n_0}$ is the element we are looking for.) Obviously, $\{f_n\}$ is a bounded subset of $A(X, l^\infty(\{1/E_n\}))$ and this implies, in view of Theorem 3.32, that $\{f_n\}$ is relatively compact in X. Hence, there is a subsequence $\{f_{n_k}\}_{k=1}^\infty$ which converges in X. From (4.1) it follows that the limit f satisfies $E_m^X(f) = E_m$ for all m (since $E_m^X(\,.\,) : X \to \mathbb{R}$ is continuous for all m; see Lemma 3.9). ▌

If $A(X, S_1)$ and $A(X, S_2)$ are approximation spaces (w.r.t. the same $\{A_n\}$) with $\mathcal{D}(S_1) = \mathcal{D}(S_2)$, where

$$\mathcal{D}(S) := \{\{a_n\} \in S : a_0 \geq a_1 \geq \dots \geq 0\}\,, \tag{4.2}$$

then the set $A(X, S_1)$ is identical with the set $A(X, S_2)$. Theorem 4.1 shows that, under certain conditions, the counterdirection is also true:

Corollary 4.2 *Let the assumptions of Theorem 4.1 be fulfilled and let S_1 and S_2 be two admissible sequence spaces with $\{1\} \notin S_1$ and $\{1\} \notin S_2$. Then the set $A(X, S_1)$ is identical with the set $A(X, S_2)$ if and only if $\mathcal{D}(S_1) = \mathcal{D}(S_2) =: \mathcal{D}$. In this case, the norms of $A(X, S_1)$ and $A(X, S_2)$ are equivalent if and only if $\|\{a_n\}\|_{S_1} \sim \|\{a_n\}\|_{S_2}$ for all $\{a_n\} \in \mathcal{D}$.*

Proof. We only have to remark that $\{1\} \notin S_i$ implies $\lim_{n\to\infty} a_n = 0$ for all $\{a_n\} \in \mathcal{D}(S_i)$ (see (3.18)). Now the assertions follow from Theorem 4.1 and the definition of approximation spaces. ▌

4.2. Extension of unbounded operators

Let X be a quasi-normed space and let $\{0\} = A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots \subseteq X$ satisfy (3.2) and (3.6). In all what follows, we consider a linear operator $\mathcal{A} : \bigcup A_n \to Y$, where Y is a quasi-Banach space. $\mathcal{A}$ may be unbounded. As "degree of unboundedness" one can view the behaviour of the "norms"

$$\|\mathcal{A}\|_n := \sup_{0 \neq f_n \in A_n} \frac{\|\mathcal{A} f_n\|_Y}{\|f_n\|_X} \quad (:= 0 \text{ if } A_n = \{0\})\,.$$

If one is able to estimate these numbers, then one can use approximation spaces to obtain mapping properties of $\mathcal{A}$. More precisely, we will show that, in many cases, the sequence $\{\|\mathcal{A}\|_n\}$ defines an approximation space $A(X, S)$, which can be viewed as domain of definition of $\mathcal{A}$. This means that

$\mathcal{A}$ can be uniquely extended to a bounded linear operator $\mathcal{A} : A(X,S) \to Y$ (Theorem 4.3) and, in a certain class of approximation spaces, $A(X,S)$ is the largest space with this property (Corollary 4.13). Moreover, the reiteration theorem can be used to obtain mapping properties of $\mathcal{A}$ in scales of approximation spaces based on $(X, \{A_n\})$ and $(Y, \{\mathcal{A}(A_n)\})$, respectively (Corollary 4.6).

Theorem 4.3 *Let $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$ be a strictly increasing sequence with $a_0 = 0$, such that $\lim_{n\to\infty} a_n = \infty$ and $a_{K(n)+1} \le \mathrm{const}\, a_n$ for all $n \in \mathbb{N}$. Moreover, let $p \in (0,1]$ such that Y can be equipped with an equivalent p-norm $|||\,.\,|||_Y$ (i.e. (3.5) holds true in Y). Then the following assertions are equivalent.*

(i) $\|\mathcal{A}\|_n \le \mathrm{const}\, a_n$ *for all* $n \in \mathbb{N}$.

(ii) There exists a bounded linear extension $\mathcal{A} : A(X, l^p(\mathcal{B})) \to Y$, where

$$\mathcal{B} = \{b_n\} \quad \textit{with} \quad b_n = a_{n+1}^{1-(1/p)} \left(a_{n+1} - a_n\right)^{1/p} . \tag{4.3}$$

The space $l^p(\mathcal{B})$ is admissible and does not contain the sequence $\{1\}_{n=0}^{\infty}$ (i.e. $\|\mathcal{B}\|_p = \infty$). Especially, $\bigcup_{n=0}^{\infty} A_n$ is dense in $A(X, l^p(\mathcal{B}))$ and, hence, the extension $\mathcal{A} : A(X, l^p(\mathcal{B})) \to Y$ is uniquely determined if (i) is satisfied.

Remark 4.4 *If $a_n = a(n)$ with some $a \in C^1((0,\infty)) \cap C([0,\infty))$ which satisfies $0 < a'(t) \sim a'(n+1)$ for all $t \in [n, n+1]$ and all $n \in \mathbb{N}$, then $a_{n+1} - a_n \sim a'(n+1)$, i.e. b_n can be replaced by $a_{n+1}^{1-(1/p)} a'(n+1)^{1/p}$ in this case.*

Since the proof of Theorem 4.3 is rather long, we will give it later in Section 5. Here we only mention that assertion (ii) is equivalent to

$$\|\mathcal{A}f\|_Y \le \mathrm{const}\, \|f\|_{X, l^p(\mathcal{B})} \quad \text{for all } f \in \bigcup A_n \, . \tag{4.4}$$

(A Lipschitz continuous function $F : U \to Y$ on a dense subset U of a metrical space possesses an uniquely determined continuous extension onto the whole space. Here we use that Y is complete.) The relation

$$a_n \sim \|\{b_m\}_{m=0}^{n-1}\|_p \, , \quad n \in \mathbb{N} \tag{4.5}$$

(which can be easily proved; see (5.3) and (5.4)) suggests a reformulation of the theorem in the following sense: We start with a given sequence $\{b_n\}_{n=0}^{\infty} \subset (0,\infty)$ and set $a_n := \|\{b_m\}_{m=0}^{n-1}\|_p$ ($:= 0$ for $n = 0$). If we suppose that $a_{K(n)+1} \le \mathrm{const}\, a_n$ ($n \in \mathbb{N}$) and, consequently, $a_{n+1} \le \mathrm{const}\, a_n$, then the equation

$$a_{n+1} - a_n = \left(a_{n+1}^p\right)^{1/p} - (a_n^p)^{1/p} = \frac{1}{p}\,\xi_n^{(1/p)-1} b_n^p \quad (\xi_n \in (a_n^p, a_{n+1}^p)) \quad (4.6)$$

shows that $a_{n+1}^{1-(1/p)}\,(a_{n+1} - a_n)^{1/p} \sim b_n$, so that Theorem 4.3 now reads as follows:

Theorem 4.5 *Let $p \in (0,1]$ be defined as in Theorem 4.3 and let $\mathcal{B} = \{b_n\}_{n=0}^{\infty}$ be a sequence of positive numbers with the properties*

$$\|\mathcal{B}\|_p = \infty \;\; \textit{and} \;\; \|\{b_m\}_{m=0}^{K(n)}\|_p \le \mathrm{const}\,\|\{b_m\}_{m=0}^{n-1}\|_p \;\; \textit{for all} \;\; n \in \mathbb{N}. \quad (4.7)$$

Then $l^p(\mathcal{B})$ is admissible and the following assertions are equivalent.

(i) $\|\mathcal{A}\|_n \le \mathrm{const}\,\|\{b_m\}_{m=0}^{n-1}\|_p$ *for all $n \in \mathbb{N}$.*

(ii) (4.4) holds true.

(iii) There exists an unique bounded linear extension $\mathcal{A}\colon A(X, l^p(\mathcal{B})) \to Y$.

Corollary 4.6 *Let the assumptions of Theorem 4.5 be satisfied. Then, for all admissible sequence spaces $\mathcal{S}$, the extension $\mathcal{A} : A(X, l^p(\mathcal{B})) \to Y$ defines a bounded linear operator from $A(X, S(l^p(\mathcal{B}), \mathcal{S}))$ into $A(Y, \mathcal{S}, \{\mathcal{A}(A_n)\})$. ($S(l^p(\mathcal{B}), \mathcal{S})$ is defined in Theorem 3.35.)*

Proof. Obviously, a bounded linear operator $B : X \to Y$ is also bounded from $A(X, \mathcal{S})$ into $A(Y, \mathcal{S}, \{B(A_n)\})$. (Use that $E_n^Y(Bf) \le \|Bf - Bf_n\|_Y \le \mathrm{const}\,\|f - f_n\|_X$ for all $f_n \in A_n$.) Applying this with $A(X, l^p(\mathcal{B}))$ instead of X and using the reiteration theorem, we get the assertion. ▮

Example 4.7 *If $K(n) = Kn$ ($K \in \mathbb{N}$) and $\|\mathcal{A}\|_n \le \mathrm{const}\, n^s$ with some $s > 0$, then $\mathcal{A} \in \mathcal{L}(A_p^s(X), Y)$ (with the p from Theorem 4.3) and $\mathcal{A} \in \mathcal{L}(A_q^{s+\delta}(X), A_q^{\delta}(Y, \{\mathcal{A}(A_n)\}))$ for all $\delta > 0$ and $q \in (0, \infty]$. (Here we used $A_q^{\delta}(A_p^s(X)) = A_q^{s+\delta}(X)$; see the end of Example 3.36. The notation $\mathcal{A} \in \mathcal{L}(F, G)$ means that $\mathcal{A}$ is linear and bounded from F into G.)*

Example 4.8 *The Cauchy singular integral operator*

$$(\mathcal{A}u)(x) := \int_{-1}^{1} \frac{u(t)}{t - x}\,\frac{dt}{\varphi(t)}\,, \quad \varphi(t) := \sqrt{1 - t^2}\,,$$

has the property $\|\varphi \mathcal{A} P_n\|_{L^\infty(-1,1)} \le \mathrm{const}\,\|P_n\|_{L^\infty(-1,1)} \ln(n+1)$ for all $n \in \mathbb{N}$ and all $P_n \in \mathcal{P}_n$ (see [9], Corollary 4.5). Now, Theorem 4.3 and Remark 4.4 yield

$$\mathcal{A} \in \mathcal{L}(B_1^0(L^\infty), L_{1/2,1/2}^\infty)\,, \;\; \textit{where} \;\; B_1^0(L^\infty) := A_1^0(L^\infty(-1,1), \{\mathcal{P}_n\}). \quad (4.8)$$

Together with $\mathcal{A}(\mathcal{P}_n) = \mathcal{P}_{n-1}$ (cf. [14], Theorem 9.9), (4.8) can be used to obtain mapping properties of $\mathcal{A}$ in scales of approximation spaces based on $(L^\infty, \{\mathcal{P}_n\})$ and $(L^\infty_{1/2,1/2}, \{\mathcal{P}_n\})$, respectively. For details and generalizations we refer to [9] and [12].

4.3. Continuous embeddings as consequences of the extension theorem

Take the notations and assumptions from the beginning of Subsection 4.2. We want to apply Theorems 4.3 and 4.5 with the embedding operator $\mathcal{A} = I$. Of course, we have to suppose $\mathcal{A}(\bigcup A_n) = \bigcup A_n \subseteq Y$, i.e. $\bigcup A_n$ is a subset of $X \cap Y$. Moreover, we assume that X and Y are continuously embedded into some topological Hausdorff space V (usually, $V = X$). This ensures that the extension $\mathcal{A} : A(X, l^p(\mathcal{B})) \to Y$ from Theorem 4.5 is nothing else than the embedding operator. (Use the density of $\bigcup A_n$ in $A(X, l^p(\mathcal{B}))$.) The following Theorem is an immediate consequence of Theorem 4.5 and Corollary 4.6.

Theorem 4.9 *Let $p \in (0, 1]$ be defined as in Theorem 4.3 and let $\mathcal{B} = \{b_n\}_{n=0}^\infty$ be a sequence of positive numbers, such that (4.7) is satisfied. Then the following assertions are equivalent.*

(i) $A(X, l^p(\mathcal{B})) \hookrightarrow Y$.

(ii) $\|f_n\|_Y \le \mathrm{const}\, \|\{b_m\}_{m=0}^{n-1}\|_p \|f_n\|_X$ *for all* $f_n \in A_n$ *and all* $n \in \mathbb{N}$.

If (ii) is satisfied, then $A(X, S(l^p(\mathcal{B}), \mathcal{S})) \hookrightarrow A(Y, \mathcal{S})$ for all admissible sequence spaces $\mathcal{S}$. ($S(l^p(\mathcal{B}), \mathcal{S})$ is defined in Theorem 3.35.)

Example 4.10 *If $K(n) = K\,n$ ($K \in \mathbb{N}$) and $b_n = (n+1)^{s-(1/p)}$ with some $s > 0$, then we obtain from Theorem 4.9 and the reiteration theorem $A^\delta_q(A^s_p(X)) = A^{s+\delta}_q(X)$ (see the end of Example 3.36) that*

$$\|f_n\|_Y \le \mathrm{const}\, n^s \|f_n\|_X \quad \textit{for all } f_n \in A_n \textit{ and all } n \in \mathbb{N}$$

implies the embeddings $A^s_p(X) \hookrightarrow Y$ (p from Theorem 4.3) and $A^{s+\delta}_q(X) \hookrightarrow A^\delta_q(Y)$ for all $\delta > 0$ and all $q \in (0, \infty]$. This is the famous classical embedding theorem.

The above theorem makes it possible to prove embeddings between Besov type spaces if these spaces can be described as approximation spaces (cf. Theorem 2.2). For example, we obtain the following

Proposition 4.11 *Let $1 \le p < r \le \infty$, $1 \le q \le \infty$, $s > \frac{1}{p} - \frac{1}{r}$, $\alpha, \beta > -\frac{1}{p}$ and denote by $k(p)$ the smallest natural number greater or equal than $p/2$.*

Moreover, set

$$\begin{aligned} \rho &= \max\left\{\alpha + \frac{1}{2}\left(\frac{1}{p} - \frac{1}{r}\right), \frac{\alpha\, p}{r} - \frac{1}{k(p)}\left(1 - \frac{p}{r}\right)\left(\frac{1}{4} - \varepsilon\right), \frac{\varepsilon - 1}{r}\right\}, \\ \tau &= \max\left\{\beta + \frac{1}{2}\left(\frac{1}{p} - \frac{1}{r}\right), \frac{\beta\, p}{r} - \frac{1}{k(p)}\left(1 - \frac{p}{r}\right)\left(\frac{1}{4} - \varepsilon\right), \frac{\varepsilon - 1}{r}\right\}, \end{aligned}$$

where $\varepsilon \in (0, 1/4]$ *is an arbitrary small constant. Then the space* $B_q^s(L^p_{\alpha,\beta})$ *is continuously embedded into the space* $B_q^{s-(1/p)+(1/r)}(L^r_{\rho,\tau})$.

Proof. First we assume that $1 \le p \le 2$. Set $\gamma = \max\{2\alpha + \frac{1}{p} - \frac{1}{2}, -1 + 2\varepsilon\}$, $\delta = \max\{2\beta + \frac{1}{p} - \frac{1}{2}, -1 + 2\varepsilon\}$, $s = \frac{\gamma}{2} + \frac{1}{4}$, $t = \frac{\delta}{2} + \frac{1}{4}$. Then it is well-known that the orthonormal polynomials $p_n^{\gamma,\delta} \in \mathcal{P}_{n+1}$ w.r.t. the scalar product $\langle f, g\rangle_{\gamma,\delta} = \int_{-1}^1 f(x)\, \overline{g}(x)\, v^{\gamma,\delta}(x)\, dx$ satisfy $\|v^{s,t} p_n^{\gamma,\delta}\|_{\infty,n} \le$ const, where $\|\,.\,\|_{p,n} := \|\,.\,\|_{L^p(-1+(2n)^{-2}, 1-(2n)^{-2})}$ ([5], Theorem 1.1). Together with

$$\|v^{a,b} P_n\|_{L^p(-1,1)} \sim \|v^{a,b} P_n\|_{p,n}\,, \; P_n \in \mathcal{P}_n \quad (p \ge 1,\, a, b \ge \tfrac{\varepsilon-1}{p} \text{ fixed}) \tag{4.9}$$

(cf. [8], Theorem 8.4.8) it follows, for $P = \sum_{k=0}^{n-1} \langle P, p_k^{\gamma,\delta}\rangle_{\gamma,\delta}\, p_k^{\gamma,\delta} \in \mathcal{P}_n$,

$$\begin{aligned} \|v^{s,t} P\|_{\infty,n} &\le c \sum_{k=0}^{n-1} |\langle P, p_k^{\gamma,\delta}\rangle_{\gamma,\delta}| \le c \left(\sum_{k=0}^{n-1} |\langle P, p_k^{\gamma,\delta}\rangle_{\gamma,\delta}|^2\right)^{1/2} \left(\sum_{k=0}^{n-1} 1\right)^{1/2} \\ &\le c\sqrt{n}\, \|v^{\gamma/2,\delta/2} P\|_{2,n} \le c\sqrt{n}\, \|\, |v^{s,t} P|^{2-p} |v^{\alpha,\beta} P|^p \|_{1,n}^{1/2} \\ &\le c\sqrt{n}\, \|v^{s,t} P\|_{\infty,n}^{1-(p/2)} \|v^{\alpha,\beta} P\|_p^{p/2}\,, \end{aligned}$$

i.e. $\|v^{s,t} P\|_{\infty,n} \le c\, n^{1/p} \|v^{\alpha,\beta} P\|_p$ ($\|\,.\,\|_p$ denotes the $L^p(-1,1)$-norm). In case $r < \infty$ this leads to

$$\begin{aligned} \|v^{\rho,\tau} P\|_{r,n} &\le c \|(v^{s,t} P)^{r-p} (v^{\alpha,\beta} P)^p\|_{1,n}^{1/r} \le c \|v^{s,t} P\|_{\infty,n}^{1-(p/r)} \|v^{\alpha,\beta} P\|_p^{p/r} \\ &\le c\, n^{\frac{1}{p} - \frac{1}{r}} \|v^{\alpha,\beta} P\|_p\,. \end{aligned}$$

Together with (4.9) we obtain $\|v^{\rho,\tau} P\|_r \le c\, n^{\frac{1}{p} - \frac{1}{r}} \|v^{\alpha,\beta} P\|_p$ if $p \in [1, 2]$ and $r \in (p, \infty]$. If $p > 2$, then we set $k := k(p)$ and apply this estimate with P^k, $k\rho$, $k\tau$, $k\alpha$, $k\beta$, r/k, and p/k instead of P, ρ, τ, α, β, r, and p. Finally, we obtain the Nikolskii type inequality

$$\|v^{\rho,\tau} P\|_r \le c\, n^{\frac{1}{p} - \frac{1}{r}} \|v^{\alpha,\beta} P\|_p\,, \quad P \in \mathcal{P}_n\,, \tag{4.10}$$

for all $1 \le p < r \le \infty$. Now the assertion follows from Example 4.10 and Theorem 2.2. ∎

Now we consider embeddings of the type $A(X,S_1) \hookrightarrow A(X,S_2)$. Such embeddings are closely connected with the sets $\mathcal{D}(S)$ (see (4.2)). Indeed, if we write

$$\begin{aligned}&\mathcal{D}(S_1) \hookrightarrow \mathcal{D}(S_2)\\ &\quad :\Longleftrightarrow \mathcal{D}(S_1) \subseteq \mathcal{D}(S_2) \text{ and } \|\{a_n\}\|_{S_2} \le c\|\{a_n\}\|_{S_1} \text{ for all } \{a_n\} \in \mathcal{D}(S_1),\end{aligned}$$

then it is clear that $\mathcal{D}(S_1) \hookrightarrow \mathcal{D}(S_2)$ implies $A(X,S_1) \hookrightarrow A(X,S_2)$. If X is a Banach space, all A_n are finite-dimensional linear subspaces, such that A_n is a proper subset of A_{n+1}, and $\{1\} \notin S_1$, then the counterdirection is also true (see Theorem 4.1 and the proof of Corollary 4.2).

Theorem 4.12 *Let $0 < p \le q \le \infty$, $\mathcal{B} = \{b_n\} \subset (0,\infty)$, and $\mathcal{C} = \{c_n\} \subset (0,\infty)$. If $\mathcal{B}$ satisfies $\|\mathcal{B}\|_p = \infty$ and $b_{n+1} \le \mathrm{const}\,\|\{b_m\}_{m=0}^n\|_p$, then $\mathcal{D}(l^p(\mathcal{B})) \hookrightarrow \mathcal{D}(l^q(\mathcal{C}))$ if and only if*

$$\|\{c_m\}_{m=0}^n\|_q \le \mathrm{const}\,\|\{b_m\}_{m=0}^n\|_p \quad \textit{for all } n \in \mathbb{N}_0\,. \tag{4.11}$$

If even (4.7) is satisfied, then $l^p(\mathcal{B})$ is admissible. Especially, $A(X,l^p(\mathcal{B})) \hookrightarrow A(X,l^q(\mathcal{C}))$ if (4.11) and (4.7) are fulfilled and $l^q(\mathcal{C})$ is admissible. The implication "$\mathcal{D}(l^p(\mathcal{B})) \hookrightarrow \mathcal{D}(l^q(\mathcal{C})) \Rightarrow$ (4.11)" is even true for arbitrary $0 < p,q \le \infty$ and $\mathcal{B},\mathcal{C} \subset [0,\infty)$.

Proof. The admissibility of $l^p(\mathcal{B})$ will be proved at the end of the paper (see Remark 5.9). It is clear that $\mathcal{D}(l^p(\mathcal{B})) \hookrightarrow \mathcal{D}(l^q(\mathcal{C}))$ implies $\|\{1\}_{m=0}^n\|_{l^q(\mathcal{C})} \le c\,\|\{1\}_{m=0}^n\|_{l^p(\mathcal{B})}$, i.e. (4.11), and it remains to prove the counterdirection. So, let (4.11) be satisfied. First we consider the case $p = 1$, $q \ge 1$. In this case we choose some Banach space X, some sequence $\{0\} = A_0 \subset A_1 \subset \dots$ of finite-dimensional linear subspaces with $A_n \ne A_{n+1}$, and apply Theorem 4.9 with $Y := A(X,l^q(\mathcal{C}))$ (which is a Banach space). Obviously, we have $\|f_n\|_Y \le \|\{c_m\}_{m=0}^{n-1}\|_q\|f_n\|_X \le \mathrm{const}\,\|\{b_m\}_{m=0}^{n-1}\|_q\|f_n\|_X$ for all $f_n \in A_n$ and Theorem 4.9 yields $A(X,l^1(\mathcal{B})) \hookrightarrow A(X,l^q(\mathcal{C}))$, which is equivalent to $\mathcal{D}(l^1(\mathcal{B})) \hookrightarrow \mathcal{D}(l^q(\mathcal{C}))$. Now we consider the cases $0 < p \le q \le \infty$ with $p < \infty$. We have already proved that our assumption $\|\{c_m^p\}_{m=0}^n\|_{q/p} \le \mathrm{const}\,\|\{b_m^p\}_{m=0}^n\|_1$ implies $\mathcal{D}(l^1(\mathcal{B}^p)) \hookrightarrow \mathcal{D}(l^{q/p}(\mathcal{C}^p))$, where $\mathcal{B}^p := \{b_n^p\}$. But this means that, for $a_0 \ge a_1 \ge \dots \ge 0$,

$$\|\{a_n\}\|_{l^q(\mathcal{C})}^p = \|\{a_n^p\}\|_{l^{q/p}(\mathcal{C}^p)} \le c\,\|\{a_n^p\}\|_{l^1(\mathcal{B}^p)} = c\,\|\{a_n\}\|_{l^p(\mathcal{B})}^p\,,$$

i.e. $\mathcal{D}(l^p(\mathcal{B})) \hookrightarrow \mathcal{D}(l^q(\mathcal{C}))$. It remains to consider the case $p = q = \infty$. For every $n \in \mathbb{N}_0$, set $\widetilde{b}_n := \max_{m \le n} b_m$ and choose $m(n) \le n$ such that $b_{m(n)} = \max_{m \le n} b_m$. Then the estimate

$$a_n\widetilde{b}_n = a_n b_{m(n)} \le a_{m(n)}b_{m(n)} \le \sup_{m \ge 0} a_m b_m \text{ for } a_0 \ge a_1 \ge \dots \ge 0 \tag{4.12}$$

shows that $\mathcal{D}(l^\infty(\mathcal{B})) = \mathcal{D}(l^\infty(\widetilde{\mathcal{B}}))$ and $\mathcal{D}(l^\infty(\mathcal{C})) = \mathcal{D}(l^\infty(\widetilde{\mathcal{C}}))$. Obviously (4.11), i.e. $\widetilde{c}_n \le \text{const}\, \widetilde{b}_n$, implies $\mathcal{D}(l^\infty(\widetilde{\mathcal{B}})) \hookrightarrow \mathcal{D}(l^\infty(\widetilde{\mathcal{C}}))$, i.e. $\mathcal{D}(l^\infty(\mathcal{B})) \hookrightarrow \mathcal{D}(l^\infty(\mathcal{C}))$. ▮

Corollary 4.13 *If $\|\mathcal{A}\|_n \sim a_n$, then the space $A(X, l^p(\mathcal{B}))$ from Theorem 4.3 is, in some sense, the largest approximation space of the form $A(X, l^q(\mathcal{C}))$, $0 < q \le p$, on which a bounded extension $\mathcal{A} : A(X, l^q(\mathcal{C})) \to Y$ can be defined. More precisely: If $\mathcal{C} = \{c_n\}_{n=0}^\infty \subset (0, \infty)$ satisfies $c_{n+1} \le \text{const}\, \|\{c_m\}_{m=0}^n\|_q$ and $\|\mathcal{A}f\|_Y \le \text{const}\, \|f\|_{X, l^q(\mathcal{C})}$ for all $f \in \bigcup A_n$ and some $q \in (0, p]$, then $\mathcal{D}(l^q(\mathcal{C})) \hookrightarrow \mathcal{D}(l^p(\mathcal{B}))$.*

Proof. If $\mathcal{C}$ satisfies the above conditions, then $\|\mathcal{C}\|_q = \infty$, since otherwise $\|\mathcal{A}f\|_Y \le \text{const}\, \|f\|_{X, l^q(\mathcal{C})} \sim \|f\|_X$ for $f \in \bigcup A_n$, in contradiction to $\|\mathcal{A}\|_n \sim a_n \to \infty$. Now the assertion follows from Theorem 4.12, since $\|\mathcal{A}\|_n \sim a_n \sim \left\|\{b_m\}_{m=0}^{n-1}\right\|_p$ (see (4.5)) and, obviously, $\|\mathcal{A}f_n\|_Y \le c \left\|\{c_m\}_{m=0}^{n-1}\right\|_q \|f_n\|_X$, $f_n \in A_n$, i.e. $\|\mathcal{A}\|_n \le c \left\|\{c_m\}_{m=0}^{n-1}\right\|_q$. ▮

Corollary 4.14 *Let $0 < p \le \infty$ and let $\mathcal{B} = \{b_n\}_{n=0}^\infty$ be a sequence of positive numbers with $\|\mathcal{B}\|_p = \infty$ and $b_{n+1} \le \text{const}\, \|\{b_m\}_{m=0}^n\|_p$. If $K(n) < K(n+1)$ for all n, then $S = l^p(\mathcal{B})$ is admissible if and only if $\|\{b_m\}_{m=0}^{K(n)}\|_p \le \text{const}\, \|\{b_m\}_{m=0}^n\|_p$.*

Proof. If $l^p(\mathcal{B})$ is admissible then, for $a_0 = 1, a_1 = 1, \ldots, a_{K(m)} = 1, a_{K(m)+1} = 0, a_{K(m)+2} = 0, \ldots$, $\|\{a_n\}_{n=0}^\infty\|_{l^p(\mathcal{B})} \le c \|\{a_{K(n)}\}_{n=0}^\infty\|_{l^p(\mathcal{B})}$ means $\|\{b_n\}_{n=0}^{K(m)}\|_p \le c \|\{b_n\}_{n=0}^m\|_p$. The counterdirection is contained in Theorem 4.12. ▮

5. PROOF OF THEOREM 4.3. THE REPRESENTATION THEOREM

Proof of Theorem 4.3. We choose some $k > 1$ such that $a_{n+1} \le k\, a_n$ for all $n \in \mathbb{N}$. (This is possible, since $a_{n+1} \le a_{K(n)+1} \le \text{const}\, a_n$.) Moreover, we may assume that $a_1 = 1$. (Take $\{a_n/a_1\}$ instead of $\{a_n\}$.) Now we set $n(j) := \max\{n \in \mathbb{N} : a_n \le k^j\}$, $j = 0, 1, 2, \ldots$. Then we have

$$k^{j-1} < a_{n(j)} \le k^j, \quad j = 0, 1, 2, \ldots. \tag{5.1}$$

Indeed, $a_{n(j)} \le k^{j-1}$ would imply $a_{n(j)+1} \le k\, a_{n(j)} \le k^j$ in contradiction to the definition of $n(j)$. From (5.1) it follows that $1 = n(0) < n(1) < n(2) < \ldots$. Now, let (i) be satisfied and let $f \in \bigcup A_n$, i.e. $f \in A_{n_0}$ for some n_0. Choose $f_n \in A_n$ $(n = 0, 1, \ldots)$ such that $f_n = f$ for $n \ge n_0$ and

$$\|f - f_n\|_X^p \le E_n^X(f)^p + (2^n a_{n+1})^{-1} \|f\|_X^p \quad \text{for } n < n_0\,.$$

Then we have $\mathcal{A}f - \mathcal{A}f_{n(1)} = \sum_{j=1}^{\infty} \mathcal{A}(f_{n(j+1)} - f_{n(j)})$ (finite sum) and

$$\begin{aligned}\|f_{n(j+1)} - f_{n(j)}\|_X^p &\le c_X^p(\|f_{n(j+1)} - f\|_X + \|f - f_{n(j)}\|_X)^p \\ &\le c\left(E_{n(j)}^X(f)^p + \frac{\|f\|_X^p}{2^{n(j)}a_{n(j)}}\right).\end{aligned}$$

If we set $n(-1) := 0$, then it follows, in view of (5.1) and $a_{K(n)} \le c\, a_n$,

$$\begin{aligned}&\|\mathcal{A}f - \mathcal{A}f_{n(1)}\|_Y^p \\ &\quad\le c\sum_{j=1}^{\infty} \|\mathcal{A}(f_{n(j+1)} - f_{n(j)})\|_Y^p \le c\sum_{j=1}^{\infty} a_{K(n(j+1))}^p \|f_{n(j+1)} - f_{n(j)}\|_X^p \\ &\quad\le c\sum_{j=1}^{\infty} a_{n(j+1)}^p E_{n(j)}^X(f)^p + c\sum_{j=1}^{\infty} 2^{-n(j)}\|f\|_X^p \\ &\quad\le c\sum_{j=1}^{\infty} a_{n(j+1)}^{p-1} E_{n(j)}^X(f)^p \sum_{n=n(j-2)}^{n(j)-1}(a_{n+1} - a_n) + c\|f\|_X^p \qquad (5.2)\\ &\quad\le c\sum_{j=1}^{\infty}\sum_{n=n(j-2)}^{n(j)-1} a_{n+1}^{p-1}E_n^X(f)^p(a_{n+1} - a_n) + c\|f\|_X^p \\ &\quad\le c\sum_{n=0}^{\infty} a_{n+1}^{p-1}E_n^X(f)^p(a_{n+1} - a_n) + c\|f\|_X^p \le c\|f\|_{X,l^p(\mathcal{B})}^p.\end{aligned}$$

Consequently, $\|\mathcal{A}f\|_Y \le c(\|\mathcal{A}f - \mathcal{A}f_{n(1)}\|_Y + \|\mathcal{A}f_{n(1)}\|_Y) \le c\|f\|_{X,l^p(\mathcal{B})}$, i.e. (4.4) holds true and for the proof of (ii) it remains to show that $l^p(\mathcal{B})$ is admissible and that $\|\mathcal{B}\|_p = \infty$ (which implies that $\bigcup A_n$ is dense in $A(X, l^p(\mathcal{B}))$; cf. Theorem 3.17). The admissibility of $l^p(\mathcal{B})$ will be proved later and $\|\mathcal{B}\|_p = \infty$ is clear, since

$$\|\{b_n\}_{n=0}^m\|_p^p = \sum_{n=0}^m a_{n+1}^{p-1}(a_{n+1} - a_n) \ge a_{m+1}^{p-1}\sum_{n=0}^m (a_{n+1} - a_n) = a_{m+1}^p. \qquad (5.3)$$

Now we prove that (4.4) implies (i). Thus, let (4.4) be satisfied and let $f_m \in A_m$ ($m \in \mathbb{N}$). Then we have $E_n^X(f_m) = 0$ for $n \ge m$ and, consequently,

$$\|\mathcal{A}f_m\|_Y \le c\|f_m\|_{X,l^p(\mathcal{B})} \le c\|\{b_n\}_{n=0}^{m-1}\|_p\|f_m\|_X.$$

Using $a_{n+1}^p - a_n^p = p\,\xi_n^{p-1}(a_{n+1} - a_n)$ ($\xi_n \in (a_n, a_{n+1})$), we may estimate

$$\|\{b_n\}_{n=0}^{m-1}\|_p^p = \sum_{n=0}^{m-1} a_{n+1}^{p-1}(a_{n+1} - a_n) \le \frac{1}{p}\sum_{n=0}^{m-1}(a_{n+1}^p - a_n^p) = \frac{a_m^p}{p}. \qquad (5.4)$$

Thus, $\|\mathcal{A}\|_m \le \mathrm{const}\, a_m$. It remains to show that $l^p(\mathcal{B})$ is admissible. We will do this at the end of the paper (see Theorem 5.7 and Remark 5.9). ▌

The estimates (5.1) and (5.2) show that, for every fixed $p \in (0,\infty)$,

$$\sum_{j=0}^{\infty} k^{pj} \alpha_{n(j)}^p \le c\,\|\{\alpha_n\}_{n=0}^{\infty}\|_{l^p(\mathcal{B})}^p \quad \text{for all } \alpha_0 \ge \alpha_1 \ge \ldots \ge 0 \tag{5.5}$$

if $\mathcal{B}$ and $n(j)$ are defined by (4.3) and $n(j) := \max\{n \in \mathbb{N} : a_n/a_1 \le k^j\}$, respectively, and $0 = a_0 < a_1 < \ldots$ satisfy $a_{n+1} \le k\, a_n$ $(n \in \mathbb{N})$ and $\lim_{n\to\infty} a_n = \infty$. If we apply (5.5) with $\{\alpha_1, \alpha_1, \alpha_2, \alpha_3, \ldots\}$, then we see that even $\sum_{j=0}^{\infty} k^{pj} \alpha_{n(j)}^p \le c\,\|\{\alpha_n\}_{n=1}^{\infty}\|_{l^p(\mathcal{B})}^p$. The counterdirection is also true, i.e. we have

$$\sum_{j=0}^{\infty} k^{pj} \alpha_{n(j)}^p \sim \|\{\alpha_n\}_{n=1}^{\infty}\|_{l^p(\mathcal{B})}^p \quad \text{for all } \alpha_1 \ge \alpha_2 \ge \ldots \ge 0\,. \tag{5.6}$$

Indeed, using $k^j \sim a_{n(j)} \le a_{n+1} \le a_{n(j+1)} \sim k^j$ for all $n \in [n(j), n(j+1))$ (see (5.1)), one can estimate $\sum_{j=0}^{\infty} \sum_{n=n(j)}^{n(j+1)-1} a_{n+1}^{p-1} \alpha_n^p (a_{n+1} - a_n)$ by

$$c \sum_{j=0}^{\infty} k^{(p-1)j} \alpha_{n(j)}^p \sum_{n=n(j)}^{n(j+1)-1} (a_{n+1} - a_n) \le c \sum_{j=0}^{\infty} k^{pj} \alpha_{n(j)}^p\,,$$

which shows that $\|\{\alpha_n\}_{n=1}^{\infty}\|_{l^p(\mathcal{B})}^p \le c \sum_{j=0}^{\infty} k^{pj} \alpha_{n(j)}^p$ for $\alpha_1 \ge \alpha_2 \ge \ldots \ge 0$.

The numbers a_n do not appear explicitly in (5.6). For this reason, it is natural to start with the sequence $\mathcal{B} \subset (0,\infty)$, which leads to the following reformulation of (5.6).

Proposition 5.1 *Let $p \in (0,\infty]$ be fixed and let $\mathcal{B} = \{b_n\}_{n=0}^{\infty}$ be a sequence of positive numbers with the properties*

$$\|\mathcal{B}\|_p = \infty \quad \text{and} \quad \|\{b_m\}_{m=0}^{n+1}\|_p \le \mathrm{const}\,\|\{b_m\}_{m=0}^{n}\|_p\,. \tag{5.7}$$

Choose numbers $0 < a_1 < a_2 \le a_3 \le a_4 \le \ldots$ such that, for all sufficiently large $n \ge n_0$,

$$a_n \sim \|\{b_m\}_{m=0}^{n-1}\|_p\,,\quad a_{n+1}^{p-1}(a_{n+1} - a_n) \sim b_n^p \ (\text{read } "\infty \sim \infty" \text{ if } p = \infty),$$
$$\text{and } a_{n+1} \le k\, a_n \text{ with some constant } k > 1.$$

(For example, $0 < a_1 < \|\{b_m\}_{m=0}^{1}\|_p$ and $a_n = \|\{b_m\}_{m=0}^{n-1}\|_p$ for $n \ge 2$; cf. (4.6). In this case, k is the constant from (5.7).) If the sequence

$\{n(j)\}_{j=0}^{\infty} \subset \mathbb{N}$ is defined by

$$n(j) := \max\{n \in \mathbb{N} : a_n/a_1 \le k^j\} \tag{5.8}$$

(with the k from $a_{n+1} \le k\, a_n$), then

$$\left\|\{k^j \alpha_{n(j)}\}_{j=0}^{\infty}\right\|_p \sim \|\{\alpha_n\}_{n=1}^{\infty}\|_{l^p(\mathcal{B})} \quad \textit{for all } \alpha_1 \ge \alpha_2 \ge \ldots \ge 0\,. \tag{5.9}$$

Proof. If $n_0 = 1$ then it remains to consider the case $p = \infty$. In this case we have $a_n \sim \max_{1 \le m < n} b_m$ $(n \ge 2)$, which implies $\|\{\alpha_n\}_{n=1}^{\infty}\|_{l^\infty(\mathcal{B})} \sim \|\{\alpha_n\}_{n=1}^{\infty}\|_{l^\infty(\{a_{n+1}\})}$ (cf. (4.12)). Together with the relations after (5.6) we get the assertion. (Remark that we do not need $a_n < a_{n+1}$ in the proof of (5.1) and that $a_1 < a_2$ ensures $n(0) = 1$.) Now, let $n_0 > 1$. Set $\widetilde{a}_n = a_n$ for $n \ge n_0$ and choose $a_1 \le \widetilde{a}_1 < \ldots < \widetilde{a}_{n_0-1} < a_{n_0}$ such that $\widetilde{a}_{n+1} \le k\,\widetilde{a}_n$ for $1 \le n < n_0$. If $\widetilde{n}(j)$ is defined as $n(j)$ but w.r.t. $\{\widetilde{a}_n\}$, then we have

$$\widetilde{n}(j) = \max\{n \in \mathbb{N} : a_n/\widetilde{a}_1 \le k^j\} \quad \text{for all sufficiently large } j \ge j_0\,,$$

since $a_{n_0}/\widetilde{a}_1 \le k^j$ for all $j \ge j_0$. It follows

$$a_{\widetilde{n}(j)} \le \widetilde{a}_1 k^j \le a_1 k^{j+k_0}\,, \quad j \ge j_0\,, \quad \text{with some constant } k_0 \in \mathbb{N}\,.$$

Consequently, $\widetilde{n}(j) \le n(j + k_0)$ for $j \ge j_0$. Moreover, the estimate $a_{n(j)} \le a_1 k^j \le \widetilde{a}_1 k^j$ shows that $n(j) \le \widetilde{n}(j)$ for $j \ge j_0$. Now we obtain

$$\begin{aligned}
\|\{k^j \alpha_{n(j)}\}_{j=0}^{\infty}\|_p &\sim \alpha_1 + \|\{k^j \alpha_{n(j)}\}_{j=j_0}^{\infty}\|_p \\
&\ge \alpha_1 + \|\{k^j \alpha_{\widetilde{n}(j)}\}_{j=j_0}^{\infty}\|_p \\
&\ge \alpha_1 + k^{-k_0}\|\{k^{j+k_0}\alpha_{n(j+k_0)}\}_{j=j_0}^{\infty}\|_p \sim \|\{k^j \alpha_{n(j)}\}_{j=0}^{\infty}\|_p\,.
\end{aligned}$$

The expression in the second line is equivalent to $\|\{k^j \alpha_{\widetilde{n}(j)}\}_{j=0}^{\infty}\|_p$. But we already proved $\|\{k^j \alpha_{\widetilde{n}(j)}\}_{j=0}^{\infty}\|_p \sim \|\{\alpha_n\}_{n=1}^{\infty}\|_{l^p(\mathcal{B})}$ (case $n_0 = 1$). ▮

Example 5.2 *In the classical case $b_n = (n+1)^{s-(1/p)}$ $(s > 0)$ we have $\|\{b_m\}_{m=0}^{n-1}\|_p \sim a_n := n^s$ and (in view of Remark 4.4) $a_{n+1}^{p-1}(a_{n+1} - a_n) \sim b_n^p$. We may choose $k = l^s$ with some fixed $l \in \mathbb{N}\backslash\{1\}$ and Proposition 5.1 yields the well-known relation*

$$\|\{l^{sj}\alpha_{l^j}\}_{j=0}^{\infty}\|_p \sim \|\{n^{s-(1/p)}\alpha_n\}_{n=1}^{\infty}\|_p \quad \textit{for all } \alpha_1 \ge \alpha_2 \ge \ldots \ge 0. \tag{5.10}$$

Example 5.3 *In case $b_n = (n+1)^{-1/p}$, $p < \infty$ (cf. Example 4.8) we have $\|\{b_m\}_{m=0}^{n-1}\|_p \sim a_n := \log_2^{1/p}(n+1)$ and (in view of Remark 4.4) $a_{n+1}^{p-1}(a_{n+1} - a_n) \sim b_n^p$. We may choose $k = 2^{1/p}$ and Proposition 5.1 yields*

$$\|\{n^{-1/p}\alpha_n\}_{n=1}^{\infty}\|_p \sim \|\{2^{j/p}\alpha_{2^{2^j}-1}\}_{j=0}^{\infty}\|_p \quad \textit{for all } \alpha_1 \ge \alpha_2 \ge \ldots \ge 0\,.$$

Another equivalent form of the same expression can be obtained as follows: $\|\{n^{-1/p}\alpha_n\}_{n=1}^{\infty}\|_p$ *can be written as* $\|\{n^{1-(1/p)}\alpha_n/n\}_{n=1}^{\infty}\|_p$, *so that (5.10) (with* $l=2$ *and* $s=1$*) gives*

$$\|\{n^{-1/p}\alpha_n\}_{n=1}^{\infty}\|_p \sim \|\{\alpha_{2^j}\}_{j=0}^{\infty}\|_p \quad \textit{for all } \alpha_1 \geq \alpha_2 \geq \ldots \geq 0 \,.$$

The last considerations lead to the following

Remark 5.4 *Take the assumptions and notations of Proposition 5.1 and let* $d_1 \geq d_2 \geq \ldots \geq 0$ *be fixed. If we set* $\mathcal{C} := \{b_n d_n\}$, *then (5.9) yields*

$$\|\{k^j d_{n(j)}\alpha_{n(j)}\}_{j=0}^{\infty}\|_p \sim \|\{\alpha_n\}_{n=1}^{\infty}\|_{l^p(\mathcal{C})} \quad \textit{for all } \alpha_1 \geq \alpha_2 \geq \ldots \geq 0 \quad (5.11)$$

(since $\beta_n = d_n\alpha_n$ *is decreasing).*

This Remark can be used to obtain embeddings of the type $\mathcal{D}(l^p(\mathcal{B})) \hookrightarrow \mathcal{D}(l^q(\mathcal{C}))$ (which implies $A(X, l^p(\mathcal{B})) \hookrightarrow A(X, l^q(\mathcal{C}))$), where $p > q$:

Corollary 5.5 *Take the notations and assumptions of Proposition 5.1 and suppose that* $\{a_n\}_{n=1}^{\infty}$ *is choosen strictly increasing (in case* $p=\infty$*). If* $p > q > 0$, $c_n := (a_{n+1}^q - a_n^q)^{1/q}$ *(*$a_0 := 0$*), and* $d_n := \ln^{-t}(a_{n+2}/a_1)$ *with some* $t > \frac{1}{q} - \frac{1}{p}$, *then* $\mathcal{D}(l^p(\mathcal{B})) \hookrightarrow \mathcal{D}(l^q(\{c_n d_n\}))$.

Proof. First we remark that $\|\{c_m\}_{m=0}^{n-1}\|_q = a_n$, so that the conditions of Proposition 5.1 are also satisfied for $\mathcal{C} = \{c_n\}$ and q instead of $\mathcal{B}$ and p, where the numbers a_n, k, and $n(j)$ remain exactly the same. With the help of Hölder's inequality, the left hand side of (5.11) (applied with $\mathcal{C}$ and q instead of $\mathcal{B}$ and p) can be estimated by

$$\|\{(k^j d_{n(j)}\alpha_{n(j)})^q\}_{j=0}^{\infty}\|_1^{1/q} \leq \|\{(k^j\alpha_{n(j)})^q\}_{j=0}^{\infty}\|_{p/q}^{1/q}\, \|\{d_{n(j)}^q\}_{j=0}^{\infty}\|_{p/(p-q)}^{1/q} \,.$$

But $d_{n(j)}^q \sim (j+1)^{-tq}$ (see relation (5.1), which now holds true for $j \geq j_0$ and for a_n/a_1 instead of a_n) and it follows

$$\begin{aligned}\|\{\alpha_n\}_{n=1}^{\infty}\|_{l^q(\{c_n d_n\})} &\sim \|\{k^j d_{n(j)}\alpha_{n(j)}\}_{j=0}^{\infty}\|_q \\ &\leq c\|\{k^j\alpha_{n(j)}\}_{j=0}^{\infty}\|_p \sim \|\{\alpha_n\}_{n=1}^{\infty}\|_{l^p(\mathcal{B})}\end{aligned}$$

in view of (5.11) and (5.9). ▮

Proposition 5.1 also implies a surprising assertion about the structure of the spaces $A(X, l^p(\mathcal{B}))$:

Remark 5.6 *We have seen in Example 5.2, that every approximation space* $A_p^s(X)$ *(*$s>0$, $0 < p \leq \infty$*) can be written in the form*

$$A_p^s(X) = A(X, l^p(\{l^{js}\}), \{A_{l^j}\}_{j=-1}^{\infty}) \qquad (A_{1/l} := \{0\})\,,$$

where l is an arbitrary fixed natural number with $l > 1$. But Proposition 5.1 shows that, for every fixed $s > 0$, all spaces $A(X, l^p(\mathcal{B}))$ with $\mathcal{B}$ satisfying (5.7) can be viewed as

$$A(X, l^p(\mathcal{B})) = A(X, l^p(\{l^{js}\}), \{A_{n(j)}\}_{j=-1}^{\infty}) \qquad (n(-1) := 0)$$

if l is chosen big enough and $n(j) = n_{s,l}(j)$ is defined by (5.8) with $k = l^s$. If we set $B_0 = B_{1/l} := \{0\}$ and $B_m := A_{n([\log_l m])}$, $m \in \mathbb{N}$ ($[r]$ denotes the integer part of r), then we obtain

$$A(X, l^p(\mathcal{B}), \{A_n\}_{n=0}^{\infty}) = A(X, l^p(\{l^{js}\}), \{B_{l^j}\}_{j=-1}^{\infty}) = A_p^s(X, \{B_m\}_{m=0}^{\infty}) .$$

This shows that the $A_p^s(X, \{B_m\})$-spaces with "subsequences" $\{B_m\}$ of $\{A_n\}$ (and with fixed $s > 0$) cover almost all spaces of the type $A(X, l^p(\mathcal{B}), \{A_n\})$. W.r.t. the admissibility of the sequence spaces: see Remark 5.9.

Another consequence of Proposition 5.1 is the following representation theorem.

Theorem 5.7 *Take the assumptions and notations of Proposition 5.1 and suppose that $K(n(j)) \leq n(j+1)$ for all sufficiently large $j \geq j_0$. Then the sequence space $l^p(\mathcal{B})$ is admissible and, for every $f \in X$, the following assertions are equivalent:*

(i) $f \in A(X, l^p(\mathcal{B}))$.

(ii) There exists a sequence $\{g_j\}_{j=0}^{\infty}$ of elements $g_j \in A_{n(j)}$, such that $f = \sum_{j=0}^{\infty} g_j$ in X and $\left\| \{k^j \|g_j\|_X\}_{j=0}^{\infty} \right\|_p < \infty$.

Moreover, the expression

$$\|f\|_{X,l^p(\mathcal{B})}^{\mathrm{rep}} := \inf \left\{ \left\| \{k^j \|g_j\|_X\}_{j=0}^{\infty} \right\|_p : g_j \in A_{n(j)} ,\ f = \sum_{j=0}^{\infty} g_j \ \text{in } X \right\}$$

($\inf \emptyset := \infty$) is equivalent to $\|f\|_{X,l^p(\mathcal{B})}$ for all $f \in X$.

Proof. Using $k^{j+1}\alpha_{n(j+1)} \leq k\, k^j \alpha_{K(n(j))}$ ($j \geq j_0$, $\alpha_0 \geq \alpha_1 \geq \ldots \geq 0$) and (5.9), it is easy to prove that $l^p(\mathcal{B})$ is admissible. Let (i) be satisfied. For $j \geq j_0$ choose $f_j \in A_{n(j-1)}$ with $\|f - f_j\|_X \leq E_{n(j-1)}^X(f) + k^{-2(j+1)}\|f\|_X$. For $j < j_0$ set $f_j := 0$. Then we have $f_j \to f$ in X (see (3.18)) and, consequently, $f = \sum_{j=0}^{\infty} g_j$, where $g_j := f_j - f_{j-1} \in A_{n(j)}$. From $\|g_j\|_X \leq c_X(\|f_j - f\|_X + \|f - f_{j-1}\|_X) \leq c(E_{n(j-2)}^X(f) + k^{-2j}\|f\|_X)$ $(n(-1) = n(-2) := 0)$ and (5.9) it follows

$$\left\| \{k^j \|g_j\|_X\}_{j=0}^{\infty} \right\|_p \leq c \left\| \left\{ k^{j-2} E_{n(j-2)}^X(f) \right\}_{j=0}^{\infty} \right\|_p + c\|f\|_X \leq c\|f\|_{X,l^p(\mathcal{B})} .$$

Now, let (ii) be satisfied. Using $K(n(j)) \leq n(j+1)$ $(j \geq j_0)$ one can prove by induction that $g_{j_0} + \ldots + g_{j-1} \in A_{n(j)}$ for all $j > j_0$. Moreover, $g_0 + \ldots + g_{j_0-1} \in A_{n(0)} + \ldots + A_{n(j_0-1)} \subseteq A_{k_0}$ with some constant k_0. Consequently, $g_0 + \ldots + g_{j-1} \in A_{k_0} + A_{n(j)} \subseteq A_{n(j)} + A_{n(j)} \subseteq A_{K(n(j))} \subseteq A_{n(j+1)}$ for all sufficiently large $j \geq j_1$. Now it follows, with some constant $\nu = \nu(X) \in (0,1]$ (see Remark 3.1),

$$E^X_{n(j+1)}(f)^\nu \leq \left\| f - \sum_{i=0}^{j-1} g_i \right\|_X^\nu = \left\| \sum_{i=j}^{\infty} g_i \right\|_X^\nu \leq c \sum_{i=j}^{\infty} \|g_i\|_X^\nu, \quad j \geq j_1 .$$

(Use that $\||\,.\,\||_X : X \to \mathbb{R}$ is continuous.) If $p \leq \nu$, then the sum on the right hand side can be estimated by $\|\{\|g_i\|_X\}_{i=j}^\infty\|_p^\nu$ (Jensen's inequality). If $p > \nu$, then we define q by $\frac{\nu}{p} + \frac{1}{q} = 1$ and Hölder's inequality yields

$$\begin{aligned} \sum_{i=j}^{\infty} \|g_i\|_X^\nu &\leq \|\{k^{-i\nu/2}\}_{i=j}^\infty\|_q \, \|\{k^{i\nu/2}\|g_i\|_X^\nu\}_{i=j}^\infty\|_{p/\nu} \\ &\leq c\, k^{-j\nu/2} \|\{k^{i/2}\|g_i\|_X\}_{i=j}^\infty\|_p^\nu . \end{aligned} \tag{5.12}$$

Set $\alpha := 0$ if $p \leq \nu$ and $\alpha := 1/2$ if $p > \nu$. Then it follows

$$\begin{aligned} \|\{k^{j+1} E^X_{n(j+1)}(f)\}_{j=j_1}^\infty\|_p &\leq c \left\| \left\{ k^{j(1-\alpha)} \|\{k^{i\alpha}\|g_i\|_X\}_{i=j}^\infty\|_p \right\}_{j=j_1}^\infty \right\|_p \\ &= c \left\| \left\{ k^{i\alpha}\|g_i\|_X \|\{k^{j(1-\alpha)}\}_{j=j_1}^i\|_p \right\}_{i=j_1}^\infty \right\|_p \leq c \left\| \left\{ k^i\|g_i\|_X \right\}_{i=j_1}^\infty \right\|_p . \end{aligned}$$

Moreover, we have $E^X_{n(j)}(f)^\nu \leq \|f\|_X^\nu \leq c\sum_{i=0}^\infty \|g_i\|_X^\nu \leq c\|\{k^i\|g_i\|_X\}_{i=0}^\infty\|_p^\nu$ for all $j \leq j_1$ (use again Jensen's inequality if $p \leq \nu$ and Hölder's inequality ((5.12) with $j=0$) if $p > \nu$). Together with (5.9) we obtain

$$\|f\|_{X,l^p(\mathcal{B})} \sim \|\{k^j E^X_{n(j)}(f)\}_{j=0}^\infty\|_p + \|f\|_X \leq c\|\{k^i\|g_i\|_X\}_{i=0}^\infty\|_p$$

and the theorem is proved. ∎

Example 5.8 *If $K(n) = 2n$ and if we choose $k = 2^s$ in Example 5.2 (which means $n(j) = 2^j$), then the assumptions of Theorem 5.7 are satisfied and we obtain the famous classical representation theorem*

$$f \in A_p^s(X) \;\Leftrightarrow\; f = \sum_{j=0}^{\infty} g_j \text{ with } g_j \in A_{2^j} \text{ such that } \left\| \left\{ 2^{sj}\|g_j\|_X \right\}_j \right\|_p < \infty.$$

Remark 5.9 *If, in addition to the assumptions of Proposition 5.1,* $a_{K(n)} \leq k\,a_n$, $n \geq n_0$ *(with the same* k *as in* $a_{n+1} \leq k\,a_n$*), then the condition* $K(n(j)) \leq n(j+1)$ *(i.e.* $a_{K(n(j))}/a_1 \leq k^{j+1}$*) from Theorem 5.7 is satisfied for all* j *with* $n(j) \geq n_0$*. Especially, the numbers* $n(j) = n_{s,l}(j)$ *from Remark 5.6 have this property for all* j *if* l *is large enough and* $\mathcal{B}$ *satisfies*

$$\|\mathcal{B}\|_p = \infty \quad \text{and} \quad \|\{b_m\}_{m=0}^{K(n)+1}\|_p \leq \text{const}\,\|\{b_m\}_{m=0}^{n}\|_p \,. \qquad (5.13)$$

It follows that every space $A(X, l^p(\mathcal{B}))$ *with* $\mathcal{B} \subset (0,\infty)$ *satisfying (5.13) is an approximation space (i.e.* $l^p(\mathcal{B})$ *is admissible), which can be written in the form* $A_p^s(X, \{B_m\}_{m=0}^{\infty})$ *(for arbitrary fixed* $s > 0$*), where* $\{B_m\}$ *is a subsequence of* $\{A_n\}$ *(in the sense* $B_m = A_{k(m)}$, $0 = k(0) \leq k(1) \leq \ldots$*) with the property* $B_m + B_m \subseteq B_{lm}$ *(*l *big enough such that* $a_{K(n)} \leq l^s a_n$ *for all* $n \in \mathbb{N}$*). Indeed,*

$$B_m + B_m \subseteq A_{K(n([\log_l m]))} \subseteq A_{n([\log_l m]+1)} = A_{n([\log_l lm])} = B_{lm} \,.$$

(This property of $\{B_m\}$ *is not surprising, since, at least in case of strictly increasing* $\{K(n)\}$*, we already know that* $\{(n+1)^{s-(1/p)}\}$ *is admissible if and only if* $K(n) \leq \text{const}\, n$*; see Corollary 4.14.) But this does not mean that all theorems about* $A(X, l^p(\mathcal{B}))$*-spaces go back to the corresponding theorems for classical approximation spaces. For example, an embedding of the type* $A(X, l^p(\mathcal{B})) \hookrightarrow A(X, l^q(\mathcal{C}))$ *cannot be proved in this way if it is not possible to find the same subsequence* $\{B_m\}$ *of* $\{A_n\}$ *for both spaces.*

In practice, often the following linear representation theorem is more useful than Theorem 5.7.

Theorem 5.10 *Take the assumptions and notations of Proposition 5.1 and suppose that there exist uniformly bounded linear projections* P_n *from* X *onto* A_n *(especially, all* A_n *have to be closed linear subspaces of* X*). If we set* $Q_j := P_{n(j)} - P_{n(j-1)}$ *(*$n(-1) := 0$*), then, for every* $f \in X$*, the following assertions are equivalent:*

(i) $f \in A(X, l^p(\mathcal{B}))$.

(ii) $f = \sum_{j=0}^{\infty} Q_j f$ *in* X *(i.e.* $f \in \text{clos}_X \bigcup A_n$*) and* $\left\| \{ k^j \|Q_j f\|_X \}_{j=0}^{\infty} \right\|_p < \infty$.

Moreover, the expression $\left\| \{ k^j \|Q_j f\|_X \}_{j=0}^{\infty} \right\|_p$ *defines an equivalent quasi-norm in* $A(X, l^p(\mathcal{B}))$.

Proof. If (ii) is satisfied, then, by Theorem 5.7, $f \in A(X, l^p(\mathcal{B}))$ and $\|f\|_{X,l^p(\mathcal{B})} \sim \|f\|^{\text{rep}}_{X,l^p(\mathcal{B})} \leq \left\| \{ k^j \|Q_j f\|_X \}_{j=0}^{\infty} \right\|_p$. If (i) is fulfilled, then

(ii) and $\left\| \left\{ k^j \|Q_j f\|_X \right\}_{j=0}^{\infty} \right\|_p \leq c\|f\|_{X,l^p(\mathcal{B})}$ can be proved in a similar way as the corresponding assertions of Theorem 5.7 (replace g_j by $Q_j f$). ∎

REFERENCES

1. J. M. Almira, *Approximation Theory in Approximation Spaces*, Doctoral Thesis, University of La Laguna (Tenerife), 1999.
2. J. M. Almira and U. Luther, Approximation algebras and applications, to appear in the proceedings of the conference "Trends in approximation theory", Nashville, May 2000.
3. J. M. Almira and U. Luther, Compactness and generalized approximation spaces, to appear.
4. J. M. Almira and U. Luther, Reiteration and interpolation of generalized approximation spaces, to appear.
5. V. M. Badkow, Convergence in the mean and almost everywhere of Fourier series in polynomials orthogonal on an interval, *Math. USSR-Sb.* **24** (1974), 223-256 (in Russian).
6. R. A. DeVore and G. G. Lorentz, *Constructive Approximation*, Springer-Verlag, 1993.
7. Z. Ditzian and V. Totik, Remarks on Besov spaces and best polynomial approximation, *Proc. Amer. Math. Soc.* **104** (1988), 1059-1066.
8. Z. Ditzian and V. Totik, *Moduli of Smoothness*, Springer-Verlag, 1987.
9. P. Junghanns and U. Luther, Cauchy singular integral equations in spaces of continuous functions and methods for their numerical solution, *J. Comp. Appl. Math.* **77** (1997), 201-237.
10. S. Li, The equivalence relations between the Ditzian-Totik moduli of smoothness and best polynomial approximation in Besov spaces, *J. Math. Anal. Appl.* **215** (1997), 1-14.
11. U. Luther, *Generalized Besov Spaces and Cauchy Singular Integral Equations*, Doctoral Thesis, Technical University of Chemnitz (Germany), 1998.
12. U. Luther and M. G. Russo, Boundedness of the Hilbert transformation in some weighted Besov spaces, *Integr. Equ. Oper. Theory* **36** (2000), 220-240.
13. A. Pietsch, Approximation spaces, *J. Approx. Theory* **32** (1981), 115-134.
14. S. Prössdorf and B. Silbermann, *Numerical Analysis for Integral and Related Operator Equations*, Akademie Verlag, Berlin, 1991.
15. A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Pergamon Press Ltd., Oxford, 1963.

Submitted to *Journal of Approximation Theory.*