Approximation Algebras and Applications

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Abstract. In this paper we prove that certain approximation spaces are quasi-Banach algebras and give several applications.

§1. Introduction

Let X be a quasi-normed space and let $\{A_n\}_{n=0}^{\infty}$ be a sequence of subsets of X. We say that the pair $(X, \{A_n\}_{n=0}^{\infty})$ defines an approximation scheme if the following hold:

(C1) $\{0\} = A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$,
(C2) $\lambda A_n \subseteq A_n$ for all scalars λ and all $n \in \mathbf{N}$,
(C3) $A_n + A_n \subseteq A_{K(n)}$, where $K(0) = 0$ and $K(n) \geq n$ for all $n \in \mathbf{N}$.

Approximation spaces are defined in terms of the speedness of convergence to zero of the sequences of errors $\{E(f, A_n)\}_{n=0}^{\infty}$ associated to a certain approximation scheme $(X, \{A_n\}_{n=0}^{\infty})$, where

$$E(f, A_n) = E_n(f) := \inf_{a_n \in A_n} ||f - a_n|| \tag{1}$$

denotes the error of best approximation of f when we approximate with elements of A_n. Especially, the classical approximation spaces (see [9]) are defined for each $0 < s < \infty$ and $0 < q \leq \infty$ by the formula

$$\mathbf{A}_q^s(X) = \{f \in X : \|f\|_{A_q^s} = \left\| \left\{ (n+1)^{s-(1/q)} E_n(f) \right\}_{n=0}^{\infty} \right\|_q < \infty\}, \tag{2}$$

where $\| \, . \, \|_q$ denotes the l^q-norm. (Sometimes we also write $\mathbf{A}_q^s(X, \{A_n\})$, but usually the sequence $\{A_n\}$ is viewed as fixed so that it is not necessary to include it in the notations.) The reason because the weights $(n+1)^{s-(1/q)}$ appear in this definition is that they already appear in the Stechkin inequality, and this gives us the possibility to see these spaces as real interpolation spaces. Moreover, certain classical Besov-type spaces $B_q^s(L^p)$ can be viewed as $\mathbf{A}_q^s$-spaces (in the sense of equivalent norms). For example, if $1 \leq p, q \leq \infty$, then the spaces $B_q^s(L_{2\pi}^p)$ (Besov spaces of 2π-periodic functions) and $B_q^s(L_{\rho,\tau}^p)$

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(weighted Besov spaces; $L^p_{\rho,\tau} := \{f : f(x)(1-x)^\rho(1+x)^\tau \in L^p(-1,1)\}$, $\rho,\tau \geq (\varepsilon - 1)/p$), which are defined with the help of the classical and the Ditzian-Totik moduli of smoothness, respectively, are equal to $\mathbf{A}^s_q(L^p_{2\pi}, \{\mathbf{T}_n\})$ and $\mathbf{A}^s_q(L^p_{\rho,\tau}, \{\mathbf{P}_n\})$, respectively (cf. [4], 7.Theo.9.2 and [5], Theo.3.1), where $\mathbf{T}_n$ and $\mathbf{P}_n$ denote the sets of all polynomials of degree less than n (trigonometric and algebraic, respectively).

Recently there appeared applications in which more general approximation spaces are needed (see e.g. [7], [8], [2], [1]). More precisely, instead of the sequence $\{(n+1)^{s-(1/q)}\}$ $(s > 0)$, more general sequences are considered in the definiton of approximation spaces. This means that the weighted l^q-norm $\|\{(n+1)^{s-(1/q)} \, . \}\|_q$ is replaced by another quasi-norm on a certain sequence space. Instead of weighted l^q-spaces one can also use other sequence spaces S to define approximation spaces $\mathbf{A}(X,S)$. To ensure that $\| \, . \, \|_{A(X,S)}$ remains a quasi-norm, we have to give certain assumptions on S.

Definition. *Let S be a real linear space of sequences $\{a_n\}_{n=0}^\infty \subseteq \mathbf{R}$ (with elementwise defined operations), equipped with a quasi-norm $\| \, . \, \|_S$. S is called admissible sequence space if the following assumptions are satisfied:*

(A1) All finite sequences $\{a_n\}_{n=0}^N$ belong to S.
(A2) If $0 \leq a_n \leq b_n$ for all $n = 0, 1, 2, \ldots$ and $\{b_n\}_{n=0}^\infty \in S$, then $\{a_n\}_{n=0}^\infty \in S$ and $\|\{a_n\}\|_S \leq \|\{b_n\}\|_S$.
(A3) If $a_0 \geq a_1 \geq \ldots \geq 0$ and $\{a_{K(n)}\}_{n=0}^\infty \in S$ then $\{a_n\} \in S$ and $\|\{a_n\}_{n=0}^\infty\|_S \leq C_S \, \|\{a_{K(n)}\}_{n=0}^\infty\|_S$, where C_S only depends on S and $\{K(n)\}_{n=0}^\infty$.

(In **(A1)** we used the notation $\{a_n\}_{n=0}^N := \{a_n^*\}_{n=0}^\infty$, $a_n^* := \left\{ \begin{array}{l} a_n, \ n \leq N \\ 0, \ n > N \end{array} \right\}$.)

Definition. *Let $(X, \{A_n\})$ be an approximation scheme and let S be an admissible sequence space. Then the space $\mathbf{A}(X,S) := \{f \in X : \{E_n(f)\}_{n=0}^\infty \in S\}$ endowed with $\|f\|_{X,S} := \|\{E_n(f)\}_{n=0}^\infty\|_S$ is called (generalized) approximation space.*

Remark (cf. [2], Rem.3.5). *If $K(n) = K\,n$ with some constant $K \in \mathbf{N}$, then the sequence spaces $S = l^q(\{(n+1)^{s-(1/q)}\})$ $(0 < s < \infty, \, 0 < q \leq \infty)$, which appear in the definition of the classical approximation spaces $\mathbf{A}^s_q(X) = \mathbf{A}(X,S)$, are admissible. (Here we used the following notation: If $\{b_n\}_{n=0}^\infty$ is a sequence of positive numbers, then $l^q(\{b_n\})$ denotes the space of all sequences $\{a_n\}_{n=0}^\infty$, for which $\|\{a_n\}\|_{l^q(\{b_n\})} := \|\{a_n b_n\}\|_q < \infty$.)*

The main goal of this paper is not to study the general theory of approximation spaces (this has been done by the authors in [2]) but just one aspect of this theory, which is to know when these spaces are quasi-normed algebras, and to give several applications. Hence, in this section we will only mention the following three properties of generalized approximation spaces, without proof.

P1 $\mathbf{A}(X,S)$ is a quasi-normed space, which is continuously imbedded into X. If X and S are normed spaces and all A_n are linear subspaces of X, then $\mathbf{A}(X,S)$ is even a normed space.

P2 The space $\mathbf{A}(X,S)$ is complete, under certain reasonable hypotheses on X and S. To be more precise: Suppose that S has the property

$$\|\{a_n\}_{n=0}^{\infty}\|_S \le C \lim_{k\to\infty} \|\{a_n\}_{n=0}^{k}\|_S \text{ for all } \{a_n\} \in S \text{ with } a_0 \ge a_1 \ge \ldots \ge 0,$$

where C is a constant depending only on S. If X is complete and if one of the following conditions (a), (b) or (c) is satisfied, then $\mathbf{A}(X,S)$ is also complete.
(a) X is a Banach space, all A_n are linear subspaces, $\|\{a_n\}\|_S = \|\{|a_n|\}\|_S$ for all $\{a_n\} \in S$, and every Cauchy sequence $\{\mathbf{a}_m\}$ in S, which consists of decreasing sequences $\mathbf{a}_m = \{a_{mn}\}_{n=0}^{\infty} \subseteq [0,\infty)$, is convergent in S.
(b) A decreasing sequence $\{a_n\}_{n=0}^{\infty} \subseteq [0,\infty)$ belongs to S if and only if the limit $\lim_{k\to\infty} \|\{a_n\}_{n=0}^{k}\|_S$ $(= \sup_k \|\{a_n\}_{n=0}^{k}\|_S)$ is finite.
(c) A decreasing sequence $\{a_n\}_{n=0}^{\infty} \subseteq [0,\infty)$ belongs to S if and only if $\lim_{l\to\infty} \lim_{k\to\infty} \|\{a_n\}_{n=l}^{k}\|_S = 0$. Moreover, $\lim_{k\to\infty} \|\{1\}_{n=0}^{k}\|_S = \infty$.
(Of course, $\{a_n\}_{n=l}^{k} := \{a_n\}_{n=0}^{k} - \{a_n\}_{n=0}^{l-1}$.)

P3 If $\{1\}_{n=0}^{\infty} \notin S$ and if

$$\lim_{l\to\infty} ||\{a_n\}_{n=l}^{\infty}||_S = 0 \text{ for all decreasing } \{a_n\} \subseteq [0,\infty) \text{ with } \{a_n\} \in S \quad (3)$$

then $\bigcup A_n$ is dense in $\mathbf{A}(X,S)$. If even (3) and $\lim_{k\to\infty} ||\{1\}_{n=0}^{k}||_S = \infty$ are satisfied, then $\mathbf{A}(X,S)$ is complete if X and S are complete.

§2. Quasi-normed Approximation Algebras

Let $\mathbf{A}(X,S)$ be an approximation space based on a quasi-normed algebra X. We will show that, under a certain additional condition on $\{A_n\}$, $\mathbf{A}(X,S)$ is also a quasi-normed algebra and that, under more restrictive assumptions, $\mathbf{A}(X,S)$ is an inversely closed subalgebra of X. But first we have to make clear what we mean by "quasi-normed algebra":

Definition. *An algebra $\mathcal{A}$ is called quasi-normed algebra, if $\mathcal{A}$ is endowed with a quasi-norm $||\,.\,||$, such that $||fg|| \le \text{const}\, ||f||\,||g||$ for all $f,g \in \mathcal{A}$.*

Theorem 1. *Let $\mathbf{A}(X,S)$ be an approximation space based on a quasi-normed algebra X, and suppose that $A_nA_n \subseteq A_{K(n)}$ for all $n \in \mathbf{N}$. Then $\mathbf{A}(X,S)$ is a quasi-normed algebra (a normed algebra, if X and S are a normed spaces and the sets A_n are linear subspaces of X for all n) with the product from X.*

Proof: Let $f,g \in \mathbf{A}(X,S)$. It follows from the inequality $||\{E_n(\cdot)\}||_S \le C_S||\{E_{K(n)}(\cdot)\}||_S$ that we are mainly interested in the estimation of the errors $E_{K(n)}(fg)$. Now,

$$\begin{aligned} E_{K(n)}(fg) &\le ||fg - p_nq_n||_X \le c_X\,(||(f-p_n)g||_X + ||p_n(g-q_n)||_X) \\ &\le c\,[||f-p_n||_X||g||_X + (||p_n-f||_X + ||f||_X)\,||g-q_n||_X] \end{aligned}$$

for all $p_n, q_n \in A_n$. Hence, $E_{K(n)}(fg)$ can be estimated by

$$c\,[E_n(f)||g||_X + ||f||_X E_n(g)] \le C\,[E_n(f)||g||_{X,S} + ||f||_{X,S} E_n(g)]$$

and, consequently, $fg \in \mathbf{A}(X,S)$ and $||fg||_{X,S} \le \text{const}\, ||f||_{X,S}||g||_{X,S}$ (see **(A2)** and **(A3)**). □

Remark. *If there appear different numbers $K_1(n)$ and $K_2(n)$ in the relations $A_n + A_n \subseteq A_{K_1(n)}$ and $A_n A_n \subseteq A_{K_2(n)}$ then we have to take $K(n) = \max\{K_1(n), K_2(n)\}$ in Theorem 1 and S has to be admissible w.r.t. these $K(n)$. (We may call this algebra admissible. This is more than usual admissibility, i.e. admissibility w.r.t. $\{K_1(n)\}$.)*

Now we come to the main theorem of the present paper:

Theorem 2. *Let the assumptions of Theorem 1 be satisfied and suppose that X is a complex commutative quasi-Banach algebra with unit $e \neq 0$. Moreover, assume that $e \in \mathbf{A}(X,S)$, that S satisfies (3) and $\{1\} \notin S$, and that the subspace $\mathcal{A}_0 := \{\lambda e : \lambda \in \mathbf{C}\} + \bigcup A_n$ has the following properties:*

$$\mathcal{A}_0 \text{ is dense in } X \text{ and invertibility of } f \in \mathcal{A}_0 \text{ in } X \text{ implies } f^{-1} \in \mathbf{A}(X,S). \quad (4)$$

If $\mathbf{A}(X,S)$ is complete (cf. **P2** *and* **P3**), *then $\mathbf{A}(X,S)$ is an inversely closed subalgebra of X, i.e. $f \in \mathbf{A}(X,S)$ is invertible in $\mathbf{A}(X,S)$ if and only if it is invertible in X.*

Proof: If $\mathcal{A}$ is a quasi-normed algebra with $||xy|| \leq c(\mathcal{A})||x||\,||y||$ for all $x, y \in \mathcal{A}$, then we denote by $|\,.\,|$ the equivalent quasi-norm, which is defined by $|\,.\,| := c(\mathcal{A})||\,.\,||$. Obviously, we have $|xy| \leq |x|\,|y|$ for all $x, y \in \mathcal{A}$.

We need the following well-known facts from the theory of complex commutative algebras $\mathcal{A}$ with unit $e \neq 0$, which are of purely algebraic nature:

(i) $x \in \mathcal{A}$ is invertible if and only if $x \notin M$ for all maximal ideals $M \subset \mathcal{A}$.
(ii) An ideal $J \neq \mathcal{A}$ is maximal if and only if all non-zero elements of $\mathcal{A}/J$ are invertible.
(iii) Let $\mathcal{M} = \mathcal{M}(\mathcal{A})$ denote the set of all multiplicative functionals $\varphi \not\equiv 0$. If $\varphi \in \mathcal{M}$ and $M = \ker\varphi$, then M is a maximal ideal and $\mathcal{A}$ can be written as direct sum $\mathcal{A} = \{\lambda e : \lambda \in \mathbf{C}\} \oplus M$. More precisely, for every $x \in \mathcal{A}$ we have $x = \lambda e + m$, where $\lambda \in \mathbf{C}$ and $m \in M$ are uniquely determined by $\lambda = \varphi(x)$ and $m = x - \varphi(x)e$.

Now we generalize some useful facts from Banach algebra theory to the case of complex commutative quasi-Banach algebras $\mathcal{A}$ with unit $e \neq 0$:

(iv) If $||x|| < 1$ then $e - x$ is invertible and $(e-x)^{-1} = \sum_{n=0}^{\infty} x^n$ (conv. in $\mathcal{A}$).
(v) The set $\mathcal{G}$ of all invertible elements of $\mathcal{A}$ is open.
(vi) Every multiplicative functional $\varphi \in \mathcal{M}$ is continuous with $||\varphi|| \leq 1$.
(vii) Every maximal ideal $M \subset \mathcal{A}$ is closed.
(viii) If $J \neq \mathcal{A}$ is a closed ideal, then $\mathcal{A}/J$ is a quasi-Banach algebra with $||x + J|| := \inf\{||x + j|| : j \in J\}$.
(ix) For every $a \in \mathcal{A}$, the limit $r(a) := \lim_{n\to\infty}(|a^n|)^{1/n} = \inf_n(|a^n|)^{1/n}$ exists.
(x) If $a \in \mathcal{G}$ and $a_k \to a$, then $a_k \in \mathcal{G}$ for all $k \geq k_0$ and $a_k^{-1} \to a^{-1}$.
(xi) For all $x \in \mathcal{A}$, the spectrum $\sigma(x)$ is non-empty.
(xii) $M \subseteq \mathcal{A}$ is a maximal ideal if and only if $M = \ker\varphi$ with some $\varphi \in \mathcal{M}$.

First of all, we recall that for every quasi-normed space $(X, ||\cdot||)$, there exists a constant $p = p(X) \in (0,1]$ and an equivalent quasi-norm $|||\cdot|||$ which

satisfies the inequality $|||x+y|||^p \le |||x|||^p + |||y|||^p$ for all $x, y \in X$ ($|||\cdot|||$ is called a p-norm). Now, the convergence of the series in (iv) follows from

$$\left|\left|\sum_{n=k}^{l} x^n\right|\right|^p \sim \left|\left|\left|\sum_{n=k}^{l} x^n\right|\right|\right|^p \le \sum_{n=k}^{l} |||x^n|||^p \le c\sum_{n=k}^{\infty} |x^n|^p \le C\sum_{n=k}^{\infty} \|x\|^{np}$$

The limit is $(e-x)^{-1}$, since $(e-x)\sum_{n=0}^{m} x^n = e - x^{m+1}$ converges to e for $m \to \infty$. Assertion (v) is an immediate consequence of (iv) (use that $y = x(e - x^{-1}(x-y))$ for $x \in \mathcal{G}$ and $y \in \mathcal{A}$). The continuity of $\varphi \in \mathcal{M}$ can also be proved with the help of (iv): If we assume that $||\varphi|| > 1$, then there exists some $a \in \mathcal{A}$ such that $|\varphi(a)| > ||a||$. Now we set $x := a/\varphi(a)$. Then we have $||x|| < 1$ and $\varphi(x) = 1$. Hence, there exists $b := (e-x)^{-1} = \sum_{n=0}^{\infty} x^n$. From $xb + e = b$ and $\varphi(x) = 1$ it follows

$$\varphi(b) = \varphi(e + xb) = \varphi(e) + \varphi(x)\varphi(b) = 1 + \varphi(b),$$

which is impossible. Consequently, $||\varphi|| \le 1$. If we use that the multiplication is a continuous map from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$ (this follows easyly from $||xy|| \le \text{const}\,||x||\,||y||$), then it is easy to show that the closure $\overline{M}$ of an ideal M is again an ideal. If M is maximal, then $\overline{M} \ne \mathcal{A}$, since $M \subseteq \mathcal{A}\backslash\mathcal{G}$ (this is obvious; otherwise we would have $M = \mathcal{A}$) implies $\overline{M} \subseteq \mathcal{A} \setminus \mathcal{G}$ in view of (v). Hence, $M = \overline{M}$ and (vii) is proved. If J is an ideal then it is well-known that $\mathcal{A}/J$ is an algebra. Now it is easy to prove that the expression $||x + J||$ from (viii) defines a quasi-norm on $\mathcal{A}/J$ if J is closed. Also the completeness of $\mathcal{A}/J$ can be proved in the same way as for Banach algebras. For the proof of (ix) we only have to remark that $0 \le |a^{n+m}| \le |a^n|\,|a^m|$, which implies $\lim_{n\to\infty}(|a^n|)^{1/n} = \inf_n(|a^n|)^{1/n}$ by a general convergence theorem (cf. [6], Satz 25.7). Let $a \in \mathcal{G}$. From (iv) and its proof it follows $b = a[e - (a-b)a^{-1}] \in \mathcal{G}$ and

$$||b^{-1} - a^{-1}|| = \left|\left|a^{-1}(a-b)a^{-1}\sum_{n=1}^{\infty}[(a-b)a^{-1}]^{n-1}\right|\right| \le K||a^{-2}||\,||a-b||$$

for all $b \in \mathcal{A}$ with $||(a-b)a^{-1}||^p \le 1/2$, where $K = K(\mathcal{A})$ is a certain constant. Obviously, this implies (x). In [3], Theo.1.7, one can find a non-standart proof of (xi) for the case of Banach algebras, which can be generalized to the quasi-Banach algebra setting as follows: Let $\omega_k = \mathrm{e}^{2\pi\mathrm{i}\,(k-1)/n}$, $k = 1, \ldots, n$, be the zeros of the polynomial $1 - z^n$. Then the partial fraction decomposition of $(1-z^n)^{-1}$ is given by $(1-z^n)^{-1} = (1/n)\sum_{k=1}^{n} \omega_k(\omega_k - z)^{-1}$. It is a fact of purely algebraic nature, that this implies $e - a^n \in \mathcal{G}$ and $(e-a^n)^{-1} = (1/n)\sum_{k=1}^{n} \omega_k(\omega_k e - a)^{-1}$ for all $a \in \mathcal{A}$ with $\omega_k e - a \in \mathcal{G}$ $(k = 1, \ldots, n)$. Now we assume that there exists an $x \in \mathcal{A}$ with $\sigma(x) = \emptyset$. Then x is invertible and from $0 < 1 \le |e| \le |x^n|\,|x^{-1}|^n$ it follows $r(x) \ge |x^{-1}|^{-1} > 0$. Without loss of generality we may suppose that $r(x) = 1$. (Replace x by $x/r(x)$.) From $\sigma(x) = \emptyset$ it follows that the element $a = \lambda_n x$, $\lambda_n = (n|x^n|)^{-1/n}$, has the property $\omega_k e - a \in \mathcal{G}$ $(k = 1, \ldots, n)$ and, consequently,

$$c_n := [e - (n|x^n|)^{-1}x^n]^{-1} = \frac{1}{n}\sum_{k=1}^{n} \omega_k(\omega_k e - \lambda_n x)^{-1}\,.$$

Moreover, in view of (x), the mapping $\omega \in \mathbf{T} = \{z : |z| = 1\} \to (\omega e - x)^{-1}$ is continuous and, consequently, $\{(\omega e - x)^{-1} : \omega \in \mathbf{T}\}$ is bounded in $\mathcal{A}$. Now, the above estimate for $||b^{-1} - a^{-1}||$ (with $b = \omega_k e - \lambda_n x$ and $a = \omega_k e - x$) shows that $\max_{k=1,\ldots,n} |||(\omega_k e - \lambda_n x)^{-1} - (\omega_k e - x)^{-1}|||^p < \varepsilon$ for all sufficiently large $n \geq n_0(\varepsilon)$. Thus, $|||c_n - (e - x^n)^{-1}|||^p = |||c_n - (1/n)\sum_{k=1}^n \omega_k(\omega_k e - x)^{-1}|||^p < \varepsilon$ for $n \geq n_0$. Together with (x) it follows $\lim_{n\to\infty}(e - x^n)^{-1} = \lim_{n\to\infty} c_n = \lim_{n\to\infty}[e - (n|x^n|)^{-1}x^n]^{-1} = e$, which implies $e - x^n \to e$, i.e. $x^n \to 0$. But $|x^n| \geq r(x)^n = 1$. So we have a contradiction and (xi) is proved. Now, let M be a maximal ideal. In view of (vii) and (viii), $\mathcal{A}/M$ is a quasi-Banach algebra. Moreover, all non-zero elements of $\mathcal{A}/M$ are invertible (see (ii)). Together with (xi) it follows that, for every $x + M \in \mathcal{A}/M$, there exists a $\lambda = \lambda(x) \in \mathbf{C}$ such that $(x - \lambda e) + M = M$, i.e. $x - \lambda e \in M$. λ is uniquely determined, since $e \notin M$. Obviously, the mapping $x + M \in \mathcal{A}/M \to \lambda(x) \in \mathbf{C}$ is an algebra isomorphism. Hence, the mapping $x \in \mathcal{A} \to \lambda(x) \in \mathbf{C}$ is a multiplicative functional with kernel M. On the other hand, if $\varphi \in \mathcal{M}$, then $\ker\varphi$ is a maximal ideal by (iii) and (xii) is proved.

After these preparations we are now able to prove the Theorem. Let M be a maximal ideal in $\mathbf{A}(X,S)$, i.e. $M = \ker\varphi$ with some $\varphi \in \mathcal{M}(\mathbf{A}(X,S))$. Using the continuity of the multiplication and the density of $\mathbf{A}(X,S)$ in X, it is easy to prove that the closure $\overline{M}$ of M in X is an ideal in X. We will show that $\overline{M} \neq X$. Thus, let us assume $\overline{M} = X$. In this case there exist

$$m_k \in M \quad \text{with} \quad m_k \to e \ \text{ in } \ X \ \ (\text{for } k \to \infty).$$

Moreover, $\bigcup A_n$ is dense in $\mathbf{A}(X,S)$ (see **P3**) and, consequently, there are

$$a_k \in \bigcup A_n \quad \text{with} \quad ||a_k - m_k||_{X,S} \leq 2^{-k}\,.$$

Together with (vi) it follows that $\varphi(a_k) = \varphi(a_k - m_k)$ converges to zero. Now we conclude that $f_k := a_k - \varphi(a_k)\,e \in \mathcal{A}_0 \cap M$ converges to e in X. But this implies that f_k is invertible in X if k is large enough (see (v)). Hence, $f_k^{-1} \in \mathbf{A}(X,S)$, since $f_k \in \mathcal{A}_0$. But f_k also belongs to the ideal M so that $e = f_k^{-1}f_k$ must be an element of M, which is impossible. (This would imply $M = \mathbf{A}(X,S)$.) So we have proved $\overline{M} \neq X$. Now it follows from Zorn's lemma that $\overline{M}$ is contained in some maximal ideal J of X. Obviously, $J \cap \mathbf{A}(X,S) \supseteq M$ is a proper ideal in $\mathbf{A}(X,S)$ (since $e \notin J$). Consequently, $J \cap \mathbf{A}(X,S) = M$, since M is maximal. On the other hand, if J is a maximal ideal in X, i.e. $J = \ker\varphi$ with some $\varphi \in \mathcal{M}(X)$, then $\varphi_{|A(X,S)} \in \mathcal{M}(\mathbf{A}(X,S))$ (obviously) and, consequently, $\ker\varphi_{|A(X,S)} = J \cap \mathbf{A}(X,S)$ is a maximal ideal in $\mathbf{A}(X,S)$. So we have proved the following nice fact:

(xiii) M is a maximal ideal in $\mathbf{A}(X,S)$ if and only if $M = J \cap \mathbf{A}(X,S)$ with some maximal ideal J of X.

Now the assertion follows from (i) and (xiii). □

Remark. *The proof of Theorem 2 shows that even the following generalization holds true: Let $\mathcal{B}$ be a commutative complex quasi-Banach algebra with*

unit $e \neq 0$ and let $\mathcal{A}$ be a dense subalgebra of $\mathcal{B}$ with $e \in \mathcal{A}$. Moreover, let $\mathcal{A}$ be equipped with a quasi-norm which turns $\mathcal{A}$ into a quasi-Banach algebra which is continuously embedded into $\mathcal{B}$. If there exists a dense linear subspace $\mathcal{A}_0 \supseteq \{e\}$ of $\mathcal{A}$ such that $f \in \mathcal{A}_0$ is invertible in $\mathcal{A}$ if and only if it is invertible in $\mathcal{B}$, then $\mathcal{A}$ is an inversely closed subalgebra of $\mathcal{B}$.

From (i) and (xii) it follows that an element x of a complex commutative quasi-Banach algebra $\mathcal{A}$ with unit $e \neq 0$ is invertible if and only if $\varphi(x) \neq 0$ for all $\varphi \in \mathcal{M}(\mathcal{A})$. This shows that it is important to know the multiplicative functionals on $\mathcal{A}$. In the setting of Theorem 2 it is possible to determine the multiplicative functionals on $\mathbf{A}(X,S)$ by those on X:

Corollary 1. *Let the assumptions of Theorem 2 be satisfied. Then the multiplicative functionals on $\mathbf{A}(X,S)$ are nothing else than the restrictions onto $\mathbf{A}(X,S)$ of the multiplicative functionals on X.*

Proof: It is clear that $\varphi \in \mathcal{M}(X)$ implies $\varphi_{|A(X,S)} \in \mathcal{M}(\mathbf{A}(X,S))$. Now, let $\varphi_0 \in \mathcal{M}(\mathbf{A}(X,S))$. In view of (xii) and (xiii) there exists some $\varphi \in \mathcal{M}(X)$ such that $\ker \varphi \cap \mathbf{A}(X,S) = \ker \varphi_0$. From (iii) it follows that, for $x \in \mathbf{A}(X,S)$, the element $\lambda_0 = \varphi_0(x)$ in the representation

$$x = \lambda_0 e + (x - \lambda_0 e) \in \{\lambda e\} + \ker \varphi_0 \subseteq \{\lambda e\} + \ker \varphi$$

must be equal to $\varphi(x)$. Thus, φ_0 is the restriction of φ onto $\mathbf{A}(X,S)$. □

Another important consequence of Theorem 2 is the following

Corollary 2. *Let the assumptions of Theorem 2 be satisfied. Then the $\mathbf{A}(X,S)$-spectrum $\sigma_{X,S}(f)$ of an element $f \in \mathbf{A}(X,S)$ is equal to its spectrum $\sigma(f)$ in X. More precisely, $\sigma_{X,S}(f) = \sigma(f) = \{\varphi(f) : \varphi \in \mathcal{M}(X)\}$.*

Proof: From Theorem 2 it follows that $f - \lambda e$ is not invertible in $\mathbf{A}(X,S)$ if and only if it is not invertible in X. Hence, $\sigma_{X,S}(f) = \sigma(f)$. Moreover, $f - \lambda e$ is not invertible in X if and only if there exists some $\varphi \in \mathcal{M}(X)$ with $\varphi(f - \lambda e) = \varphi(f) - \lambda = 0$ (see the considerations before Corollary 1). □

§3. Applications

3.1. Invertibility in Besov Spaces of Continuous Functions

Obviously, the (complex) spaces $B^s_q(L^\infty_{2\pi}) = \mathbf{A}^s_q(L^\infty_{2\pi}, \{\mathbf{T}_n\})$ and $B^s_q(L^\infty_{0,0}) = \mathbf{A}^s_q(L^\infty_{0,0}, \{\mathbf{P}_n\})$ ($0 < s < \infty$, $1 \leq q \leq \infty$; see §1) can be identified with the corresponding Besov spaces $B^s_q(C_{2\pi}) = \mathbf{A}^s_q(C_{2\pi}, \{\mathbf{T}_n\})$ and $B^s_q(C) = \mathbf{A}^s_q(C, \{\mathbf{P}_n\})$ of continuous functions, where $C_{2\pi}$ denotes the Banach algebra of all continuous 2π-periodic functions $f : \mathbf{R} \to \mathbf{C}$ and C stands for the (complex) Banach algebra $C = C[-1,1]$. (Use that the elements of $B^s_q(L^\infty_{2\pi})$ and $B^s_q(L^\infty_{0,0})$ can be approximated uniformly by polynomials (trigonometric and algebraic, respectively) and, hence, have to be continuous.) In view of Theorem 1, $B^s_q(C_{2\pi})$ and $B^s_q(C)$ are Banach algebras for all $0 < s < \infty$ and

all $1 \leq q \leq \infty$. (Remark that $\mathbf{T}_n\mathbf{T}_n \subseteq \mathbf{T}_{2n}$ and $\mathbf{P}_n\mathbf{P}_n \subseteq \mathbf{P}_{2n}$.) If $q < \infty$ (i.e. (3) is satisfied for $S = l^q(\{(n+1)^{s-(1/q)}\})$), then Theorem 2 is applicable for these algebras, since the inverse of a non-vanishing (trigonometric or algebraic) polynomial is a C^∞-function and, consequently, belongs to every space B_q^s. So we obtain the following well-known result (see [10]).

Theorem 3. *Let $f \in B_q^s(C_{2\pi})$ ($\in B_q^s(C)$) with some $s > 0$ and some $q \in [1, \infty)$. If $f(x) \neq 0$ for all x, then $1/f \in B_q^s(C_{2\pi})$ ($\in B_q^s(C)$).*

3.2. Wiener-type Theorems

In this subsection we prove a generalization of Wiener's Theorem on the absolute summability of the Fourier coeffients of a function. The classical result claims that if the Fourier coefficients of a continuous non-vanishing 2π-periodic function f satisfy $\{c_k(f)\} \in l^1(\mathbf{Z})$, then the Fourier coefficients of $1/f$ also satisfy $\{c_k(1/f)\} \in l^1(\mathbf{Z})$ (see [11]). This result was proved in a very easy way by Gelfand, as an application of (complex) commutative Banach algebra theory.

In all what follows we set $A_n = \{(a_k)_{k=-\infty}^{\infty} : a_k = 0 \text{ for } |k| \geq n\}$. (The entries of the sequences are complex numbers, i.e. we only deal with complex spaces.) We will study approximation spaces $\mathbf{A}(l^1, S) = \mathbf{A}(l^1, S, \{A_n\})$ based on the Banach algebra $l^1 = l^1(\mathbf{Z})$. Hence, the product we are interested in is the usual convolution of sequences: $(a_n)(b_n) := (c_n)$, where $c_n = \sum_{k=-\infty}^{\infty} a_{n-k}b_k$ (i.e. c_n are the Fourier coefficients of $(\sum a_k z^k) \cdot (\sum b_k z^k) \in C(\mathbf{T})$; $\mathbf{T}$: unit circle). It is clear that $\{A_n\}_{n=0}^{\infty}$ satisfies (C1), (C2), (C3), and $A_nA_n \subseteq A_{K(n)}$ with $K(n) = 2n$, so that algebra admissibility of a sequence space S (see the Remark after Theorem 1) means that S satisfies **(A1)**, **(A2)**, and **(A3)** with $K(n) = 2n$.

From Theorem 1, Theorem 2, and Corollary 1 we obtain

Theorem 4. *Let S be a complete algebra admissible sequence space, such that (3) is satisfied and $\lim_{k\to\infty} ||\{1\}_{n=0}^{k}||_S = \infty$. If all sequences $\mathbf{a} = \{a_n\}_{n=0}^{\infty}$ with $0 \leq a_n \leq \text{const}\,\delta(\mathbf{a})^n$ for some $\delta(\mathbf{a}) \in (0,1)$ belong to S, then $\mathbf{A}(l^1, S)$ is an inversely closed subalgebra of l^1. Furthermore, the multiplicative functionals on $\mathbf{A}(l^1, S)$ are nothing else than the restrictions onto $\mathbf{A}(l^1, S)$ of the multiplicative functionals on l^1.*

Proof: In view of **P3**, $\mathbf{A}(l^1, S)$ is complete. Thus, it remains to prove that $\mathcal{A}_0 = \bigcup A_n$ satisfies (4). It is well-known that the sequence $\mathbf{a} = \{a_n\}_{n=-\infty}^{\infty} \in \mathcal{A}_0$ is an invertible element of $l^1(\mathbf{Z})$ if and only if the trigonometric polynomial $h(t) = \sum_{n=-\infty}^{+\infty} a_n \mathrm{e}^{int}$ does not vanish in $[0, 2\pi]$. Then $1/h$ is analytic in a neighbourhood of $[0, 2\pi]$ and, consequently, the $C[0, 2\pi]$-errors $E_n(1/h)$ of approximation by trigonometric polynomials of degree n at most satisfy $E_n(1/h) \leq c\,\delta^n$, where $c > 0$ and $\delta \in (0, 1)$ are certain constants (cf. [4], Chapter 7, Theorem 8.1). In other words, the best approximations $g_n \in \text{span}\,\{z^k : |k| \leq n\}$ of $1/g(z)$, $g(\mathrm{e}^{it}) := h(t)$, in the $C(\mathbf{T})$-norm satisfy $\|(1/g) - g_n\|_{C(\mathbf{T})} \leq c\,\delta^n$. But this implies that the Fourier coefficients $c_n(1/g)$

of $1/g$ also have this property:

$$|c_n(1/g)| = \left| \int_{\mathbf{T}} (1/g)(z)\, z^n\, dt \right| = \left| \int_{\mathbf{T}} [(1/g) - g_{|n|-1}](z)\, z^n\, dt \right| \leq C\, \delta^{|n|}$$

for all $n \in \mathbf{Z}$. But $\{c_n(1/g)\}_{n=-\infty}^{\infty}$ is clearly the inverse of the sequence $\{a_n\}_{n=-\infty}^{\infty} = \{c_n(g)\}_{n=-\infty}^{\infty}$ with respect to the convolution of sequences. Moreover, it is clear that $E_m(\mathbf{a}) = \inf_{\mathbf{a}_m \in A_m} \|\mathbf{a} - \mathbf{a}_m\|_1$ is given by

$$E_m(\mathbf{a}) = \sum_{|k| \geq m} |a_k| \tag{5}$$

so that $E_m(\{c_n(1/g)\}) \leq C \sum_{|k|\geq m} \delta^{|k|} \leq \mathrm{const}\, \delta^m$. Now, our assumptions on S ensure that $\{E_m(\{c_n(1/g)\})\}_{m=0}^{\infty} \in S$, i.e. $\{c_n(1/g)\}_{n=-\infty}^{\infty} \in \mathbf{A}(l^1, S)$. □

It is clear that Theorem 4 implies the following Wiener-type result:

Corollary 3. *Let the assumptions of Theorem 4 be satisfied and let $f(z)$ be a continuous function on the unit circle $\mathbf{T}$ such that the sequence $\{c_k(f)\}_{k=-\infty}^{\infty}$ of its Fourier coefficients belongs to $\mathbf{A}(l^1, S)$. If $f(z) \neq 0$ for all $z \in \mathbf{T}$, then $\{c_k(1/f)\}_{k=-\infty}^{+\infty} \in \mathbf{A}(l^1, S)$.*

Proof: The multiplicative functionals on $l^1(\mathbf{Z})$ are well-known (see e.g. [11]). By Theorem 4, the multiplicative functionals on $\mathbf{A}(l^1, S)$ have the same form: φ is a non-trivial multiplicative functional on $\mathbf{A}(l^1, S)$ if and only if there exists some $z \in \mathbf{T}$ such that $\varphi((c_n)) = \sum_{n=-\infty}^{\infty} c_n z^n$ for all $(c_n) \in \mathbf{A}(l^1, S)$. Consequently, $(c_n) \in \mathbf{A}(l^1, S)$ is invertible in $\mathbf{A}(l^1, S)$ if and only if $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ does not vanish on $\mathbf{T}$ (see the considerations before Corollary 1). In this case, the inverse of (c_n) is the sequence of the Fourier coefficients of $1/f$. □

Of course, Corollary 3 is only useful if one is able to describe the elements of $\mathbf{A}(l^1, S)$. We will rewrite Corollary 3 for two cases in which this is possible:

Theorem 5. *Let $\{b_n\}_{n=0}^{\infty}$ be a strictly increasing sequence of positive numbers such that $b_{2n+1} \leq c\, b_n$ and $b_n \leq c\,(n+1)^s$ with some constants $c, s > 0$. Moreover, let $f(z)$ be a continuous function on $\mathbf{T}$ with $f(z) \neq 0$ for all $z \in \mathbf{T}$.*

(i) *If $\lim_{n\to\infty} b_n \sum_{|k|\geq n} |c_k(f)| = 0$ then $\lim_{n\to\infty} b_n \sum_{|k|\geq n} |c_k(1/f)| = 0$.*
(ii) *If $\sum_{k=-\infty}^{\infty} |c_k(f)|\, b_{|k|} < \infty$ then $\sum_{k=-\infty}^{\infty} |c_k(1/f)|\, b_{|k|} < \infty$.*

Proof: We may assume that $\lim_{n\to\infty} b_n = \infty$ since otherwise (i) and (ii) are nothing else than the well-known classical Wiener theorem. For the proof of (i) we consider the Banach space $S := \{\{a_n\} : \lim_{n\to\infty} a_n b_n = 0\}$ with $\|\{a_n\}\|_S = \sup_n |a_n b_n|$. Obviously, (3) is satisfied, $\lim_{k\to\infty} \|\{1\}_{n=0}^{k}\|_S = \infty$, and all sequences $\mathbf{a} = \{a_n\}_{n=0}^{\infty}$ with $0 \leq a_n \leq \mathrm{const}\, \delta(\mathbf{a})^n$ for some $\delta(\mathbf{a}) \in (0,1)$ belong to S. Moreover, S is algebra admissible (see [2], Rem.3.5 and Theo.3.17). Now the assertion (i) follows from Corollary 3 and (5). For the

proof of (ii) we set $d_n := b_n - b_{n-1}$ ($b_{-1} := 0$) and consider the Banach space $S = \{\{a_n\} : \|\{a_n\}\|_S := \|\{a_n d_n\}\|_1 < \infty\}$. Again it is clear that (3) is satisfied, $\lim_{k\to\infty} ||\{1\}_{n=0}^k||_S = \infty$, and all sequences $\mathbf{a} = \{a_n\}_{n=0}^\infty$ with $0 \leq a_n \leq \text{const}\, \delta(\mathbf{a})^n$ for some $\delta(\mathbf{a}) \in (0,1)$ belong to S. S is also algebra admissible (see [2], Theo.4.3). Moreover, in view of (5),

$$\|\mathbf{a}\|_{\mathbf{A}(l^1,S)} = \sum_{n=0}^{\infty} d_n \sum_{|k|\geq n} |a_k| = \sum_{k=-\infty}^{\infty} |a_k| \sum_{n=0}^{|k|} d_n = \sum_{k=-\infty}^{\infty} |a_k|\, b_{|k|}$$

for all $\mathbf{a} \in \mathbf{A}(l^1, S)$. Thus, assertion (ii) also follows from Corollary 3. □

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