Finite Sections in an Algebra of Convolution and Multiplication Operators on $L^\infty(\mathbb{R})$

M. Lindner and B. Silbermann

May 25, 2000

ABSTRACT. This text is concerned with the applicability of the finite section method to a certain type of operators on $L^\infty(\mathbb{R})$. These operators belong to the smallest closed subalgebra of $\mathcal{L}(L^\infty(\mathbb{R}))$ generated by convolution operators with symbol in the Wiener algebra and operators of multiplication by L^∞ functions with limits at $+\infty$ and $-\infty$. We use Banach algebra techniques to show that the finite section method is applicable to some operator A in that algebra if and only if A and an associated operator $\tilde{A}$ are invertible operators.

1 Introduction

For a function $k \in L^1(\mathbb{R})$ let Fk refer to its Fourier transform

$$(Fk)(x) = \int_{-\infty}^{+\infty} k(t)e^{itx}\, dt$$

and denote the set of functions $\{c + Fk : c \in \mathbb{C},\ k \in L^1(\mathbb{R})\}$ by $\mathbb{C} + FL^1$ which is also called the *Wiener algebra*.

With every Wiener function $a = c + Fk$ one can associate a *convolution operator* $\mathring{W}_a$ acting on $L^\infty(\mathbb{R})$ by

$$(\mathring{W}_a\varphi)(t) := c\varphi(t) + \int_{-\infty}^{\infty} k(t-s)\varphi(s)\, ds$$

and say that the function a is the *symbol* of the operator $\mathring{W}_a$. Further for any function $b \in L^\infty(\mathbb{R})$ let M_b refer to the multiplication operator on $L^\infty(\mathbb{R})$

$$(M_b\varphi)(t) = b(t)\varphi(t).$$

In what follows we are especially interested in L^∞ functions b that possess limits at $+\infty$ and $-\infty$. The set $L^\infty(\mathbb{R}) \cap C(\{-\infty, +\infty\})$ will be abbreviated to $L^\infty(\overline{\mathbb{R}})$.

Before defining our algebra of operators we introduce some other operators and structures. If U is some arbitrary subset of the real axis $\mathbb{R}$ we denote by P_U the multiplication operator by the characteristic function of U and in the case $U = [-\tau, \tau]$ we abbreviate

$P_{[-\tau,\tau]}$ to P_τ. The operators P_τ are called finite section projectors and we denote the complementary projector $I - P_\tau$ by Q_τ.

When investigating the finite section method (e.g. in L^2 or ℓ^2) the ideal of compact operators often plays an important role. We will use the set $\mathcal{K}$ instead which is defined by

$$\mathcal{K} := \Big\{ \; K \in \mathcal{L}(L^\infty) \; : \; \|KQ_\tau\| \to 0, \; \|Q_\tau K\| \to 0 \text{ as } \tau \to \infty \; \Big\}.$$

A much weaker demand is as follows:

$$\sup_{U,V\subset\mathbb{R}} \|P_U A P_V\| \to 0 \quad \text{as} \quad d(U,V) \to \infty$$

where $d(U,V)$ is defined as the infimum of $|u-v|$ for $u \in U$ and $v \in V$. All operators A that are subject to that condition are referred to as operators of local type and we will denote the set of all these operators by $\mathfrak{A}$. This set turns out to be a Banach algebra containing the identity I.

Note that operators $A \in \mathfrak{A}$ obviously have the property

$$\|P_\omega A Q_\tau\| \to 0 \text{ and } \|Q_\tau A P_\omega\| \to 0 \qquad \text{as } \tau \to \infty \text{ for each fixed } \omega$$

which we will make extensive use of. So one easily checks that $\mathcal{K}$ is a closed two-sided ideal in $\mathfrak{A}$. Indeed,

$$\|Q_\tau(KA)\| \le \|Q_\tau K\| \; \|A\| \quad \text{and} \quad \|Q_\tau(AK)\| \le \|Q_\tau A P_\omega\| \; \|K\| + \|Q_\tau A\| \; \|Q_\omega K\|$$

tend to zero as $\tau \to \infty$ or can be made as small as desired by choosing (the fixed) ω large enough for every $K \in \mathcal{K}$ and $A \in \mathfrak{A}$. And obviously the same holds for Q_τ from the right.

2 The operator algebra $\mathcal{A}$

Now we are in the position to define our operator algebra as

$$\mathcal{A} := \mathbf{alg}_{\mathcal{L}(L^\infty(\mathbb{R}))} \Big\{ \mathring{W}_a, \; M_b, \; \mathcal{K} \; : \; a \in \mathbb{C} + FL^1, \; b \in L^\infty(\overline{\mathbb{R}}) \Big\}$$

which is the smallest Banach algebra containing all convolution operators with Wiener symbol, all multiplicators by functions in $L^\infty(\overline{\mathbb{R}})$ and the set $\mathcal{K}$. Since multiplicators and convolution operators are operators of local type we have the inclusion $\mathcal{A} \subset \mathfrak{A}$ and so $\mathcal{K}$ is also a closed two-sided ideal in $\mathcal{A}$.

If we define $P := P_{\mathbb{R}_+}$ and $Q := I - P$ then both P and Q are operators in the algebra $\mathcal{A}$ as well as the so called Wiener-Hopf operators

$$W_a := P\mathring{W}_a P \; (+Q)$$

with symbol a in the Wiener algebra $\mathbb{C} + FL^1$.

The finite section projectors P_ω with fixed ω are multiplicators by the characteristic function of the interval $[-\omega, \omega]$. Obviously such functions are always in $L^\infty(\overline{\mathbb{R}})$ and so all the projectors P_ω are in our operator algebra $\mathcal{A}$ as well. They are even contained in the ideal $\mathcal{K}$ as all multiplicators M_b by L^∞ functions with finite support are. But there is another important class of operators in $\mathcal{K}$. Let therefore

$$[A, B] \;:=\; AB - BA$$

denote the so called <u>*commutator*</u> of two operators A and B.

Proposition 2.1 *All commutators $[\mathring{W}_a, M_b]$ of operators $\mathring{W}_a, M_b \in \mathcal{A}$ are in $\mathcal{K}$.*

Proof. Let $a = c + Fk \in \mathbb{C} + FL^1$ and $b \in L^\infty(\overline{\mathbb{R}})$ and put $K := [\mathring{W}_a, M_b]$. Then

$$(K\varphi)(t) = \int_{-\infty}^{+\infty} k(t-s)(b(s) - b(t))\varphi(s)\; ds.$$

We show $\|KQ_\tau\| \to 0$ as $\tau \to \infty$. The assertion $\|Q_\tau K\| \to 0$ is very similar to prove.

$$\begin{aligned} \|KQ_\tau\varphi\|_\infty &= \Big\| \int_{|s|>\tau} k(t-s)(b(s) - b(t))\varphi(s)\; ds \Big\|_\infty \\ &\leq \Big\| \int_{|s|>\tau} k(t-s)(b(s) - b(t))\; ds \Big\|_\infty \; \|\varphi\|_\infty \end{aligned}$$

So we have $\|KQ_\tau\| \leq \|f_\tau\|_\infty$ where we put

$$f_\tau(t) := \int_{|s|>\tau} k(t-s)(b(s) - b(t))\; ds.$$

Now fix some $\varepsilon > 0$. Take $\tau > 0$ large enough that

(1) $|b(x) - b(-\infty)| < \varepsilon/(4\|k\|_1)$ for all $x < -\frac{\tau}{2}$,
(2) $|b(x) - b(+\infty)| < \varepsilon/(4\|k\|_1)$ for all $x > \frac{\tau}{2}$ and
(3) $\int_{-\infty}^{-\frac{\tau}{2}} |k(x)|\; dx < \varepsilon/(4\|b\|_\infty)$ and $\int_{\frac{\tau}{2}}^{\infty} |k(x)|\; dx < \varepsilon/(4\|b\|_\infty)$

Then for $t > \frac{\tau}{2}$ one has

$$\begin{aligned} |f_\tau(t)| &\leq \int_{|s|>\tau} |k(t-s)|\; |b(s) - b(t)|\; ds \\ &< 2\|b\|_\infty \int_{-\infty}^{-\tau} |k(t-s)|\; ds \;+\; 2\varepsilon/(4\|k\|_1) \int_{\tau}^{+\infty} |k(t-s)|\; ds \\ &< 2\|b\|_\infty\; \varepsilon/(4\|b\|_\infty) \;+\; 2\varepsilon/(4\|k\|_1)\; \|k\|_1 \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

The same idea shows $|f_\tau(t)| < \varepsilon$ for all $t < -\frac{\tau}{2}$. And for $t \in [-\frac{\tau}{2}, \frac{\tau}{2}]$ one simply has $|t - s| > \frac{\tau}{2}$ and thus

$$\begin{aligned} |f_\tau(t)| &\leq \int_{|s|>\tau} |k(t-s)|\; |b(s) - b(t)|\; ds \\ &< 2\|b\|_\infty \int_{|s|>\tau} |k(t-s)|\; ds \\ &< 2\|b\|_\infty\; 2\varepsilon/(4\|b\|_\infty) \;=\; \varepsilon \end{aligned}$$

which gives $\|KQ_\tau\| \leq \|f_\tau\|_\infty \leq \varepsilon$ whenever τ is large enough. ■

3 Finite sections of operators in our algebra $\mathcal{A}$

3.1 Approximation methods - a general overview

If one has to solve an equation

$$Ax = b \tag{1}$$

where the operator $A \in \mathcal{A}$ and the right hand side $b \in L^\infty(\mathbb{R})$ are given and $x \in L^\infty(\mathbb{R})$ is to be determined, one might try to approximate A by a sequence of operators

$$A_\tau : \text{im}P_\tau \to \text{im}P_\tau$$

and solve the somewhat easier equations

$$A_\tau x_\tau = P_\tau b \tag{2}$$

instead, hoping these are uniquely solvable - at least for all sufficiently large τ - and that the solutions $x_\tau \in \text{im}P_\tau$ of (2) in some sense tend to a solution x of (1) as τ goes to infinity.

This is the idea of approximation methods. A quite natural choice of the operator sequence $\{A_\tau\}$ is $A_\tau := P_\tau A P_\tau$. In this case the approximation method

$$(P_\tau A P_\tau)\ x_\tau = P_\tau b \tag{3}$$

is referred to as *finite section method* for obvious reasons.

We still have to specify which type of convergence of A_τ to A and x_τ to x is appropriate to the L^∞ case. In many function spaces E one can choose the sequence $\{A_\tau\}$ such that $A_\tau \to A$ *strongly* (i.e. $\|A_\tau\varphi - A\varphi\|_E \to 0$ as $\tau \to \infty\ \forall \varphi \in E$) and can expect x_τ to converge to x in the norm of E. Both fails in our setting $E = L^\infty(\mathbb{R})$ as already the most trivial examples show.

Firstly, $\|Q_\tau \varphi\|_\infty$ does in general not converge to zero as $\tau \to \infty$ (only if $\varphi(t) \to 0$ as $|t| \to \infty$) whence Q_τ does not converge strongly to 0, P_τ not to I and thus in general a sequence $\{A_\tau\}$ with $A_\tau : \text{im}P_\tau \to \text{im}P_\tau$ will not converge strongly at all.

Secondly, consider the finite section method for the identity operator $A = I$. Then $x_\tau = P_\tau b$ which is the restriction of $b \in L^\infty(\mathbb{R})$ to the interval $[-\tau, \tau]$ does in general not converge uniformly to $x = b$ since $\|x - x_\tau\|_\infty = \|b - P_\tau b\|_\infty = \|Q_\tau b\|_\infty$.

So we have to formulate both convergence processes in a somewhat weaker sense. For A_τ and A this will be done in section 3.2. Concerning x_τ and x we will expect x_τ to converge uniformly on every finite interval of $\mathbb{R}$ to x. Obviously this is equivalent to the condition

$$\|P_\omega x_\tau - P_\omega x\|_\infty \to 0 \text{ as } \tau \to \infty \text{ for all fixed } \omega > 0.$$

Something similar will be done with A_τ and A.

3.2 $\mathcal{P}$-convergent operator sequences

In what follows let $\mathcal{P}$ refer to the sequence $\{P_\tau\}_{\tau>0}$ of finite section projectors. Note, that $\mathcal{P}$ has the properties

$$\bigcap_{\tau>0} \ker P_\tau = \bigcap_{\tau>0} \text{im}Q_\tau = \{0\} \tag{4}$$

$$\bigcup_{\tau>0} \mathrm{im} P_\tau = \bigcup_{\tau>0} \ker Q_\tau = L^\infty(\mathbb{R}) \tag{5}$$

Definition 3.1 *Let $\{A_\tau\}$ be some sequence of operators on $L^\infty(\mathbb{R})$. We say that this sequence converges $\mathcal{P}$-strongly to the operator A on $L^\infty(\mathbb{R})$ if for all fixed $\omega > 0$*

$$\|P_\omega(A_\tau - A)\| \to 0 \quad \textit{and} \quad \|(A_\tau - A)P_\omega\| \to 0$$

holds for $\tau \to \infty$ and we write $A = \mathcal{P}\text{-}\lim A_\tau$.

In what follows we abbreviate uniform convergence $\|C_\tau\| \to 0$ to $C_\tau \rightrightarrows 0$.
Property (4) and (5) ensure that $\mathcal{P}$-limits are unique if they exist:

If $P_\omega(A_\tau - A) \rightrightarrows 0$ and $P_\omega(A_\tau - B) \rightrightarrows 0$ then $P_\omega(A_\tau - A - A_\tau + B) = P_\omega(B - A) \rightrightarrows 0$ which means $P_\omega(B - A) = 0\ \forall \omega > 0$. So $B - A = Q_\omega(B - A)\ \forall \omega > 0$ whence

$$\mathrm{im}(B - A) \subseteq \bigcap_{\omega>0} \mathrm{im} Q_\omega = \{0\}.$$

And if $(A_\tau - A)P_\omega \rightrightarrows 0$ and $(A_\tau - B)P_\omega \rightrightarrows 0$ then $(B - A)P_\omega = 0\ \forall \omega > 0$. So $B - A = (B - A)Q_\omega\ \forall \omega > 0$ whence

$$\ker(B - A) \supseteq \bigcup_{\omega>0} \ker Q_\omega = L^\infty(\mathbb{R}).$$

In both cases we get $B - A = 0$. So even each one of the conditions in Definition 3.1 guarantees the uniqueness of A.

Having this notion of convergence we can reformulate the property of local type operators. For all operators $A \in \mathfrak{A}$ the sequences AQ_τ and $Q_\tau A$ $\mathcal{P}$-strongly converge to zero and those operators where this convergence is even in the operator norm are collected in the set $\mathcal{K}$ which turns out to be a Banach algebra.

If the sequence $\{A_\tau\}$ is bounded (i.e. $\sup\limits_\tau \|A_\tau\| < \infty$, which will always be the case in our setting) one has another characterization of $\mathcal{P}$-limits.

Proposition 3.2 *Let $\{A_\tau\}$ be some bounded sequence of operators in $\mathcal{A}$. Then $A = \mathcal{P}\text{-}\lim A_\tau$ if and only if for all operators $K \in \mathcal{K}$*

$$\|K(A_\tau - A)\| \to 0 \quad \textit{and} \quad \|(A_\tau - A)K\| \to 0 \tag{6}$$

holds for $\tau \to \infty$.

Proof. Let $A = \mathcal{P}\text{-}\lim A_\tau$. Then for all ω one has

$$\|K(A_\tau - A)\| \le \|K(P_\omega(A_\tau - A))\| + \|(KQ_\omega)(A_\tau - A)\|$$

where the first term tends to zero and the second one is as small as desired if ω is large enough. The second inequality of (6) is shown completely symmetric. Conversely, if (6) holds for all $K \in \mathcal{K}$ then it holds for all $K = P_\omega$ since $\mathcal{P} \subset \mathcal{K}$. ■

Note that if one defines $\mathcal{P}$-limits by (6) every $\mathcal{P}$-strongly converging sequence is automatically bounded starting from some $\tau_0 > 0$. We will nevertheless use Definition 3.1.

Proposition 3.3 *Let $\{A_\tau\}, \{B_\tau\}$ be bounded sequences of operators in $\mathcal{A}$ with $A = \mathcal{P}\text{-}\lim A_\tau \in \mathcal{A}$ and $B = \mathcal{P}\text{-}\lim B_\tau \in \mathcal{A}$. Then*

$$\|A\| \le \liminf \|A_\tau\| \le \sup \|A_\tau\| < \infty \tag{7}$$

$$\mathcal{P}\text{-}\lim A_\tau B_\tau = AB. \tag{8}$$

Proof. With every operator $A \in \mathcal{A}$ one can associate two operators on $\mathcal{K}$ by

$$A^l : K \mapsto AK \qquad \text{and} \qquad A^r : K \mapsto KA$$

and one has the norm equality $\|A\| = \|A^l\|_{\mathcal{L}(\mathcal{K})} = \|A^r\|_{\mathcal{L}(\mathcal{K})}$ for all $A \in \mathcal{A}$. (Clearly $\|A^l\|_{\mathcal{L}(\mathcal{K})} \le \|A\|$ holds and the reverse inequality can be shown by a Hahn-Banach argument). So $\mathcal{A}$ can be isometrically embedded into $\mathcal{L}(\mathcal{K})$ by each of the mappings $A \mapsto A^l$ and $A \mapsto A^r$. Consequently, $\|(A_\tau - A)K\| \to 0$ for all $K \in \mathcal{K}$ is equivalent to the strong convergence of A^l_τ to A^l on $\mathcal{K}$ and assertion (7) follows immediately from Proposition 3.2 and the Banach-Steinhaus theorem. Assertion (8) can be shown using Proposition 3.2 and the multiplicativity of strong limits - or we show it directly: For every fixed $\mu > 0$

$$\|P_\omega(A_\tau B_\tau - AB)\| \le \|(P_\omega(A_\tau - A))B_\tau\| + \|P_\omega A(P_\mu(B_\tau - B))\| + \|(P_\omega A Q_\mu)(B_\tau - B)\|$$

holds where the first two terms tend to zero as $\tau \to \infty$ and the third one becomes as small as desired if we choose μ large enough (note that $\{B_\tau\}$ is bounded).
$(A_\tau B_\tau - AB)P_\omega \rightrightarrows 0$ is shown completely symmetric using the boundedness of $\{A_\tau\}$. ∎

3.3 Applicability of approximation methods in $L^\infty(\mathbb{R})$

In what follows we often regard the approximating operator sequence $\{A_\tau\}$ in (2) itself as the approximation method.

Suppose we are given a bounded sequence of operators A_τ on $\mathrm{im}P_\tau$ which is $\mathcal{P}$-strongly converging to an operator A on $L^\infty(\mathbb{R})$.

Definition 3.4 *We say that the approximation method $\{A_\tau\}$ is applicable to the operator A if for all large enough τ, say $\tau > \tau_0$ the operators A_τ are invertible on $\mathrm{im}P_\tau$ and if for every right hand side $b \in L^\infty(\mathbb{R})$ the sequence of (unique) solutions x_τ of (2) is uniformly bounded and tends to a solution x of (1) uniformly on each finite interval of $\mathbb{R}$ as τ goes to infinity.*

We say that the finite section method is applicable to an operator A on $L^\infty(\mathbb{R})$ if the approximation method $\{P_\tau A P_\tau\}$ is applicable to A.

It can be shown that for the applicability of $\{A_\tau\}$ to A one has the same criterion as in the cases of strongly converging approximation methods:

Theorem 3.5 *Suppose $\{A_\tau\}$ is a bounded sequence of operators in $\mathcal{A}$ with $A_\tau : \mathrm{im}P_\tau \to \mathrm{im}P_\tau$ and $\mathcal{P}\text{-}\lim A_\tau =: A \in \mathcal{A}$.*

Then the approximation method $\{A_\tau\}$ is applicable to A if and only if A is invertible and for some τ_0 all operators A_τ with $\tau > \tau_0$ are invertible on $\mathrm{im}P_\tau$ and $\sup\limits_{\tau > \tau_0} \|A_\tau^{-1}\| < \infty$.

This theorem is proven in [3, 4.41f]. Note that the notion of $\mathcal{P}$-limits is even more general there than in our situation.

3.4 Algebraization and stability

Having so much to do with operator sequences it seems to be a good idea to introduce a structure whose elements are operator sequences. Once more this structure turns out to be an algebra.

Let therefore $\mathcal{F}$ denote the set of all operator sequences $\{A_\tau\}$ with $A_\tau : \mathrm{im}P_\tau \to \mathrm{im}P_\tau$ that are subject to the condition

$$\sup_\tau \|A_\tau\| < \infty,$$

namely, the bounded sequences.

Equipped with elementwise operations $\{A_\tau\}+\{B_\tau\} := \{A_\tau+B_\tau\}$, $\{A_\tau\}\{B_\tau\} := \{A_\tau B_\tau\}$ and $\lambda\{A_\tau\} := \{\lambda A_\tau\}$ and the norm

$$\|\{A_\tau\}\|_{\mathcal{F}} := \sup_\tau \|A_\tau\|$$

$\mathcal{F}$ becomes a Banach algebra. An important closed two-sided ideal in $\mathcal{F}$ is the set of all sequences tending to zero with respect to the operator norm. Let

$$\mathcal{N} := \Big\{ \ \{C_\tau\} \in \mathcal{F} \ : \ \|C_\tau\| \to 0 \text{ as } \tau \to \infty \ \Big\}.$$

If a sequence $\{A_\tau\} \in \mathcal{F}$ is invertible in $\mathcal{F}$ this means that all operators A_τ are invertible on $\mathrm{im}P_\tau$ and $\sup\limits_{\tau>0} \|A_\tau^{-1}\| < \infty$. This is not exactly the property we are familiar with from Theorem 3.5

$$\text{For some } \tau_0 \text{ all } A_\tau \text{ with } \tau > \tau_0 \text{ are invertible on } \mathrm{im}P_\tau \text{ and } \sup_{\tau>\tau_0} \|A_\tau^{-1}\| < \infty. \qquad (9)$$

but it definitely has something to do with it.

Definition 3.6 *A sequence $\{A_\tau\}$ is said to be stable if it is subject to condition (9).*

So Theorem 3.5 essentially says "applicability = invertibility of A + stability of $\{A_\tau\}$". One easily proves the following proposition which is due to A. V. Kozak [2].

Proposition 3.7 (KOZAK) *A sequence $\{A_\tau\} \in \mathcal{F}$ is stable if and only if the coset $\{A_\tau\} + \mathcal{N}$ is invertible in the factor algebra $\mathcal{F}/\mathcal{N}$.*

This is the key observation that makes the question of applicability of approximation methods accessible to Banach algebra techniques in terms of the invertibility problem in the algebra $\mathcal{F}/\mathcal{N}$.

3.5 The finite section algebra $\mathcal{B}$

To study finite sections $P_\tau A P_\tau$ of operators $A \in \mathcal{A}$ we construct an algebra that contains all these operator sequences. So let

$$\mathcal{B} := \mathbf{alg}_{\mathcal{F}}\Big\{ \ \{P_\tau A P_\tau\} + \{C_\tau\} \ : \ A \in \mathcal{A}, \ \|C_\tau\| \to 0 \ \Big\}.$$

$\mathcal{B}$ is a unital subalgebra of $\mathcal{F}$. The identity in $\mathcal{B}$ is the sequence $\{P_\tau\}$. Similar to the finite section projectors P_τ a second family of operators will be introduced now. For each $\tau > 0$ we define the so called <u>*reflection operator*</u> R_τ by

$$(R_\tau\varphi)(t) := \begin{cases} \varphi(\tau - t) & \text{for} \quad t \in [0, \tau] \\ \varphi(-\tau - t) & \text{for} \quad t \in [-\tau, 0) \\ 0 & \qquad\text{otherwise} \end{cases} .$$

Note that $R_\tau R_\tau = P_\tau$ holds as well as $P_\tau R_\tau = R_\tau = R_\tau P_\tau$.

Now we define two mappings of type "sequence $\mapsto$ operator" on $\mathcal{B}$,

$$\mathcal{W} : \{A_\tau\} \mapsto \mathcal{P}\text{-}\lim P_\tau A_\tau P_\tau \qquad \text{and} \qquad \widetilde{\mathcal{W}} : \{A_\tau\} \mapsto \mathcal{P}\text{-}\lim R_\tau A_\tau R_\tau .$$

Proposition 3.8 *The mappings $\mathcal{W}$ and $\widetilde{\mathcal{W}}$ are continuous Banach algebra homomorphisms from $\mathcal{B}$ to $\mathcal{A}$.*

We prove this assertion in several steps. If a function a is given on $\mathbb{R}$ then by $\tilde{a}$ we denote the reflected function $\tilde{a}(x) = a(-x)$. Further we will write $C_\tau \rightrightarrows 0$ if we mean $\|C_\tau\| \to 0$.

Lemma 3.9 *For all $K \in \mathcal{K}$ the identity $\mathcal{P}\text{-}\lim R_\tau K R_\tau = 0$ holds.*

Proof. Indeed, fix some $\omega > 0$. Then for all $\tau > \omega$

$$(R_\tau K R_\tau - 0)P_\omega = R_\tau K R_\tau P_\omega = R_\tau K(P_{\tau-\omega} R_\tau P_\omega) + R_\tau(K Q_{\tau-\omega}) R_\tau P_\omega$$

holds where $P_{\tau-\omega} R_\tau P_\omega = 0$ and $K Q_{\tau-\omega} \rightrightarrows 0$ as $\tau \to \infty$.
The same idea works for P_ω from the left. ■

Lemma 3.10 *$\mathcal{W}$ and $\widetilde{\mathcal{W}}$ are defined on sequences $\{P_\tau A P_\tau\}$ for all $A \in \mathcal{A}$. In detail $\mathcal{W}(\{P_\tau A P_\tau\}) = A$ and $\widetilde{\mathcal{W}}(\{P_\tau A P_\tau\}) = \tilde{A}$ with*

$$\tilde{A} \;=\; \begin{cases} P\mathring{W}_{\tilde{a}}P + Q\mathring{W}_{\tilde{a}}Q & \text{for} \quad A = \mathring{W}_a \\ b(-\infty)Q + b(+\infty)P & \text{for} \quad A = M_b \\ 0 & \text{for} \quad A \in \mathcal{K} \\ \tilde{B} + \tilde{C} & \text{for} \quad A = B + C \\ \tilde{B}\tilde{C} & \text{for} \quad A = BC \\ \lim\limits_{n\to\infty} \tilde{B}_n & \text{for} \quad A = \lim\limits_{n\to\infty} B_n \end{cases} . \tag{10}$$

Proof. To show by definition that the $\mathcal{P}$-lim exist we fix some $\omega > 0$ and let $\tau \to \infty$. Then for all $\tau > \omega$ we have $(P_\tau A P_\tau - A)P_\omega = P_\tau A P_\omega - A P_\omega = -Q_\tau A P_\omega \rightrightarrows 0 \quad \forall A \in \mathcal{A}$. The case with P_ω from the left is proven completely symmetric here and in what follows. Now turn over to $\widetilde{\mathcal{W}}$ and equation (10). Using the integral representation of $\mathring{W}_a$ one easily checks the following

$$((R_\tau \mathring{W}_a R_\tau - P_\tau(P\mathring{W}_{\tilde{a}}P + Q\mathring{W}_{\tilde{a}}Q)P_\tau)P_\omega\varphi)(t) = \begin{cases} \int\limits_0^\omega k(s - t - 2\tau)\varphi(s)\,ds\,, & t \in [-\tau, 0) \\ \int\limits_{-\omega}^0 k(t - s + 2\tau)\varphi(s)\,ds\,, & t \in [0, \tau] \\ \qquad 0 & \text{otherwise} \end{cases}$$

and as in the proof of Proposition 2.1 one shows (using $k \in L^1(\mathbb{R})$)

$$(R_\tau \mathring{W}_a R_\tau - P_\tau(P\mathring{W}_{\tilde{a}}P + Q\mathring{W}_{\tilde{a}}Q)P_\tau)P_\omega \rightrightarrows 0 \quad \text{as} \quad \tau \to \infty.$$

This and $(P_\tau BP_\tau - B)P_\omega \rightrightarrows 0$ for all $B \in \mathcal{A}$ show the assertion for $A = \mathring{W}_a$. For $A = M_b$ $R_\tau M_b R_\tau = M_{R_\tau b}$ holds. This and a little thought shows that for $\tau > \omega$ one has

$$(R_\tau M_b R_\tau \; - \; P_\tau(b(-\infty)Q + b(+\infty)P)P_\tau)P_\omega = (M_{R_\tau b} \; - \; (b(-\infty)Q + b(+\infty)P))P_\omega \rightrightarrows 0$$

as $\tau \to \infty$. So we're done with $A = M_b$. The assertion $\tilde{A} = 0$ for $A \in \mathcal{K}$ is nothing but Lemma 3.9. The case $A = B + C$ is trivial and for $A = BC$

$$(R_\tau BCR_\tau - \tilde{B}\tilde{C})P_\omega = ((R_\tau BR_\tau)(R_\tau CR_\tau) - \tilde{B}\tilde{C})P_\omega + R_\tau BQ_\tau CR_\tau P_\omega$$

holds. The first term tends to zero by (8) and the second one can be written as

$$R_\tau BQ_\tau CR_\tau P_\omega = R_\tau B(Q_\tau CP_\omega)R_\tau P_\omega + R_\tau BQ_\tau C(Q_\omega R_\tau P_\omega).$$

This sequence also tends to zero as $\tau \to \infty$ since $Q_\tau CP_\omega \rightrightarrows 0$ as $\tau \to \infty$ and $Q_\omega R_\tau P_\omega = 0$ for all $\tau > 2\omega$.

Now we have shown that $\widetilde{\mathcal{W}}(\{P_\tau AP_\tau\})$ exists for all A being finite sum-products of convolution and multiplication operators and operators in $\mathcal{K}$. Since (see (7))

$$\|\widetilde{\mathcal{W}}(\{P_\tau AP_\tau\})\| = \|\mathcal{P} - \lim R_\tau AR_\tau\| \le \|\{R_\tau AR_\tau\}\|_{\mathcal{F}} \le \|A\|$$

holds, $\widetilde{\mathcal{W}}$ is defined for all $\{P_\tau AP_\tau\}$ with $A \in \mathcal{A}$ and we're finished with (10). ■

Now we can turn over to sums and products of sequences $\{P_\tau AP_\tau\}$ and finally prove Proposition 3.8.

Proof. For $\{A_\tau\}, \{B_\tau\} \in \mathcal{B}$ let $A := \mathcal{W}(\{A_\tau\})$, $B := \mathcal{W}(\{B_\tau\})$, $\tilde{A} := \widetilde{\mathcal{W}}(\{A_\tau\})$ and $\tilde{B} := \widetilde{\mathcal{W}}(\{B_\tau\})$. Then

$$(P_\tau A_\tau B_\tau P_\tau - AB)P_\omega = ((P_\tau A_\tau P_\tau)(P_\tau B_\tau P_\tau) - AB)P_\omega \rightrightarrows 0$$

shows $\mathcal{W}(\{A_\tau B_\tau\}) = AB$. And finally $\widetilde{\mathcal{W}}(\{A_\tau B_\tau\}) = \tilde{A}\tilde{B}$ follows from

$$(R_\tau A_\tau B_\tau R_\tau - \tilde{A}\tilde{B})P_\omega = ((R_\tau A_\tau R_\tau)(R_\tau B_\tau R_\tau) - \tilde{A}\tilde{B})P_\omega \rightrightarrows 0$$

(note that $A_\tau P_\tau = A_\tau = P_\tau A_\tau$). So with Lemma 3.10 everything is clear for $\{A_\tau\}$ being a finite sum-product of sequences $\{P_\tau AP_\tau\}$ with $A \in \mathcal{A}$. To complete this proof just verify the following two inequalities (again see (7))

$$\|\mathcal{W}(\{A_\tau\})\| = \|\mathcal{P} - \lim P_\tau A_\tau P_\tau\| \le \|\{P_\tau A_\tau P_\tau\}\|_{\mathcal{F}} \le \|\{A_\tau\}\|_{\mathcal{F}} \qquad (11)$$

$$\|\widetilde{\mathcal{W}}(\{A_\tau\})\| = \|\mathcal{P} - \lim R_\tau A_\tau R_\tau\| \le \|\{R_\tau A_\tau R_\tau\}\|_{\mathcal{F}} \le \|\{A_\tau\}\|_{\mathcal{F}} \qquad (12)$$

and it follows that $\mathcal{W}$ and $\widetilde{\mathcal{W}}$ are continuous homomorphisms from $\mathcal{B}$ to $\mathcal{A}$. ■

3.6 A necessary stability condition

From Theorem 3.5 we have learnt that stability is an essential ingredient of applicability of approximation sequences. So we are looking for sufficient and necessary conditions for the stability of sequences in the algebra $\mathcal{B}$. We start with a necessary condition.

Proposition 3.11 *For a sequence $\{A_\tau\} \in \mathcal{B}$ put $A := \mathcal{W}(\{A_\tau\})$ and $\tilde{A} := \widetilde{\mathcal{W}}(\{A_\tau\})$. If $\{A_\tau\}$ is stable then both A and $\tilde{A}$ are invertible operators (in $\mathcal{L}(L^\infty)$).*

Proof. If $\{A_\tau\}$ is stable then there exists some τ_0 such that the sequence $\{A_\tau\}_{\tau>\tau_0}$ is invertible in $\mathcal{F}$. Now [3, Corollary4.40] is applicable to the sequence $\{A_\tau\}_{\tau>\tau_0}$ and tells that its $\mathcal{P}$- lim which is the operator A is injective.
For the (pre)adjoint operators A_τ^* resp. A^* on L^1 one has (by the same argument) that the invertibility of the bounded sequence $\{A_\tau^*\}_{\tau>\tau_0}$ implies the injectivity of A^* and thus A is an invertible operator.

Now the stability of $\{A_\tau\}$ implies that of $\{R_\tau A_\tau R_\tau\}$. Indeed, $(R_\tau A_\tau R_\tau)^{-1} = R_\tau A_\tau^{-1} R_\tau$ and thus $\|(R_\tau A_\tau R_\tau)^{-1}\| \leq \|A_\tau^{-1}\|$. The analogue argumentation as above shows that also $\tilde{A}$, the $\mathcal{P}$- lim of $\{R_\tau A_\tau R_\tau\}$, is an invertible operator. ■

Finding a sufficient stability condition is a bit more delicate and will be managed in the following section.

3.7 Essentialization - The ideal $\mathcal{J}$.

If we want to verify stability of sequences in $\mathcal{B}$ we have to study elements of $\mathcal{F}/\mathcal{N}$. Unfortunately it is almost impossible to tackle the invertibility problem in this algebra directly. So we are going to do something else.

We introduce another, somewhat larger ideal - say $\mathcal{J}$ - in $\mathcal{B}$ where $\mathcal{B}/\mathcal{J}$ is in some sense "nicer" than $\mathcal{F}/\mathcal{N}$ and invertibility in $\mathcal{B}/\mathcal{J}$ still has something to do with stability.

Remember the operator ideal $\mathcal{K}$. Now let

$$\mathcal{J} := \Big\{ \ \{P_\tau K_1 P_\tau + R_\tau K_2 R_\tau + C_\tau\} \in \mathcal{B} \ : \ K_{1/2} \in \mathcal{K}, \ C_\tau \rightrightarrows 0 \ \Big\}.$$

Before showing that $\mathcal{J}$ is an ideal in $\mathcal{B}$ we want to see what our homomorphisms $\mathcal{W}$ and $\widetilde{\mathcal{W}}$ do with sequences from $\mathcal{J}$.

Proposition 3.12 *Let $\{A_\tau\} = \{P_\tau K_1 P_\tau + R_\tau K_2 R_\tau + C_\tau\} \in \mathcal{J}$. Then $\mathcal{W}(\{A_\tau\}) = K_1$ and $\widetilde{\mathcal{W}}(\{A_\tau\}) = K_2$.*

Proof. The assertions follow immediately from $\mathcal{P}\text{-}\lim P_\tau K P_\tau = K$ and Lemma 3.9. ■

Proposition 3.13 *$\mathcal{J}$ is a closed two-sided ideal in $\mathcal{B}$.*

Proof. Let $\{B_\tau\} \in \mathcal{J}$. For the ideal property it suffices to show, that for all operators $A \in \mathcal{A}$ the sequences $\{B_\tau\}\{P_\tau A P_\tau\}$ and $\{P_\tau A P_\tau\}\{B_\tau\}$ are in $\mathcal{J}$ again.

For $B_\tau = P_\tau K P_\tau$ with $K \in \mathcal{K}$ one has

$$\begin{aligned} B_\tau(P_\tau A P_\tau) = (P_\tau K P_\tau)(P_\tau A P_\tau) = P_\tau K P_\tau A P_\tau &= P_\tau(KA)P_\tau - P_\tau(KQ_\tau)AP_\tau \\ &= P_\tau K' P_\tau + C_\tau \end{aligned} \tag{13}$$

with $K' = KA \in \mathcal{K}$ and $C_\tau = -P_\tau(KQ_\tau)AP_\tau \rightrightarrows 0$ since $KQ_\tau \rightrightarrows 0$.

For $B_\tau = R_\tau K R_\tau$ with $K \in \mathcal{K}$ we have

$$B_\tau(P_\tau A P_\tau) = (R_\tau K R_\tau)(P_\tau A P_\tau) = R_\tau K R_\tau A P_\tau = R_\tau(K R_\tau A R_\tau) R_\tau. \tag{14}$$

Let's have a closer look at the term in brackets. For all fixed $\omega > 0$

$$K R_\tau A R_\tau = K P_\omega (R_\tau A R_\tau) + (K Q_\omega) R_\tau A R_\tau$$

holds. The second term on the right can be made as small as desired by choosing ω large enough and with $\tilde{A} := \widetilde{\mathcal{W}}(\{P_\tau A P_\tau\}) = \mathcal{P}\text{-}\lim R_\tau A R_\tau$ and the definition of the $\mathcal{P}$-limit it is clear that $P_\omega(R_\tau A R_\tau)$ differs from $P_\omega \tilde{A}$ only by a zero sequence $C_\tau \rightrightarrows 0$. This means

$$K R_\tau A R_\tau = K P_\omega \tilde{A} + C'_\tau = K' + C'_\tau$$

with $K' = K P_\omega \tilde{A} \in \mathcal{K}$ and $C'_\tau \rightrightarrows 0$. Together with (13) and (14) this shows, that $\{B_\tau\}\{P_\tau A P_\tau\}$ is in $\mathcal{J}$ whenever $\{B_\tau\}$ is in $\mathcal{J}$. The same can be shown easily for $\{P_\tau A P_\tau\}\{B_\tau\}$. $\mathcal{J} \subseteq \mathcal{B}$ holds by definition but we still have to show that $\mathcal{J}$ is closed.

(11), (12) and Propos. 3.12 show, that for $\{A_\tau\} = \{P_\tau K_1 P_\tau + R_\tau K_2 R_\tau + C_\tau\} \in \mathcal{J}$ we have $\|K_1\| = \|\mathcal{W}(\{A_\tau\})\| \le \|\{A_\tau\}\|$ and $\|K_2\| = \|\widetilde{\mathcal{W}}(\{A_\tau\})\| \le \|\{A_\tau\}\|$.

So if $\left\{\{A_\tau^{(n)}\}_\tau\right\}_n = \left\{\{P_\tau K_1^{(n)} P_\tau + R_\tau K_2^{(n)} R_\tau + C_\tau^{(n)}\}_\tau\right\}_n$ is Cauchy sequence in $\mathcal{J}$, then $\{K_1^{(n)}\}_n$ and $\{K_2^{(n)}\}_n$ are so in $\mathcal{K}$ and thus converge in $\mathcal{K}$ to say, K_1 and K_2 respectively. Since $C_\tau^{(n)} = A_\tau^{(n)} - P_\tau K_1^{(n)} P_\tau - R_\tau K_2^{(n)} R_\tau$ it is clear that also $\left\{\{C_\tau^{(n)}\}_\tau\right\}_n$ is a Cauchy sequence in $\mathcal{N}$ and thus converges to some $\{C_\tau\} \in \mathcal{N}$. If you take these three limit objects and construct the sequence

$$\{A_\tau\} := \{P_\tau K_1 P_\tau + R_\tau K_2 R_\tau + C_\tau\}$$

then $\{A_\tau\}$ is obviously in $\mathcal{J}$ and on the other hand one has $\|\{A_\tau^{(n)} - A_\tau\}\| \to 0$ as $n \to \infty$. ■

Now we're going to find out what invertibility in $\mathcal{B}/\mathcal{J}$ has to do with stability. This is the point where the so called lifting theorem enters the scene. For $\{A_\tau\} \in \mathcal{B}$ we abbreviate the coset $\{A_\tau\} + \mathcal{J}$ to $\{A_\tau\}^{\mathcal{J}}$.

Remark. We defined the ideal $\mathcal{J}$ as intersection of the set of all $\{P_\tau K_1 P_\tau + R_\tau K_2 R_\tau + C_\tau\}$ (with $K_{1/2} \in \mathcal{K}$, $C_\tau \rightrightarrows 0$) and $\mathcal{B}$ to make it a subset of $\mathcal{B}$. What we could have done instead is putting **all** these sequences into $\mathcal{J}$ and making $\mathcal{B}$ a bit larger such that it contains them all. In both cases $\mathcal{B}/\mathcal{J}$ is the same algebra.

Theorem 3.14 (Lifting Theorem) *Let $\{A_\tau\}$ be some arbitrary sequence in $\mathcal{B}$. Put $A := \mathcal{W}(\{A_\tau\})$ and $\tilde{A} := \widetilde{\mathcal{W}}(\{A_\tau\})$. If $\{A_\tau\}^{\mathcal{J}}$ is invertible in $\mathcal{B}/\mathcal{J}$ and the two operators A and $\tilde{A}$ are invertible (in $\mathcal{L}(L^\infty)$) then $\{A_\tau\}$ is stable.*

Proof. Let $\{A_\tau\}^{\mathcal{J}}$, A and $\tilde{A}$ be invertible. Since $A, \tilde{A} \in \mathcal{A} \subset \mathfrak{A}$ are invertible operators and $\mathfrak{A}$ is inverse closed in $\mathcal{L}(L^\infty)$ (see [3, Theorem4.39]) the inverses A^{-1} and $\tilde{A}^{-1}$ are operators of local type again.

Since $\{A_\tau\}^{\mathcal{J}}$ is invertible in $\mathcal{B}/\mathcal{J}$ there is a $\{B_\tau\} \in \mathcal{B}$ such that

$$\{A_\tau\}\{B_\tau\} = \{P_\tau + P_\tau K_1 P_\tau + R_\tau K_2 R_\tau + C_\tau\}$$

holds for some $K_{1/2} \in \mathcal{K}$ and $C_\tau \rightrightarrows 0$. Now one easily checks that

$$\{B_\tau\} - \{P_\tau A^{-1} K_1 P_\tau + R_\tau \tilde{A}^{-1} K_2 R_\tau\}$$

is a right inverse of $\{A_\tau\}$ modulo $\mathcal{N}$. We demonstrate this for one term :

$$\begin{aligned} R_\tau K_2 R_\tau - A_\tau R_\tau \tilde{A}^{-1} K_2 R_\tau & \\ = \quad & R_\tau K_2 R_\tau - R_\tau((R_\tau A_\tau R_\tau)\tilde{A}^{-1} P_\omega) K_2 R_\tau - A_\tau R_\tau \tilde{A}^{-1}(Q_\omega K_2) R_\tau \\ = \quad & R_\tau K_2 R_\tau - R_\tau(I P_\omega + C_\tau^{'}) K_2 R_\tau - A_\tau R_\tau \tilde{A}^{-1}(Q_\omega K_2) R_\tau \\ = \quad & R_\tau((I - P_\omega) K_2) R_\tau + C_\tau^{''} - A_\tau R_\tau \tilde{A}^{-1}(Q_\omega K_2) R_\tau \end{aligned}$$

But $C_\tau^{''} \rightrightarrows 0$ and the norms of the remaining two terms can be made as small as desired by choosing ω large enough since $\|A_\tau\|$ is bounded and $\|(I - P_\omega)K_2\| = \|Q_\omega K_2\| \to 0$ as $\omega \to \infty$. The proof of the left invertibility is symmetric to this one. ■

Our goal was to find a sufficient stability condition. With Theorem 3.14 we are much closer to finding that condition than before. The invertibility of A and $\tilde{A}$ has already proven to be necessary. So we are left with invertibility in the factor algebra $\mathcal{B}/\mathcal{J}$. We start thinking in $\mathcal{B}/\mathcal{J}$ with a tiny example. Let

$$\{A_\tau\} := \{P_\tau \mathring{W}_{a_1} P_\tau\}\{P_\tau M_{b_1} \mathring{W}_{a_2} P_\tau\}\{P_\tau M_{b_2} P_\tau\}$$

and let $A := \mathcal{W}(\{A_\tau\}) = \mathring{W}_{a_1} M_{b_1} \mathring{W}_{a_2} M_{b_2}$ (by Lemma 3.10). What makes the sequences $\{A_\tau\}$ and $\{P_\tau A P_\tau\}$ different from each other are the several P_τ's in the middle of $\{A_\tau\}$. But we will see that this doesn't change anything when thinking modulo $\mathcal{J}$. The key observation is

Proposition 3.15 *For all $a, b \in \mathbb{C} + FL^1$ one has $\{P_\tau \mathring{W}_{ab} P_\tau\}^{\mathcal{J}} = \{P_\tau \mathring{W}_a P_\tau \mathring{W}_b P_\tau\}^{\mathcal{J}}$.*

Proof. Since $\mathring{W}_{ab} = \mathring{W}_a \mathring{W}_b$ we just have to show $\{P_\tau \mathring{W}_a Q_\tau \mathring{W}_b P_\tau\} \in \mathcal{J}$.

On associating each function $\varphi \in L^\infty(\mathbb{R})$ with the vector $(P\varphi, \widetilde{Q\varphi})^T$ we can identify $L^\infty(\mathbb{R})$ with $L^\infty(\mathbb{R}_+) \times L^\infty(\mathbb{R}_+)$ and in this identification the operator $\mathring{W}_a$ corresponds to the 2×2 matrix $\begin{bmatrix} W_a & H_a \\ H_{\tilde{a}} & W_{\tilde{a}} \end{bmatrix}$ where W_a is the Wiener-Hopf operator and

$$(H_a\varphi)(t) = \int_0^\infty k(t+s)\varphi(s)ds$$

is the so called <u>*Hankel operator*</u> with symbol $a = c + Fk \in \mathbb{C} + FL^1$, both acting on $L^\infty(\mathbb{R}_+)$. Further by $\hat{P}_\tau$, $\hat{Q}_\tau$ and $\hat{R}_\tau$ we denote the restrictions of P_τ, Q_τ and R_τ to

$L^\infty(\mathbb{R}_+)$ respectively. Then

$$\begin{aligned} P_\tau \mathring{W}_a Q_\tau \mathring{W}_b P_\tau &= P_\tau \begin{bmatrix} W_a & H_a \\ H_{\tilde a} & W_{\tilde a} \end{bmatrix} Q_\tau \begin{bmatrix} W_b & H_b \\ H_{\tilde b} & W_{\tilde b} \end{bmatrix} P_\tau \\ &= \begin{bmatrix} \hat P_\tau W_a \hat Q_\tau W_b \hat P_\tau + \hat P_\tau H_a \hat Q_\tau H_{\tilde b} \hat P_\tau & \hat P_\tau W_a \hat Q_\tau H_b \hat P_\tau + \hat P_\tau H_a \hat Q_\tau W_{\tilde b} \hat P_\tau \\ \hat P_\tau H_{\tilde a} \hat Q_\tau W_b \hat P_\tau + \hat P_\tau W_{\tilde a} \hat Q_\tau H_{\tilde b} \hat P_\tau & \hat P_\tau H_{\tilde a} \hat Q_\tau H_b \hat P_\tau + \hat P_\tau W_{\tilde a} \hat Q_\tau W_{\tilde b} \hat P_\tau \end{bmatrix} \end{aligned}$$

holds. One easily checks

$$\|H_a \hat Q_\tau\| \to 0 \quad \text{and} \quad \|\hat Q_\tau H_a\| \to 0 \quad \text{as} \quad \tau \to \infty \quad \text{for all} \quad a \in \mathbb{C} + FL^1. \tag{15}$$

An important observation by Widom shows that $\hat P_\tau W_a \hat Q_\tau W_b \hat P_\tau = \hat R_\tau H_a H_{\tilde b} \hat R_\tau$ holds for all $a, b \in \mathbb{C} + FL^1$ and so we have

$$\begin{aligned} P_\tau \mathring{W}_a Q_\tau \mathring{W}_b P_\tau &= \begin{bmatrix} \hat R_\tau H_a H_{\tilde b} \hat R_\tau & 0 \\ 0 & \hat R_\tau H_{\tilde a} H_b \hat R_\tau \end{bmatrix} + C_\tau \\ &= R_\tau \begin{bmatrix} H_a H_{\tilde b} & 0 \\ 0 & H_{\tilde a} H_b \end{bmatrix} R_\tau + C_\tau \end{aligned}$$

with some $C_\tau \rightrightarrows 0$. But the operator in $\mathcal{L}(L^\infty(\mathbb{R}))$ that corresponds to $\begin{bmatrix} H_a H_{\tilde b} & 0 \\ 0 & H_{\tilde a} H_b \end{bmatrix}$ is in $\mathcal{K}$ by (15) and we are finished. ■

So in our example

$$\{A_\tau\}^{\mathcal{J}} = \{P_\tau \mathring{W}_{a_1} P_\tau M_{b_1} \mathring{W}_{a_2} P_\tau M_{b_2} P_\tau\}^{\mathcal{J}}$$

we can first (by Proposition 2.1) commute M_{b_1} and $\mathring{W}_{a_2}$ and bring all multiplicators to one end of the term, then apply Proposition 3.15 to the convolution operators and so remove all the P_τ's in the middle of the term to finally bring the multiplication operators back to their old position. This results in

$$\begin{aligned} \{A_\tau\}^{\mathcal{J}} &= \{P_\tau \ \mathring{W}_{a_1} \ P_\tau \ M_{b_1} \mathring{W}_{a_2} \ P_\tau \ M_{b_2} \ P_\tau\}^{\mathcal{J}} \\ &= \{P_\tau \ \mathring{W}_{a_1} \ P_\tau \ \mathring{W}_{a_2} M_{b_1} \ P_\tau \ M_{b_2} \ P_\tau\}^{\mathcal{J}} \\ &= \{P_\tau \ \mathring{W}_{a_1} \ P_\tau \ \mathring{W}_{a_2} \ P_\tau \ M_{b_1} M_{b_2} \ P_\tau\}^{\mathcal{J}} \\ &= \{P_\tau \ \mathring{W}_{a_1} \mathring{W}_{a_2} \ P_\tau \ M_{b_1} M_{b_2} \ P_\tau\}^{\mathcal{J}} \\ &= \{P_\tau \ \mathring{W}_{a_1} M_{b_1} \mathring{W}_{a_2} M_{b_2} \ P_\tau\}^{\mathcal{J}} \\ &= \{P_\tau \ A \ P_\tau\}^{\mathcal{J}}. \end{aligned}$$

Now we are optimistic that

Proposition 3.16 *For all $\{A_\tau\} \in \mathcal{B}$ the identity $\{A_\tau\}^{\mathcal{J}} = \{P_\tau \ \mathcal{W}(\{A_\tau\}) \ P_\tau\}^{\mathcal{J}}$ holds.*

Proof. As demonstrated in our example the assertion holds for finite products of sequences $\{P_\tau A P_\tau\}$ where $A \in \mathcal{A}$ is a finite product of convolution and multiplication operators. (For operators $K \in \mathcal{K}$ we have $\{P_\tau K P_\tau\}^{\mathcal{J}} = 0$.) And since taking sums and limits of such sequences does not change the result we're done. ■

This Proposition almost yields that the invertibility of the class $\{A_\tau\}^{\mathcal{J}}$ already follows from the invertibility of $A = \mathcal{W}(\{A_\tau\})$ in the algebra $\mathcal{A}$. But even more is true.

Proposition 3.17 *Let $\{A_\tau\}$ be some arbitrary sequence in $\mathcal{B}$. If $A := \mathcal{W}(\{A_\tau\})$ is an invertible operator (in $\mathcal{L}(L^\infty)$) then the coset $\{A_\tau\}^{\mathcal{J}}$ is invertible in $\mathcal{B}/\mathcal{J}$.*

Proof. Let $A \in \mathcal{A}$ be an invertible operator. Unfortunately we do not know whether the inverse A^{-1} is in $\mathcal{A}$ again but in our situation it is enough to show invertibility in $\mathcal{A}$ modulo $\mathcal{K}$.

Lemma 3.18 *If $A \in \mathcal{A}$ is an invertible operator (in $\mathcal{L}(L^\infty)$) then there is an operator $B \in \mathcal{A}$ such that both $AB - I$ and $BA - I$ are contained in $\mathcal{K}$ (i.e. $A + \mathcal{K}$ is invertible in $\mathcal{A}/\mathcal{K}$).*

Proof. By Λ denote the Banach algebra of all operators of local type that commute with each multiplicator M_b ($b \in L^\infty(\overline{\mathbb{R}})$) modulo $\mathcal{K}$. Then $\mathcal{A} \subset \Lambda \subset \mathfrak{A}$ and $\mathcal{K}$ is closed two-sided ideal in each of the three algebras. Write $\Lambda^{\mathcal{K}} := \Lambda/\mathcal{K}$ and $A^{\mathcal{K}} := A + \mathcal{K}$ for each $A \in \Lambda$.
The algebra $Z := \{M_b^{\mathcal{K}} : b \in L^\infty(\overline{\mathbb{R}})\}$ is closed, contains the identity $I^{\mathcal{K}}$ and is contained in the center of $\Lambda^{\mathcal{K}}$. Let $\mathcal{M}$ denote the set of maximal ideals of Z and for each $M \in \mathcal{M}$ put $J_M := \mathbf{closid}_{\Lambda^{\mathcal{K}}} M$ and $x^M := x + J_M$ for all $x \in \Lambda^{\mathcal{K}}$ and finally put $(\Lambda^{\mathcal{K}})^M := \Lambda^{\mathcal{K}}/J_M$.

If $A \in \mathcal{A} \subset \Lambda \subset \mathfrak{A}$ is an invertible operator then the inverse A^{-1} is even in Λ since not only $\mathfrak{A}$ is inverse closed in $\mathcal{L}(L^\infty)$ (see [3, Theorem4.39]) but also Λ is clearly inverse closed in $\mathfrak{A}$. So the coset $A^{\mathcal{K}}$ is surely invertible in $\Lambda^{\mathcal{K}}$.

The coset $A^{\mathcal{K}}$ can be approximated (in the norm of $\Lambda^{\mathcal{K}}$) as good as desired by finite sum-products of the form

$$A_\varepsilon^{\mathcal{K}} := (\sum_{i=1}^{m} \prod_{j=1}^{n} \mathring{W}_{a_{ij}} M_{b_{ij}})^{\mathcal{K}} \in \mathcal{A}/\mathcal{K}$$

where $a_{ij} \in \{1, 1 - Fl\}$ with $l(t) = \begin{cases} 2e^{-t} & \text{for } t > 0 \\ 0 & \text{otherwise} \end{cases}$ and $b_{ij} \in L^\infty(\overline{\mathbb{R}})$.
Since $A^{\mathcal{K}}$ is invertible also $A_\varepsilon^{\mathcal{K}}$ is invertible in $\Lambda^{\mathcal{K}}$ if it is close enough to $A^{\mathcal{K}}$. Then all the local classes

$$(A_\varepsilon^{\mathcal{K}})^M = \Big((\sum_i \prod_j \mathring{W}_{a_{ij}} M_{b_{ij}})^{\mathcal{K}}\Big)^M = (\sum_i \prod_j c_{ij,M} \mathring{W}_{a_{ij}}^{\mathcal{K}})^M \tag{16}$$

with $c_{ij,M} = (M_{b_{ij}}^{K})^M \in \mathbb{C}$ are invertible in their local algebras $(\Lambda^{\mathcal{K}})^M$ for all $M \in \mathcal{M}$.

However, if we put $(Uf)(t) := f(t) - 2\int_{-\infty}^t e^{s-t} f(s)ds$ and denote by $QC(U)$ the quasi-commutator ideal of the algebra $\mathbf{alg}(U, U^{-1}, P)$ (see [3, sections 4.65ff]) then the spectrum of the coset $(\sum_i \prod_j c_{ij,M} \mathring{W}_{a_{ij}}) + QC(U)$ in the factor algebra $\mathbf{alg}(U, U^{-1}, P)/QC(U)$ is thin and thus the spectrum of $\sum_i \prod_j c_{ij,M} \mathring{W}_{a_{ij}}^{\mathcal{K}}$ in $\mathcal{A}/\mathcal{K}$ is thin as well since it is contained in the previous one (because $\mathbf{alg}(U, U^{-1}, P) \subset \mathcal{A}$ and $QC(U) \subset \mathcal{K}$, see [3, 4.88]). But then the local spectra of $(\sum_i \prod_j c_{ij,M} \mathring{W}_{a_{ij}}^{\mathcal{K}})^M$ in $(\mathcal{A}/\mathcal{K})^M$ are thin and so are the spectra of $(A_\varepsilon^{\mathcal{K}})^M$ in $(\mathcal{A}/\mathcal{K})^M$ since the latter coincide by (16).
Now the spectrum of $(A_\varepsilon^{\mathcal{K}})^M$ in the algebra $(\mathcal{A}/\mathcal{K})^M$ could differ from the spectrum in $(\Lambda^{\mathcal{K}})^M$ only by massive components. So these spectra coincide and thus $(A_\varepsilon^{\mathcal{K}})^M$ is not

only invertible in $(\Lambda^{\mathcal{K}})^M$ but even in $(\mathcal{A}/\mathcal{K})^M$ for all $M \in \mathcal{M}$. Allan's local principle (see [1, Theorem1.34]) tells now that $A_\varepsilon^{\mathcal{K}}$ is invertible in $\mathcal{A}/\mathcal{K}$, i.e. $(A_\varepsilon^{\mathcal{K}})^{-1} \in \mathcal{A}/\mathcal{K}$.

But since $(A_\varepsilon^{\mathcal{K}})^{-1} \to (A^{\mathcal{K}})^{-1}$ in the norm of $\Lambda^{\mathcal{K}}$ as $\varepsilon \to 0$ and since $\mathcal{A}/\mathcal{K}$ is closed in that norm the inverse $(A^{\mathcal{K}})^{-1}$ is also contained in $\mathcal{A}/\mathcal{K}$. ■

Now we can continue proving Proposition 3.17 by just adding the facts from Lemma 3.18 and Proposition 3.16:

If $B \in \mathcal{A}$ is the operator from Lemma 3.18 then the sequence $\{P_\tau B P_\tau\}$ is an inverse in $\mathcal{B}$ modulo $\mathcal{J}$ of $\{A_\tau\}$. Indeed,

$$\begin{array}{rcll} \{A_\tau\}^{\mathcal{J}}\{P_\tau B P_\tau\}^{\mathcal{J}} & = & \{P_\tau A P_\tau\}^{\mathcal{J}}\{P_\tau B P_\tau\}^{\mathcal{J}} & \text{(by Proposition 3.16)} \\ & = & \{P_\tau A B P_\tau\}^{\mathcal{J}} & \text{(by Proposition 3.16)} \\ & = & \{P_\tau\}^{\mathcal{J}} & \text{(by Lemma 3.18)} \end{array}$$

holds and it is clear that $\{P_\tau B P_\tau\}^{\mathcal{J}}$ is also a left inverse. ■

Now it is obvious that our necessary stability condition from Proposition 3.11 is also sufficient and we can state the following Theorem.

Theorem 3.19 *Let $\{A_\tau\}$ be some arbitrary sequence in $\mathcal{B}$.*
Put $A := \mathcal{W}(\{A_\tau\})$ and $\tilde{A} := \widetilde{\mathcal{W}}(\{A_\tau\})$. Then the sequence $\{A_\tau\}$ is stable if and only if both operators A and $\tilde{A}$ are invertible (in $\mathcal{L}(L^\infty)$).

Proof. If $\{A_\tau\}$ is stable then by Proposition 3.11 the two operators are invertible.
If conversely, A and $\tilde{A}$ are invertible then by Proposition 3.17 the coset $\{A_\tau\}^{\mathcal{J}}$ is invertible in $\mathcal{B}/\mathcal{J}$ and the Lifting Theorem 3.14 finally shows that $\{A_\tau\}$ is stable. ■

An immediate consequence of this and Theorem 3.5 is

Corollary 3.20 *For a sequence $\{A_\tau\} \in \mathcal{B}$ put $A := \mathcal{W}(\{A_\tau\})$ and $\tilde{A} := \widetilde{\mathcal{W}}(\{A_\tau\})$.*
Then the approximation method $\{A_\tau\}$ is applicable to the operator A if and only if both operators A and $\tilde{A}$ are invertible (in $\mathcal{L}(L^\infty)$).

4 Applications

The main results of the previous section show that we are left with the invertibility of two operators, A and $\tilde{A}$. From (10) we conclude that for the "pure" finite section method $\{A_\tau\} = \{P_\tau A P_\tau\}$ the operator $\tilde{A}$ is always of the form $P\mathring{W}_e P + Q\mathring{W}_f Q$ with Wiener functions e and f. This can be read as the direct sum of two Wiener-Hopf operators: one on im P and one on im Q. For those one has a very effective invertibility criterion which tells that $\tilde{A}$ is invertible if and only if both functions, e and f, have no zeros on the one-point compactification $\dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ of $\mathbb{R}$ and their images (which are continuous and closed curves) have winding number zero with respect to the origin O of the complex plane. So the invertibility of $\tilde{A}$ is readily verified once e and f are computed.
The invertibility of operator A is often guaranteed by "the physics" of the problem itself. So in the case of the finite section method Theorem 3.19 and Corollary 3.20 are effectively verifiable criteria for the stability/applicability of this method.

Example. The finite section method is applicable to $A = M_b \mathring{W}_a$ if and only if A and $\tilde{A} = P\mathring{W}_e P + Q\mathring{W}_f Q$ are invertible, where $e(x) = b(+\infty)a(-x)$ and $f(x) = b(-\infty)a(-x)$. $\tilde{A}$ is invertible if and only if the curves $e(\dot{\mathbb{R}}) = b(+\infty)a(\dot{\mathbb{R}})$ and $f(\dot{\mathbb{R}}) = b(-\infty)a(\dot{\mathbb{R}})$ do not contain the origin O and have winding number zero with respect to O.
So $O \notin a(\dot{\mathbb{R}})$, $wind(a, O) = 0$ and the additional condition, that $O \notin b(\overline{\mathbb{R}})$ is thus necessary and sufficient for the invertibility of $\tilde{A}$ and A and so for the applicability of the finite section method to $A = M_b \mathring{W}_a$.

4.1 An approximation method in continuous function spaces

Denote by BC the algebra $C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ of bounded continuous functions and by $C(\overline{\mathbb{R}})$ the algebra of continuous functions with limits at plus and minus infinity. So we have $C(\overline{\mathbb{R}}) = C(\mathbb{R}) \cap L^\infty(\overline{\mathbb{R}})$.

Then we can study operators of the form

$$A = I + \sum_{i=1}^{n} M_{b_i} \mathring{W}_{a_i} M_{c_i} \tag{17}$$

where $b_i \in C(\overline{\mathbb{R}})$, $c_i \in L^\infty(\overline{\mathbb{R}})$, $a_i \in FL^1$ and n is some integer. The following simple Lemma shows that A maps BC into BC.

Lemma 4.1 *Put A as in equation (17). If we take some arbitrary $\varphi \in BC$ then $A\varphi$ is contained in BC as well. But if $\varphi \notin BC$ then also $A\varphi \notin BC$.*

Proof. It remains to show that $\mathring{W}_a \varphi$ is continuous whenever $a = Fk$ with $k \in L^1$ and φ is bounded. Obviously

$$\begin{aligned} |(\mathring{W}_a\varphi)(t_1) - (\mathring{W}_a\varphi)(t_2)| &= \left| \int_{-\infty}^{\infty} (k(t_1 - s) - k(t_2 - s))\varphi(s)ds \right| \\ &\leq \|\varphi\|_\infty \int_{-\infty}^{\infty} |k(x + \varepsilon) - k(x)|\, dx \end{aligned}$$

holds with $\varepsilon = t_1 - t_2$. But the latter integral tends to zero as $\varepsilon \to 0$ since $k \in L^1$ can be approximated by piecewise constant functions with a finite number of jumps and for those the assertion is clear. ■

Again we are looking for solutions x of the equation

$$Ax = b \tag{18}$$

which is nothing but the integral equation

$$x(t) \;+\; \sum_{i=1}^{n} b_i(t) \int_{-\infty}^{\infty} k_i(t-s)c_i(s)x(s)ds \;=\; b(t), \quad (t \in \mathbb{R}) \tag{19}$$

for each right hand side b. But this time b is bounded and continuous (i.e. $b \in BC$) and so we are searching for x in BC as well.

In this continuous setting one uses an approximation method which is slightly different from what we did earlier. We are now looking for solutions $x_\tau \in BC$ of

$$(I + (A - I)P_\tau)x_\tau = b \tag{20}$$

which in our situation means

$$x_\tau \;+\; \sum_{i=1}^{n} M_{b_i} \mathring{W}_{a_i} M_{c_i} P_\tau x_\tau \;=\; b \tag{21}$$

and hope that the sequence $\{x_\tau\}$ of solutions of (20) (i.e. (21)) tells us something about the solution x of (18). Note that this time the support of x_τ is not limited to some interval. So all one changes from (18) to (20) is reducing the range of integration from $\mathbb{R}$ to $[-\tau, \tau]$:

$$x_\tau(t) \;+\; \sum_{i=1}^{n} b_i(t) \int_{-\tau}^{\tau} k_i(t-s)c_i(s)x_\tau(s)ds \;=\; b(t), \quad (t \in \mathbb{R}) \tag{22}$$

For the stability of approximation method (20) one has the following fact.

Proposition 4.2 *The sequence $\{I + (A - I)P_\tau\}$ is stable if and only if $\{P_\tau A P_\tau\}$ is so.*

Proof. First note that $I + (A - I)P_\tau$ are operators on the whole axis while we regard $P_\tau A P_\tau$ as operators on $\mathrm{im}P_\tau$ of course. Then

$$\begin{aligned} I + (A - I)P_\tau &= P_\tau \;+\; P_\tau(A - I)P_\tau \;+\; Q_\tau + \;Q_\tau(A - I)P_\tau \\ &= P_\tau A P_\tau \;+\; Q_\tau + \;Q_\tau(A - I)P_\tau \\ &= (P_\tau A P_\tau \;+\; Q_\tau)(I \;+\; Q_\tau(A - I)P_\tau) \end{aligned}$$

holds where the second factor is always invertible with $(I + Q_\tau(A-I)P_\tau)^{-1} = I - Q_\tau(A - I)P_\tau$ and so $\|(I \;+\; Q_\tau(A - I)P_\tau)^{-1}\| \le 1 + \|A - I\|$ and the first factor is invertible for $\tau > \tau_0$ with an uniformly bounded sequence of inverses if and only if $\{P_\tau A P_\tau\}$ is stable. ■

Since $I + (A - I)P_\tau$ converges $\mathcal{P}$-strongly to A and since operators A given by (17) are always contained in our algebra $\mathcal{A}$ we can use the machinery of section 3 to finally obtain an applicability criterion for the approximation method (20).

Proposition 4.3 *The approximation method (20) is applicable to the operator A given by equation (17) if and only if both A and $\tilde{A}$ are invertible operators (in $\mathcal{L}(L^\infty)$), where*

$$\tilde{A} = I \;+\; \sum_{i=1}^{n} \Big(b_i(+\infty)c_i(+\infty)P\mathring{W}_{\tilde{a}_i}P \;+\; b_i(-\infty)c_i(-\infty)Q\mathring{W}_{\tilde{a}_i}Q \Big). \tag{23}$$

Proof. Method (20) is applicable if and only if $A = \mathcal{P}\text{-}\lim(I + (A - I)P_\tau)$ is invertible in $\mathcal{L}(BC)$ and $\{I + (A - I)P_\tau\}$ is stable. This stability is (by the preceding proposition) equivalent to the stability of $\{P_\tau A P_\tau\}$ and Theorem 3.19 says that this happens if and only if both A and $\widetilde{\mathcal{W}}(\{P_\tau A P_\tau\})$ are invertible on L^∞. Of course identity $\tilde{A} = \widetilde{\mathcal{W}}(\{P_\tau A P_\tau\})$ follows immediately from (17) and (10). Finally, Lemma 4.1 shows that A is automatically invertible on BC if it is so on L^∞. ■

4.2 A more elementary proof of Proposition 4.3

Note that we presented this answer to the applicability question of method (20) as an application of our theory in section 3. It is possible to prove Proposition 4.3 directly, without using the ideas of subsection 3.7.

2^{nd} *proof of Proposition 4.3.* As seen in the first proof we are done if we can show that the sequence $\{P_\tau A P_\tau\}$ is stable if and only if A and $\tilde{A}$ are invertible operators on L^∞. The *only if* portion is Proposition 3.11 but the *if* portion can be shown directly:
So let A and $\tilde{A}$ be invertible on L^∞. Then the following equations hold.

$$\begin{aligned}
A &= I+\sum_{i=1}^{n} M_{b_i}\mathring{W}_{a_i}M_{c_i} \\
&= P\Big(I+\sum_{i=1}^{n} M_{b_i}\mathring{W}_{a_i}M_{c_i}\Big)P + Q\Big(I+\sum_{i=1}^{n} M_{b_i}\mathring{W}_{a_i}M_{c_i}\Big)Q + T_1 \qquad (T_1\in\mathcal{K}) \\
&= P\Big(I+\sum_{i=1}^{n} b_i(+\infty)c_i(+\infty)(I+M_{(\frac{b_i}{b_i(+\infty)}-1)})\mathring{W}_{a_i}(I+M_{(\frac{c_i}{c_i(+\infty)}-1)})\Big)P \\
&\quad +Q\Big(I+\sum_{i=1}^{n} b_i(-\infty)c_i(-\infty)(I+M_{(\frac{b_i}{b_i(-\infty)}-1)})\mathring{W}_{a_i}(I+M_{(\frac{c_i}{c_i(-\infty)}-1)})\Big)Q + T_1
\end{aligned}$$

Note that $M_{(\frac{f}{f(+\infty)}-1)}P$ and $M_{(\frac{f}{f(-\infty)}-1)}Q$ are in $\mathcal{K}$ for $f\in L^\infty(\overline{\mathbb{R}})$ and define symbols $s_1,\ s_2$ by $s_1 := 1+\sum_{i=1}^{n} b_i(+\infty)c_i(+\infty)a_i$ and $s_2 := 1+\sum_{i=1}^{n} b_i(-\infty)c_i(-\infty)a_i$. Then

$$A = P\mathring{W}_{s_1}P + Q\mathring{W}_{s_2}Q + T_2 \qquad (T_2\in\mathcal{K}).$$

Remember that by $\tilde{a}$ we denote the function $\tilde{a}(x)=a(-x)$. Then one easily checks

$$\tilde{A} = P\mathring{W}_{\tilde{s}_1}P + Q\mathring{W}_{\tilde{s}_2}Q.$$

Finally we introduce another operator which is

$$A' := P\mathring{W}_{s_1^{-1}}P + Q\mathring{W}_{s_2^{-1}}Q.$$

From the invertibility of $\tilde{A}$ we can conclude the invertibility of A' modulo $\mathcal{K}$. To see this let J refer to the flip operator $a\mapsto\tilde{a}$ and check the identities $J^2=I,\ JP=QJ,\ JQ=PJ$ and $J\mathring{W}_aJ=\mathring{W}_{\tilde{a}}$. Since $\tilde{A}$ is invertible also

$$\begin{aligned}
J\tilde{A}J &= JP\mathring{W}_{\tilde{s}_1}PJ + JQ\mathring{W}_{\tilde{s}_2}QJ \\
&= Q\mathring{W}_{s_1}Q + P\mathring{W}_{s_2}P \\
&= \mathring{W}_{s_1s_2}(A'+T)
\end{aligned}$$

is invertible where T is some operator in $\mathcal{K}$ (and $\mathring{W}_{s_1s_2}^{-1} = \mathring{W}_{s_1^{-1}s_2^{-1}}$).

The Wiener-Hopf operator A' can be identified with each of the operators $Q_\tau A' Q_\tau$ $(\tau > 0)$ and since $\{Q_\tau T Q_\tau\}$ is contained in the ideal $\mathcal{N}$ of sequences tending to zero with respect to the operator norm if $T \in \mathcal{K}$, the invertibility of $A' + T$ implies the invertibility of the sequence $\{Q_\tau A' Q_\tau\}$ in $\mathcal{F}$ modulo $\mathcal{N}$, i.e. its stability.

But obviously AA' and $A'A$ are of the form $I + T$ with $T \in \mathcal{K}$ and so A' differs from A^{-1} only by an operator in $\mathcal{K}$. Thus also the sequence $\{Q_\tau A^{-1} Q_\tau\}$ is stable.

The following formula can be easily checked by multiplying by $P_\tau A P_\tau$:

$$(P_\tau A P_\tau)^{-1} = P_\tau A^{-1} P_\tau - P_\tau A^{-1} Q_\tau (Q_\tau A^{-1} Q_\tau)^{-1} Q_\tau A^{-1} P_\tau$$

It tells, that if the sequence $\{Q_\tau A^{-1} Q_\tau\}$ is stable, then so is $\{P_\tau A P_\tau\}$ and we're done. ∎

4.3 An outlook to other approximation methods

Although we have restricted ourselves to the finite section method there are some other approximation methods that are "close enough" to it to handle the question of their applicability with our results about the finite section method. An example:

Instead of the finite section projectors P_τ one can consider projections P^n $(n \in \mathbb{N})$ onto spaces of polynomial splines. These splines are defined on a mesh with increasing density as n increases and have a finite support $[-\tau(n), \tau(n)]$ on the real axis with $\tau(n) \to \infty$ as $n \to \infty$.

The resulting Galerkin method $\{P^n A P^n\}_{n=1}^\infty$ is close enough to the finite section method $\{P_{\tau(n)} A P_{\tau(n)}\}_{n=1}^\infty$ to handle its applicability question as a consequence of Theorem 3.19.

References

[1] A. Böttcher and B. Silbermann: *Analysis of Toeplitz Operators*, Akademie-Verlag, Berlin, 1989 and Springer Verlag, Berlin, Heidelberg, New York 1990.

[2] A. V. Kozak: *A Local Principle in the Theory of Projection Methods*, Dokl. Akad. Nauk SSSR, 212 (1973) 6, 1287-1289 (russian).

[3] S. Prössdorf and B. Silbermann: *Numerical Analysis for Integral and Related Operator Equations*, Akademie-Verlag, Berlin, 1991 and Birkhäuser Verlag, Basel, Boston, Berlin 1991.

Authors' address:

Fakultät für Mathematik
TU Chemnitz
D - 09107 Chemnitz
Germany

Marko.Lindner@mathematik.tu-chemnitz.de
Bernd.Silbermann@mathematik.tu-chemnitz.de