Asymptotic expansions of integral functionals of vector valued ε-correlated processes

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Abstract

In the paper asymptotic expansions for second-order moments of integral functionals of a family of complex vector valued wide-sense stationary and ε-correlated random processes are considered. The asymptotic expansions are derived for $\varepsilon \to 0$ and extend the results for real valued functionals to the complex vector valued case.

1 Introduction

In this paper asymptotic expansions for second-order moments of integral functionals of the type

$$^{\varepsilon}r(\omega) = \int_{\mathcal{D}} Q(s)\, {}^{\varepsilon}f(s,\omega)\, ds, \qquad \varepsilon > 0, \tag{1}$$

are considered, where $Q(\cdot)$ is a deterministic complex $n \times n$-matrix valued function on $\mathcal{D} \subset \mathbf{R}$, $n \in \mathbf{N}$, and $({}^{\varepsilon}f(\cdot))$, $\varepsilon > 0$, denotes a family of complex vector valued wide-sense stationary and ε-correlated random processes. The given results extend the corresponding expansions for real valued processes in [7]. In the case examined here the noncommutativity of matrix multiplication has to be taken into account. In the paper vectors are assumed to be columns and * denotes the adjoint of a vector or matrix.

Analogously to [7] we will suppose

Assumption 1

1. $({}^{\varepsilon}f(\cdot)), \varepsilon > 0$, *are complex vector valued wide-sense stationary processes with matrix correlation functions*

$$\mathbf{E}\{{}^{\varepsilon}f(s)\, {}^{\varepsilon}f^*(t)\} = {}^{\varepsilon}R(t-s);$$

2. $({}^{\varepsilon}f(\cdot)), \varepsilon > 0$, *are centered processes, i.e.* $\mathbf{E}\{{}^{\varepsilon}f(s)\} = 0$;

3. $(\,^{\varepsilon}f(\cdot)), \varepsilon > 0$, *are ε-correlated processes, i.e.* $^{\varepsilon}R(s) = 0$ *for* $|s| \geq \varepsilon$;

4. *the matrix correlation functions* $^{\varepsilon}R(\cdot), \varepsilon > 0$, *are generated by a matrix correlation function* $R(\cdot)$ *of a complex vector valued 1–correlated wide-sense stationary process, i.e.*

$$^{\varepsilon}R(s) = R\left(\frac{s}{\varepsilon}\right),\ s \in \mathbf{R},\ \varepsilon > 0;$$

5. *the matrix correlation function* $R(\cdot)$ *is continuous on* $\mathbf{R}$, *hence the processes* $(\,^{\varepsilon}f(\cdot)), \varepsilon > 0$, *are continuous in mean square on* $\mathbf{R}$.

The integral in (1) is assumed to exist in mean square sense, under weak conditions it coincides a.s. with the pathwise integral. From condition 2 of Assumption 1 it follows that the random vectors $^{\varepsilon}r(\cdot), \varepsilon > 0$, are centered, i.e. $\mathbf{E}\{\,^{\varepsilon}r\} = 0$. Also in the vector valued case correlation moments of the processes play a crucial role.

Definition 1 *Let* $R(\cdot) = (R_{ij}(\cdot))_{i,j=1,\ldots,n}$ *be a continuous matrix correlation function of a complex vector valued wide-sense stationary process (not necessarily 1–correlated) and* $j \in \mathbf{N}_0 = \{0, 1, 2, \ldots\}$ *with*

$$\int_{-\infty}^{\infty} |s|^j |R_{ij}(s)|\, ds < \infty, \qquad i, j = 1, \ldots, n.$$

Then the matrix

$$\mu_j = \int_{-\infty}^{\infty} s^j R(s)\, ds$$

is called the correlation moment matrix of j-th order *of the correlation function or the random process and the matrix*

$$\nu_j = \int_{-\infty}^{\infty} |s|^j R(s)\, ds$$

is called the absolute correlation moment matrix of j-th order.

With the notation

$$\mu_j^+ = \nu_j^+ = \int_0^{\infty} s^j R(s)\, ds$$

we find from the relation $R(s) = R^*(-s)$

$$\begin{aligned} \mu_j &= \nu_j^+ + (-1)^j [\nu_j^+]^*, \\ \nu_j &= \nu_j^+ + [\nu_j^+]^*, \end{aligned}$$

hence it also holds

$$\mu_j^* = (-1)^j \mu_j, \qquad \nu_j^* = \nu_j.$$

In the following treatment the function $Q(\cdot)$ satisfies

Assumption 2 *The deterministic complex matrix valued function $Q(\cdot)$ is $N-$ times continuously differentiable on $\mathcal{D}$, $N \in \mathbf{N}_0$, $Q^{(N)}(\cdot)$ is absolutely continuous on $\mathcal{D}$ and the derivatives of $Q(\cdot)$ up to the order $(N+1)$ belong to the space $L^2(\mathcal{D}) \cap L^1(\mathcal{D})$ of matrix valued functions.*

For such functions the Taylor expansion formula with exact integral representation of the remainder (cf. e.g. [2], Sec. 5.4) remains valid, for the integration by parts formula for absolutely continuous function see e.g. [3], Ch. IX, §7.

2 Expansion of covariance matrices

2.1 The case $\mathcal{D} = \mathbf{R}$

For functionals of the type

$$ {}^{\varepsilon}r = \int_{-\infty}^{\infty} Q(s)\, {}^{\varepsilon}f(s)\, ds $$

we get

$$ \begin{aligned} \mathbf{E}\{ {}^{\varepsilon}r\, {}^{\varepsilon}r^* \} &= \varepsilon \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(t-\varepsilon u) R(u) Q^*(t)\, dt\, du \\ &= \varepsilon \int_{-1}^{1} \int_{-\infty}^{\infty} Q(t-\varepsilon u) R(u) Q^*(t)\, dt\, du. \end{aligned} $$

From the Taylor expansion

$$ Q(t-\varepsilon u) = \sum_{j=0}^{N} Q^{(j)}(t) \frac{(-\varepsilon u)^j}{j!} + \tilde{\rho}_{N+1}(t,u,\varepsilon) $$

with

$$ \tilde{\rho}_{N+1}(t,u,\varepsilon) = \frac{1}{N!} \int_{t}^{t-\varepsilon u} Q^{(N+1)}(v)(t-\varepsilon u - v)^N\, dv $$

we find

Theorem 1 *Let $({}^{\varepsilon}f(s);\, s \in \mathbf{R}), \varepsilon > 0$, be a family of $\mathbf{C}^n$-valued random processes satisfying Assumption 1 and $Q(\cdot)$ a matrix valued function satisfying Assumption 2 on $\mathcal{D} = \mathbf{R}$. Then it holds*

$$ \mathbf{E}\{ {}^{\varepsilon}r\, {}^{\varepsilon}r^* \} = \sum_{j=0}^{N} \frac{\varepsilon^{j+1}}{j!} q_j + \rho_{N+1}(\varepsilon) $$

with

$$ q_j = \int_{-\infty}^{\infty} Q^{(j)}(t) \mu_j^* Q^*(t)\, dt $$

and the remainder term

$$ \rho_{N+1}(\varepsilon) = \frac{\varepsilon}{N!} \int_{-1}^{1} \int_{-\infty}^{\infty} \int_{t}^{t-\varepsilon u} Q^{(N+1)}(v)(t-\varepsilon u - v)^N R(u) Q^*(t)\, dv\, dt\, du. $$

Integration by parts with respect to the quantities q_j leads for $k = 0, \ldots, j$ to

$$q_j = (-1)^k \int_{-\infty}^{\infty} Q^{(j-k)}(t) \mu_j^* [Q^{(k)}(t)]^* \, dt,$$

especially

$$\begin{aligned} q_{2k} &= (-1)^k \int_{-\infty}^{\infty} Q^{(k)}(t) \mu_{2k}^* [Q^{(k)}(t)]^* \, dt, \qquad k = 0, \ldots, \left[\frac{N}{2}\right], \\ q_{2k+1} &= (-1)^k \int_{-\infty}^{\infty} Q^{(k+1)}(t) \mu_{2k+1}^* [Q^{(k)}(t)]^* \, dt, \quad k = 0, \ldots, \left[\frac{N-1}{2}\right]. \end{aligned}$$

2.2 The case $\mathcal{D} = \mathbf{R}_+$

For functionals of the type

$${}^{\varepsilon}r = \int_0^{\infty} Q(s) \, {}^{\varepsilon}f(s) \, ds$$

we get

$$\begin{aligned} \mathbf{E}\{ {}^{\varepsilon}r \, {}^{\varepsilon}r^* \} &= \int_0^{\infty} \int_0^{\infty} Q(s) R\left(\frac{t-s}{\varepsilon}\right) Q^*(t) \, ds \, dt \\ &= \varepsilon \int_{-\infty}^{0} \int_0^{\infty} Q(t - \varepsilon u) R(u) Q^*(t) \, dt \, du \\ &\quad + \varepsilon \int_0^{\infty} \int_{\varepsilon u}^{\infty} Q(t - \varepsilon u) R(u) Q^*(t) \, dt \, du \\ &= \varepsilon \int_{-\infty}^{0} \int_0^{\infty} Q(t - \varepsilon u) R(u) Q^*(t) \, dt \, du \\ &\quad + \varepsilon \int_0^{\infty} \int_0^{\infty} Q(t) R(u) Q^*(t + \varepsilon u) \, dt \, du \\ &= \varepsilon \int_{-1}^{0} \int_0^{\infty} Q(t + \varepsilon |u|) R(u) Q^*(t) \, dt \, du \\ &\quad + \varepsilon \int_0^{1} \int_0^{\infty} Q(t) R(u) Q^*(t + \varepsilon |u|) \, dt \, du \\ &= \left[\varepsilon \int_0^{1} \int_0^{\infty} Q(t) R(u) Q^*(t + \varepsilon u) \, dt \, du \right]^* \\ &\quad + \varepsilon \int_0^{1} \int_0^{\infty} Q(t) R(u) Q^*(t + \varepsilon u) \, dt \, du. \end{aligned}$$

Taylor expansion gives

$$Q^*(t + \varepsilon u) = \sum_{j=0}^{N} [Q^{(j)}(t)]^* \frac{(\varepsilon u)^j}{j!} + \tilde{\rho}_{N+1}(t, u, \varepsilon)$$

with

$$\tilde{\rho}_{N+1}(t, u, \varepsilon) = \frac{1}{N!} \int_t^{t+\varepsilon u} [Q^{(N+1)}(v)]^* (t + \varepsilon u - v)^N \, dv.$$

Theorem 2 *Let $({}^{\varepsilon}f(s);\, s \in \mathbf{R}_+), \varepsilon > 0$, be a family of $\mathbf{C}^n$-valued random processes satisfying Assumption 1 and $Q(\cdot)$ a matrix valued function satisfying Assumption 2 on $\mathcal{D} = \mathbf{R}_+$. Then it holds*

$$\mathbf{E}\{{}^{\varepsilon}r\, {}^{\varepsilon}r^*\} \;=\; \sum_{j=0}^{N} \frac{\varepsilon^{j+1}}{j!}\{q_j + q_j^*\} + \rho_{N+1}(\varepsilon)$$

with

$$q_j \;=\; \int_0^\infty Q(t)\nu_j^+[Q^{(j)}(t)]^*\, dt$$

and the remainder term

$$\begin{aligned}\rho_{N+1}(\varepsilon) \;=\; & \frac{\varepsilon}{N!}\Bigg\{\int_0^1\int_0^\infty\int_t^{t+\varepsilon u} Q(t)(t+\varepsilon u - v)^N R(u)[Q^{(N+1)}(v)]^* dv\, dt du \\ & + \left[\int_0^1\int_0^\infty\int_t^{t+\varepsilon u} Q(t)(t+\varepsilon u - v)^N R(u)[Q^{(N+1)}(v)]^* dv\, dt du\right]^*\Bigg\}.\end{aligned}$$

Integration by parts with respect to the quantities q_j leads for $k = 1, \ldots, j$ to

$$q_j = \sum_{l=0}^{k-1}(-1)^{l+1}Q^{(l)}(0)\nu_j^+[Q^{(j-1-l)}(0)]^* + (-1)^k\int_0^\infty Q^{(k)}(t)\nu_j^+[Q^{(j-k)}(t)]^*\, dt,$$

especially

$$\begin{aligned} q_{2k} \;=\; & \sum_{l=0}^{k-1}(-1)^{l+1}Q^{(l)}(0)\nu_{2k}^+[Q^{(2k-1-l)}(0)]^* \\ & + (-1)^k\int_0^\infty Q^{(k)}(t)\nu_{2k}^+[Q^{(k)}(t)]^*\, dt, \qquad k = 0, \ldots, \left[\frac{N}{2}\right], \\ q_{2k+1} \;=\; & \sum_{l=0}^{k-1}(-1)^{l+1}Q^{(l)}(0)\nu_{2k+1}^+[Q^{(2k-l)}(0)]^* + \\ & + (-1)^k\int_0^\infty Q^{(k)}(t)\nu_{2k+1}^+[Q^{(k+1)}(t)]^*\, dt, \quad k = 0, \ldots, \left[\frac{N-1}{2}\right]. \end{aligned}$$

2.3 The case $\mathcal{D} = [a, b]$

For functionals of the type

$${}^{\varepsilon}r = \int_a^b Q(s)\, {}^{\varepsilon}f(s)\, ds$$

with finite values $a, b \in \mathbf{R}$, $b - a \geq \varepsilon$, we get

$$\begin{aligned}
\mathbf{E}\{{}^{\varepsilon}r\,{}^{\varepsilon}r^*\} &= \int_a^b \int_a^b Q(s) R\left(\frac{t-s}{\varepsilon}\right) Q^*(t)\, ds\, dt \\
&= \varepsilon \int_{\frac{a-b}{\varepsilon}}^0 \int_a^{b+\varepsilon u} Q(t-\varepsilon u) R(u) Q^*(t)\, dt\, du + \\
&\quad + \varepsilon \int_0^{\frac{b-a}{\varepsilon}} \int_{a+\varepsilon u}^b Q(t-\varepsilon u) R(u) Q^*(t)\, dt\, du \\
&= \varepsilon \int_{-1}^0 \int_a^{b-\varepsilon|u|} Q(t+\varepsilon|u|) R(u) Q^*(t)\, dt\, du + \\
&\quad + \varepsilon \int_0^1 \int_a^{b-\varepsilon|u|} Q(t) R(u) Q^*(t+\varepsilon|u|)\, dt\, du \\
&= \left[\varepsilon \int_0^1 \int_a^{b-\varepsilon u} Q(t) R(u) Q^*(t+\varepsilon u)\, dt\, du\right]^* \\
&\quad + \varepsilon \int_0^1 \int_a^{b-\varepsilon u} Q(t) R(u) Q^*(t+\varepsilon u)\, dt\, du.
\end{aligned}$$

Defining for $\varepsilon \geq 0, 0 \leq u \leq 1$ the matrix valued function

$$\phi(\varepsilon, u) := \int_a^{b-\varepsilon u} Q(t) R(u) Q^*(t+\varepsilon u)\, dt$$

we find from Assumption 2

$$\phi(\varepsilon, u) = \sum_{j=0}^N \frac{\varepsilon^j}{j!} \frac{\partial^j \phi(0,u)}{\partial \varepsilon^j} + \frac{1}{N!} \int_0^\varepsilon \frac{\partial^{N+1}\phi(v,u)}{\partial v^{N+1}} (\varepsilon - v)^N\, dv.$$

Calculation of the matrices $\frac{\partial^j \phi(0,u)}{\partial \varepsilon^j}$, $0 \leq u \leq 1$, gives

$$\begin{aligned}
\phi(\varepsilon, u) &= \int_a^{b-\varepsilon u} Q(t) R(u) Q^*(t+\varepsilon u)\, dt, \\
\phi(0, u) &= \int_a^b Q(t) R(u) Q^*(t)\, dt, \\
\frac{\partial \phi(\varepsilon, u)}{\partial \varepsilon} &= \int_a^{b-\varepsilon u} Q(t) u R(u) [Q'(t+\varepsilon u)]^*\, dt - Q(b-\varepsilon u) u R(u) Q^*(b), \\
\frac{\partial \phi(0, u)}{\partial \varepsilon} &= \int_a^b Q(t) u R(u) [Q'(t)]^*\, dt - Q(b) u R(u) Q^*(b), \\
\frac{\partial^2 \phi(\varepsilon, u)}{\partial \varepsilon^2} &= \int_a^{b-\varepsilon u} Q(t) u^2 R(u) [Q''(t+\varepsilon u)]^*\, dt - Q(b-\varepsilon u) u^2 R(u) [Q'(b)]^* \\
&\quad + Q'(b-\varepsilon u) u^2 R(u) Q^*(b), \\
\frac{\partial^2 \phi(0, u)}{\partial \varepsilon^2} &= \int_a^b Q(t) u^2 R(u) [Q''(t)]^*\, dt - Q(b) u^2 R(u) [Q'(b)]^* \\
&\quad + Q'(b) u^2 R(u) Q^*(b),
\end{aligned}$$

in general for $j = 0, 1, \ldots, N+1$

$$\begin{aligned}
\frac{\partial^j \phi(\varepsilon, u)}{\partial \varepsilon^j} &= \int_a^{b-\varepsilon u} Q(t) u^j R(u) [Q^{(j)}(t+\varepsilon u)]^* \, dt \\
&\quad - \sum_{l=0}^{j-1} (-1)^l Q^{(l)}(b-\varepsilon u) u^j R(u) [Q^{(j-1-l)}(b)]^*, \\
\frac{\partial^j \phi(0, u)}{\partial \varepsilon^j} &= \int_a^{b} Q(t) u^j R(u) [Q^{(j)}(t)]^* \, dt \\
&\quad - \sum_{l=0}^{j-1} (-1)^l Q^{(l)}(b) u^j R(u) [Q^{(j-1-l)}(b)]^*.
\end{aligned}$$

Theorem 3 *Let* $({}^\varepsilon f(s);\ s \in [a,b]), \varepsilon > 0$, *be a family of* $\mathbf{C}^n$*-valued random processes satisfying Assumption 1 and* $Q(\cdot)$ *a matrix valued function satisfying Assumption 2 on* $\mathcal{D} = [a,b]$. *Then it holds*

$$\mathbf{E}\{{}^\varepsilon r\, {}^\varepsilon r^*\} = \sum_{j=0}^{N} \frac{\varepsilon^{j+1}}{j!} \{q_j + q_j^*\} + \rho_{N+1}(\varepsilon)$$

with

$$q_j = \int_a^b Q(t) \nu_j^+ [Q^{(j)}(t)]^* \, dt - \sum_{l=0}^{j-1} (-1)^l Q^{(l)}(b) \nu_j^+ [Q^{(j-1-l)}(b)]^*$$

and the remainder term

$$\begin{aligned}
\rho_{N+1}(\varepsilon) &= \frac{\varepsilon}{N!} \left\{ \int_0^1 \int_0^\varepsilon \frac{\partial^{N+1} \phi(v,u)}{\partial v^{N+1}} (\varepsilon - v)^N \, dv \, du \right. \\
&\qquad \left. + \left[\int_0^1 \int_0^\varepsilon \frac{\partial^{N+1} \phi(v,u)}{\partial v^{N+1}} (\varepsilon - v)^N \, dv \, du \right]^* \right\}.
\end{aligned}$$

3 Expansion of matrix correlation functions

Considering the complex vector valued wide-sense stationary processes

$${}^\varepsilon g(t) = \int_{-\infty}^{t} Q(t-s)\, {}^\varepsilon f(s) \, ds = \int_0^\infty Q(u)\, {}^\varepsilon f(t-u) \, du, \qquad \varepsilon > 0,$$

its matrix correlation functions can be written as

$$\begin{aligned}
{}^\varepsilon R_{gg}(\tau) &= \mathbf{E}\{{}^\varepsilon g(t)\, {}^\varepsilon g^*(t+\tau)\} \\
&= \int_{-\infty}^{t} \int_{-\infty}^{t+\tau} Q(t-s_1) \mathbf{E}\{{}^\varepsilon f(s_1)\, {}^\varepsilon f^*(s_2)\} Q^*(t+\tau-s_2) \, ds_2 \, ds_1 \\
&= \int_0^\infty \int_0^\infty Q(u_1)\, {}^\varepsilon R_{ff}(\tau + u_1 - u_2) Q^*(u_2) \, du_2 \, du_1
\end{aligned}$$

and it can be obtained

$$
\begin{aligned}
{}^{\varepsilon}R_{gg}(\tau) &= \int_{-\infty}^{0}\int_{-v_1}^{\infty} Q(v_1+v_2)\,{}^{\varepsilon}R_{ff}(\tau+v_1)Q^*(v_2)\,dv_2\,dv_1 + \\
&\quad + \int_{0}^{\infty}\int_{0}^{\infty} Q(v_1+v_2)\,{}^{\varepsilon}R_{ff}(\tau+v_1)Q^*(v_2)\,dv_2\,dv_1 \\
&= \int_{-\infty}^{0}\int_{0}^{\infty} Q(v_2)\,{}^{\varepsilon}R_{ff}(\tau+v_1)Q^*(v_2-v_1)\,dv_2\,dv_1 + \\
&\quad + \int_{0}^{\infty}\int_{0}^{\infty} Q(v_1+v_2)\,{}^{\varepsilon}R_{ff}(\tau+v_1)Q^*(v_2)\,dv_2\,dv_1 \\
&= \int_{-\infty}^{\infty}\int_{0}^{\infty} Q(v_2+v_1^+)\,{}^{\varepsilon}R_{ff}(\tau+v_1)Q^*(v_2+v_1^-)\,dv_2\,dv_1 \\
&= \int_{-\infty}^{\infty}\int_{0}^{\infty} Q(u+(v-\tau)^+)\,{}^{\varepsilon}R_{ff}(v)Q^*(u+(v-\tau)^-)\,du\,dv \\
&= \int_{-\infty}^{\infty}\int_{0}^{\infty} Q(u+(v-\tau)^+)R\left(\frac{v}{\varepsilon}\right)Q^*(u+(v-\tau)^-)\,du\,dv \\
&= \varepsilon\int_{-1}^{1}\int_{0}^{\infty} Q(u+(\varepsilon v-\tau)^+)R(v)Q^*(u+(\varepsilon v-\tau)^-)\,du\,dv.
\end{aligned}
$$

Now, Taylor expansion of the matrix function $Q(\cdot)$ in the neighbourhoods of the points $u+|\tau|$ is considered. In the case $\tau=0$ it yields

$$
\begin{aligned}
{}^{\varepsilon}R_{gg}(0) &= \varepsilon\int_{-1}^{0}\int_{0}^{\infty} Q(u)R(v)Q^*(u+|\varepsilon v|)\,du\,dv \\
&\quad +\varepsilon\int_{0}^{1}\int_{0}^{\infty} Q(u+|\varepsilon v|)R(v)Q^*(u)\,du\,dv \\
&= +\left[\varepsilon\int_{0}^{1}\int_{0}^{\infty} Q(u+\varepsilon v)R(v)Q^*(u)\,du\,dv\right]^* \\
&\quad +\varepsilon\int_{0}^{1}\int_{0}^{\infty} Q(u+\varepsilon v)R(v)Q^*(u)\,du\,dv \\
&= \sum_{j=0}^{N}\frac{\varepsilon^{j+1}}{j!}\left\{\left[\int_{0}^{1}\int_{0}^{\infty} Q^{(j)}(u)v^jR(v)Q^*(u)\,du\,dv\right]^*\right. \\
&\quad \left.+\int_{0}^{1}\int_{0}^{\infty} Q^{(j)}(u)v^jR(v)Q^*(u)\,du\,dv\right\}+\rho_{N+1}(\varepsilon,0) \\
&= \sum_{j=0}^{N}\frac{\varepsilon^{j+1}}{j!}\left\{q_j+q_j^*\right\}+\rho_{N+1}(\varepsilon,0)
\end{aligned}
$$

with

$$
q_j=\int_{0}^{\infty} Q^{(j)}(u)\nu_j^+Q^*(u)\,du, \quad j=0,\dots,N,
$$

and the remainder term $\rho_{N+1}(\varepsilon,0)$.
For $\tau>0$ we obtain

$$
{}^{\varepsilon}R_{gg}(\tau) \;=\; \varepsilon\int_{-1}^{\frac{\tau}{\varepsilon}\wedge 1}\int_{0}^{\infty} Q(u)R(v)Q^*(u+\tau-\varepsilon v)\,du\,dv
$$

$$+\varepsilon \int_{\frac{\tau}{\varepsilon}\wedge 1}^{1} \int_0^\infty Q(u+\varepsilon v-\tau)R(v)Q^*(u)\,du\,dv,$$

where $a \wedge b := \min(a,b)$. Taylor expansion in the neighbourhoods of the points $u+\tau$ results in

$$\begin{aligned}
Q^*(u+\tau-\varepsilon v) &= \sum_{j=0}^N [Q^{(j)}(u+\tau)]^* \frac{(-\varepsilon v)^j}{j!} + \tilde\rho^1_{N+1}(u,v,\varepsilon,\tau),\\
Q(u+\varepsilon v-\tau) &= \sum_{j=0}^N Q^{(j)}(u+\tau)\frac{\varepsilon^j}{j!}\left(v-\frac{2\tau}{\varepsilon}\right)^j + \tilde\rho^2_{N+1}(u,v,\varepsilon,\tau),
\end{aligned}$$

hence

$$\begin{aligned}
{}^\varepsilon R_{gg}(\tau) &= \sum_{j=0}^N \frac{\varepsilon^{j+1}}{j!}\left\{\int_{-1}^{\frac{\tau}{\varepsilon}\wedge 1}\int_0^\infty Q(u)(-v)^j R(v)[Q^{(j)}(u+\tau)]^*\,du\,dv\right.\\
&\quad \left. + \int_{\frac{\tau}{\varepsilon}\wedge 1}^{1}\int_0^\infty Q^{(j)}(u+\tau)\left(v-\frac{2\tau}{\varepsilon}\right)^j R(v)Q^*(u)\,du\,dv\right\} + \rho_{N+1}(\varepsilon,\tau)\\
&= \sum_{j=0}^N \frac{\varepsilon^{j+1}}{j!}\left\{\int_{-1}^{1}\int_0^\infty Q(u)(-v)^j R(v)[Q^{(j)}(u+\tau)]^*\,du\,dv\right.\\
&\quad - \int_{\frac{\tau}{\varepsilon}\wedge 1}^{1}\int_0^\infty Q(u)(-v)^j R(v)[Q^{(j)}(u+\tau)]^*\,du\,dv\\
&\quad \left. + \int_{\frac{\tau}{\varepsilon}\wedge 1}^{1}\int_0^\infty Q^{(j)}(u+\tau)\left(v-\frac{2\tau}{\varepsilon}\right)^j R(v)Q^*(u)\,du\,dv\right\} + \rho_{N+1}(\varepsilon,\tau).
\end{aligned}$$

Using the relation $(-1)^j\mu_j = \mu_j^*$ we conclude

Theorem 4 *Let $({}^\varepsilon f(s);\ s\in\mathbf{R}), \varepsilon>0$, be a family of $\mathbf{C}^n$-valued random processes satisfying Assumption 1 and $Q(\cdot)$ a matrix valued function satisfying Assumption 2 on $\mathcal{D}=\mathbf{R}_+$. Then it holds for $\tau>0$*

$${}^\varepsilon R_{gg}(\tau) = \sum_{j=0}^N \frac{\varepsilon^{j+1}}{j!}\left\{q_j(\tau) + \mathbf{1}_{\{0<\tau<\varepsilon\}}(\tau)c_j(\tau)\right\} + \rho_{N+1}(\varepsilon,\tau)$$

with

$$\begin{aligned}
q_j(\tau) &= \int_0^\infty Q(u)\mu_j^*[Q^{(j)}(u+\tau)]^*\,du, & j=0,\ldots,N,\\
c_j(\tau) &= \int_0^\infty Q^{(j)}(u+\tau)\int_{\frac{\tau}{\varepsilon}}^1\left(v-\frac{2\tau}{\varepsilon}\right)^j R(v)\,dv\,Q^*(u)\,du &\\
&\quad + (-1)^{j+1}\int_0^\infty Q(u)\int_{\frac{\tau}{\varepsilon}}^1 v^j R(v)\,dv\,[Q^{(j)}(u+\tau)]^*\,du, & j=0,\ldots,N,
\end{aligned}$$

$$
\begin{aligned}
\rho_{N+1}(\varepsilon,\tau) &= \varepsilon\left\{\int_{-1}^{\frac{\tau}{\varepsilon}\wedge 1}\int_0^\infty Q(u)R(v)\tilde{\rho}^1_{N+1}(u,v,\varepsilon,\tau)\,du\,dv\right.\\
&\qquad\left.+\int_{\frac{\tau}{\varepsilon}\wedge 1}^{1}\int_0^\infty \tilde{\rho}^2_{N+1}(u,v,\varepsilon,\tau)R(v)Q^*(u)\,du\,dv\right\},\\
\tilde{\rho}^1_{N+1}(u,v,\varepsilon,\tau) &= \frac{1}{N!}\int_{u+\tau}^{u+\tau-\varepsilon v}[Q^{(N+1)}(w)]^*(u+\tau-\varepsilon v-w)^N\,dw,\\
\tilde{\rho}^2_{N+1}(u,v,\varepsilon,\tau) &= \frac{1}{N!}\int_{u+\tau}^{u+\varepsilon v-\tau}Q^{(N+1)}(w)(u+\varepsilon v-\tau-w)^N\,dw.
\end{aligned}
$$

For $\tau = 0$ it holds

$$
{}^\varepsilon R_{gg}(0) = \sum_{j=0}^{N}\frac{\varepsilon^{j+1}}{j!}\left\{q_j+q_j^*\right\}+\rho_{N+1}(\varepsilon,0)
$$

with

$$
q_j = \int_0^\infty Q^{(j)}(u)\nu_j^+Q^*(u)\,du, \quad j=0,\dots,N,
$$

and the remainder term

$$
\begin{aligned}
\rho_{N+1}(\varepsilon,0) &= \frac{\varepsilon}{N!}\left\{\left[\int_0^1\int_0^\infty\int_u^{u+\varepsilon v}Q(u)R(v)Q^{N+1}(w)(u+\varepsilon v-w)^N\,dw\,du\,dv\right]^*\right.\\
&\qquad\left.+\int_0^1\int_0^\infty\int_u^{u+\varepsilon v}Q(u)R(v)Q^{N+1}(w)(u+\varepsilon v-w)^N\,dw\,du\,dv\right\}.
\end{aligned}
$$

The values for $\tau < 0$ can be calculated from the relation ${}^\varepsilon R_{gg}(\tau) = {}^\varepsilon R^*_{gg}(-\tau)$.

Example Let us consider a system of n ordinary differential equations of first order with constant coefficients and an ε-correlated wide-sense stationary inhomogeneous term

$$
{}^\varepsilon\dot{x} = A\,{}^\varepsilon x + {}^\varepsilon f.
$$

The matrix A is assumed to be stable (i.e. all eigenvalues have negative real parts) and diagonalizable, i.e. $A = V\Lambda V^{-1}$ with $\Lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_n)$. Then a stationary solution

$$
{}^\varepsilon x(t) = \int_{-\infty}^t e^{A(t-s)}\,{}^\varepsilon f(s)\,ds
$$

exists (see e.g. [1], [4], [6]). For fixed τ the asymptotic expansions can be calculated as follows.

For $\tau = 0$ it holds

$$
R_{gg}(0) = \sum_{j=0}^{N}\frac{\varepsilon^{j+1}}{j!}\{q_j+q_j^*\}+o(\varepsilon^{N+1})
$$

with

$$
\begin{aligned}
q_j &= V\Lambda^j \int_0^\infty e^{\Lambda u} B_j e^{\Lambda^* u} du \; V^* = -V \left(\frac{b_{jkl}\lambda_k^j}{\lambda_k + \overline{\lambda_l}} \right)_{k,l=1,2,\ldots,n} V^*, \\
B_j &= (b_{jkl})_{k,l=1,2,\ldots,n} = V^{-1}\nu_j^+ [V^{-1}]^*,
\end{aligned}
$$

and for $\tau > 0$ it results

$$
R_{gg}(\tau) = \sum_{j=0}^{N} \frac{\varepsilon^{j+1}}{j!} q_j(\tau) + o(\varepsilon^{N+1})
$$

with

$$
\begin{aligned}
q_j(\tau) &= V \int_0^\infty e^{\Lambda u} C_j e^{\Lambda^* u} du \, [\Lambda^j]^* e^{\Lambda^* \tau} V^* = -V \left(\frac{c_{jkl} e^{\overline{\lambda_l}\tau} \overline{\lambda_l}^j}{\lambda_k + \overline{\lambda_l}} \right)_{k,l=1,2,\ldots,n} V^*, \\
C_j &= (c_{jkl})_{k,l=1,2,\ldots,n} = V^{-1}\mu_j^* [V^{-1}]^*.
\end{aligned}
$$

4 Expansion for scalar complex processes

In this section asymptotic expansions as $\varepsilon \to 0$ for the special case of complex valued wide-sense stationary processes and scalar complex valued kernel functions are given. Here, the correlation functions are defined by

$$
{}^\varepsilon R(s) = \mathbf{E}\{ {}^\varepsilon f(t) \overline{{}^\varepsilon f(t+s)} \} = R\left(\frac{s}{\varepsilon}\right), \quad \varepsilon > 0,
$$

according to condition 4 of Assumption 1. Now, for the correlation moments we get

$$
\begin{aligned}
\mu_j &= \int_{-\infty}^{\infty} s^j R(s)\, ds = \begin{cases} 2\int_0^\infty s^j \mathrm{Re} R(s)\, ds & \text{for } j \text{ even}, \\ 2i \int_0^\infty s^j \mathrm{Im} R(s)\, ds & \text{for } j \text{ odd}, \end{cases} \\
\nu_j &= \int_{-\infty}^{\infty} |s|^j R(s)\, ds = 2\int_0^\infty s^j \mathrm{Re} R(s)\, ds, \\
\nu_j^+ &= \int_0^\infty s^j R(s)\, ds,
\end{aligned}
$$

hence $\overline{\mu_j} = (-1)^j \mu_j, \overline{\nu_j} = \nu_j$.
With analogous assumptions as in the real or vector valued case the following asymptotic expansions are valid.

- For ${}^{\varepsilon}r = \int\limits_{-\infty}^{\infty} Q(s)\,{}^{\varepsilon}f(s)\,ds$ it holds

$$\begin{aligned}\mathbf{E}\{{}^{\varepsilon}r\,\overline{{}^{\varepsilon}r}\} &= \mathbf{E}\{|{}^{\varepsilon}r|^2\} = \sum_{j=0}^{N} \frac{\varepsilon^{j+1}}{j!}\overline{\mu_j}q_j + \rho_{N+1}(\varepsilon) \\ \text{with} \qquad q_j &= \int\limits_{-\infty}^{\infty} Q^{(j)}(t)\overline{Q(t)}\,dt.\end{aligned}$$

- For ${}^{\varepsilon}r = \int\limits_{0}^{\infty} Q(s)\,{}^{\varepsilon}f(s)\,ds$ it holds

$$\begin{aligned}\mathbf{E}\{{}^{\varepsilon}r\,\overline{{}^{\varepsilon}r}\} &= \sum_{j=0}^{N} \frac{\varepsilon^{j+1}}{j!}\left\{\nu_j^+ q_j + \overline{\nu_j^+}\,\overline{q_j}\right\} + \rho_{N+1}(\varepsilon) \\ \text{with} \qquad q_j &= \int_0^{\infty} Q(t)\overline{Q^{(j)}(t)}\,dt.\end{aligned}$$

- For ${}^{\varepsilon}r = \int\limits_{a}^{b} Q(s)\,{}^{\varepsilon}f(s)\,ds$ it holds

$$\begin{aligned}\mathbf{E}\{{}^{\varepsilon}r\,\overline{{}^{\varepsilon}r}\} &= \sum_{j=0}^{N} \frac{\varepsilon^{j+1}}{j!}\left\{\nu_j^+ q_j + \overline{\nu_j^+}\,\overline{q_j}\right\} + \rho_{N+1}(\varepsilon) \\ \text{with} \qquad q_j &= \int_a^{b} Q(t)\overline{Q^{(j)}(t)}\,dt - \sum_{l=0}^{j-1}(-1)^l Q^{(l)}(b)\overline{Q^{(j-1-l)}(b)}.\end{aligned}$$

- For the correlation functions ${}^{\varepsilon}R_{gg}(\tau) = \mathbf{E}\{{}^{\varepsilon}g(t)\,\overline{{}^{\varepsilon}g(t+\tau)}\}$ of the stationary processes

$$ {}^{\varepsilon}g(t) = \int_{-\infty}^{t} Q(t-s)\,{}^{\varepsilon}f(s)\,ds$$

it holds for $\tau = 0$

$$\begin{aligned}{}^{\varepsilon}R_{gg}(0) &= \sum_{j=0}^{N} \frac{\varepsilon^{j+1}}{j!}\{\nu_j^+ q_j + \overline{\nu_j^+}\,\overline{q_j}\} + \rho_{N+1}(\varepsilon, 0) \\ \text{with} \qquad q_j &= \int_0^{\infty} Q^{(j)}(u)\overline{Q(u)}\,du,\end{aligned}$$

for $\tau > 0$ it results

$$\begin{aligned}{}^{\varepsilon}R_{gg}(\tau) &= \sum_{j=0}^{N} \frac{\varepsilon^{j+1}}{j!}\left\{\overline{\mu_j}q_j(\tau) + \mathbf{1}_{\{0<\tau<\varepsilon\}}(\tau)c_j(\tau)\right\} + \rho_{N+1}(\varepsilon, \tau) \\ \text{with } q_j(\tau) &= \int_0^{\infty} Q(u)\overline{Q^{(j)}(u+\tau)}\,du, \\ c_j(\tau) &= \overline{q_j(\tau)}\int_{\frac{\tau}{\varepsilon}}^{1}\left(v - \frac{2\tau}{\varepsilon}\right)^j R(v)dv + (-1)^{j+1}q_j(\tau)\int_{\frac{\tau}{\varepsilon}}^{1} v^j R(v)dv,\end{aligned}$$

and the values for $\tau < 0$ can be found from the relation ${}^{\varepsilon}R_{gg}(\tau) = \overline{{}^{\varepsilon}R_{gg}(-\tau)}$.

5 Conclusion

The results of [7] concerning asymptotic expansions for $\varepsilon \to 0$ of variances or correlation functions of real valued integral functionals involving ε-correlated wide-sense stationary random processes are extended to the case of complex vector valued random functions. These expansions can be used in the investigation of random vibrations, e.g. given by a system of ordinary differential equations with a random inhomogeneous term. In the paper also the corresponding asymptotic expansions for scalar complex valued processes are given. An extension of the results to integral functionals of scalar or vector valued random fields and certain classes of nonstationary processes will be considered in a subsequent paper.

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