Approximation of stationary random functions with fractional rational spectral density *

Fellenberg, B. Starkloff, H.-J. Wunderlich, R.

Abstract

The paper deals with the approximation of stationary random functions with a prescribed fractional rational spectral density. The approximations are found from stationary solutions of so-called form filter equations which are ODEs with an inhomogeneous term containing a given random function. Here, this random function is weakly correlated and ideas of form filter theory are combined with expansions of spectral densities of stationary solutions with respect to the correlation length. Further, the smoothing of derivatives of the approximated functions is considered. Some examples concerning modeling of random road and railway profiles are given.

1 Introduction

Random functions are widely used to model random phenomena in nature, technology and economy. An important class of these functions is characterized by the property of weak or short range dependence of its values and known as weakly correlated functions. The mathematical theory of so-called weakly correlated random functions as well as a large number of applications to stochastic equations of mathematical physics can be found in [11, 9] . Equations containing weakly correlated random functions can be treated using classical concepts of calculus for the stochastic analysis of pathwise solutions.

Beside the case of short range dependence also long range dependence has to be modeled in many applications. To treat this problem with the theoretical framework of weakly correlated functions a tool is needed, which transforms weakly correlated into long range correlated or dependent functions. The present paper introduces such a tool for the case of stationary random functions with fractional rational spectral density. This class of functions contains a large number of random functions of practical importance.

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Section 2 describes the basic idea of such transformation tools. It consists in the analysis of solutions of ordinary differential equations with constant coefficients whose right hand sides contain a given random function as well as its derivatives. In the literature such equations are known as so-called form filters and they are used to transform white noise into long range correlated functions (see Pugachev, Sinitsyn [7], Doob [2], Krylov [3]).

Section 3 provides the main results of weakly correlated functions and expansions of integral functionals of weakly correlated functions with respect to the correlation length. These expansions are applied to find approximations of spectral densities and correlation functions of stationary solutions of form filter equations containing weakly correlated functions.

A procedure for deriving the coefficients of a form filter equation to a given fractional-rational spectral density is described in section 4. Applications to stationary random functions used in modeling random road or railway profiles are given in section 5. Section 6 deals with a modification of stationary solution of form filter equations in order to find integral functional representations of high-order derivatives of stationary solutions.

The results of this paper can also be used to model homogeneous random functions of several variables e.g. random surfaces (see [13]).

2 Form filter equations

In this section the existence and some properties of stationary solutions of the differential equation of order n for a real-valued random function $f(t)$

$$\sum_{k=0}^{n} a_k f^{(k)} = \sum_{k=0}^{m} b_k h^{(k)} \tag{1}$$

with $n > m \geq 0$ and a real-valued, centered and stationary random function $h : \mathbb{R} \times \Omega \to \mathbb{R}$ defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ are investigated. Here, $\mathcal{A}$ denotes the σ-algebra of subsets of Ω on which a probability measure $\mathbf{P}$ is defined. Further, $a_0, \ldots, a_n, b_0, \ldots, b_m$ denote real coefficients, without loss of generality $a_n = 1$ is assumed. Denote

$$Q_n(s) = \sum_{k=0}^{n} a_k s^k \text{ and } P_m(s) = \sum_{k=0}^{m} b_k s^k, \ s \in \mathbb{C},$$

where $\mathbb{C}$ denotes the set of complex numbers. The random function h is assumed to be m times a.s. as well as in mean square continuously differentiable. If the differentiability of h holds only in the sense of generalized functions the subsequent assertions have to be modified. A stationary solution is a stationary random function which satisfies the equation (1).

Equation (1) is known in the literature as form filter of the input function h. Frequently, white noise is used as input function (see Pugachev, Sinitsyn [7], Doob [2], Krylov [3]).

Equation (1) corresponds to so-called ARMA (n, m)-models of time series theory

$$\sum_{k=0}^{n} \tilde{a}_k f(t_{j+k}) = \sum_{k=0}^{m} \tilde{b}_k h(t_{j-k}) \ , \quad j = 0, \pm 1, \pm 2, \dots$$

with some real coefficients $\tilde{a}_k$, $\tilde{b}_k$ (see Priestley [6]. Such models can be derived from (1) if the derivatives are approximated by the corresponding finite differences with respect to a given discretization of the real axis.

2.1 Existence of a stationary solution

To investigate stationary solutions the equation (1) is transformed by

$$z_1 := f \ , \quad z_2 := f^{(1)} \ , \quad \dots \ , \quad z_n := f^{(n-1)}$$

into a system of first-order differential equations for the functions $z_1, ..., z_n$

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= -\sum_{k=0}^{n-1} a_k z_{k+1} + \sum_{k=0}^{m} b_k h^{(k)}. \end{aligned}$$

Using the notations

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & \dots & -a_{n-2} & -a_{n-1} \end{pmatrix}, \mathbf{z} = \begin{pmatrix} z_1 \\ \vdots \\ \vdots \\ \vdots \\ z_n \end{pmatrix}, \mathbf{g} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ \phi \end{pmatrix}$$

with $\phi = \sum\limits_{k=0}^{m} b_k h^{(k)}$ a linear first-order system

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{g} \tag{2}$$

for $\mathbf{z}$ is obtained. A sufficient condition for the existence of a stationary solution of system (2) (and consequently of equation (1)) is the stability of the system matrix $\mathbf{A}$ (see Balakrishnan [1]). $\mathbf{A}$ is stable, if the eigenvalues of $\mathbf{A}$ possess negative real parts, only. Considering the characteristic polynomial of $\mathbf{A}$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = (-1)^n \sum_{k=0}^{n} a_k \lambda^k = (-1)^n Q_n(\lambda) \ , \quad \lambda \in \mathbb{C},$$

the matrix $\mathbf{A}$ is stable, if the zeros $s_1, ..., s_n \in \mathbb{C}$ of the polynomial $Q_n(s)$ fulfill the condition

$$\text{Re } s_k < 0 \ , \quad k = 1, ..., n \quad . \tag{3}$$

Then, the stationary solution of (2) can be represented as

$$\mathbf{z}(t) = \int_{-\infty}^{t} \exp(\mathbf{A}(t-s))\mathbf{g}(s)ds \ . \tag{4}$$

2.2 Spectral density of stationary solution

The spectral density matrix of $\mathbf{z}$

$$\mathbf{S}_{\mathbf{zz}}(\alpha) = \left\{ \mathbf{S}_{z_i z_j}(\alpha) \right\}_{i,j=1}^{n} \ , \quad \alpha \in \mathbb{R},$$

can be derived by applying the spectral method (see [10], Soong [8], Preumont [5]). It holds

$$\mathbf{S}_{\mathbf{zz}}(\alpha) = \mathbf{H}(-i\alpha)\mathbf{S}_{\mathbf{gg}}(\alpha)\mathbf{H}^T(i\alpha) \tag{5}$$

with the spectral density matrix of $\mathbf{g}$

$$\mathbf{S}_{\mathbf{gg}}(\alpha) = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & S_{\phi\phi}(\alpha) \end{pmatrix} \tag{6}$$

and the complex frequency response matrix function

$$\mathbf{H}(s) = \{H_{ij}(s)\}_{i,j=1}^{n} = (s\mathbf{I} - \mathbf{A})^{-1} \ , \quad s \in \mathbb{C} \ .$$

Here, $\mathbf{H}(s)$ can be expressed by

$$\mathbf{H}(s) = \frac{1}{Q_n(s)} \times \tag{7}$$

$$\begin{pmatrix}
\frac{1}{s}Q^+_{n0}(s) & \frac{1}{s^2}Q^+_{n1}(s) & \frac{1}{s^3}Q^+_{n2}(s) & \dots & \frac{1}{s^{n-2}}Q^+_{n,n-3}(s) & \frac{1}{s^{n-1}}Q^+_{n,n-2}(s) & 1 \\
-Q^-_{n0}(s) & \frac{1}{s}Q^+_{n1}(s) & \frac{1}{s^2}Q^+_{n2}(s) & \dots & & \frac{1}{s^{n-2}}Q^+_{n,n-2}(s) & s \\
-sQ^-_{n0}(s) & -Q^-_{n1}(s) & \frac{1}{s}Q^+_{n2}(s) & \dots & & & s^2 \\
\vdots & \vdots & \vdots & & & & \\
 & & & \ddots & \ddots & & \\
-s^{n-3}Q^-_{n0}(s) & -s^{n-4}Q^-_{n1}(s) & -s^{n-5}Q^-_{n2}(s) & & -Q^-_{n,n-3}(s) & \frac{1}{s}Q^+_{n,n-2} & s^{n-1} \\
-s^{n-2}Q^-_{n0}(s) & -s^{n-3}Q^-_{n1}(s) & -s^{n-4}Q^-_{n2}(s) & & -sQ^-_{n,n-3}(s) & -Q^-_{n,n-2}(s) & s^n
\end{pmatrix}$$

where

$$Q^-_{nj}(s) = \sum_{k=0}^{j} a_k s^k \quad \text{and} \quad Q^+_{nj}(s) = \sum_{k=j+1}^{n} a_k s^k .$$

From (5), (6) and (7) for the spectral density of $f^{(k)} = z_{k+1}$, $k = 0, ..., n-1$, it follows

$$\begin{aligned}
S_{f^{(k)}f^{(k)}}(\alpha) &= S_{z_{k+1}z_{k+1}}(\alpha) = H_{k+1,n}(-i\alpha)S_{\phi\phi}(\alpha)H_{k+1,n}(i\alpha) \\
&= \frac{(-i\alpha)^k}{Q_n(-i\alpha)}\frac{(i\alpha)^k}{Q_n(i\alpha)}S_{\phi\phi}(\alpha) = \frac{\alpha^{2k}}{|Q_n(i\alpha)|^2}S_{\phi\phi}(\alpha). \qquad (8)
\end{aligned}$$

Taking into account the representation $\phi = \sum\limits_{k=0}^{m} b_k h^{(k)}$ with a m-times mean square differentiable function h the spectral density of ϕ can be derived

$$\begin{aligned}
S_{\phi\phi}(\alpha) &= \sum_{k,l=0}^{m} b_k b_l S_{h^{(k)}h^{(l)}}(\alpha) \\
&= \sum_{k,l=0}^{m} b_k b_l (-1)^k (i\alpha)^{k+l} S_{hh}(\alpha) \\
&= \left(\sum_{k=0}^{m} b_k(-i\alpha)^k\right)\left(\sum_{l=0}^{m} b_l(i\alpha)^l\right) S_{hh}(\alpha) \\
&= P_m(-i\alpha)P_m(i\alpha)S_{hh}(\alpha) \\
&= |P_m(i\alpha)|^2 S_{hh}(\alpha) \ . \qquad (9)
\end{aligned}$$

Finally, with (9) and (8) it yields

$$S_{f^{(k)}f^{(k)}}(\alpha) = \alpha^{2k} \left| \frac{P_m(i\alpha)}{Q_n(i\alpha)} \right|^2 S_{hh}(\alpha), \quad k = 0, \dots, n-1. \tag{10}$$

2.3 Integral functional representation of stationary solutions

The components of $\mathbf{z}$ in representation (4) of the stationary solution of the linear sytem (2) are given by

$$\begin{aligned} f^{(k)}(t) = z_{k+1}(t) &= \int_{-\infty}^{t} \left[\exp(\mathbf{A}(t-s))\mathbf{g}(s)\right]_{k+1} ds \\ &= \int_{-\infty}^{t} \left[\exp(\mathbf{A}(t-s))\right]_{k+1,n} \left(\sum_{l=0}^{m} b_l h^{(l)}(s,\omega) \right) ds \ , \quad k = 0, ..., n-1 \ , \end{aligned}$$

where $[\ .\ ]_i$ denotes the $i-$th component of a vector and $[\ .\ ]_{i,j}$ the element of a matrix in row i and column j.

Subsequently, other integral functional representations of f and its derivatives up to order $n-m-1$ will be derived. In these representations the derivatives $h^{(k)}$, $k \geq 1$, vanish. To this end the substitutions

$$\begin{aligned} y_1 &:= f - r_0 h \\ y_2 &:= \dot{y}_1 - r_1 h &&= f^{(1)} - r_0 h^{(1)} - r_1 h \\ y_3 &:= \dot{y}_2 - r_2 h &&= f^{(2)} - \sum_{l=0}^{2} r_{2-l} h^{(l)} \\ &\vdots \\ y_n &:= \dot{y}_{n-1} - r_{n-1} h &&= f^{(n-1)} - \sum_{l=0}^{n-1} r_{n-1-l} h^{(l)} \end{aligned}$$

are considered, where $r_0, ...r_{n-1}$ are real constants. It yields

$$f^{(k-1)} = y_k + \sum_{l=0}^{k-1} r_{k-1-l} h^{(l)}, \quad k = 1, ..., n \tag{11}$$

and

$$\dot{y}_k = y_{k+1} + r_k h, \quad k = 1, ..., n-1 \ . \tag{12}$$

For $\dot{y}_n$ from equation (11) it follows

$$\dot{y}_n \ = \ \frac{d}{dt} \left(f^{(n-1)} - \sum_{l=0}^{n-1} r_{n-1-l} h^{(l)} \right) = f^{(n)} - \sum_{l=0}^{n-1} r_{n-1-l} h^{(l+1)}$$

and by using the identity

$$f^{(n)} = -\sum_{k=0}^{n-1} a_k f^{(k)} + \sum_{k=0}^{m} b_k h^{(k)}$$

which can be derived from equation (1) the equation

$$\dot{y}_n = -\sum_{k=0}^{n-1} a_k \left(y_{k+1} + \sum_{l=0}^{k} r_{k-l} h^{(l)} \right) + \sum_{k=0}^{m} b_k h^{(k)} - \sum_{l=0}^{n-1} r_{n-1-l} h^{(l+1)} \tag{13}$$

is obtained. Pugachev, Sinitsyn (see [7], p. 43) have shown that by choosing the constants r_k according to

$$r_k = \begin{cases} 0 & k = 0, ..., n-m-1 \\ b_m & k = n-m \\ b_{n-k} - \sum\limits_{l=n-m}^{k-1} a_{n-k+l} r_l, & k = n-m+1, ..., n \end{cases} \tag{14}$$

the equation (13) is transformed into

$$\dot{y}_n = -\sum_{k=0}^{n-1} a_k y_{k+1} + r_n h. \tag{15}$$

Here, the right hand side of this equation does not contain derivatives of h. From (12) and (15) it follows, that the vector function $\mathbf{y} = (y_1, ..., y_n)^T$ satisfies the equation

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{r}h \tag{16}$$

with $\mathbf{r} = (r_1, ..., r_n)^T$ and the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & \dots & -a_{n-2} & -a_{n-1} \end{pmatrix}.$$

It can be seen that the first-order systems (2) and (16) possess identical system matrices. Because of the choice $r_0 = ... = r_{n-m-1} = 0$ for the first $n-m$ components of $\mathbf{y}$ the identities

$$y_k = z_k = f^{(k-1)}, \quad k = 1, ..., n-m, \tag{17}$$

yield. The representation of the stationary solution of (16)

$$\begin{aligned}\mathbf{y}(t) &= \int\limits_{-\infty}^{t} \exp(\mathbf{A}(t-s))\,\mathbf{r}\,h(s,\omega)\,ds \\ &= \int\limits_{-\infty}^{t} \mathbf{d}(t-s)\,h(s,\omega)ds\end{aligned}$$

with the vector function $\mathbf{d}(t) := \exp(\mathbf{A}(t))\,\mathbf{r}$ leads to

$$f^{(k)}(t) = y_{k+1}(t) = \int\limits_{-\infty}^{t} d_{k+1}(t-s)\,h(s,\omega)\,ds,\;\; k = 0, ..., n-m-1\;. \tag{18}$$

Higher-order derivatives of f result in additional terms containing the input function as well as its derivatives. For $k = n-m, ..., n-1$ it yields

$$\begin{aligned}f^{(k)}(t) &= y_{k+1}(t) + \sum_{l=0}^{k} r_{k-l}h^{(l)}(t) \\ &= \int\limits_{-\infty}^{t} d_{k+1}(t-s)\,h(s,\omega)\,ds + \sum_{l=0}^{k} r_{k-l}h^{(l)}(t)\;.\end{aligned} \tag{19}$$

3 Form filter equations with weakly correlated functions

3.1 Weakly correlated functions

The essential property of a weakly correlated function $h_\varepsilon(.,\omega)$ is that the influence of the values of the function does not reach far. That means, the values of $h_\varepsilon(.,\omega)$ at two points s and t are independent (and especially not correlated) if the distance $|t-s|$ exceeds a certain quantity $\varepsilon > 0$. The so-called correlation lenght ε is assumed to be sufficiently small to apply limit theorems and expansions in powers of ε of distribution characteristics of integral functionals

$$g(t,\omega) = \int\limits_{-\infty}^{t} F(t-u)h_\varepsilon(u,\omega)\,du, \quad t \in \mathbb{R}.$$

Here, the kernel function $F(.) \in L_1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$ is non-random. The complete definition of weakly correlated functions (which contains that h_ε is centered), limit theorems and expansions in powers of ε for distribution characteristics of integral functionals can be found in [11] and [9]. For example, the process $\frac{1}{\sqrt{\varepsilon}}g(t,\omega)$ converges in distribution to a Gaussian process as $\varepsilon \downarrow 0$.

Here, only second-order moments of f will be reviewed . A simplified approach to these expansions can be found in [12]. An important quantity needed for expansions of second-order moments is the intensity which is in case of a stationary function h_ε defined by

$$a = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} R_{h_\varepsilon h_\varepsilon}(\tau)\, d\tau,$$

where $R_{h_\varepsilon h_\varepsilon}(\tau), \tau \in \mathbb{R}$, denotes the correlation function of h_ε. The above definition can be rewritten as

$$\int_{-\varepsilon}^{\varepsilon} R_{h_\varepsilon h_\varepsilon}(\tau)\, d\tau = a\varepsilon + o(\varepsilon).$$

and for the spectral density of h_ε it follows

$$S_{h_\varepsilon h_\varepsilon}(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{h_\varepsilon h_\varepsilon}(\tau) e^{i\alpha\tau} d\tau = \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} R_{h_\varepsilon h_\varepsilon}(\tau) \cos(\alpha\tau) d\tau$$

Using Taylor's theorem $\cos(x) = 1 - \frac{\theta^2}{2}$ for $0 \leq |x| \leq \delta < \infty$ and some θ with $|\theta| \leq \delta$ it follows for $|\alpha| \leq \alpha_{\max} < \infty$ and $\varepsilon \downarrow 0$

$$\begin{aligned} S_{h_\varepsilon h_\varepsilon}(\alpha) &= \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} R_{h_\varepsilon h_\varepsilon}(\tau) \left(1 - \frac{\tilde{\theta}^2(\tau)}{2}\right) d\tau \\ &= \frac{a\varepsilon}{2\pi} + o(\varepsilon) \quad , \qquad (20) \end{aligned}$$

since it holds $|\tilde{\theta}(\tau)| \leq |\alpha\tau| \leq \alpha_{\max}\varepsilon$ and $\int_{-\varepsilon}^{\varepsilon} \frac{\tilde{\theta}^2}{2} \leq \varepsilon^3 \alpha_{\max}^2 = o(\varepsilon)$.

For the correlation function of the integral functional $f(t,\omega)$ the following expansion holds (see [9])

$$\begin{aligned} R_{gg}(\tau) = \mathbf{E}\{g(t)g(t+\tau)\} &= \int_{-\infty}^{t} \int_{-\infty}^{t+\tau} F(t-u)F(t+\tau-v)\mathbf{E}\{h_\varepsilon(u)h_\varepsilon(v)\}\, dudv \\ &= \varepsilon R_{gg}^1(\tau) + o(\varepsilon), \qquad (21) \end{aligned}$$

$$\text{where} \quad R_{gg}^1(\tau) = a \int_{-\infty}^{t} F(u)F(u+|\tau|)\, du, \quad \tau \in \mathbb{R}.$$

An expansion of the spectral density of f for $|\alpha| \leq \alpha_{\max} < \infty$ can be found using Fourier transformation

$$
\begin{aligned}
S_{gg}(\alpha) &= \frac{1}{\pi}\int_0^\infty \cos(\alpha\tau)R_{gg}(\tau)\,d\tau = \varepsilon S^1_{gg}(\alpha) + o(\varepsilon) \qquad (22)\\
\text{where} \quad S^1_{gg}(\alpha) &= \frac{1}{\pi}\int_0^\infty \cos(\alpha\tau)R^1_{gg}(\tau)\,d\tau.
\end{aligned}
$$

3.2 Expansion of spectral density and correlation function

Now, the function h which is involved in the right hand side of the form filter equation is assumed to be a weakly correlated function. Then, Eq. (1) reads as

$$
\sum_{k=0}^{n} a_k f^{(k)} = \sum_{k=0}^{m} b_k h_\varepsilon^{(k)} \quad ,
$$

where h_ε is a weakly correlated random function with correlation length $\varepsilon > 0$ which is stationary and m-times a.s. and mean square differentiable. From Eq. (18) it can be seen that the function f as well as its derivatives up to the order $n - m - 1$ can be represented as integral functionals of h_ε. For the higher-order derivatices $f^{(k)}$ with $k > n-m-1$ these representations (see Eq. (19)) contain additional terms where h_ε as well as its derivatives are involved.

Subsequently, the expansions of spectral density and correlation function of f and of those derivatives of f which admit an integral functional representation are given. From Eq. (10) it follows for the spectral density of $f^{(k)}$, $k = 0, \ldots, n-m-1$

$$
S_{f^{(k)}f^{(k)}}(\alpha) = \alpha^{2k}\left|\frac{P_m(i\alpha)}{Q_n(i\alpha)}\right|^2 S_{h_\varepsilon h_\varepsilon}(\alpha) \; . \qquad (23)
$$

Substituting the spectral density $S_{h_\varepsilon h_\varepsilon}(\alpha)$ by its expansion (20) the following expansions of $S_{f^{(k)}f^{(k)}}(\alpha)$

$$
S_{f^{(k)}f^{(k)}}(\alpha) = \frac{a\varepsilon}{2\pi}\alpha^{2k}\left|\frac{P_m(i\alpha)}{Q_n(i\alpha)}\right|^2 + o(\varepsilon) \qquad (24)
$$

can be derived for $|\alpha| < \alpha_{\max} < \infty$.

An expansion of the correlation function of $f^{(k)}$, $k = 0, ..., n-m-1$, with respect to ε can be derived from the corresponding expansion of the matrix correlation function $\mathbf{R_{yy}}(\tau)$ of the vector function $\mathbf{y}$, which satisfies the first order system

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{r}h_\varepsilon. \tag{25}$$

This follows from the identities

$$f^{(k)} = y_{k+1} \ , \quad k = 0, ..., n-m-1$$

(see Eqs. (16) and (17)).

Let

$$\mathbf{C} := \int_0^\infty \exp(\mathbf{A}u)\mathbf{r}\mathbf{r}^T(\exp(\mathbf{A}u))^T du \ ,$$

then, in [10], pp. 112, it has been proved, that for $\tau \geq 0$ it holds

$$\mathbf{R_{yy}}(\tau) = \varepsilon \cdot a\mathbf{C}\exp(\mathbf{A}\tau)^T + o(\varepsilon) \quad \text{and} \quad \mathbf{R_{yy}}(-\tau) = \mathbf{R}^T_{\mathbf{yy}}(\tau).$$

If the matrix $\mathbf{A}$ is diagonalizable, then the eigenvalue decomposition $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1}$, where $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ with eigenvalues $\lambda_1, ..., \lambda_n$ beeing the zeros of $Q_n(\lambda)$ (see section 2.1) leads to

$$\mathbf{R_{yy}}(\tau) = \varepsilon \cdot a\mathbf{C}\mathbf{V}\exp(\Lambda^*\tau)\mathbf{V}^{-1} + o(\varepsilon)$$

for $\tau \geq 0$. For the first $n-m$ components the representation

$$R_{f^{(k)}f^{(k)}}(\tau) = R_{y_{k+1}y_{k+1}}(\tau) = \varepsilon \cdot a\sum_{i=1}^{n}\beta_{ki}e^{\overline{\lambda_i}|\tau|} + o(\varepsilon) \ , \quad k = 0, ..., n-m-1,$$

with

$$\beta_{ki} = [\mathbf{C}\mathbf{V}]_{k+1,i}[\mathbf{V}^{-1}]_{i,k+1}$$

can be derived.

4 Fractional-rational spectral densities

The results of preceding sections are now used to find approximations of real-valued, centered, stationary random functions $\xi(t, \omega)$, $t \in \mathbf{R}$ with a prescribed spectral density

$$S_{\xi\xi}(\alpha) = S_0^2\frac{\tilde{P}_M(\alpha)}{\tilde{Q}_N(\alpha)} = S_0^2\frac{\sum\limits_{k=0}^{M} p_k\alpha^k}{\sum\limits_{k=0}^{N} q_k\alpha^k} \ , \quad \alpha \in \mathbf{R}, \tag{26}$$

by integral functionals of the form $\int\limits_{-\infty}^{t} F(t-s)h_\varepsilon(s,\omega)ds$. $\tilde{P}_M$ and $\tilde{Q}_N$ are polynomials of order M and N with real coefficients p_k and q_k, S_0^2 is a positive real number, h_ε a stationary, weakly correlated random function with correlation length ε and intensity a and $F(.)$ is a real-valued non-random function which has to be determined.

In section 2 and 3 it has been derived, that the spectral density of the stationary solution of the form filter equation

$$\sum_{k=0}^{n} a_k f^{(k)} = \sum_{k=0}^{m} b_k h_\varepsilon^{(k)}$$

can be expanded in powers of ε as follows

$$S_{ff}(\alpha) = \frac{a\varepsilon}{2\pi}\left|\frac{P_m(i\alpha)}{Q_n(i\alpha)}\right|^2 + o(\varepsilon)$$

with

$$P_m(s) = \sum_{k=0}^{m} b_k s^k \quad \text{and} \quad Q_n(s) = \sum_{k=0}^{n} a_k s^k \quad , \qquad s \in \mathbb{C}.$$

Therefore, the stationary solution f which can be represented in form of an integral functional (18) will be used as approximation of ξ. To this end intensity a and correlation length ε are chosen so that $S_0^2 = \frac{a\varepsilon}{2\pi}$. It remains to check conditions on the polynomials $\tilde{P}_M(\alpha)$ and $\tilde{Q}_N(\alpha)$ which ensure the existence of a representation

$$\frac{\tilde{P}_M(\alpha)}{\tilde{Q}_N(\alpha)} = \left|\frac{P_m(i\alpha)}{Q_n(i\alpha)}\right|^2 = \left|\frac{\sum\limits_{k=0}^{m} b_k(i\alpha)^k}{\sum\limits_{k=0}^{n} a_k(i\alpha)^k}\right|^2$$

with real coefficients a_k and b_k, $a_n = 1$, $n > m \geq 0$, and finally, to determine the polynomials P_m and Q_n .

First, from the uniqueness of representation (26) the following conditions are derived

(F1) $p_M = q_N = 1$

(F2) $\tilde{P}_M$ and $\tilde{Q}_N$ possess no common zeros.

Second, the usual properties of the spectral density of a real valued random process ξ

(i) $S_{\xi\xi}(\alpha) \geq 0$
(ii) $S_{\xi\xi}(\alpha) < \infty$
(iii) $S_{\xi\xi}(\alpha) = S_{\xi\xi}(-\alpha)$
(iv) $\int\limits_{-\infty}^{\infty} S_{\xi\xi}(\alpha) d\alpha < \infty$

hold, if the subsequent conditions

(F3) $p_0 \geq 0,\ q_0 > 0$

(F4) $N > M \geq 0$

(F5) $\tilde{P}_M$ and $\tilde{Q}_N$ are even functions (where $M = 2m$ and $N = 2n$)

(F6) $\tilde{Q}_N$ has no real zeros

(F7) If $s \in \mathbf{C}$ is a zero of $\tilde{Q}_N$, then $-s, \bar{s}, -\bar{s}$ are zeros, too. (If s is an imaginary zero, i.e. $\bar{s} = -s$, then s and $\bar{s}$ may be simple zeros.)

(F8) For the zeros $\tilde{P}_M$ holds (F7) and additionally: if s is a real zero, i.e. $s = \bar{s}$, then the multiplicity of this zero is an even number

are fulfilled. From (F4), (F5) and (F7) follows, that the $N = 2n \geq 2$ zeros of the polynomial $\tilde{Q}_N$ may be divided into 2 classes

– $2n_1 \geq 0$ imaginary zeros $\pm i\mu_k$ with $\mu_k > 0,\ k = 1, ..., n_1$

– $4n_2 \geq 0$ complex zeros (with non-zero real and imaginary parts) $\pm\eta_k,\ \pm\bar{\eta}_k$ with $\mathrm{Re}\,\eta_k,\ \mathrm{Im}\,\eta_k > 0,\ k = 1, ..., n_2$.

It holds $n_1 + 2n_2 = n = \frac{N}{2}$. Higher multiplicities are treated as several equal zeros.

Using the linear factor representation of $\tilde{Q}_N$

$$\tilde{Q}_N(\alpha) = \prod_{k_1=1}^{n_1} (\alpha - i\mu_{k_1})(\alpha + i\mu_{k_1}) \prod_{k_2=1}^{n_2} (\alpha - \eta_{k_2})(\alpha + \eta_{k_2})(\alpha - \bar{\eta}_{k_2})(\alpha + \bar{\eta}_{k_2})$$

and applying the identities

$$\begin{aligned}
(\alpha - i\mu)(\alpha + i\mu) &= (i\alpha + \mu)(-i\alpha + \mu) \\
(\alpha - \eta)(\alpha + \eta)(\alpha - \bar{\eta})(\alpha + \bar{\eta}) &= [(\alpha - \eta)(\alpha + \bar{\eta})][(\alpha + \eta)(\alpha - \bar{\eta})] \\
&= [\alpha^2 - (\eta - \bar{\eta})\alpha - |\eta|^2][\alpha^2 + (\eta - \bar{\eta})\alpha - |\eta|^2] \\
&= [\alpha^2 - 2\mathrm{Im}\,\eta \cdot i\alpha - |\eta|^2][\alpha^2 + 2\mathrm{Im}\,\eta \cdot i\alpha - |\eta|^2] \\
&= [(i\alpha)^2 + 2\mathrm{Im}\,\eta \cdot i\alpha + |\eta|^2][(-i\alpha)^2 + 2\mathrm{Im}\,\eta(-i\alpha) + |\eta|^2]
\end{aligned}$$

it follows the representation

$$\tilde{Q}_N(\alpha) = Q_n(i\alpha) Q_n(-i\alpha)$$

with the polynomial of order n

$$Q_n(s) = \prod_{k_1=1}^{n_1} (s + \mu_{k_1}) \prod_{k_2=1}^{n_2} (s^2 + 2\mathrm{Im}\ \eta_{k_2} s + |\eta_k|^2) \ , \quad s \in \mathbb{C} \quad .$$

Multiplying out leads to the representation

$$Q_n(s) = \sum_{k=0}^{n} a_k s^k \tag{27}$$

with real, non-negative (because of $\mu_{k_1}, \mathrm{Im}\,\eta_{k_2} > 0$) coefficients $a_0, \ldots, a_n$ and $a_n = 1$. Hence, it holds

$$Q_n(\bar{s}) = \sum_{k=0}^{n} a_k \bar{s}^k = \overline{\sum_{k=0}^{n} a_k s^k} = \overline{Q_n(s)}$$

and

$$\tilde{Q}_N(\alpha) = Q_n(i\alpha) Q_n(-i\alpha) = |Q_n(i\alpha)|^2 \quad .$$

The polynomial $Q_n(s)$ possesses the $n = n_1 + 2n_2$ zeros

$$-\mu_1, ..., -\mu_{n_1}, \quad -\mathrm{Im}\,\eta_1 \pm i\mathrm{Re}\,\eta_1, ..., -\mathrm{Im}\,\eta_{n_2} \pm i\mathrm{Re}\ \eta_{n_2}, \tag{28}$$

having negative real parts. It satisfies condition (3), which ensures the existence of a stationary solution of form filter equation (1) with the coefficients $a_0, ..., a_n$ from (27).

Similarly, a corresponding representation of the numerator polynomial $\tilde{P}_M(\alpha)$ can be derived. Additionally, the real zeros of an even multiplicity (see condition (F8)) have to be taken into account.

In case of $M = 2m = 0$, i.e. $\tilde{P}_0(\alpha) = 1$ (see (F1)), the polynomial $P_0(s) = 1$ is used. For $M = 2m \geq 2$ the following three classes of zeros are considered

- $2m_2 \geq 0$ imaginary zeros $\pm i v_k$ with $v_k > 0$, $k = 1, ..., n_1$
- $4m_2 \geq 0$ complex zeros (with non-zero real parts) $\pm\theta_k$, $\pm\bar{\theta}_k$ with $\mathrm{Re}\,\theta_k > 0$, $\mathrm{Im}\,\theta_k \geq 0$, $k = 1, ..., n_2$. This class contains the real zeros ($\neq 0$) with even multiplicity ($\mathrm{Im}\ \theta_k = 0$) in consequence of condition (F8)
- zero 0 with multiplicity $2m_3$.

It holds $m_1 + 2m_2 + m_3 = m = \frac{M}{2}$. Higher multiplicities are considered as several equal zeros. For the additional term of the linear factor representation of $\tilde{P}_M(\alpha)$ corresponding to the last class of zeros it holds

$$\alpha^{2m_3} = (i\alpha)^{m_3} (-i\alpha)^{m_3} \quad .$$

Like in the case of $\tilde{Q}_M$ (see above) it holds the representation

$$\tilde{P}_M(\alpha) = P_m(i\alpha)P_m(-i\alpha) = |P_m(i\alpha)|^2$$

with a polynomial of order m for $s \in \mathbb{C}$

$$\begin{aligned} P_m(s) &= \prod_{k_1=1}^{m_1}(s+v_{k_1})\prod_{k_2=1}^{m_2}(s^2+2\mathrm{Im}\,\theta_{k_2}s+|\theta_{k_2}|^2)\cdot s^{m_3} \\ &= \sum_{k=1}^{m} b_k s^k \end{aligned}$$

where the real non-negative coefficients $b_0, ..., b_m$, ($b_m = 1$) are found by multiplying out.

5 Applications

In this section the idea of approximating stationary random functions with a fractional-rational spectral density by integral functionals of weakly correlated functions is applied to some practical important cases. Table 1 gives some examples of spectral densities and correlation functions of stationary random functions which are widely used in engineering applications e.g. for modeling random road or railway profiles (see Popp, Schiehlen [4], p. 200 ff).

5.1 $S_{\xi\xi}(\alpha) = \frac{\sigma^2}{\pi}\frac{\gamma}{\alpha^2+\gamma^2}$, $\gamma > 0$

The spectral density of this example can be written as

$$S_{\xi\xi}(\alpha) = S_0^2\frac{\tilde{P}_M(\alpha)}{\tilde{Q}_N(\alpha)}$$

(see representation (26)), where

$$S_0^2 = \frac{\sigma^2}{\pi}\gamma,\ M=0,\ N=2,\ \tilde{P}_M(\alpha)=1,\ \tilde{Q}_N(\alpha)=\alpha^2+\gamma^2.$$

Hence, the form filter equation is of order $n = \frac{N}{2} = 1$ and contains no derivatives of h_ε ($m = \frac{M}{2} = 0$). Its coefficients are found from the corresponding coefficients of the polynomials P_m and Q_n. It holds $P_m(s) = P_0(s) = 1$ while $Q_n(s)$ is found from the imaginary zeros $\pm i\gamma$ of $\tilde{Q}_N(\alpha) = \alpha^2+\gamma^2$. According to section 4 it holds $n_1 = 1, \mu_1 = \gamma, n_2 = 0$ and $Q_n(s) = Q_1(s) = s+\gamma$.

	$S_{\xi\xi}(\alpha)$	$R_{\xi\xi}(\tau)$	parameter
(1)	$\frac{\sigma^2}{\pi}\frac{\gamma}{\alpha^2+\gamma^2}$	$\sigma^2 e^{-\gamma\|\tau\|}$	$\gamma > 0$
(2)	$\frac{2\sigma^2\gamma}{\pi}\frac{\beta^2}{(\alpha^2-\beta^2)^2+4\gamma^2\alpha^2}$	$\sigma^2 e^{-\gamma\|\tau\|}(\cos p\tau + \frac{\gamma}{p}\sin p\|\tau\|)$	$\beta > \gamma > 0$ $p = \sqrt{\beta^2-\gamma^2}$
(3)	$\frac{2\sigma^2\gamma}{\pi}\frac{\alpha^2}{(\alpha^2-\beta^2)^2+4\gamma^2\alpha^2}$	$\sigma^2 e^{-\gamma\|\tau\|}(\cos p\tau - \frac{\gamma}{p}\sin p\|\tau\|)$	see (2)
(4)	$\frac{\sigma^2\gamma}{\pi}\frac{\alpha^2+\beta^2}{(\alpha^2-\beta^2)^2+4\gamma^2\alpha^2}$	$\sigma^2 e^{-\gamma\|\tau\|}\cos p\tau$	see (2)

Table 1: Examples of spectral densities and correlation functions

Hence, the derived approximation is found from the stationary solution of the equation

$$\dot{f} + \gamma f = h_\varepsilon . \tag{29}$$

Then, for the spectral density of f then it holds

$$\begin{aligned} S_{ff}(\alpha) &= \frac{a\varepsilon}{2\pi}\frac{1}{\alpha^2+\gamma^2} + o(\varepsilon) \\ &= \frac{\sigma^2}{\pi}\frac{\gamma}{\alpha^2+\gamma^2} + o(\varepsilon) \quad \text{if } a\varepsilon = 2\sigma^2\gamma \ . \end{aligned}$$

A representation of Equation (29) in form of (16)

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{r}h_\varepsilon$$

is obtained by setting $\mathbf{y} = f$, $\mathbf{A} = -\gamma$ and $\mathbf{r} = r_{n-m} = b_m = 1$ (see (14)). The stationary solution is found to be the integral functional of h_ε

$$\mathbf{y}(t) = f(t) = \int_{-\infty}^{t} e^{-\gamma(t-s)} h_\varepsilon(s,\omega) ds \ .$$

Further, from Equation (16) it follows

$$\dot{\mathbf{y}}(t) = \dot{f}(t) = \mathbf{A}\mathbf{y} + \mathbf{r}h_\varepsilon = -\gamma \int_{-\infty}^{t} e^{-\gamma(t-s)} h_\varepsilon(s,\omega) ds + h_\varepsilon(t,\omega) \ .$$

5.2 $S_{\xi\xi}(\alpha) = \frac{2\sigma^2\gamma}{\pi}\frac{\beta^2}{(\alpha^2-\beta^2)^2+4\gamma^2\alpha^2}$, $\beta > \gamma > 0$

The representation (26) of the spectral density is obtained by setting

$$S_0^2 = \frac{2\sigma^2\gamma\beta^2}{\pi},\ M = 0,\ N = 4,\ \tilde{P}_M(\alpha) = 1$$

and

$$\tilde{Q}_N(\alpha) = (\alpha^2 - \beta^2)^2 + 4\gamma^2\alpha^2 = \alpha^4 + \left(4\gamma^2 - 2\beta^2\right)\alpha^2 + \beta^4.$$

Since $n = \frac{N}{2} = 2$ a second-order form filter equation has to be studied. $\tilde{Q}_N(\alpha)$ has 4 complex zeros $\eta_1 = \sqrt{\beta^2 - \gamma^2} + i\gamma,\ \ -\eta_1,\ \overline{\eta_1},\ \ -\overline{\eta_1}$ and according to section 4 with $n_1 = 0$ and $n_2 = 1$ it follows

$$Q_2(s) = s^2 + 2\mathrm{Im}\,\eta_1 s + |\eta_1|^2 = s^2 + 2\gamma s + \beta^2.$$

Hence, ξ can be approximated by the stationary solution of

$$\ddot{f} + 2\gamma\dot{f} + \beta^2 f = h_\varepsilon. \tag{30}$$

For the spectral density of f the expansion

$$\begin{aligned} S_{f^{(k)}f^{(k)}}(\alpha) &= \frac{a\varepsilon}{2\pi}\frac{\alpha^{2k}}{(\alpha^2-\beta^2)^2+4\gamma^2\alpha^2} + o(\varepsilon) \\ &= \frac{2\sigma^2\gamma}{\pi}\frac{\beta^2\cdot\alpha^{2k}}{(\alpha^2-\beta^2)^2+4\gamma^2\alpha^2} + o(\varepsilon) \quad \text{if } a\varepsilon = 4\sigma^2\gamma\beta^2, \end{aligned}$$

with respect to ε holds for $k = 0, 1$. The first-order system (16) is derived by setting

$$\begin{aligned} \mathbf{y} &= (f, \dot{f})^T \quad (\text{since } y_2 = \dot{f} - r_1 h_\varepsilon \text{ and } r_1 = r_{n-m-1} = 0) \\ \mathbf{A} &= \begin{pmatrix} 0 & 1 \\ -\beta^2 & -2\gamma \end{pmatrix} \\ \mathbf{r} &= (r_1, r_2)^T = (0, 1)^T \quad (\text{since } r_2 = r_{n-m} = b_m = b_0 = 1) \end{aligned}$$

(see (14)). For the computation of the stationary solution

$$\mathbf{y}(t) = \int_{-\infty}^{t} \exp(\mathbf{A}(t-s))\mathbf{r}h_\varepsilon(s,\omega)ds$$

the eigenvalue decomposition $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1}$ and the relation

$$\exp(\mathbf{A}t) = \mathbf{V}\exp(\Lambda t)\mathbf{V}^{-1},$$

which holds in case of simple eigenvalues, (see section 3) is used. In the present case it holds

$$\begin{aligned}
\Lambda &= \mathrm{diag}(\lambda, \bar{\lambda}) \quad \text{where } \lambda = -\gamma + ip \text{ and } p := \sqrt{\beta^2 - \gamma^2} \\
\mathbf{V} &= \begin{pmatrix} \bar{\lambda} & \lambda \\ -\beta^2 & -\beta^2 \end{pmatrix} \\
\mathbf{V}^{-1} &= \frac{1}{-\beta^2(\bar{\lambda} - \lambda)} \begin{pmatrix} -\beta^2 & -\lambda \\ \beta^2 & \bar{\lambda} \end{pmatrix} = \frac{i}{2\beta^2 p} \begin{pmatrix} \beta^2 & \lambda \\ -\beta^2 & -\bar{\lambda} \end{pmatrix} \\
\exp(\mathbf{A}t) &= \frac{1}{p} e^{-\gamma t} \begin{pmatrix} p\cos(pt) + \gamma \sin(pt) & \sin(pt) \\ -\beta^2 \sin(pt) & p\cos(pt) - \gamma \sin(pt) \end{pmatrix}. \qquad (31)
\end{aligned}$$

It yields

$$\exp(\mathbf{A}t)\mathbf{r} = \exp(\mathbf{A}t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{p} e^{-\gamma t} \begin{pmatrix} \sin(pt) \\ p\cos(pt) - \gamma \sin(pt) \end{pmatrix}$$

and for f and $\dot{f}$ the following integral functionals of h_ε are found

$$\begin{aligned}
f(t) = y_1(t) &= \int_{-\infty}^{t} \frac{1}{p} e^{-\gamma(t-s)} \sin(p(t-s)) h_\varepsilon(s, \omega) ds \\
\dot{f}(t) = y_2(t) &= \int_{-\infty}^{t} \frac{1}{p} e^{-\gamma(t-s)} \left[p\cos(p(t-s)) - \gamma \sin(p(t-s))\right] h_\varepsilon(s, \omega) ds.
\end{aligned}$$

Then, the second derivative of f can be written as

$$\begin{aligned}
\ddot{f}(t) = \dot{y}_2(t) &= a_{21} y_1(t) + a_{22} y_2(t) + r_2 h_\varepsilon(t, \omega) \\
&= -\beta^2 f(t) - 2\gamma \dot{f}(t) + h_\varepsilon(t, \omega) \\
&= \int_{-\infty}^{t} \frac{1}{p} e^{-\gamma(t-s)} \left[-2\gamma p \cos(p(t-s)) + (2\gamma^2 - \beta^2) \sin(p(t-s))\right] h_\varepsilon(s, \omega) ds \\
&\quad + h_\varepsilon(t, \omega) \\
&= \int_{-\infty}^{t} \frac{1}{p} e^{-\gamma(t-s)} \left[-2\gamma p \cos(p(t-s)) + (\gamma^2 - p^2) \sin(p(t-s))\right] h_\varepsilon(s, \omega) ds \\
&\quad + h_\varepsilon(t, \omega).
\end{aligned}$$

5.3 $S_{\xi\xi}(\alpha) = \frac{2\sigma^2\gamma}{\pi} \frac{\alpha^2}{(\alpha^2 - \beta^2)^2 + 4\gamma^2\alpha^2}$, $\beta > \gamma > 0$

For representation (26) of $S_{\xi\xi}(\alpha)$ the settings

$$S_0^2 = \frac{2\sigma^2\gamma}{\pi},\ M = 2,\ N = 4,\ \tilde{P}_M(\alpha) = \alpha^2$$

and

$$\tilde{Q}_N(\alpha) = (\alpha^2 - \beta^2)^2 + 4\gamma^2\alpha^2$$

are used. As in the preceding example it holds $n = 2$ and $Q_n(s) = s^2 + 2\gamma s + \beta^2$ while $m = \frac{M}{2} = 1$ and $P_1(s) = s$ since $\tilde{P}_M(\alpha)$ has two real zeros 0 and according to section 4 it is $m_1 = m_2 = 0$, $m_3 = 1$ and $P_m(s) = s^{m_3} = s$. Again, a second-order form filter equation is obtained, it reads as

$$\ddot{f} + 2\gamma\dot{f} + \beta^2 f = \dot{h}_\varepsilon .$$

In contrast to the preceding example the derivative $\dot{h}_\varepsilon$ instead of h_ε is involved on the right hand side. It can be seen, that this equation can be derived from the corresponding form filter equation of the preceding example by differentiating and the substitution $\dot{f} \to f$. For the spectral density of the stationary solution f it holds

$$\begin{aligned} S_{ff}(\alpha) &= \frac{a\varepsilon}{2\pi}\frac{\alpha^2}{(\alpha^2-\beta^2)^2+4\gamma^2\alpha^2} + o(\varepsilon) \\ &= \frac{2\sigma^2\gamma}{\pi}\frac{\alpha^2}{(\alpha^2-\beta^2)^2+4\gamma^2\alpha^2} + o(\varepsilon) \quad \text{if } a\varepsilon = 4\sigma^2\gamma \quad . \end{aligned}$$

With the help of Equation (14) and the settings

$$\begin{aligned} \mathbf{y} &= (y_1, y_2)^T = (f, \dot{f} - r_1 h_\varepsilon)^T = (f, \dot{f} - h_\varepsilon)^T \\ \mathbf{A} &= \begin{pmatrix} 0 & 1 \\ -\beta^2 & -2\gamma \end{pmatrix} \\ r_1 &= r_{n-m} = b_m = b_1 = 1 \\ r_2 &= b_{n-2} - \sum_{l=n-m}^{2-1} a_{n-2+l} r_l = b_0 - \sum_{l=1}^{1} a_l r_l = b_0 - a_1 r_1 = -2\gamma \\ \mathbf{r} &= (1, -2\gamma)^T \end{aligned}$$

the first-order system (16) reads as

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\beta^2 & -2\gamma \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -2\gamma \end{pmatrix} h_\varepsilon .$$

Its stationary solution is given by

$$\mathbf{y}(t) = \int_{-\infty}^{t} \exp(\mathbf{A}(t-s))\mathbf{r} h_\varepsilon(s,\omega) ds$$

and the function $\mathbf{d}(t)$ is found to be

$$\mathbf{d}(t) = \exp(\mathbf{A}t)\mathbf{r} = \frac{1}{p}e^{-\gamma t}\begin{pmatrix} p\cos(pt) - \gamma\sin(pt) \\ -2\gamma p\cos pt + (2\gamma^2 - \beta^2)\sin pt \end{pmatrix} .$$

Hence, it holds the subsequent representation of f and $\dot f$ by integral functionals of h_ε

$$
\begin{aligned}
f(t) &= y_1(t) = \int\limits_{-\infty}^{t} \frac{1}{p} e^{-\gamma(t-s)} \left[p\cos(p(t-s)) - \gamma\sin(p(t-s))\right] h_\varepsilon(s,\omega)ds \\
\dot f(t) &= y_2(t) + r_1 h_\varepsilon(t,\omega) \\
&= \int\limits_{-\infty}^{t} \frac{1}{p} e^{-\gamma(t-s)} \left[-2\gamma p\cos(p(t-s)) + (\gamma^2 - p^2)\sin(p(t-s))\right] h_\varepsilon(s,\omega)ds \\
&\quad + h_\varepsilon(t,\omega) \quad .
\end{aligned}
$$

For the second derivative $\ddot f$ from the form filter equation it follows

$$
\begin{aligned}
\ddot f(t) &= -\beta^2 f(t) - 2\gamma \dot f(t) + h_\varepsilon(t,\omega) \\
&= \int\limits_{-\infty}^{t} \frac{1}{p} e^{-\gamma(t-s)} \left[(4\gamma^2 - \beta^2)p\cos(p(t-s))\right. \\
&\qquad \left. -\gamma(\beta^2 - 2(\gamma^2 + p^2))\sin(p(t-s))\right] h_\varepsilon(s,\omega)ds \\
&\quad -2\gamma h_\varepsilon(t,\omega) + \dot h_\varepsilon(t,\omega).
\end{aligned}
$$

5.4 $S_{\xi\xi}(\alpha) = \frac{\sigma^2\gamma}{\pi}\frac{\alpha^2+\beta^2}{(\alpha^2-\beta^2)^2+4\gamma^2\alpha^2}$, $\beta > \gamma > 0$

For this example

$$
S_0^2 = \frac{\sigma^2\gamma}{\pi},\ M = 2,\ N = 4,\ \tilde P_M(\alpha) = \alpha^2 + \beta^2
$$

and

$$
\tilde Q_N(\alpha) = (\alpha^2 - \beta^2)^2 + 4\gamma^2\alpha^2
$$

have to be used to represent $S_{\xi\xi}(\alpha)$ in form (26). Again it holds $n = 2$ and $Q_n(s) = s^2 + 2\gamma s + \beta^2$ while the numerator polynomial possesses the imaginary zeros $\pm i\beta$. Hence, with $m_1 = 1$, $m_2 = m_3 = 0$ and $\nu_1 = \beta$ the polynomial $P_m(s)$ reads as

$$
P_m(s) = \prod_{k_1=1}^{m_1} (s + \nu_{k_1}) = (s + \beta).
$$

Then, for the stationary solution of the form filter equation

$$
\ddot f + 2\gamma \dot f + \beta^2 f = \beta h_\varepsilon + \dot h_\varepsilon
$$

it holds

$$\begin{aligned} S_{ff}(\alpha) &= \frac{a\varepsilon}{2\pi}\frac{\alpha^2+\beta^2}{(\alpha^2-\beta^2)^2+4\gamma\alpha^2}+o(\varepsilon) \\ &= \frac{\sigma^2\gamma}{\pi}\frac{\alpha^2+\beta^2}{(\alpha^2-\beta^2)^2+4\gamma\alpha^2}+(\varepsilon) \quad \text{if } a\varepsilon = 2\sigma^2\gamma \quad . \end{aligned}$$

Using

$$\begin{aligned} \mathbf{y} &= (y_1,y_2)^T = (f,\dot{f}-r_1h_\varepsilon)^T = (f,\dot{f}-h_\varepsilon)^T \\ \mathbf{A} &= \begin{pmatrix} 0 & 1 \\ -\beta^2 & -2\gamma \end{pmatrix} \\ \mathbf{r} &= (1,\beta-2\gamma)^T \end{aligned}$$

where the coefficients r_1 and r_2 follow from (14)

$$\begin{aligned} r_1 &= r_{n-m} = b_m = b_1 = 1 \\ r_2 &= b_{n-2}\sum_{l=n-m}^{2-1} a_{n-2+l}r_l = b_0 - a_1r_1 = \beta - 2\gamma \end{aligned}$$

the first-order system (16) reads as

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\beta^2 & -2\gamma \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 1 \\ \beta-2\gamma \end{pmatrix} h_\varepsilon \ ,$$

and its stationary solution is

$$\mathbf{y}(t) = \int_{-\infty}^{t} \exp(\mathbf{A}(t-s))\mathbf{r}h_\varepsilon(s,\omega)ds$$

where

$$\mathbf{d}(s) = \exp(\mathbf{A}(t))r = \frac{1}{p}e^{-\gamma t}\begin{pmatrix} p\cos(pt)+(\beta-\gamma)\sin(pt) \\ (\beta-2\gamma)p\cos(pt)-(\beta-\gamma)(\beta+2\gamma)\sin p(t) \end{pmatrix}.$$

Then, the following integral functional representations

$$f(t) = y_1(t) = \int_{-\infty}^{t} \tfrac{1}{p}e^{-\gamma(t-s)}\left[p\cos(p(t-s))+(\beta-\gamma)\sin(p(t-s))\right]h_\varepsilon(s,\omega)ds$$

$$\begin{aligned} \dot{f}(t) &= y_1(t) + r_1h_\varepsilon(t,\omega) \\ &= \int_{-\infty}^{t} \tfrac{1}{p}e^{-\gamma(t-s)}\left[(\beta-2\gamma)p\cos(p(t-s))\right. \\ &\qquad \left. -(\beta-\gamma)(\beta+2\gamma)\sin(p(t-s))\right]h_\varepsilon(s,\omega)ds + h_\varepsilon(t,\omega) \end{aligned}$$

for f and $\dot{f}$ are derived. Finally, from the form filter equation

$$
\begin{aligned}
&\ddot{f}(t,\omega) = -\beta^2 f(t) - 2\gamma\dot{f}(t) + \beta h_\varepsilon(t,\omega) + \dot{h}_\varepsilon(t,\omega) \\
&= \int_{-\infty}^{t} \frac{1}{p} e^{-\gamma(t-s)} \big[-\beta^2 - 2\gamma(\beta - 2\gamma)p\cos(p(t-s)) \\
&\quad + (-\beta^2)(\beta-\gamma) - 2\gamma(-(\beta-\gamma)(\beta+2\gamma))\sin p(t-s)\big]\, h_\varepsilon(s,\omega)\, ds \\
&\quad +\beta h_\varepsilon(t,\omega) + \dot{h}_\varepsilon(t,\omega) \\
&= \int_{-\infty}^{t} \frac{1}{p} e^{-\gamma(t-s)} \left[(3\gamma-\beta)(\gamma+\beta)p\cos(p(t-s))\right. \\
&\quad \left. -(\beta-\gamma)(\beta^2 - 2\beta\gamma - 4\gamma^2)\sin p(t-s)\right] h_\varepsilon(s,\omega)\, ds + \beta h_\varepsilon(t,\omega) + \dot{h}_\varepsilon(t,\omega)
\end{aligned}
$$

is obtained.

6 Smoothing

In the preceding sections a stationary random function $\xi(t,\omega)$ with a prescribed fractional rational spectral density has been approximated by an integral functional of a weakly correlated random function h_ε

$$f(t,\omega) = \int_{-\infty}^{t} F(t-s) h_\varepsilon(s,\omega)\, ds.$$

The kernel function F is found to be (see Eq. (18))

$$
\begin{aligned}
F(t) &= [\exp(\mathbf{A}t)\mathbf{r}]_1 = \left[\mathbf{V}\exp(\Lambda t)\mathbf{V}^{-1}\mathbf{r}\right]_1 \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} V_{1i}[V^{-1}]_{ij} r_j \right) e^{\lambda_i t}
\end{aligned}
$$

where the eigenvalue decomposition $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1}$ with $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and the assumption of simple eigenvalues has been used.

From section 2.1 it is known that the eigenvalues of $\mathbf{A}$ are equal to the zeros of the polynomial Q_n which are denoted by (see (28)) $-\mu_1, \ldots, -\mu_{n_1}, -\mathrm{Im}\,\eta_1 \pm i\mathrm{Re}\,\eta_1, \ldots, -\mathrm{Im}\,\eta_{n_2} \pm i\mathrm{Re}\,\eta_{n_2}$ with $n_1 + 2n_2 = n$. Then, there exists a representation of F of the form

$$F(t) = \sum_{k=1}^{n_1+n_2} e^{-\gamma_k t}(u_k\cos(\beta_k t) + v_k\sin(\beta_k t)) \qquad (32)$$

with non-negative coefficients

$$
\begin{aligned}
\gamma_k &= \begin{cases} \mu_k, & k = 1, \dots, n_1 \\ \operatorname{Im} \eta_{\mathrm{k}-\mathrm{n}_1}, & k = n_1 + 1, \dots, n_1 + n_2 \end{cases}, \\
\beta_k &= \begin{cases} 0, & k = 1, \dots, n_1 \\ \operatorname{Re} \eta_{\mathrm{k}-\mathrm{n}_1}, & k = n_1 + 1, \dots, n_1 + n_2, \end{cases}
\end{aligned}
$$

and some real coefficients u_k, v_k, $k = 1, \dots, n_1 + n_2$, which can be determined from $\mathbf{V}, \mathbf{V}^{-1}$ and $\mathbf{r}$. Hence, F is infinitely often differentiable and bounded, it holds

$$
|F(t)| \leq \sum_{k=1}^{n_1+n_2} |u_k| + |v_k| =: M_F. \tag{33}
$$

Eqs. (18) and (19) give a representation of f as well as of its derivatives. It can be observed that also derivatives of f up to an order $n - m - 1$ can be written as integral functionals while for higher order derivatives additional terms in which the weakly correlated function h_ε and its derivatives are involved has to be taken into account.

The aim of this section is to smooth $f(t, \omega)$ in order to ensure representations of higher order derivatives $f^{(k)}$ as integral functionals, only. The latter property is very helpful in stochastic analysis of the stationary response of vibrating systems with an external excitation containing random functions with fractional rational spectral density. Then, depending on modeling also first- and second-order derivatives (e.g. velocities and accelerations) of excitation functions can be involved. If, for example, second-order derivatives f'' have to be modeled as integral functionals a form filter equation with $n - m - 1 \geq 2$, i.e. at least an equation of order $n = 3$, has to be used. In case of examples considered in section 5 the function $\ddot{f}$ cannot be represented by an integral functional.

The essential idea of the considered smoothing procedure is to slightly modify the kernel function F by multiplying with a sufficiently smooth function denoted by $F_0(., \delta)$ which increases from 0 to 1 in a small interval $[0, \delta]$ and is equal to 1 in (δ, ∞). It will be seen that this modification for small δ approximately preserves the correlation function and the spectral density of the integral functional for f (while the corresponding moment functions of derivatives $f^{(k)}$, $k = 1, \dots, n - m - 1$ may be changed essentially).

Denote the modified kernel function by F_δ, i.e. $F_\delta(t) := F_0(t, \delta) F(t)$, and the resulting integral functional by f_δ, i.e.

$$
f_\delta(t, \omega) := \int_{-\infty}^{t} F_\delta(t - s) h_\varepsilon(s, \omega) \, ds.
$$

Then, the differentiation of the above integral functional leads to

$$f_\delta^{(k)}(t,\omega) = \int\limits_{-\infty}^{t} F_\delta^{(k)}(t-s)h_\varepsilon(s,\omega)\,ds + \sum_{l=0}^{k-1} F_\delta^{(l)}(0)\,h_\varepsilon^{(k-1-l)}, \quad k=1,2,\dots,$$

provided h_ε is $k-1$ times a.s. differentiable and $F_\delta \in C^k(\mathbb{R}_+)$.

A sufficient condition for the vanishing of the sum on the right hand side is $F_\delta^{(l)}(0)=0,\ l=0,\dots,k-1$. Since

$$F_\delta^{(l)}(t) = \sum_{j=0}^{l} \binom{l}{j} F_0^{(j)}(t,\delta) F^{(l-j)}(t)$$

this condition is fulfilled obviously if $F_0^{(l)}(0,\delta)=0,\ l=0,\dots,k-1$.

The order of differentialibity of F_δ can be controlled by the corresponding order of differentialibilty of F_0 since F is a $C^\infty(\mathbb{R}_+)$-function (see Eq. (32)). Hence, if $F_0(.,\delta)\in C^k(\mathbb{R}_+)$ then $F_\delta \in C^k(\mathbb{R}_+)$. Since $F_0(t)=1$ for $t>\delta$ the conditions

$$F_0(\delta,\delta)=1 \quad \text{and additionally} \quad F_0^{(l)}(\delta,\delta)=0, \quad l=1,\dots,k,$$

have to be fulfilled.

Summarizing, there are k conditions for the value of F_0 and its derivatives for $t=0$ and further $k+1$ conditions for $t=1$. Functions satisfiying these conditions can be found in the class of polynomials with real coefficients of order $2k$.

Now expansions of second-order moment functions of f and its modification f_δ in powers of the correlation length ε are compared. From section 3.1 it is known that there hold the expansions

$$\begin{aligned}
R_{ff}(\tau) &= \varepsilon R^1_{ff}(\tau) + o(\varepsilon), \quad \tau\in\mathbb{R}\\
S_{ff}(\alpha) &= \varepsilon S^1_{ff}(\alpha) + o(\varepsilon), \quad \alpha\in\mathbb{R},\ |\alpha|\le\alpha_{max}<\infty\\
\text{where} \quad R^1_{ff}(\tau) &= a\int\limits_0^\infty F(u)F(u+|\tau|)\,d\tau\\
\text{and} \quad S^1_{ff}(\alpha) &= \frac{1}{\pi}\int\limits_0^\infty \cos(\alpha\tau)\,R^1_{ff}(\tau)\,d\tau.
\end{aligned}$$

In case of modified integral functional f_δ where the kernel function F_δ is used instead of F the corresponding leading expansion terms are denoted by $R^1_{f_\delta f_\delta}(\tau)$ and $S^1_{f_\delta f_\delta}(\alpha)$, respectively. Then, the following limit theorem holds for $\delta\downarrow 0$.

Theorem

For the expansion terms $R^1_{f_\delta f_\delta}(\tau)$ and $S^1_{f_\delta f_\delta}(\alpha)$ of the expansion of second-order moment functions $R_{f_\delta f_\delta}(\tau)$ and $S_{f_\delta f_\delta}(\alpha)$ in powers of the correlation length ε it holds

i) $\lim\limits_{\delta\downarrow 0} R^1_{f_\delta f_\delta}(\tau) = R^1_{ff}(\tau) \quad$ *for* $\tau \in \mathbb{R}$,

ii) $\lim\limits_{\delta\downarrow 0} S^1_{f_\delta f_\delta}(\alpha) = S^1_{ff}(\alpha) \quad$ *for* $\alpha \in \mathbb{R}$.

Proof: It holds

$$
\begin{aligned}
R^1_{f_\delta f_\delta}(\tau) &= a\int_0^\infty F_\delta(u)F_\delta(u+|\tau|)\,du \\
&= a\left\{\int_0^\delta F_\delta(u)F_\delta(u+|\tau|)\,du + \int_\delta^\infty F(u)F(u+|\tau|)\,du\right. \\
&\quad \left. + \left(\int_0^\delta F(u)F(u+|\tau|)\,du - \int_0^\delta F(u)F(u+|\tau|)\,du\right)\right\}.
\end{aligned}
$$

Then, for $\tau \in \mathbb{R}$ holds

$$
\begin{aligned}
\left|R^1_{f_\delta f_\delta}(\tau) - R^1_{ff}(\tau)\right| &= a\left|\int_0^\delta \left(F_\delta(u)F_\delta(u+|\tau|) - F(u)F(u+|\tau|)\right)du\right| \\
&\leq \delta\cdot a \sup_{0\leq s\leq\delta} |F(u)F(u+|\tau|)|\ |1 - F_0(u,\delta)F_0(u+|\tau|,\delta)| \\
&\leq \delta\cdot aM_F^2\cdot 1 \quad \underset{\delta\downarrow 0}{\to} 0 \qquad (34)
\end{aligned}
$$

where the upper bound M_F defined in (33) has been used.

The proof of the second statement makes use of Lebesgue's Theorem on dominated convergence applied to the function $g_\delta(\tau) := \cos(\alpha\tau)R^1_{f_\delta f_\delta}(\tau)$. It holds

$$
\lim_{\delta\downarrow 0} g_\delta(\tau) = \cos(\alpha\tau)R^1_{ff}(\tau) =: g(\tau)
$$

since the estimate (34) implies

$$
\begin{aligned}
|g_\delta(\tau) - g(\tau)| &= |cos(\alpha\tau)||(R^1_{f_\delta f_\delta}(\tau) - R^1_{ff}(\tau))| \\
&\leq \delta\cdot aM_F^2 \quad \underset{\delta\downarrow 0}{\to} 0, \quad \forall\tau\in\mathbb{R}.
\end{aligned}
$$

Further, it holds

$$\begin{aligned}
|g_\delta(\tau)| &\le |\cos(\alpha\tau)| a \int_0^\infty |F_\delta(u) F_\delta(u+|\tau|)\, du \\
&\le 1 \cdot a \int_0^\infty |F(u)F(u+|\tau|)|\, du \\
&\le a \int_0^\infty \left(\sum_{k=1}^{n_1+n_2} e^{-\gamma_k u}(|u_k|+|v_k|) \right)\left(\sum_{k=1}^{n_1+n_2} e^{-\gamma_k (u+|\tau|)}(|u_k|+|v_k|) \right) du \\
&\le aU^2(n_1+n_2)^2 \int_0^\infty e^{-\gamma u} e^{-\gamma(u+|\tau|)}\, du \\
&= aU^2(n_1+n_2)^2 \frac{1}{2\gamma}\, e^{-\gamma|\tau|} =: \varphi(\tau)
\end{aligned}$$

where the representation (32) of the kernel function F and the notation

$$U := \max_{1\le k\le n_1+n_2} |u_k| + |v_k| \quad \text{and} \quad \gamma := \min_{1\le k\le n_1+n_2} \gamma_k$$

have been used. Then, for the majorizing function $\varphi(\tau)$ it holds

$$\int_0^\infty \varphi(\tau)\, d\tau = \frac{aU^2(n_1+n_2)^2}{2\gamma} \cdot \frac{1}{\gamma} < \infty$$

From Lebesgue's Theorem on dominated convergence then follows

$$\lim_{\delta\downarrow 0} \int_0^\infty g_\delta(\tau)\, d\tau = \int_0^\infty g(\tau)\, d\tau$$

and for $\alpha \in \mathbb{R}$ it holds

$$\begin{aligned}
\lim_{\delta\downarrow 0} S^1_{f_\delta f_\delta}(\alpha) &= \lim_{\delta\downarrow 0} \frac{1}{\pi} \int_0^\infty \cos(\alpha\tau) R^1_{f_\delta f_\delta}(\tau)\, d\tau \\
&= \frac{1}{\pi} \int_0^\infty \cos(\alpha\tau) R^1_{ff}(\tau)\, d\tau = S^1_{ff}(\alpha)
\end{aligned}$$

∎

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