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Abstract

The paper deals with the determination of statistical characteristics of eigenvalues for a class of ordinary differential operators with random coefficients. This problem arises from the computation of eigenfrequencies for the bending vibrations of beams possessing random geometry and material properties. Representations of eigenvalues are found by applying the Ritz method and perturbation results for matrix eigenvalue problems. Approximations of the probability density function and the moments of the random eigenvalues are given by means of expansions in powers of the correlation length of weakly correlated random functions which are used for modelling the random terms. The eigenvalue statistics determined analytically are compared favourably with Monte-Carlo simulations.

1 Introduction

The paper considers the computation of eigenfrequencies for the bending vibrations of beams possessing random geometry and random material properties. This investigation leads to an eigenvalue problem for ordinary differential operators with random coefficient functions. Then the resulting eigenvalues are random variables for which certain statistical characteristics as probability density functions and moments have to be determined.

The random terms involved in the problem can be the radius of the circular cross section of the beam and the mass per unit. Both quantities are assumed to have random fluctuations around a constant mean along the length of the beam. The random fluctuations are modelled by Gaussian weakly correlated random processes. The concept of weakly correlated processes is based on the idea that these processes have no distant effect. The values of the process at two points are independent if the distance of these points exceeds a certain quantity $\varepsilon > 0$. This quantity ε is refered to as the correlation length of the random process and is assumed to be sufficiently small (see [6, 8]).

Approximations of probability density functions and moments of the random eigenvalues are found by applying the Ritz method, perturbation results and using expansions in powers of the correlation length ε. In [8] the asymptotic normality of the eigenvalues for $\varepsilon \downarrow 0$ is proved. Hence, the eigenvalue distributions may be approximated by the corresponding Gaussian limit distributions. In the following these limit distributions will

be refered to as classical or low-order approximations since they take into account only the leading terms of the expansions in powers of ε.

Results of Monte-Carlo simulations show a good coincidence with the low-order approximations if ε is near to zero. However, regarding larger values of ε significant differences between low-order approximations and estimated statistical characteristics come into consideration. In particular deviations from normality have been observed. In order to find more accurate approximations in this paper higher-order expansions are derived, i.e. expansions including not only the leading but also the following terms. The higher-order approximations will show a much improved coincidence with the results from Monte-Carlo simulation.

Section 2 deals with the boundary value problem for an ordinary differential operator which describes bending vibrations of beams. Further the model of random coefficients resulting from the random geometry and material properties of the beam is introduced. In Section 3 applying the Ritz method an eigenvalue problem for random matrices is derived. The eigenvalues are represented by perturbation series with respect to the random fluctuations.

In Section 4 these representations are used to find moments and probability density functions of the eigenvalues by means of expansions in powers of the correlation length ε of the weakly correlated processes involved in the model of the random fluctuations. Section 5 gives a sketch of the Monte-Carlo simulation procedure investigating statistical estimates of probability density functions and moments of the eigenvalues. Finally, Section 6 presents numerical results and compares statistical estimates with the approximations found in Section 4.

A further application of the presented method to buckling problems of a simply supported beam with a random geometry can be found in [7].

2 Bending vibrations of random beams

We consider a thin and simply supported beam of length $l = 1$ possessing a circular cross section with radius r (see Figure 1). Let the cross sectional area be denoted by A, the mass per unit by ρ, the bending stiffness by $E_0 I$ with the modulus of elasticity E_0 and the moment of inertia of the cross sectional area I. Bending vibrations are investigated in terms of the transverse displacement $u = u(x)$ of the beam. The squares of the eigenfrequencies α_i^2, $i = 1, 2 \ldots$, for the bending vibrations are found to be equal to the eigenvalues λ_i, $i = 1, 2 \ldots$, of the boundary value problem for u (see [2])

$$\begin{aligned} &(E_0 I u'')'' = \lambda \rho A u, \\ &u(0) = u(1) = u''(0) = u''(1) = 0. \end{aligned} \tag{1}$$

Now we describe the random geometry of the beam. The cross sectional area is assumed to be circular, where the radius r varies randomly along the length of the beam, i.e. r is a random process $r = r(x, \omega)$, $0 \leq x \leq 1$, and therefore

$$A = A(x, \omega) = \pi\, r^2(x, \omega) \quad \text{and} \quad I = I(x, \omega) = \frac{\pi}{2} r^4(x, \omega)$$

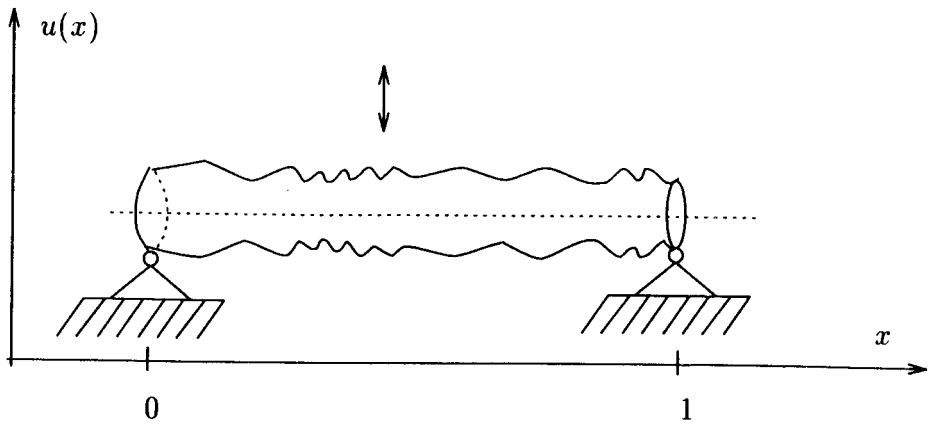


Figure 1: Simply supported random beam with circular cross section

are random processes, too. Further we consider random material properties by modelling the mass per unit as a random process $\rho = \rho(x,\omega)$ while in this paper the modulus of elasticity E_0 is assumed to be constant.

To be more precise the radius and the mass per unit are assumed to have the form of random perturbations of the constant mean values

$$r(x,\omega) = r_0 + r_\varepsilon(x,\omega) \quad \text{and} \quad \rho(x,\omega) = \rho_0 + \rho_\varepsilon(x,\omega)$$

respectively, where r_ε and ρ_ε are independent and weakly correlated random processes with correlation length ε. The characteristic property of weakly correlated processes is that they do not have a distant effect. The values of the process at two points do not correlate (are independent) if the distance between these points exceeds the correlation length ε. The correlation length is supposed to be sufficiently small. A great number of random phenomena can be modelled by these functions. The idea of weakly correlated processes goes back to Ornstein and Uhlenbeck (see [5]) who investigated the Brownian motion of a small particle surrounded by a gas. A detailed representation of the theory and a large number of applications to equations of mathematical physics can be found in [6].

The definition of weakly correlated processes contains

$$\mathbf{E}\{r_\varepsilon(x)\} = \mathbf{E}\{\rho_\varepsilon(x)\} = 0$$

and

$$\begin{aligned} \mathbf{E}\{r_\varepsilon(x)r_\varepsilon(y)\} = 0 \quad &\text{and} \quad \mathbf{E}\{\rho_\varepsilon(x)\rho_\varepsilon(y)\} = 0 \quad \text{for } |x-y| > \varepsilon \\ \mathbf{E}\{r_\varepsilon(x)r_\varepsilon(y)\} = R_\varepsilon(x,y) \quad &\text{and} \quad \mathbf{E}\{\rho_\varepsilon(x)\rho_\varepsilon(y)\} = Q_\varepsilon(x,y) \quad \text{for } |x-y| \le \varepsilon \end{aligned}$$

for the means and the correlation functions of $r_\varepsilon(x)$, $\rho_\varepsilon(x)$ respectively, where $R_\varepsilon(.,.)$ and $Q_\varepsilon(.,.)$ denote some correlation functions. Additionally, we assume that $r_\varepsilon(x,\omega)$ and $\rho_\varepsilon(x,\omega)$ are Gaussian processes. Then, all information about the distribution of r_ε and ρ_ε is contained in the first- and second-order moments, i.e. the functions $R_\varepsilon(.,.)$ and $Q_\varepsilon(.,.)$ describe the processes completely. Especially the representation of higher-order moments of Gaussian processes in terms of the correlation function can be applied.

Since the coefficients of the eigenvalue problem (1) are random processes the eigenvalues λ_i and therefore the eigenfrequencies $\alpha_i = \sqrt{\lambda_i}$, $i = 1, 2, \ldots$, are random variables. The aim of the investigation consists in the computation of the probability density function (p.d.f.) $p_{\lambda_i}(z)$ of the random eigenvalues

$$\mathbf{P}(\lambda_i(\omega) < z) = \int_{-\infty}^{z} p_{\lambda_i}(s)\, ds, \quad z \in \mathbb{R},$$

and of moments, especially the mean $m_{\lambda_i} = \mathbf{E}\{\lambda_i\}$ and the variance $\sigma^2_{\lambda_i} = \mathbf{E}\{(\lambda_i - m_{\lambda_i})^2\}$.

As a first step to find approximate results the Ritz method is applied to derive an eigenvalue problem for random matrices (see Section 3). For this end the coefficient functions of (1) are denoted by

$$\begin{aligned} f(x,\omega) &:= E_0 I(x,\omega) = E_0 \frac{\pi}{2}(r_0 + r_\varepsilon(x,\omega))^4 \\ g(x,\omega) &:= \rho(x,\omega)A(x,\omega) = \pi(\rho_0 + \rho_\varepsilon(x,\omega)(r_0 + r_\varepsilon(x,\omega))^2 \end{aligned}$$

and the corresponding fluctuations around the means $\mathbf{E}\{f(x)\}$ and $\mathbf{E}\{g(x)\}$ by

$$\begin{aligned} \bar{f}_\varepsilon(x,\omega) &:= f(x,\omega) - \mathbf{E}\{f(x)\} \\ &= \frac{\pi}{2} E_0 \left[(r_0 + r_\varepsilon(x,\omega))^4 - \mathbf{E}\left\{(r_0 + r_\varepsilon(x))^4\right\}\right] \\ \bar{g}_\varepsilon(x,\omega) &:= g(x,\omega) - \mathbf{E}\{g(x)\} \\ &= \pi\left[(\rho_0 + \rho_\varepsilon(x,\omega))(r_0 + r_\varepsilon(x,\omega))^2 - \rho_0 \mathbf{E}\left\{(r_0 + r_\varepsilon(x))^2\right\}\right]. \end{aligned} \tag{2}$$

Since r_ε and ρ_ε are independent and weakly correlated processes the processes $\bar{f}_\varepsilon$ and $\bar{g}_\varepsilon$ are weakly correlated, too. Furthermore, the vector process $(\bar{f}_\varepsilon, \bar{g}_\varepsilon)^T$ is weakly correlated connected with the correlation length ε (see [6]).

Finally, using the notations above the eigenvalue problem (1) can be written as

$$\begin{aligned} &(fu'')'' = \lambda g u, \\ &u(0) = u(1) = u''(0) = u''(1) = 0. \end{aligned} \tag{3}$$

3 Ritz method for random eigenvalue problems

Using the Ritz method a sequence of functions $\{\phi_i\}_{i=1}^{\infty}$ is chosen to be a basis of the energetic space of the differential operator $(fu'')''$ and the boundary conditions in (3). In Michlin [4] the polynomials

$$\begin{aligned} \phi_1(x) &= x - 2x^3 + x^4 \\ \phi_2(x) &= 7x - 10x^3 + 3x^5 \\ \phi_i(x) &= x^i(1-x)^3, \quad i \geq 3. \end{aligned} \tag{4}$$

are proposed which are used afterwards.

The resulting Ritz equations can be written as a symmetric matrix eigenvalue problem

$$\left(\boldsymbol{A} + \overline{\boldsymbol{A}}(\omega)\right)\, {}^n\boldsymbol{x} = {}^n\lambda \left(\boldsymbol{B} + \overline{\boldsymbol{B}}(\omega)\right)\, {}^n\boldsymbol{x} \tag{5}$$

where ${}^n\lambda \in \mathbb{R}$ denote the eigenvalues and ${}^n\boldsymbol{x} \in \mathbb{R}^n$ the eigenvectors. The elements of the $n \times n$ matrices $\boldsymbol{A}, \boldsymbol{B}, \overline{\boldsymbol{A}}(\omega)$ and $\overline{\boldsymbol{B}}(\omega)$ have the form

$$a_{ij} = \int_0^1 \mathbf{E}\{f(x)\}\ \phi_i''(x)\,\phi_j''(x)\,dx \qquad \bar{a}_{ij}(\omega) = \int_0^1 \bar{f}_\varepsilon(x,\omega)\,\phi_i''(x)\,\phi_j''(x)\,dx$$

$$b_{ij} = \int_0^1 \mathbf{E}\{g(x)\}\ \phi_i(x)\,\phi_j(x)\,dx \qquad \bar{b}_{ij}(\omega) = \int_0^1 \bar{g}_\varepsilon(x,\omega)\,\phi_i(x)\,\phi_j(x)\,dx,$$

for $1 \leq i,j \leq n$. The eigenvalues and the corresponding eigenvectors of the n-dimensional matrix eigenvalue problem (5) are denoted by ${}^n\lambda_g(\omega)$ and ${}^n\boldsymbol{x}_g$ for $g = 1,\ldots,n$, respectively, where it is assumed that

$${}^n\lambda_1(\omega) \leq {}^n\lambda_2(\omega) \leq \ldots \leq {}^n\lambda_n(\omega) \qquad \text{a.s.} \quad .$$

Then, approximations of the eigenvalues and eigenfunctions of (3) (or (1)) can be obtained by

$$\begin{aligned} \lambda_g(\omega) &\approx {}^n\lambda_g(\omega) \\ {}^n u_g(x,\omega) &\approx \sum_{i=1}^n {}^n x_{gi}(\omega)\,\phi_i(x). \end{aligned}$$

Approximations of the p.d.f. and the moments of $\lambda_g(\omega)$ can be computed by the corresponding characteristics of ${}^n\lambda_g(\omega)$. Therefore perturbation series of the eigenvalues ${}^n\lambda_g(\omega)$ with respect to the random perturbations $\overline{\boldsymbol{A}}(\omega)$ and $\overline{\boldsymbol{B}}(\omega)$ (see Eq. (5)) are derived.

In a first step the averaged matrix eigenvalue problem

$$\boldsymbol{A}\ {}^n\boldsymbol{y} = {}^n\mu\,\boldsymbol{B}\ {}^n\boldsymbol{y} \tag{6}$$

is considered which can be obtained from Eq. (5) by replacing the random matrices $\overline{\boldsymbol{A}}(\omega)$ and $\overline{\boldsymbol{B}}(\omega)$ by their zero means. The matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are real and symmetric, $\boldsymbol{B}$ is positive definite. Denote the eigenvalues of Eq. (6) by

$${}^n\mu_1 \leq {}^n\mu_2 \leq \ldots \leq {}^n\mu_n$$

and the corresponding eigenvectors by ${}^n\boldsymbol{y}_g = ({}^n y_{g1},\ldots,{}^n y_{gn})^\tau$, $g = 1,\ldots,n$. The eigenvectors ${}^n\boldsymbol{y}_g$ are assumed to be normalized by

$$\left\langle \boldsymbol{B}\ {}^n\boldsymbol{y}_g,\ {}^n\boldsymbol{y}_h \right\rangle = \delta_{gh}, \qquad g,h = 1,\ldots,n,$$

with the scalar product $\langle .,. \rangle$ of $\mathbb{R}^n$. Additionally, the eigenvalues μ_g, $g = 1,\ldots,n$, are supposed to be simple.

The eigenvalues ${}^n\lambda_g(\omega)$, $g = 1,\ldots,n$, can be represented by

$${}^n\lambda_g(\omega) = {}^n\mu_g - \sum_{k=1}^\infty {}^n\lambda_{gk}(\omega)$$

where ${}^n\lambda_{gk}(\omega)$ denote the homogeneous terms of order k with respect to the perturbations $\overline{a}_{ij}(\omega)$ and $\overline{b}_{ij}(\omega)$. Then the relations

$$
\begin{aligned}
{}^n\lambda_{g1}(\omega) &= {}^g\hat{s}_{gg}(\omega) \\
{}^n\lambda_{g2}(\omega) &= \sum_{\substack{k=1 \\ k\neq g}}^{n} \frac{1}{\mu_{kg}} \; {}^g\hat{s}^2_{gk}(\omega) \; - \hat{d}_{gg}(\omega)\, {}^g\hat{s}_{gg}(\omega) \\
{}^n\lambda_{g3}(\omega) &= \sum_{\substack{k,j=1 \\ k,j\neq g}}^{n} \frac{1}{\mu_{kg}\mu_{jg}} \; {}^g\hat{s}_{gk}(\omega)\, {}^g\hat{s}_{kj}(\omega)\, {}^g\hat{s}_{jg}(\omega) - \sum_{\substack{k=1 \\ k\neq g}}^{n} \frac{1}{\mu^2_{kg}} \; {}^g\hat{s}_{gg}(\omega)\, {}^g\hat{s}^2_{gk}(\omega) \\
&\quad - \sum_{\substack{k=1 \\ k\neq g}}^{n} \frac{1}{\mu_{kg}} \left[\hat{d}_{gg}(\omega)\, {}^g\hat{s}^2_{gk}(\omega) + 2\, {}^g\hat{s}_{gg}(\omega)\, \hat{d}_{gk}(\omega)\, {}^g\hat{s}_{gk}(\omega) \right] + \hat{d}^2_{gg}(\omega)\, {}^g\hat{s}_{gg}(\omega).
\end{aligned}
$$

are obtained (see [3]) where the notations

$$
\begin{aligned}
\mu_{kg} &:= {}^n\mu_k - {}^n\mu_g, \quad k,g = 1,\ldots,n, \\
{}^g\hat{s}_{ij}(\omega) &:= \sum_{r,s=1}^{n} {}^n y_{ir}\, {}^n y_{js} ({}^n\mu_g \overline{b}_{rs}(\omega) - \overline{a}_{rs}(\omega)), \quad i,j = 1,\ldots,n, \\
\hat{d}_{ij}(\omega) &:= \sum_{r,s=1}^{n} {}^n y_{ir}\, {}^n y_{js}\, \overline{b}_{rs}(\omega), \quad i,j = 1,\ldots,n
\end{aligned}
$$

have been used.

4 Expansion of moments and probability density function

In this section expansions of the moments and the p.d.f. of the eigenvalues ${}^n\lambda_g(\omega)$ of the matrix eigenvalue problem (3) are investigated to find approximations of the corresponding eigenvalue characteristics with respect to the original problem (1).

Using the results of Section 3 the eigenvalue ${}^n\lambda_g(\omega)$ can be represented by the perturbation series

$$
{}^n\lambda_g(\omega) = {}^n\mu_g(\omega) - ({}^n\lambda_{g1}(\omega) + {}^n\lambda_{g2}(\omega) + {}^n\lambda_{g3}(\omega) + {}^n h_g(\omega))
$$

in terms of the random variables $\overline{a}_{ij}(\omega)$ and $\overline{b}_{ij}(\omega)$. It can be seen that ${}^n\lambda_g(\omega)$ contains linear combinations of ${}^g\hat{s}_{ij}(\omega)$ and $\hat{d}_{ij}(\omega)$ as well as products of two and three of these random terms and the remainder term ${}^n h_g(\omega) = \sum_{k=4}^{\infty} {}^n\lambda_{gk}(\omega)$ consists of linear combinations of products of at least four such random terms. The random variables ${}^g\hat{s}_{ij}(\omega)$ and $\hat{d}_{ij}(\omega)$ itself are linear combinations of the random perturbations $\overline{a}_{ij}(\omega)$ and $\overline{b}_{ij}(\omega)$. Hence, the eigenvalues ${}^n\lambda_g(\omega)$ can be represented by a polynomial non-linear function $d(.)$ of the $2n^2$ random variables

$$
\overline{a}_{11}(\omega),\ldots,\overline{a}_{nn}(\omega),\ \overline{b}_{11}(\omega),\ldots,\overline{b}_{nn}(\omega)
$$

with a remainder term, i.e.

$$d(y_1,\dots,y_{2n^2}) = d_0 + \sum_{\alpha=1}^{2n^2} d_\alpha\, y_\alpha + \sum_{\alpha,\beta=1}^{2n^2} d_{\alpha\beta}\, y_\alpha y_\beta + \sum_{\alpha,\beta,\gamma=1}^{2n^2} d_{\alpha\beta\gamma}\, y_\alpha y_\beta y_\gamma + \left(\sum_{\alpha=1}^{2n^2} y_\alpha^2\right)^{\nu} \cdot h(y_1,\dots,y_n). \qquad (7)$$

This representation contains the real coefficients d_0, d_α, $d_{\alpha\beta}$, $d_{\alpha\beta\gamma}$, a real number $\nu > \frac{3}{2}$ and a function h which is bounded on $\mathcal{K}_\delta(0) = \{\mathbf{y} \in \mathbb{R}^n : ||\mathbf{y}|| \le \delta\}$ for a $\delta > 0$. In general, it is a difficult procedure to find at least approximately the distribution of $d(\bar{a}_{11},\dots,\bar{b}_{nn})$ for a given distribution of $\left(\bar{a}_{11}(\omega),\dots,\bar{b}_{nn}(\omega)\right)$. This can be established by the non-linearity of $d(.)$ and the mutual dependence of the random variables $\bar{a}_{11}(\omega),\dots,\bar{b}_{nn}(\omega)$.

In the considered case a practicable approximation of the desired distribution can be found since the random variables $\bar{a}_{11}(\omega),\dots,\bar{b}_{nn}(\omega)$ are integral functionals of the weakly correlated processes $\bar{f}_\varepsilon(x,\omega)$ and $\bar{g}_\varepsilon(x,\omega)$ given in (2), i.e. it holds

$$\bar{a}_{ij}(\omega) = \int_0^1 \bar{f}_\varepsilon(x,\omega)\,\phi_i''(x)\,\phi_j''(x)\,dx \quad \text{and} \quad \bar{b}_{ij}(\omega) = \int_0^1 \bar{g}_\varepsilon(x,\omega)\,\phi_i(x)\,\phi_j(x)\,dx$$

for $i,j = 1,\dots,n$. For these integral functionals as well as for polynomial non-linear functions (7) of integral functionals the theory of weakly correlated random functions (see [6, 3]) proves the asymptotic normality for $\varepsilon \downarrow 0$ and provides expansions of the moments and the p.d.f. in powers of the correlation length ε. For the sake of brevity we give only the reults used and refer to the literature.

First the moments of order k of $d(\omega)$ are considered which are given in terms of the "standardized" random variable

$$\tilde{d}(\omega) = \frac{t}{\sqrt{\varepsilon}}(d(\omega) - d_0) \qquad (8)$$

where the term t is determined by the condition

$$\mathbf{E}\left\{\left(\tilde{d} - \mathbf{E}\left\{\tilde{d}\right\}\right)^2\right\} = 1 + O(\varepsilon).$$

Then the moments of $\tilde{d}(\omega)$ can be derived in the form

$$\mathbf{E}\left\{\tilde{d}^k\right\} = \begin{cases} e_k + \frac{k!}{2^{\frac{k}{2}}}\left\{\frac{1}{(\frac{k-2}{2})!}R_{2,\tilde{d}} + \frac{1}{(\frac{k-4}{2})!}R_{4,\tilde{d}} + \frac{1}{(\frac{k-6}{2})!}R_{6,\tilde{d}}\right\}\cdot\varepsilon + o(\varepsilon) & k \text{ even} \\ \frac{k!}{2^{\frac{k-1}{2}}}\left\{\frac{1}{(\frac{k-1}{2})!}R_{1,\tilde{d}} + \frac{1}{(\frac{k-3}{2})!}R_{3,\tilde{d}}\right\}\cdot\sqrt{\varepsilon} + o(\varepsilon) & k \text{ odd} \end{cases} \qquad (9)$$

where

$$e_k = \begin{cases} \dfrac{k!}{2^{\frac{k}{2}}\cdot\left(\frac{k}{2}\right)!} = (k-1)\cdot(k-3)\cdot\ldots\cdot 3\cdot 1 & \text{for } k \text{ even and } k \ge 0, \\ 0 & \text{for } k \text{ odd or } k < 0 \end{cases}$$

and the terms $\frac{1}{j!}$ contained in Eq. (9) are defined to be zero for $j < 0$.

The terms $R_{i,\tilde{d}}$, $i = 1, \ldots, 4, 6$, are given in [3]. They can be determined by means of the so-called intensities

$$a_{ij}(x),\ a_{ijk}(x),\ a_{ijkl}(x), \quad i,j,k,l \in \{1,2\},$$

describing certain statistical characteristics of the involved weakly correlated processes $f_{1\varepsilon} := \bar{f}_\varepsilon$ and $f_{2\varepsilon} := \bar{g}_\varepsilon$ (see [3]). For instance, the intensity a_{ij} is defined by

$$a_{ij}(x) = \lim_{\varepsilon \downarrow 0} \int_{-\varepsilon}^{\varepsilon} \mathbf{E}\left\{f_{i\varepsilon}(x) f_{j\varepsilon}(x+z)\right\}\, dz.$$

Similarly, the intensities $a_{ijk}(x)$ and $a_{ijkl}(x)$ are given by the corresponding limits for the the third- and fourth-order moments, respectively. Since the random fluctuations of the radius $r_\varepsilon(x,\omega)$ and the random mass per unit $\rho_\varepsilon(x,\omega)$ are assumed to be Gaussian, these moments can be expressed in terms of the correlation functions $R_\varepsilon(x,y)$ of r_ε and $Q_\varepsilon(x,y)$ of ρ_ε.

Using the intensities the expressions $R_{i,\tilde{d}}$ can be found as sums of one-dimensional integrals over $[0,1]$ whose integrands contain products of the intensities of $\bar{f}_\varepsilon$ and $\bar{g}_\varepsilon$ and Ritz basis functions ϕ_i and/or its derivatives $\phi_i^{\text{“}}$, $i = 1, \ldots, n$.

Now the p.d.f. of the "standardized" variable $\tilde{d}(\omega)$ denoted by $p_{\tilde{d}}(u)$ is considered. Since for $\varepsilon \downarrow 0$ the random variable $\tilde{d}(\omega)$ is asymptotically Gaussian with zero mean and variance 1 it is convenient to study deviations from the normality for a correlation length $\varepsilon > 0$ by means of the Gram-Charlier series

$$p_{\tilde{d}}(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right) \sum_{k=0}^{\infty} (-1)^k \frac{c_k}{k!} H_k(u).$$

Thereby the functions $H_k(.)$ denote the Chebyshev-Hermite polynomials of order k defined by

$$\begin{aligned} H_k(u) &:= (-1)^k \exp\left(\frac{1}{2}u^2\right) \cdot \frac{d}{du^k} \exp\left(-\frac{1}{2}u^2\right) \\ &= \sum_{j=0}^{\left[\frac{k}{2}\right]} (-1)^j \frac{k!}{2^j j!(k-2j)!}\, u^{k-2j} \end{aligned}$$

and c_k are real coefficients which are to be determined. Applying the orthogonality of the Chebyshev-Hermite polynomials

$$\int_{-\infty}^{\infty} \exp\left(\frac{1}{2}u^2\right) H_k(u) H_l(u) du = \sqrt{2\pi} k! \delta_{kl},$$

the coefficients can be calculated as

$$\begin{aligned} c_k &= (-1)^k \int_{-\infty}^{\infty} p_{\tilde{d}}(u) H_k(u) du \\ &= \sum_{j=0}^{\left[\frac{k}{2}\right]} (-1)^{k+j} \frac{k!}{2^j j!(k-2j)!} \mathbf{E}\left\{\tilde{d}^{\,k-2j}\right\}. \end{aligned}$$

Substituting the moments $\mathbf{E}\left\{\tilde{d}^{k-2j}\right\}$ of the above representation for c_k by Eq. (9) the following expansion of the p.d.f. of $\tilde{d}(\omega)$ can be obtained containing terms of order $O(\sqrt{\varepsilon})$ and $O(\varepsilon)$

$$\begin{aligned} p_{\tilde{d}}(u) &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right) \left[1 + \left(R_{1,\tilde{d}} H_1(u) + \frac{1}{2} R_{3,\tilde{d}} H_3(u)\right)\sqrt{\varepsilon} \right. \\ &\quad \left. + \left(\frac{1}{2} R_{2,\tilde{d}} H_2(u) + \frac{1}{4} R_{4,\tilde{d}} H_4(u) + \frac{1}{8} R_{6,\tilde{d}} H_6(u)\right)\varepsilon + o(\varepsilon)\right]. \end{aligned}$$

Finally, approximations of the moments and the p.d.f. of $d(\omega)$ shall be derived. The relation (8) leads to

$$d(\omega) = \frac{\sqrt{\varepsilon}}{t}\tilde{d}(\omega) + d_0$$

and then to the mean and the variance

$$\begin{aligned} m_d = \mathbf{E}\{d\} &= d_0 + \frac{R_{1,\tilde{d}}}{t}\varepsilon + o(\varepsilon) \\ \sigma_d^2 = \mathbf{E}\left\{(d - m_d)^2\right\} &= \frac{1}{t^2}\left[\varepsilon + \left(R_{2,\tilde{d}} - R_{1,\tilde{d}}^2\right)\varepsilon^2\right] + o(\varepsilon^2). \end{aligned} \tag{10}$$

Furthermore, the p.d.f. of $d(\omega) - d_0$ can be approximated by

$$\begin{aligned} p_{d-d_0}(u) &= \frac{t}{\sqrt{\varepsilon}} p_{\tilde{d}}\left(\frac{t}{\sqrt{\varepsilon}}u\right) \\ &\approx \frac{t}{\sqrt{2\pi\varepsilon}} \exp\left(-\frac{t^2}{2\varepsilon}u\right) \left[1 + \left(R_{1,\tilde{d}} H_1\left(\frac{t}{\sqrt{\varepsilon}}u\right) + \frac{1}{2} R_{3,\tilde{d}} H_3\left(\frac{t}{\sqrt{\varepsilon}}u\right)\right)\sqrt{\varepsilon} \right. \\ &\quad \left. + \left(\frac{1}{2} R_{2,\tilde{d}} H_2\left(\frac{t}{\sqrt{\varepsilon}}u\right) + \frac{1}{4} R_{4,\tilde{d}} H_4\left(\frac{t}{\sqrt{\varepsilon}}u\right) + \frac{1}{8} R_{6,\tilde{d}} H_6\left(\frac{t}{\sqrt{\varepsilon}}u\right)\right)\varepsilon\right]. \end{aligned} \tag{11}$$

Therefore a higher-order approximation of the p.d.f. $p_{d-d_0}(u)$ has been derived. The corresponding low-order approximation

$$p_{d-d_0}(u) \approx \frac{t}{\sqrt{2\pi\varepsilon}} \exp\left(-\frac{t^2}{2\varepsilon}u\right)[1+0] = \frac{t}{\sqrt{2\pi\varepsilon}} \exp\left(-\frac{t^2}{2\varepsilon}u\right). \tag{12}$$

is obtained if all higher-order terms of the Gram-Charlier series in the representation (11) are neglected. Eq. (12) corresponds to the result of the asymptotic normality of $d(.)$ (see [8]).

Remark 1 Results for the joint distribution of a vector-valued non-linear function $\mathbf{d}(\omega) = (d_1, \ldots, d_m)^\tau$, $m \geq 1$, are also available (see [3]). These can be applied to investigate the joint distribution of several random eigenvalues.

5 Monte-Carlo simulation

The accuracy of the results above can be assessed by comparing the analytical results with estimations from Monte-Carlo simulations. For this purpose the eigenvalues of (1)

are determined numerically by means of the Ritz method for a couple of realizations of the Gaussian random processes $r_\varepsilon(x,\omega)$ and $\rho_\varepsilon(x,\omega)$.

A class of weakly correlated processes is chosen for which realizations can be generated simply. The generation is based on an equidistant partition of the domain of definition of the random processes $[0,1]$ which is slightly extended to avoid boundary effects. The values of the process at the partition nodes $\frac{i}{N}, i = -3,\dots,N+3$, $N > 1$, are obtained from a sequence of independent Gaussian random variables and the process values between the nodes are found by polynomial interpolation, i.e.

$$\begin{aligned} r_\varepsilon(x,\omega) &= \overline{q}_{ri}(x)\,\xi_i(\omega) + q_{ri}(x)\,\xi_{i+1}(\omega) \\ \rho_\varepsilon(x,\omega) &= \overline{q}_{\rho i}(x)\,\nu_i(\omega) + q_{\rho i}(x)\,\nu_{i+1}(\omega) \end{aligned}$$

for $x \in \left[\frac{i}{N}, \frac{i+1}{N}\right]$, $i = -3,\dots,N+2$. In these formulas (ξ_i) and (ν_i) denote sequences of independent zero mean Gaussian random variables with variances $\sigma_\xi^2 = \beta^2 r_0^2$ and $\sigma_\nu^2 = \gamma^2\rho_0^2$, $\beta,\gamma > 0$, respectively. By this choice the variances of the random radius and mass per unit are related to the constant means r_0 and ρ_0 and can be controlled by the parameters β and γ.

The interpolation functions $q_{ri}(x)$ and $q_{\rho i}(x)$ are defined by

$$\begin{aligned} q_{ri}(x) &= q_r(Nx-i) \quad \text{where} \quad q_r(x) := 6x^5 - 15x^4 + 10x^3 \\ q_{\rho i}(x) &= q_\rho(Nx-i) \quad \text{where} \quad q_\rho(x) := x \end{aligned}$$

for which hold $q_r(0) = q_\rho(0) = 0$ and $q_r(1) = q_\rho(1) = 1$. Furthermore we define $\overline{q}_{ri}(x) := 1 - q_{ri}(x)$ and $\overline{q}_{\rho i}(x) := 1 - q_{\rho i}(x)$. These interpolation polynomials guarantee the respective smoothness of the random coefficients of the eigenvalue problems (1) and (3), i.e. we derive continuity of the realizations of $\rho_\varepsilon(x,\omega)$ and of the second derivative of $r_\varepsilon(x,\omega)$.

Because of the independence of the random variables ξ_i and ν_i the processes $r_\varepsilon(x,\omega)$ and $\rho_\varepsilon(x,\omega)$ are independent and weakly correlated with the correlation length $\varepsilon = \frac{2}{N}$. In case of $x \in \left[\frac{i}{N}, \frac{i+1}{N}\right]$, $i = 0,\dots,N-1$, the correlation function of r_ε can be written as

$$\mathbf{E}\left\{r_\varepsilon(x)r_\varepsilon(y)\right\} = \sigma_\xi^2 \begin{cases} \overline{q}_{ri}(x)q_{ri-1}(y) & \text{for } y \in \left[\frac{i-1}{N}, \frac{i}{N}\right] \\ \overline{q}_{ri}(x)\overline{q}_{ri}(y) + q_{ri}(x)q_{ri}(y) & \text{for } y \in \left[\frac{i}{N}, \frac{i+1}{N}\right] \\ q_{ri}(x)\overline{q}_{ri+1}(y) & \text{for } y \in \left[\frac{i+1}{N}, \frac{i+2}{N}\right] \\ 0 & \text{otherwise.} \end{cases}$$

A similar representation is valid for the correlation function of ρ_ε.

Even though the random variables ξ_i, ν_i are identically distributed for $i = -3,\dots,N+3$ the processes r_ε and ρ_ε are not (weakly) stationary. This fact is caused by the interpolation between the nodes of the partition. However the processes are "periodically distributed", i.e. the values of r_ε or ρ_ε at two points x, y in $[0,1]$ are identically distributed for $|x-y| = \frac{k}{N}$, $k = 1,2,\dots$. These properties have to be taken into account for the computation of the intensities a_{ij}, a_{ijk} and a_{ijkl} mentioned in the previous section.

6 Application

For the purpose of illustration we present some results for the first (smallest) eigenvalue λ_1 of (1). Using this eigenvalue the first eigenfrequency $\alpha_1 = \sqrt{\lambda_1}$ of the beam can be computed. For instance in engineering this frequency leads to certain resonance conditions. Now moments and the p.d.f. of the random eigenvalue $\lambda_1 = \alpha_1^2$ are given. From these results subsets of the frequency domain can be determined which contain the first resonance frequency with a prescribed probability.

The results are given for the "normalized" eigenvalue

$$ {}^n\overline{\lambda}_1 := \frac{2\rho_0}{E_0 r_0^2}\, {}^n\lambda_1. $$

It is possible to study the influence of the random fluctuations of the radius and the mass per unit by variation of two parameters, namely the quantities β and γ which control the variances of the random radius and the mass per unit, respectively. For the special case $\beta = \gamma = 0$ the eigenvalue ${}^n\overline{\lambda}_1$ converges for $n \to \infty$ to the first eigenvalue of the non-perturbed differential eigenvalue problem, which is found to be $\overline{\mu}_1 = \pi^4 \approx 97.409$.

		Approximation		Simulation
		low-order	higher-order	
β	γ	$m_{n\overline{\lambda}_1} \approx {}^n\overline{\mu}_1$	$m_{n\overline{\lambda}_1} \approx {}^n\overline{\mu}_1 + \frac{\sqrt{\varepsilon}}{t} R_{1,n\overline{\lambda}_1}$	10^5 realizations
0	0.1	97.409	97.478	97.498
0.1	0.1	101.214	99.333	99.400
0.2	0.1	112.503	104.362	105.023
0.1	0	101.214	99.201	99.328
0.1	0.1	101.214	99.333	99.400
0.1	0.2	101.214	99.553	99.617

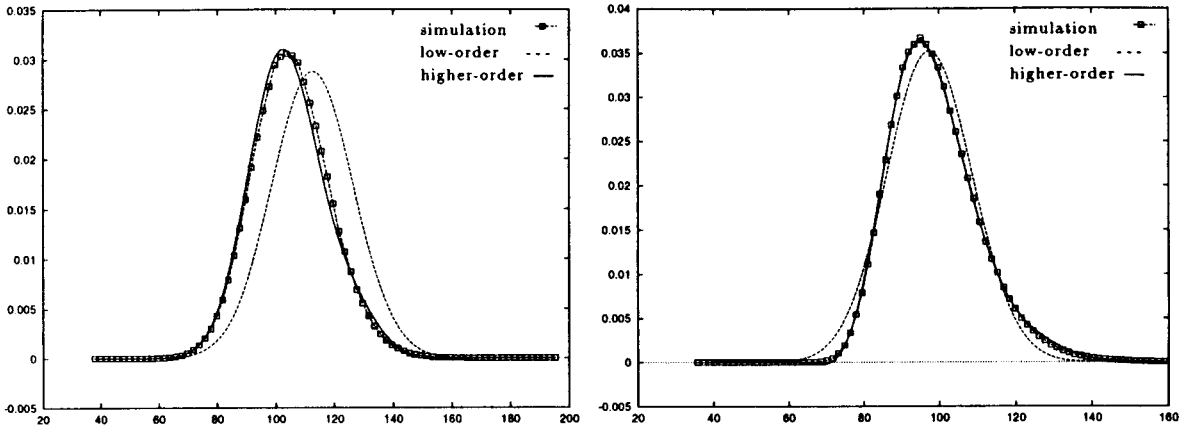
Table 1: Means of the first eigenvalue ${}^4\overline{\lambda}_1$ for $\varepsilon = 0.1$ ($N = 20$)

		Approximation		Simulation
		low-order	higher-order	
β	γ	$\sigma^2_{n\overline{\lambda}_1} \approx \frac{1}{t^2}\varepsilon$	$\sigma^2_{n\overline{\lambda}_1} \approx \frac{1}{t^2}\varepsilon + \frac{R_{2,n\overline{\lambda}_1} - R^2_{1,n\overline{\lambda}_1}}{t^2}\varepsilon^2$	10^5 realizations
0	0.1	7.116	7.157	7.158
0.1	0.1	38.582	37.617	37.455
0.2	0.1	161.122	146.759	143.797
0.1	0	30.732	30.056	29.906
0.1	0.1	38.582	37.617	37.455
0.1	0.2	62.133	60.852	60.692

Table 2: Variances of the first eigenvalue ${}^4\overline{\lambda}_1$ for $\varepsilon = 0.1$ ($N = 20$)

The dimension of the matrix eigenvalue problem (5) which results from (1) by applying the Ritz method was chosen $n = 4$. The value $n = 4$ leads to sufficiently accurate

results for the first eigenvalue. In Table 1 and 2 low- and higher-order approximations of the mean and the variance of ${}^n\overline{\lambda}_1$ are summarized which result from the expansions (10) for various values of the parameters β and γ and for a correlation length $\varepsilon = 0.1$. The analytical results are compared with Monte-Carlo simulations using a partition of $N = 20$ subintervals of $[0,1]$ (which gives $\varepsilon = \frac{2}{N} = 0.1$) and 10^5 realizations of $r_\varepsilon(x,\omega)$ and $\rho_\varepsilon(x,\omega)$.



(a) $\varepsilon = 0.1, \beta = 0.2, \gamma = 0.2$ (b) $\varepsilon = 0.2, \beta = 0, \gamma = 0.3$

Figure 2: P.d.f. of the first eigenvalue ${}^4\overline{\lambda}_1$: low- and higher-order approximation and Monte-Carlo simulation with 10^6 realizations

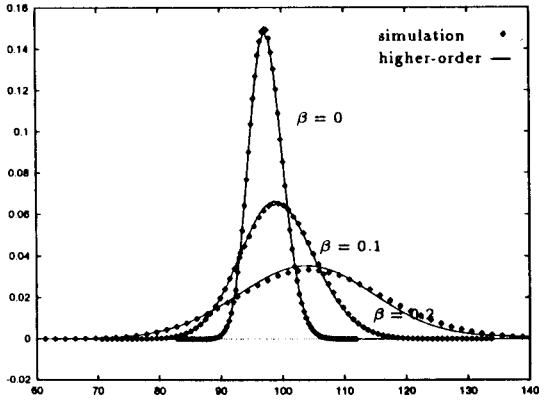


Figure 3: P.d.f. of the first eigenvalue ${}^4\overline{\lambda}_1$: higher-order approximation and Monte-Carlo simulation, $\varepsilon = 0.1$, $\gamma = 0.1$

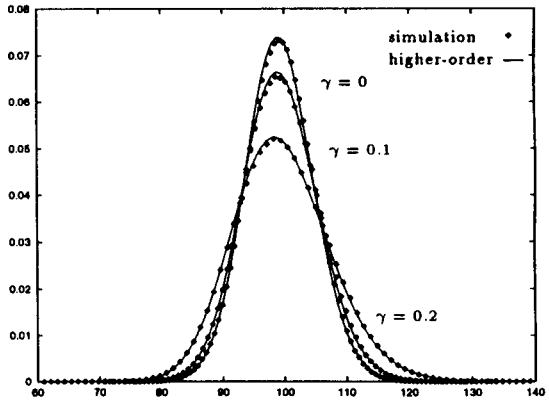


Figure 4: P.d.f. of the first eigenvalue ${}^4\overline{\lambda}_1$: higher-order approximation and Monte-Carlo simulation, $\varepsilon = 0.1$, $\beta = 0.1$

Figures 2, 3 and 4 show plots of approximated p.d.f.s of ${}^n\overline{\lambda}_1$ resulting from the low- and higher-order approximations given in (11) and (12). These results are compared with the estimated p.d.f. resulting from Monte-Carlo simulation using 10^6 realizations for various values of the parameters β, γ and the correlation length ε. From Figure 5 the dependence of p.d.f. approximations of the first eigenvalue on the variance of the mass per unit which is controlled by the parameter γ can be assessed.

From Tables 1 and 2 and Figure 2 it can be seen that the Monte-Carlo simulations confirm the higher-order approximations of the moments and the p.d.f. while the low-order approximations show significant deviations.

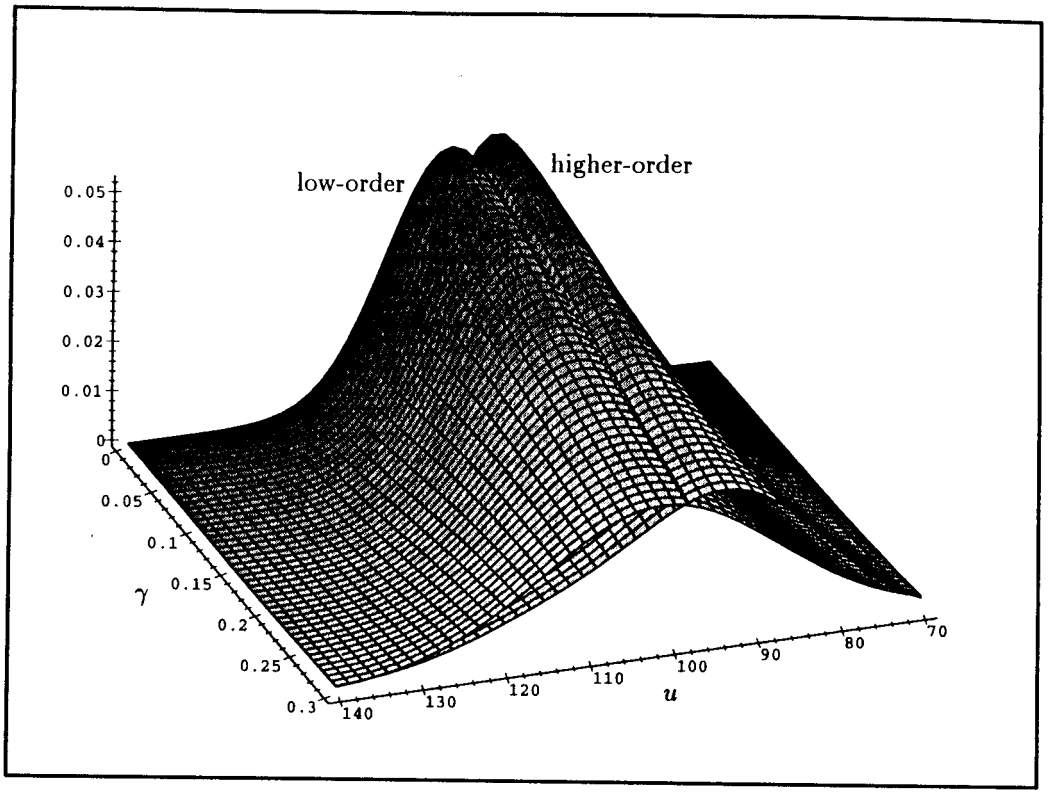


Figure 5: P.d.f. of the first eigenvalue $\overline{\lambda}_1$: low- and higher-order approximation, $\varepsilon = 0.2$, $\beta = 0.1$ und $0 \leq \gamma \leq 0.3$

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