TECHNISCHE UNIVERSITÄT CHEMNITZ

On the symmetry of second derivatives in optimal shape design and sufficient optimality conditions for shape functionals

K. Eppler

Preprint 98-11

Fakultät für Mathematik

On the symmetry of second derivatives in optimal shape design and sufficient optimality conditions for shape functionals

by

Karsten Eppler

Technical University of Chemnitz
Faculty of Mathematics
09107 Chemnitz
Germany

For some heuristic approaches of the boundary variation in shape optimization the computation of second derivatives of domain and boundary integral functionals, their symmetry and a comparison to the velocity field or material derivative method are discussed. Moreover, for some of these approaches the funcionals are Fréchet-differentiable, because an embedding into a Banach space problem is possible. This allows the discussion of sufficient condition in terms of a coercivity assumption on the second Fréchet-derivative. The theory is illustrated by a discussion of the famous Dido problem.

Key words: *optimal shape design, second directional derivatives, boundary integral equation*, .
AMS(MOS) subject classifications: 49Q10, 58C20, 49K10

1 Introduction

Shape optimization problems have been intensively studied in the literature througout the last 25 - 30 years with respect to various directions of investigation. A lot of methods for the description of the domain variation are developed and derivatives of functionals and solutions of state equations with respect to these domain or boundary variations can be computed. Moreover, necessary optimality conditions are given and numerical algorithms for a wide variety of problems are applied (cf. the surveys in [15] and [16]). Nevertheless, due to some difficulties arising from theoretical as well as from technical point of view, the study of sufficient conditions seems to be not very well developed at the moment. Only a few number of papers are concerned with related investigations [7], [1]. Therefore, it may makes sense, to discuss the most easiest case of shape functionals only, in order to apply some of the ideas for more interesting shape optimization problems.

In the paper [4], the author discussed an easy approach for the description of the boundary variation for starshaped domains by the use of polar coordinates. This allows the description of the boundary (domain) **and** the boundary(domain) perturbation in the same way by functions of the polar angle ϕ. Consequently, a Banach space embedding of the shape problem is possible, which allows the investigation of Fréchet-differentiability by use of the standard differential calculus for Banach spaces. In this way the existence of first (Fréchet)-derivatives for domain and boundary integrals of the type

$$J_1(\Omega) = \int_{\Omega} h\, dx \quad \text{and} \quad J_2(\Omega) = \int_{\Gamma} g\, dS_\Gamma\,, \;\; \Gamma = \partial\Omega, \;\; (g, h \text{ are given data}),$$

is shown, which are equivalent to formulas for first (directional) derivatives for other approaches.

As a starting point for this paper we have the following in the case of starshaped domains: Similar to first derivatives $dJ_i(\Omega)[r_1]$, $(i = 1, 2)$, second derivatives can be directly obtained in the sense of

$$d^2 J_i(\Omega_0)[r_1; r_2] = \lim_{\delta \to 0} \frac{dJ_i(\Omega_{\delta r_2})[r_1] - dJ_i(\Omega_0)[r_1]}{\delta}\,, \qquad i = 1, 2,$$

because the first derivatives can be expressed as integrals over the interval $[0, 2\pi]$, where only the integrand contains the perturbation parameter δ. For sufficiently smooth data h or g these derivatives are of Fréchet-type and therefore they have to be symmetric. Thus, the question of symmetry of second (directional) derivatives arises also for other approaches.

Following the ideas of Kirsch, Kress and Potthast, this is investigated for boundary perturbations with smooth fields for the case of two-dimensionsional domains. Although this approach allows at least a "local Banach space embedding", the computation of second derivatives is not straight forward and needs a special definition of the direction of boundary perturbation on perturbed domains (in a neibourhood of the reference surface). Symmetry is proved and the derivatives of the area and boundary arc lenght are discussed as examples. Furthermore, the normal boundary variation is investigated for the sake of completeness.

Based on this, at least second order sufficient optimality conditions are obtained for the case of starshaped domains, i.e., for the description of the boundary variation by a function of the polar angle. A comparison to other approaches is also discussed. Finally, these conditions were applied to the Dido problem.

2 Description of domain perturbations and first directional derivatives

In this paper we shall study shape optimization problems for 2-dimensional simply connected bounded domains Ω, where the domains under consideration satisfying a condition of starshapeness with respect to a neighbourhood $U_\delta(x_0) = \{y \in \mathbb{R}^2 \mid |y - x_0| < \delta\}$, with some fixed $\delta > 0$. Without loss of generality we assume in the sequel $x_0 = \mathbf{0}$. The main advantage of this assumption is that the boundary $\Gamma = \partial\Omega$ of such domains can be described by a Lipschitz continuous function $r = r(\phi)$ of the polar angle ϕ (i. e., $\Gamma := \{\gamma(\phi) = \begin{pmatrix} r(\phi)\cos\phi \\ r(\phi)\sin\phi \end{pmatrix} \mid \phi \in [0, 2\pi]\}$. Moreover, vice verca, each domain (boundary) can be identified with this describing function.

Remark 1: Due to a result of Mazja [12], the boundary function of a domain Ω, which is starsheped to an open subset U_δ, is Lipschitz continuous with a constant, depending only on δ an on

$$d_\Omega := \sup\{|x| \mid x \in \Omega\}.$$

Consequently, if we assume that all domains under consideration are **uniformly** bounded (i.e., there exists a bounded outer "security set" D), then they have **uniform** Lipschitz continuous boundaries.

Remark 2: The assumption $\Gamma \in C^k$, $(k \in \mathbb{N})$ is equivalent to

$$r(\cdot) \in C_p^k[0, 2\pi] := \{r(\cdot) \in C^k[0, 2\pi] \mid r^{(i)}(0) = r^{(i)}(2\pi), \ i = 0, \ldots, k\}. \tag{1}$$

For transformations into polar coordinates we recall well known formulas for the (local) curvature $\kappa(\cdot)$ (for $\Gamma \in C^2$), arclength $l(\cdot)$ and unscaled and scaled outer normal of the boundary, given by

$$\kappa(\phi) = \frac{2r'^2(\phi) + r^2(\phi) - r(\phi)r''(\phi)}{\sqrt{r^2(\phi) + r'^2(\phi)}^{\,3}}, \quad \text{and} \quad l(\phi) = \sqrt{r^2(\phi) + r'^2(\phi)}, \tag{2}$$

and

$$\vec{a}(\phi) = \begin{pmatrix} r(\phi)\cos\phi + r'(\phi)\sin\phi \\ r(\phi)\sin\phi - r'(\phi)\cos\phi \end{pmatrix} \text{ (unscaled) } \Rightarrow \vec{n}(\phi) = \frac{1}{\sqrt{r^2(\phi) + r'^2(\phi)}}\, \vec{a}(\phi). \tag{3}$$

In the following a reference domain $\Omega_0 \in C^1$ is given, where the boundary Γ_0 is associated with the describing function $r_0 \in C_p^1[0, 2\pi]$. Moreover, quantities, related to the reference domain (such as arclenght, normal and so on), will be denoted by subscription with 0 in the sequal.

Admissible perturbed domains (or boundaries) Ω_ε are now defined by the connection $\Gamma_\varepsilon \Leftrightarrow r_\varepsilon(\phi) = r_0(\phi) + \varepsilon r_1(\phi)$ with $r_1 \in C_p^1[0, 2\pi]$ and $\varepsilon > 0$ sufficiently small, provided that $r_\varepsilon(\phi) > \delta$, $\phi \in [0, 2\pi]$ is satisfied. Moreover, the subscript $\cdot_\varepsilon$ is used in the sequal in

order to denote quantities related to Ω_ε.
In this way, the "variables" (the admissible domains) are identified with elements of an (open) subset of the Banach space $C_p^1[0,2\pi]$, and differential calculus in Banach spaces can be applied for the study of the problem.
Remark 3: Because of

$$\vec{e}_r \cdot \vec{n}_0 = \frac{r_0}{\sqrt{r_0^2 + r_0'^2}} > 0,\ \vec{e}_r = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \text{ - the radial unit vector,}$$

the above perturbations are always regular, i. e., the perturbation field is a tangential field if and only if $r_1(\cdot) \equiv 0$. From [4] we have

Lemma 1 *Let $h \in C(D)$ and $g \in C^1(D)$ be given. Then the functionals $J_1 = \int_\Omega h\,dx$ and $J_2 = \int_\Gamma g\,dS_\Gamma$ are Fréchet-differentiable with respect to $C_p^1[0,2\pi]$ at Ω_0 with the derivatives*

$$\nabla J_1(r_0)[r_1] = \int_0^{2\pi} r_0(\phi) r_1(\phi) h(r_0(\phi),\phi)\,d\phi, \tag{4}$$

and

$$\nabla J_2(r_0)[r_1] = \int_0^{2\pi} r_1 \sqrt{r_0^2 + r_0'^2}\,\frac{\partial g}{\partial r}(r_0(\phi),\phi) + g(r_0(\phi),\phi)\,\frac{r_0 r_1 + r_0' r_1'}{\sqrt{r_0^2 + r_0'^2}}\,d\phi. \tag{5}$$

Remark 4: For the proof see [4]. Obviously, we have directional derivatives given by (4) and (5), respectively, which are linear and continuous w. r. t. r_1. Moreover, the related operator-norm of the Gateaux-derivative depends continuously on the $C_p^1[0,2\pi]$-norm of r_0. This ensures the **continuous Fréchet**-differentiability of the functionals by standard arguments from functional analysis (cf. [2]).

The description of boundary perturbations by smooth fields can be used for more general domains. Especially for 2D-problems boundaries and perturbations can be described by vector parameter functions, based on the usual cartesian coordinates, more precisely, we have for some $T > 0$

$$\Gamma_0 := \{\gamma_0(t) = \begin{pmatrix} x_0(t) \\ y_0(t) \end{pmatrix} \mid t \in [0,T]\},\ \text{ with } \gamma_0(t) = \gamma_0(t+T),\ \text{ and } \gamma_0(\cdot) \in C^2(R).$$

Moreover, we assume $\gamma_0(t_1) = \gamma_0(t_2) \Leftrightarrow t_1 = t_2,\ t_1, t_2 \in (0,T)$, i.e., the curve is free of double points. The curvature ($\Gamma_0 \in C^2$), arclenght and the normal direction are given by

$$\kappa_0(t) = \frac{\dot{x}_0 \ddot{y}_0 - \dot{y}_0 \ddot{x}_0}{\sqrt{\dot{x}_0^2 + \dot{y}_0^2}^{\,3}},\ l_0(t) = \sqrt{\dot{x}_0^2 + \dot{y}_0^2},\ \vec{a}_0(t) = (\pm)\begin{pmatrix} \dot{y}_0(t) \\ -\dot{x}_0(t) \end{pmatrix} \Rightarrow \vec{n}_0(t) = \frac{(\pm)}{l(t)}\vec{a}_0(t), \tag{6}$$

where the sign for outward normal is "+", if Γ_0 is positiv oriented for increasing t. Furthermore, differentiation with respect to arclength is connected with $\frac{d}{dt}$ by $\frac{df}{ds} = \frac{1}{l_0(t)}\frac{df}{dt} = \frac{\dot f}{\sqrt{\dot x_0^2+\dot y_0^2}}$. The description of the perturbed boundary γ_ε is similar to γ_0:

$$\Gamma_\varepsilon := \{\gamma_\varepsilon(t) = \begin{pmatrix} x_0(t) \\ y_0(t) \end{pmatrix} + \varepsilon \begin{pmatrix} d_x(t) \\ d_y(t) \end{pmatrix} \mid t \in [0,T]\}, \ (\vec d = \begin{pmatrix} d_x \\ d_y \end{pmatrix} \text{ suff. smooth}),$$

because at least for sufficiently small ε, the same parameter intervall for Ω_ε as for Ω_0 can be taken. In order to have a nontrivial perturbation we additionally assume $\binom{d_x(\cdot)}{d_y(\cdot)} \cdot \vec n_0(\cdot) \not\equiv 0$. Although, there are some problems with nonuniquiness, an additional degree of freedom and the existence of smooth tangential fields, the approach is useful and allows at least a "local" Banach space embedding in a neighbourhood of Ω_0. Formulas for first derivatives are obtained similar to Lemma 1 in terms of integrals on $[0,T]$.

Lemma 2 *Let $h \in C(D)$ and $g \in C^1(D)$ be given. Then the functionals $J_1 = \int_\Omega h\,dx$ and $J_2 = \int_\Gamma g\,dS_\Gamma$ are Fréchet-differentiable with respect to $\{C_p^1[0,T]\}^2$ at Ω_0 with the derivatives*

$$\nabla J_1(\gamma_0)[\vec d] = \int_{\Gamma_0} (\vec d \cdot \vec n)\, h\; dS_\Gamma = \int_0^T h(x_0(t), y_0(t))(d_x \dot y_0 - d_y \dot x_0)(t)\, dt, \tag{7}$$

and

$$\nabla J_2(\gamma_0)[\vec d] = \int_0^T g_0(t) \frac{\dot x_0 \dot d_x + \dot y_0 \dot d_y}{\sqrt{\dot x_0^2 + \dot y_0^2}}(t) + (\nabla g_0 \cdot \begin{pmatrix} d_x \\ d_y \end{pmatrix})(t) \sqrt{\dot x_0^2(t) + \dot y_0^2(t)}\, dt. \tag{8}$$

Remark 5: Relation (8) is directly clear from

$$J_2 = \int_{\Gamma_0} g\, dS_\Gamma = \int_0^T g(x_0(t), y_0(t)) \cdot \sqrt{\dot x_0^2 + \dot y_0^2} dt.$$

Moreover, for $\Gamma_0 \in C^2$, (8) is equivalent to (cf. (14)),

$$dJ_2(\gamma_0)[\vec d] = \int_{\Gamma_0} (\vec d \cdot \nabla g) + g\, \mathrm{div}_\Gamma \vec d\; dS_\Gamma,$$

where $\mathrm{div}_\Gamma \vec d := \mathrm{div}_\Gamma\{\vec d - (\vec n \cdot \vec d)\vec n\} + \kappa(\vec n \cdot \vec d)$ - for the definition of div_Γ (or Div) see [16] or [3].

From historical point of view the first approach (see Hadamard [?]) was the method of normal boundary perturbation by using

$$\Gamma_\varepsilon :\ \gamma_\varepsilon(t) = \gamma_0(t) + \varepsilon \rho(t) \vec n_0(t),\ t \in [0,T].$$

However, this approache allows not directly an embedding of the optimization problem into a Banach space, because each step of approximation looses one degree of smoothness. Nevertheless, directional derivatives exist for sufficiently smooth domains (boundaries).

Lemma 3 *Let $h \in C(D)$, $g \in C^1(D)$ and $\Omega_0 \in C^2$ be given. Then the functionals $J_1(\cdot)$ and $J_2(\cdot)$ are directional differentiable with respect to $\rho(\cdot) \in C^1$ at Ω_0 with the derivatives*

$$dJ_1(\gamma_0)[\rho] = \int_{\Gamma_0} \rho\, h \, dS_\Gamma = \int_0^T h(x_0(t), y_0(t))\rho_0(t)\sqrt{\dot{x}_0^2 + \dot{y}_0^2}\, dt, \tag{9}$$

and

$$dJ_2(\gamma_0)[\rho] = \int_{\Gamma_0} \rho \cdot (\frac{\partial g}{\partial n} + \frac{g}{R})\, dS_\Gamma = \int_0^T \rho(t) \left(\kappa_0(t) g_0(t) + \frac{\partial g}{\partial n_0}(t)\right) \sqrt{\dot{x}_0^2(t) + \dot{y}_0^2(t)}\, dt. \tag{10}$$

Remark 6: Because of $\vec{d} = r_1 \vec{e}_r$ and $(\vec{d} \cdot \vec{n_0}) dS_\Gamma = r_0(\phi) r_1(\phi) d\phi$ for the polar coordinates, we have the equivalence of (4) to (7), and (5) to (8), respectively. Moreover, for $\Gamma_0 \in C^2$, (5) is similar to (10), which can be seen after integration by parts.

Remark 7: The assumptions on the data fields f and g can be weakened to fields with weak singularities (see [4]). Furthermore, regularity of the boundaries can be reduced, but this will not be studied in the paper.

The next result contains some technical details, useful for the computation and the transformation of higher order derivatives.

Lemma 4 *Let Ω_0 and the perturbations be sufficiently smooth. Then it holds for the shape-derivative of the normal*

$$\begin{aligned}
\frac{d}{d\varepsilon}\vec{n}_\varepsilon(t)|_{\varepsilon=0} &= \frac{\dot{x}_0 \dot{d}_y - \dot{y}_0 \dot{d}_x}{\dot{x}_0^2 + \dot{y}_0^2}(t) \cdot \vec{\tau}_0(t) = -\langle \vec{n}_0, \frac{d}{ds}\vec{d}\rangle \cdot \vec{\tau}_0 \perp \vec{n}_0(t), \\
\frac{d}{d\varepsilon}\vec{n}_\varepsilon(\phi)|_{\varepsilon=0} &= \frac{r_0 r_1' - r_0' r_1}{r_0^2 + r_0'^2}(\phi) \cdot \vec{\tau}_0(\phi) \perp \vec{n}_0(\phi), \\
\frac{d}{d\varepsilon}\vec{n}_\varepsilon(t)|_{\varepsilon=0} &= -\frac{\dot{\rho}}{\sqrt{\dot{x}_0^2 + \dot{y}_0^2}}(t) \cdot \vec{\tau}_0(t) = -\frac{d\rho}{ds} \cdot \vec{\tau}_0 \perp \vec{n}_0(t),
\end{aligned} \tag{11}$$

where $\vec{\tau}_0(\phi/t)$ denotes the unit tangential vector on Γ_0 directed to increasing ϕ (t). Furthermore, the derivative of the curvature is given by

$$\begin{aligned}
\frac{d}{d\varepsilon}\kappa_\varepsilon|_{\varepsilon=0} &= \frac{\dot{x}_0 \ddot{d}_y + \ddot{y}_0 \dot{d}_x - \dot{y}_0 \ddot{d}_x - \ddot{x}_0 \dot{d}_y}{\sqrt{\dot{x}_0^2 + \dot{y}_0^2}^3} - 3\kappa_0 \cdot \frac{\dot{x}_0 \dot{d}_x + \dot{y}_0 \dot{d}_y}{\dot{x}_0^2 + \dot{y}_0^2}, \\
\frac{d}{d\varepsilon}\kappa_\varepsilon|_{\varepsilon=0} &= \frac{2 r_0 r_1 + 4 r_0' r_1' - r_0 r''_1 - r''_0 r_1}{\sqrt{r_0^2 + r_0'^2}^3} - 3\kappa_0 \cdot \frac{r_0 r_1 - r_0' r_1'}{r_0^2 + r_0'^2}, \\
\frac{d}{d\varepsilon}\kappa_\varepsilon|_{\varepsilon=0} &= -\frac{\ddot{\rho}}{\dot{x}_0^2 + \dot{y}_0^2} + \frac{\dot{\rho}[\dot{y}_0 \ddot{y}_0 + \dot{x}_0 \ddot{x}_0]}{(\dot{x}_0^2 + \dot{y}_0^2)^2} - \rho\kappa_0^2
\end{aligned} \tag{12}$$

$$= \frac{-\frac{d}{dt}\left[\dot{\rho}\sqrt{\dot{y}_0^2+\dot{x}_0^2}\right]+2\frac{d}{dt}\left[\rho\frac{d}{dt}(\sqrt{\dot{x}_0^2+\dot{y}_0^2})\right]}{\sqrt{\dot{x}_0^2+\dot{y}_0^2}^3} - \rho\{\kappa_0^2+2\frac{\frac{d^2}{dt^2}(\sqrt{\dot{x}_0^2+\dot{y}_0^2})}{\sqrt{\dot{x}_0^2+\dot{y}_0^2}^3}\}.$$

Remark 8: The relation $\frac{d}{d\varepsilon}\vec{n}_\varepsilon(t)|_{\varepsilon=0} \perp \vec{n}_0$ is also known for more general cases (see [16]). The last transformation of (12) needs obviously $\Omega_0 \in C^3$. Moreover, a well founded derivation of the derivative formula in the case of normal variation needs formally also $\Omega_0 \in C^3$. However, the result is valid for C^2-boundaries, too.

Similar formulas for first directional derivatives hold for the velocity field (or material derivative) method, developed by Sokolowski and Zolesio. We present for the sake of completeness the main idea of the approach (for a detailed investigation see [16]):

Given a so called "velocity field $V(t,x): V \in C(0,\varepsilon; C^k(\bar{D}, \mathbb{R}^N))$, one direction of perturbation of a reference domain Ω_0 is described by a family of domains Ω_t, defined by

$$\Omega_t := \{x(t,X) \in \mathbb{R}^N \mid \frac{dx(\tau,X)}{d\tau} = V(\tau,X),\ x(0,X) = X \in \Omega_0\}.$$

The main advantage is that the (direction of the) domain perturbation is well defined on $\bar{D}$, where $V(0)|_\Gamma$ can be viewed as the boundary perturbation in comparison to the other approaches. First directional derivatives are given by

Lemma 5 *Let $h \in C(D)$ and $g \in C^1(D)$ and $\Omega_0 \in C^2$ be given. Then the functionals $J_1(\cdot)$ and $J_2(\cdot)$ are directional differentiable with respect to $V(\cdot) \in C^1$ at Ω_0 with the derivatives*

$$dJ_1(\Omega_0)[V(0)] = \lim_{t\to 0}\frac{J_1(\Omega_t)-J_1(\Omega_0)}{t} = \int_{\Gamma_0}\langle V(0),\vec{n}\rangle\, h\, dS_\Gamma, \qquad (13)$$

and

$$dJ_2(\Omega_0)[V(0)] = \int_{\Gamma_0}\langle V(0),\nabla g\rangle + g(divV(0) - [DV(0)\vec{n},\vec{n}])\, dS_\Gamma. \qquad (14)$$

Remark 9: $DV(0)$ denotes the Jacobian of the mapping $x \in \mathbb{R}^2 \mapsto V(0,x) \in \mathbb{R}^2$. Furthermore, the following transformation of (13)

$$dJ_1(\Omega_0)[V(0)] = \int_{\Gamma_0}(\vec{n}\cdot V(0))h\, dS_\Gamma = \int_{\Omega_0}\mathrm{div}[h\cdot V(0)]\, dx, \quad h \in C^1,\ (h \in W^{1,1}),$$

shows, that the velocity method allows the definition of shape derivatives under essentially weaker assumptions on the domains. Additional degrees of freedom ($V_1(0)|_\Gamma = V_2(0)|_\Gamma \Rightarrow$ both "velocity fields" represent the "same boundary variation") cause no difficulties.

3 Symmetry of second Fréchet- and directional derivatives

As we had already announced, second shape derivatives for starshaped domains can be computed "straight forward", if the data fields are smooth enough.

Theorem 1 *Let $h \in C^1(D)$ and $g \in C^2(D)$ be given. Then the functionals $J_1 = \int_\Omega h\,dx$ and $J_2 = \int_\Gamma g\,dS_\Gamma$ are twice Fréchet-differentiable with respect to $C_p^1[0,2\pi]$ at Ω_0 with the second derivatives*

$$\nabla^2 J_1(r_0)[r_1;r_2] = \int_0^{2\pi} r_2(\phi)r_1(\phi)h(r_0,\phi) + r_0(\phi)r_1(\phi)r_2(\phi)\frac{\partial h}{\partial r}(r_0,\phi)\,d\phi, \tag{15}$$

and

$$\begin{aligned}\nabla^2 J_2(r_0)[r_1;r_2] &= \int_0^{2\pi}\Bigg\{r_2(\phi)r_1(\phi)\sqrt{r_0^2+r_0'^2}\frac{\partial^2 g}{\partial r^2} + \frac{\partial g}{\partial r}\left[r_1\frac{r_0r_2+r_0'r_2'}{\sqrt{r_0^2+r_0'^2}} + r_2\frac{r_0r_1+r_0'r_1'}{\sqrt{r_0^2+r_0'^2}}\right]\\ &\quad + g\frac{(r_1r_2+r_1'r_2')(r_0^2+r_0'^2) - (r_0r_1+r_0'r_1')(r_0r_2+r_0'r_2')}{\sqrt{r_0^2+r_0'^2}^3}\Bigg\}\,d\phi. \end{aligned}\tag{16}$$

Remark 10: Due to the Banach space embedding, the boundary variation r_2 on perturbed boundaries $\Gamma_{\delta r_1}$ and on Γ_0 is defined in the same way without any additional problem. Therefore, differentiation can be carried out and leads obviously to symmetry with respect to r_1 and r_2. Moreover, we need no additional regularity of the boundary for the definition of higher order derivatives of shape functionals.

In order to investigate higher order derivatives for the other cases, a definition of the boundary variation on perturbed boundaries is necessary. Following Potthast and Kirsch, in the case of boundary variation by smooth fields we may proceed for $N = 2$ as follows, in some sense similar to the case of polar coordinates:

We compute the derivative of $DJ_i(\vec{d})$, $(i = 1,2)$, after the transformation into an integral over the fixed interval $[0,T]$ with

$$\Gamma_\delta := \{\gamma_\delta(t) = \begin{pmatrix} x_0(t) \\ y_0(t)\end{pmatrix} + \delta\begin{pmatrix} f_x(t) \\ f_y(t)\end{pmatrix} \mid t \in [0,T]\},$$

because a smooth parametrization of the perturbed domain exists on the same interval $[0,T]$ for δ sufficiently small. The "transformation" of direction $\vec{d}$ onto Γ_δ is defined by an "unchanged translation", i.e., $\vec{d}(\gamma_\delta(t)) := \vec{d}(\gamma_0(t)) = \vec{d}(t)$. From

$$dJ_1[\vec{d}]|_\delta = \int_0^T h(x_\delta,y_\delta)(d_x\dot{y}_\delta - d_y\dot{x}_\delta)\,dt = \int_0^T h_\delta(t)\left\langle \vec{d},\vec{a}_\delta\right\rangle(t)dt$$

and

$$dJ_2[\vec{d}]|_\delta = \int_0^T g_\delta \frac{\dot{x}_\delta \dot{d}_x + \dot{y}_\delta \dot{d}_y}{\sqrt{\dot{x}_\delta^2 + \dot{y}_\delta^2}} + (\nabla g_\delta \cdot \begin{pmatrix} d_x \\ d_y \end{pmatrix}) \sqrt{\dot{x}_\delta^2 + \dot{y}_\delta^2}\, dt$$

we immediately obtain

Corollary 1 *Let $h \in C^1(D)$ and $g \in C^2(D)$ be given. Then the functionals $J_1 = \int_\Omega h\, dx$ and $J_2 = \int_\Gamma g\, dS_\Gamma$ are twice Fréchet-differentiable with respect to $\{C_p^1[0,T]\}^2$ at Ω_0 with the second derivatives*

$$\nabla^2 J_1(\gamma_0)[\vec{d};\vec{f}] = \int_0^T h_0(d_x \dot{f}_y - d_y \dot{f}_x) + \left\langle \nabla h_0, \begin{pmatrix} f_x \\ f_y \end{pmatrix} \right\rangle \cdot \left\langle \begin{pmatrix} d_x \\ d_y \end{pmatrix}, \begin{pmatrix} \dot{y}_0 \\ -\dot{x}_0 \end{pmatrix} \right\rangle dt, \tag{17}$$

and

$$\begin{aligned} \nabla^2 J_2(\gamma_0)[\vec{d};\vec{f}] &= \int_0^T \left\langle \nabla^2 g_0 \vec{f}, \vec{d} \right\rangle \sqrt{\dot{x}_0^2 + \dot{y}_0^2} + \left\langle \nabla g_0, \left[\vec{d} \cdot \frac{\dot{x}_0 \dot{f}_x + \dot{y}_0 \dot{f}_y}{\sqrt{\dot{x}_0^2 + \dot{y}_0^2}} + \vec{f} \cdot \frac{\dot{x}_0 \dot{d}_x + \dot{y}_0 \dot{d}_y}{\sqrt{\dot{x}_0^2 + \dot{y}_0^2}} \right] \right\rangle \\ &+ g_0 \cdot \frac{(\dot{f}_x \dot{d}_x + \dot{f}_y \dot{d}_y)(\dot{x}_0^2 + \dot{y}_0^2) - (\dot{x}_0 \dot{d}_x + \dot{y}_0 \dot{d}_y)(\dot{x}_0 \dot{f}_x + \dot{y}_0 \dot{f}_y)}{\sqrt{\dot{x}_0^2 + \dot{y}_0^2}^3} dt. \end{aligned} \tag{18}$$

The symmetry of $\nabla^2 J_2(\gamma_0)[\vec{d};\vec{f}]$ can be seen directly from (18). However, after integration by parts of the first part $I_1(\vec{d};\vec{f})$ of $\nabla^2 J_1(\gamma_0)[\vec{d};\vec{f}]$ we obtain (boundary terms at $t = 0$ and $t = T$ vanish, because all functions are periodic w. r. t. t)

$$\begin{aligned} I_1(\vec{d};\vec{f}) &= \int_0^T h_0(d_x \dot{f}_y - d_y \dot{f}_x)\, dt = \int_0^T -[h_0 \dot{d}_x] f_y + [h_0 \dot{d}_y] f_x\, dt \\ &= I_1(\vec{f};\vec{d}) + \int_0^T \left\langle \nabla h_0, \begin{pmatrix} \dot{x}_0 \\ \dot{y}_0 \end{pmatrix} \right\rangle (-d_x f_y + d_y f_x)\, dt. \end{aligned}$$

An easy calculation shows (with $\quad I_2(\vec{d};\vec{f}) = \int_0^T \left\langle \nabla h_0, \vec{f} \right\rangle \cdot \left\langle \vec{d}, \vec{a}_0 \right\rangle dt$)

$$\int_0^T \left\langle \nabla h_0, \begin{pmatrix} \dot{x}_0 \\ \dot{y}_0 \end{pmatrix} \right\rangle (-d_x f_y + d_y f_x)\, dt = I_2(\vec{f};\vec{d}) - I_2(\vec{d};\vec{f}),$$

i.e., symmetry holds.

Remark 11: As a natural method for the definition of domain variations on perturbed surfaces one may use any smooth extension of the boundary field $\vec{d}$, which is very close to the velocity field approach for autonomous velocity fields. However, this is not equivalent to above, because it leads to

$$\widetilde{dJ_1}[\vec{d}]|_\delta = \int_0^T h_\delta(t) \left\langle \vec{d}_\delta, \vec{a}_\delta \right\rangle (t) dt \ \Rightarrow$$

$$\widetilde{d^2 J_1}(\gamma_0)[\vec{d}; \vec{f}] = \nabla^2 J_1(\gamma_0)[\vec{d}; \vec{f}] + \int_0^T h_0(t) \left\langle \frac{d\vec{d}_\delta}{d\delta}|_{\delta=0}(t), \vec{a}_0(t) \right\rangle dt,$$

where the additional part in the derivative implies nonuniquiness (it depends on the way of extension) and destroys the symmetry of second derivatives in general.

Remark 12: For tangential directions of perturbation $\vec{d} = \alpha(t)\vec{\tau}_0$, and $\vec{f} = \beta(t)\vec{\tau}_0$ we formally obtain

$$\nabla^2 J_1(\vec{d}; \vec{f}) = \int_0^T \alpha\beta\, \kappa_0 h_0 \sqrt{\dot{x}_0^2 + \dot{y}_0^2}\, dt = \int_{\Gamma_0} \alpha\beta\, \kappa_0 h_0\, dS_\Gamma,$$

and

$$\nabla^2 J_2(\vec{d}; \vec{f}) = \int_0^T \alpha\beta[\frac{\partial^2 g_0}{\partial\tau^2} + g_0\kappa_0^2]\sqrt{\dot{x}_0^2 + \dot{y}_0^2} + [\alpha\dot{\beta} + \dot{\alpha}\beta]\frac{\partial g_0}{\partial\tau}\, dt$$

$$= \int_{\Gamma_0} \alpha\beta[\frac{\partial^2 g_0}{\partial\tau^2} + g_0\kappa_0^2] + \frac{d}{ds}(\alpha\beta)\frac{\partial g_0}{\partial\tau} dS_\Gamma.$$

For the definition of second derivatives for the normal variation approach, we use the following transformation of direction $\vec{d} = \rho \cdot \vec{n}_0$ onto $\Gamma_\delta := \Gamma_0 + \delta\nu \cdot \vec{n}_0$:
We define $\vec{d}(\gamma_\delta(t))$ by $\vec{d}(\gamma_\delta(t)) := \rho(t)\vec{n}(\gamma_\delta(t))$, where only $\rho(\cdot)$ is "unchanged translated", but the "whole direction" is perturbed. Therefore, we get

Corollary 2 *Let $h \in C^1(D)$, $g \in C^2(D)$ and $\Omega_0 \in C^2$ be given. Then the functionals $J_1(\Omega)$ and $J_2(\Omega)$ are twice directionally differentiable at Ω_0 with respect to $\rho(\cdot), \nu(\cdot) \in C^2$ with the second derivatives*

$$d^2 J_1(\gamma_0)[\rho; \nu] = \int_0^T \left[\rho\nu(h_0\kappa_0 + \frac{\partial h}{\partial n}|_0)\right] \sqrt{\dot{x}_0^2 + \dot{y}_0^2}(t)\, dt = \int_{\Gamma_0} \rho\nu \left[\frac{\partial h}{\partial n}|_0 + \frac{h_0}{R_0}\right] dS_\Gamma, \quad (19)$$

and

$$d^2 J_2(\gamma_0)[\rho; \nu] = \int_0^T \left\{\rho\nu \left[\frac{\partial^2 g}{\partial n^2}|_0 + 2\kappa_0 \frac{\partial g}{\partial n}|_0\right] + g_0 \frac{\dot{\rho}\dot{\nu}}{\dot{x}_0^2 + \dot{y}_0^2}\right\} \sqrt{\dot{x}_0^2 + \dot{y}_0^2}\, dt. \quad (20)$$

Proof: By making use of (11) and

$$\frac{d}{d\delta}\sqrt{\dot{x}_\delta^2+\dot{y}_\delta^2}|_0 = \ldots = \nu_0(t)\left\langle \tau_0(t), \frac{d}{dt}\vec{n}_0(t)\right\rangle = \nu_0(t)\kappa_0(t)\sqrt{\dot{x}_0^2+\dot{y}_0^2}$$

we obtain (19) from

$$dJ_1(\gamma_\delta)[\rho] = \int_{\Gamma_\delta} \rho\, h_\delta\, dS_\Gamma = \int_0^T h_\delta(t)\rho(t)\sqrt{\dot{x}_\delta^2+\dot{y}_\delta^2}(t)\,dt = \int_0^T h_\delta(t)\left\langle \vec{d}_\delta, \vec{a}_\delta\right\rangle(t)dt$$

and

$$d^2J_1(\rho;\nu) = \int_0^T h_0\left[\left\langle \frac{d}{d\delta}\vec{d}_\delta|_0, \vec{a}_0\right\rangle + \left\langle \vec{d}_0, \frac{d}{d\delta}\vec{a}_\delta|_0\right\rangle\right] + \langle \nabla h_0, \nu\vec{n}_0\rangle \cdot \left\langle \vec{d}_0, \vec{a}_0\right\rangle dt,$$

with $\vec{d}_0 = \rho\cdot\vec{n}_0$. For (20) we obtain by differentiation of $dJ_2(\gamma_\delta)[\rho]$

$$\begin{aligned}
d^2J_2(\rho;\nu) &= \int_{\Gamma_0} \rho\left\{\frac{d}{d\delta}\left[\kappa_\delta g_\delta + \frac{\partial g}{\partial n}_\delta\right]|_0 + \nu\left(\kappa_0 g_0 + \frac{\partial g}{\partial n}_0\right)\kappa_0\right\} dS_\Gamma \\
&= \int_0^T \rho\left\{\nu\kappa_0\langle \nabla g_0, \vec{n}_0\rangle + g_0\frac{d}{d\delta}\kappa_\delta|_0 + \right. \\
&\quad \left. +\nu\left\langle \nabla^2 g_0\vec{n}_0, \vec{n}_0\right\rangle + \left\langle \nabla g_0, \frac{d}{d\delta}\vec{n}_\delta|_0\right\rangle + \nu\kappa_0\left(\kappa_0 g_0 + \frac{\partial g}{\partial n}_0\right)\right\}\sqrt{\dot{x}_0^2+\dot{y}_0^2}\,dt \\
&= \int_0^T \rho\sqrt{\dot{x}_0^2+\dot{y}_0^2}\,dt\left\{\nu\left[\frac{\partial^2 g}{\partial n^2}|_0 + 2\kappa_0\frac{\partial g}{\partial n}|_0\right]\right. \\
&\quad \left. -\frac{\dot{\nu}}{\sqrt{\dot{x}_0^2+\dot{y}_0^2}}\left[\frac{\partial g}{\partial \tau}|_0 - g_0\cdot\frac{\dot{y}_0\ddot{y}_0+\dot{x}_0\ddot{x}_0}{\sqrt{\dot{x}_0^2+\dot{y}_0^2}^3}\right] - \frac{\ddot{\nu}g_0}{\dot{x}_0^2+\dot{y}_0^2}\right\},
\end{aligned}$$

by (11) and (12). For further transformations we split $d^2J_2(\rho;\nu) = I_1(\rho;\nu) + I_2(\rho;\nu)$ with the symmetric part

$$I_1(\rho;\nu) = \int_0^T \rho\nu\left\{\frac{\partial^2 g}{\partial n^2}|_0 + 2\kappa_0\frac{\partial g}{\partial n}|_0 - 2g_0\frac{\frac{d^2}{dt^2}(\sqrt{\dot{x}_0^2+\dot{y}_0^2})}{\sqrt{\dot{x}_0^2+\dot{y}_0^2}^3})\right\}\sqrt{\dot{x}_0^2+\dot{y}_0^2}\,dt,$$

and the (formally) nonsymmetric part

$$I_2(\rho;\nu) = -\int_0^T \rho\left\{\dot{\nu}\frac{\partial g}{\partial \tau}|_0 + g_0\frac{\frac{d}{dt}\left[\dot{\nu}\sqrt{\dot{y}_0^2+\dot{x}_0^2} - 2\nu\frac{d}{dt}(\sqrt{\dot{x}_0^2+\dot{y}_0^2})\right]}{\dot{x}_0^2+\dot{y}_0^2}\right\}dt.$$

Integration by parts of $I_2(\rho;\nu)$ leads to (boundary terms vanish)

$$I_2(\rho;\nu) = \int_0^T \frac{dg_0}{dt} \cdot \frac{\rho\left[\dot\nu\sqrt{\dot y_0^2+\dot x_0^2} - 2\nu\frac{d}{dt}(\sqrt{\dot x_0^2+\dot y_0^2})\right]}{\dot x_0^2+\dot y_0^2} - \rho\dot\nu\frac{\partial g}{\partial\tau}|_0 +$$

$$+ g_0\frac{\left[\dot\rho\sqrt{\dot y_0^2+\dot x_0^2} - 2\rho\frac{d}{dt}(\sqrt{\dot x_0^2+\dot y_0^2})\right]\cdot\left[\dot\nu\sqrt{\dot y_0^2+\dot x_0^2} - 2\nu\frac{d}{dt}(\sqrt{\dot x_0^2+\dot y_0^2})\right]}{\sqrt{\dot x_0^2+\dot y_0^2}^3}dt.$$

An easy calculation shows

$$\frac{dg_0}{dt} = \left\langle\nabla g_0, \begin{pmatrix}\dot x_0\\ \dot y_0\end{pmatrix}\right\rangle = \frac{\partial g}{\partial\tau}|_0\sqrt{\dot x_0^2+\dot y_0^2}, \text{ hence, } \frac{dg_0}{dt}\cdot\frac{\rho\dot\nu\sqrt{\dot y_0^2+\dot x_0^2}}{\dot x_0^2+\dot y_0^2} = \rho\dot\nu\frac{\partial g}{\partial\tau}|_0,$$

i. e., symmetry holds for the second derivatives of J_2. Moreover, we continue with a further transformation of

$$I_2(\rho;\nu) - \int_0^T 2\rho\nu\frac{\frac{d^2}{dt^2}(\sqrt{\dot x_0^2+\dot y_0^2})}{(\dot x_0^2+\dot y_0^2)^2}dt =$$

$$= \int_0^T g_0\left\{\frac{\dot\rho\dot\nu}{\sqrt{\dot x_0^2+\dot y_0^2}} - 2\frac{d}{dt}(\rho\nu)\frac{\frac{d}{dt}(\sqrt{\dot x_0^2+\dot y_0^2})}{(\dot x_0^2+\dot y_0^2)^2} + \rho\nu\left[\frac{4(\frac{d}{dt}(\sqrt{\dot x_0^2+\dot y_0^2}))^2}{\sqrt{\dot x_0^2+\dot y_0^2}^3} - \right.\right.$$

$$\left.\left. -2\frac{\frac{d^2}{dt^2}(\sqrt{\dot x_0^2+\dot y_0^2})}{(\dot x_0^2+\dot y_0^2)^2}\right]\right\} - 2\rho\nu\frac{dg_0}{dt}\frac{\frac{d}{dt}(\sqrt{\dot x_0^2+\dot y_0^2})}{(\dot x_0^2+\dot y_0^2)^2}dt.$$

Integration by parts of $-2\frac{d}{dt}(\rho\nu)\{\ldots\}$ shows that all terms except of the first term vanish. The transformations are formally valid only for $\Gamma_0 \in C^3$. However, for $\Gamma_0 \in C^2$ we use an easy continuation argument by an approximating sequence $\{\Gamma_n\} \subset C^3$. Hence, we arrive at (20). □

Remark 13: Now $\frac{d\vec d_\delta}{d\delta}|_{\delta=0}$ is formally present, but the related term for d^2J_1 vanishes, because of $\frac{d}{d\varepsilon}\vec n_\varepsilon(t)|_{\varepsilon=0} \perp \vec n_0$ (cf. remark 8 and Lemma 4), whereas for d^2J_2 some of such terms have opposite sign and therefore they vanish.

Remark 14: Formula (20) can be rewritten as

$$d^2J_2(\gamma_0)[\rho;\nu] = \int_{\Gamma_0}\rho\nu\left[\frac{\partial^2 g}{\partial n^2}|_0 + 2\kappa_0\frac{\partial g}{\partial n}|_0\right] + g_0\frac{d\rho}{ds}\frac{d\nu}{ds}\,dS_\Gamma.$$

Therefore, a conjecture for an extension may be the following

$$d^2J_2(\gamma_0)[\rho;\nu] = \int_{\Gamma_0}\rho\nu\left[\frac{\partial^2 g}{\partial n^2}|_0 + 2\kappa_0\frac{\partial g}{\partial n}|_0\right] + g_0\langle\nabla_\Gamma\rho,\nabla_\Gamma\nu\rangle\,dS_\Gamma,\quad \Omega_0\subset\mathbb{R}^N,\ N>2.$$

Due to the definition of velocity fields on D, second derivatives in the sense of

$$d^2 J_i(\Omega_0)[V_1; V_2] = \lim_{t\to 0} \frac{dJ_i(\Omega_{tV_2})[V_1] - dJ_i(\Omega_0)[V_1]}{t}, \quad i = 1, 2,$$

can be obtained straight forward by using the unitary extension $\mathcal{N}_0$ of the unit normal field $\vec{n}$ on Γ_0.

Corollary 3 *Let h, g and Ω_0 be sufficiently smooth. The second directional derivatives of the functionals J_1 and J_2 at Ω_0 with respect to autonomous vector fields V_1, V_2 are given by*

$$d^2 J_1(\Omega_0)[V_1; V_2] = \int_{\Gamma_0} \langle V_2, \vec{n}\rangle \operatorname{div}[h \cdot V_1]\, dS_\Gamma = \int_{\Omega_0} \operatorname{div}[\operatorname{div}[h \cdot V_1] \cdot V_2]\, dx, \tag{21}$$

and

$$d^2 J_2(\Omega_0)[V_1; V_2] = \int_{\Gamma_0} V_2 \cdot \nabla \left\{(V_1 \cdot \nabla g) + g(\operatorname{div} V_1 - [DV_1\mathcal{N}_0, \mathcal{N}_0])\right\} \tag{22}$$
$$+ \left\{(V_1 \cdot \nabla g) + g(\operatorname{div} V_1 - [DV_1\vec{n}, \vec{n}])\right\} (\operatorname{div} V_2 - (DV_2\vec{n} \cdot \vec{n}))\; dS_\Gamma.$$

Remark 15: For nonautonomuos velocity fields additional terms from $\frac{\partial V}{\partial t}|_{t=0}$ occur in the formula. Moreover, $d^2 J_i$ contain a symmetric part and one from $\frac{dV_1(\Omega_{tV_2})}{dt}$ (cf. remark 11).

4 Some examples

For the volume $J_1 = \int_\Omega dx$ of a domain we have

- $d^2 J_1[r_1; r_2] = \int_0^{2\pi} r_1(\phi) r_2(\phi)\, d\phi,$
- $d^2 J_1[\vec{d}; \vec{f}] = \int_0^T 1 \cdot (d_x(t)\dot{f}_y(t) - d_y(t)\dot{f}_x(t))\, dt,$
- $d^2 J_1[\rho; \nu] = \int_0^T \rho\nu\kappa_0 \sqrt{\dot{y}_0^2 + \dot{x}_0^2} dt = \int_{\Gamma_0} \rho\nu\kappa_0\, dS_\Gamma,$
- $d^2 J_1(V_1; V_2) = \int_{\Omega_0} \operatorname{div}[\operatorname{div} V_1 \cdot V_2]\, dx.$

The second derivative of the volume does not depend on the reference domain in the first two formulae, hence, third derivatives will vanish (for 2D-domains). This is not the case for the normal perturbation approach, because the boundary variations depend on the domain. For the velocity method the nonsymmetric part "destroys" the independence. Especially for $V_1 = \vec{d} = (1,0)^T$ (parallel shifting in x-direction) and $V_2 = \vec{f} = (0.5x^2, 0)^T$ ("blow up/shrinking" in x-direction) we get

$$0 = d^2 J_1(\vec{d}; \vec{f}) = d^2 J_1(\vec{f}; \vec{d}),$$

whereas for the velocity method it holds

$$0 = d^2 J_1(V_1; V_2) < d^2 J_1(V_2; V_1) = \int_\Omega dx.$$

Similarly for the perimeter $J_2 = \int_\Gamma dS_\Gamma$ we obtain

- $d^2 J_2[r_1; r_2] = \int_0^{2\pi} \frac{(r_1 r_2 + r_1' r_2')(r_0^2 + {r_0'}^2) - (r_0 r_1 + r_0' r_1')(r_0 r_2 + r_0' r_2')}{\sqrt{r_0^2 + {r'}_0^2}^3} d\phi,$
- $d^2 J_2[\vec{d}; \vec{f}] = \int_0^T \frac{(\dot{f}_x \dot{d}_x + \dot{f}_y \dot{d}_y)(\dot{x}_0^2 + \dot{y}_0^2) - (\dot{x}_0 \dot{d}_x + \dot{y}_0 \dot{d}_y)(\dot{x}_0 \dot{f}_x + \dot{y}_0 \dot{f}_y)}{\sqrt{\dot{x}_0^2 + \dot{y}_0^2}^3} dt,$
- $d^2 J_1[\rho; \nu] = \int_0^T \frac{\dot{\rho}\dot{\nu}}{\dot{x}_0^2 + \dot{y}_0^2} \sqrt{\dot{x}_0^2 + \dot{y}_0^2}\, dt = \int_{\Gamma_0} \frac{d\rho}{ds}\frac{d\nu}{ds}\, dS_\Gamma.$

A more general formulation in terms of boundary integrals seems to be not directly clear for the case of smooth perturbation fields (cf. remark 14).

5 Optimality conditions for volume functionals

Whereas necessary optimality conditions can be easily obtained by using directional derivatives of first and second order, the situation for sufficient conditions is more complicated in general in shape optimization. Due to the special approach for starshaped domains, standard methods are applicable. In this paper we shall study only the case of free minima. From the standard necessary condition it follows immediately ("all $r_1 \in C^2$ are admissible")

$$\nabla J_1(\Omega_0)[r_1] = \int_0^{2\pi} r_0(\phi) r_1(\phi) h(r_0, \phi)\, d\phi = 0 \Rightarrow h|_{\Gamma_0} \equiv 0. \quad (23)$$

Moreover, according to (15) we get for a domain, satisfying the necessary condition

$$\nabla^2 J_1(\Omega_0)[r; r] = \int_0^{2\pi} r^2(\phi) r_0(\phi) \frac{\partial h}{\partial r}|_0(\phi) d\phi. \quad (24)$$

Optimality can be guaranteed often by some coercivity of the second Fréchet-derivative. However, it is impossible to have coercicity with respect to C^1 (the "space of differentiation"), only an estimate

$$\nabla^2 J_1(\Omega_0)[r,r] \geq c_0 \|r\|_{L_2}^2, \quad (\text{ where } \; c_0 > 0 \; \text{ is ensured by } \; \frac{\partial h}{\partial r}|_0(\phi) > 0, \; \forall \phi)$$

can be expected. This is known from other control problems as the so-called "two-norm-discrepancy".

Remark 16: The conditions $\frac{\partial h}{\partial r}|_0 > 0$ and $\frac{\partial h}{\partial n}|_0 > 0$ are equivalent for starshaped domains (we have $(\vec{e}_r, \vec{n}) > 0 \; \forall \phi$ and $\frac{\partial h}{\partial \vec{\tau}}|_0 = 0 \;\Rightarrow\; \frac{\partial h}{\partial r}|_0 = \frac{\partial h}{\partial n}|_0 (\vec{e}_r, \vec{n})$).

Theorem 2 *For $\Omega_0 \in C^1(r_0 \in C_p^1[0,2\pi]$ and $h \in C^2$ the conditions $h|_{\Gamma_0} \equiv 0$ and $\frac{\partial h}{\partial r}|_0 > 0$ are sufficient for optimality.*

Proof: We have (from differential calculus):

$$J_1(r_0 + r) - J_1(r_0) = \frac{1}{2}\left[d^2 J_1(r_0)[r,r] + R_2(r),\right], \quad \text{where } \frac{|R_2(r)|}{\|r\|_{C^1}^2} \to 0 \;\text{ for } \|r\|_{C^1} \to 0,$$

but this is not enough to ensure optimality. Nevertheless, by a more careful estimate of the remainder $R_2(r) = d^2 J_1(r_\nu)[r,r] - d^2 J_1(r_0)[r,r]$ (where $r_\nu := r_0 + \nu r$) it follows

$$|R_2(r)| = |\int_0^{2\pi} r^2 \left[h(r_\nu,\phi) - 0 + (r_\nu)\frac{\partial h}{\partial r}|_\nu - r_0 \frac{\partial h}{\partial r}|_0\right] d\phi|$$

$$\leq \max|r(\phi)| \int_0^{2\pi} r^2 [c_1(h,\eta) + c_2(h,\eta) + c_3(h,\eta)]\, d\phi$$

$$\leq c(h,\eta)\|r\|_C \|r\|_{L_2}^2, \text{ with } \|r\|_C < \eta.$$

We arrive at (for η sufficiently small)

$$J_1(r_0 + r) - J_1(r_0) \geq \frac{c_0}{2}\|r\|_{L_2}^2, \text{ if } \|r\|_{C^2} < \eta,$$

which ensures the optimality of Ω_0. □

Remark 17: The easy situation allows an interpretation as follows:
From the necessary and sufficient condition we have for the data field h

$$(i) h|_{\Gamma_0} = 0, \; (ii)\; h(x) > 0, \forall x \in U_\delta(\Gamma_0) \setminus \bar{\Omega}_0, \; (iii) h < 0, \forall x \in U_\delta(\Gamma_0) \cap \Omega_0.$$

Therefore, each perturbation of the boundary increases the functional value.

Remark 18: The same discussion is obviously possible using the second derivatives for normal variation. After the transformation of the second derivatives for the smooth field approach we see

$$d^2 J_1(\Omega_0)[\vec{d};\vec{d}] = \int_0^T \left\langle \nabla h_0, \vec{d} \right\rangle \cdot d_n \, (\sqrt{\dot{y}_0^2 + \dot{x}_0^2} dt) =$$

$$\int_0^T d_n \left(\frac{\partial h}{\partial n}|_0 d_n + \frac{\partial h}{\partial \tau}|_0 d_\tau \right) (\sqrt{\dot{y}_0^2 + \dot{x}_0^2} dt) = \int_0^T d_n^2 \frac{\partial h}{\partial n}|_0 \, (\sqrt{\dot{y}_0^2 + \dot{x}_0^2} dt),$$

because of $\frac{\partial h}{\partial \tau}|_0 \equiv 0$. Hence, the second order sufficient conditions are similar for a free minimum, whereas the second derivatives are different.

6 Optimality conditions for boundary functionals

The necessary condition for a free minimum follows directly from the "normal derivative"

$$dJ_2(\Omega_0)[\rho] = \int_{\Gamma_0} \rho \cdot (\frac{\partial g}{\partial n} + \frac{g}{R}) \, dS_\Gamma = 0 \Rightarrow \quad \left[\frac{\partial g}{\partial n} + g\kappa \right] |_0 \equiv 0.$$

Nevertheless, this can be also obtained from derivative formula for the other approaches. We have (for $\Omega_0 \in C^2$)

$$\nabla J_2(\gamma_0)[r_1] = \int_0^{2\pi} r_1 \sqrt{r_0^2 + r'^2_0} \frac{\partial g}{\partial r}|_0 + g_0 \frac{r_0 r_1 + r'_0 r'_1}{\sqrt{r_0^2 + r'^2_0}} \, d\phi =$$

$$= \int_0^{2\pi} r_1 \left\{ \left[\frac{r_0^2}{\sqrt{r_0^2 + r'^2_0}} \frac{\partial g}{\partial r}|_0 - \frac{r'_0}{\sqrt{r_0^2 + r'^2_0}} \frac{\partial g}{\partial \phi}|_0 \right] + g_0 r_0 \frac{r_0^2 + 2r'^2_0 - r_0 r''_0}{\sqrt{r_0^2 + r'^2_0}^3} \right\} dS_\Gamma,$$

$$\text{where} \quad \left[\frac{r_0^2}{\sqrt{r_0^2 + r'^2_0}} \frac{\partial g}{\partial r}|_0 - \frac{r'_0}{\sqrt{r_0^2 + r'^2_0}} \frac{\partial g}{\partial \phi}|_0 \right] = r_0 \frac{\partial g}{\partial n}|_0,$$

and analogously

$$\nabla J_2(\Omega_0)[\vec{d}] = \int_0^T g_0 \frac{\dot{x}_0 \dot{d}_x + \dot{y}_0 \dot{d}_y}{\sqrt{\dot{x}_0^2 + \dot{y}_0^2}} + (\nabla g_0 \cdot \begin{pmatrix} d_x \\ d_y \end{pmatrix}) \sqrt{\dot{x}_0^2 + \dot{y}_0^2} \, dt =$$

$$= \int_0^T g_0 (\vec{\tau} \cdot \frac{d}{dt} \vec{d}) + (\nabla g_0 \cdot [d_n \vec{n} + d_\tau \vec{\tau}]) \sqrt{\dot{x}_0^2 + \dot{y}_0^2} \, dt = \int_\Gamma d_n (g_0 \kappa_0 + \frac{\partial g}{\partial n}|_0) \, dS_\Gamma.$$

For the derivation of sufficient condition we investigate the second derivative $d^2J_2(\Omega_0)[r;r]$.

$$d^2J_2(\Omega_0)[r;r] = \nabla^2 J_2(r_0)[r;r] = \int_0^{2\pi} r^2\sqrt{r_0^2+r_0'^2}\frac{\partial^2 g}{\partial r^2} + 2r\frac{\partial g}{\partial r}\frac{r_0 r + r_0' r'}{\sqrt{r_0^2+r_0'^2}} + g\frac{(r_0' r - r_0 r')^2}{\sqrt{r_0^2+r_0'^2}^3}\,d\phi.$$

By integration by parts of the "mixed terms" $r'r \cdot f(\phi)$ we arrive at

$$\nabla^2 J_2(r_0)[r;r] = \int_0^{2\pi} r^2 \cdot f_1(\nabla^2 g, \nabla g, g, r_0) + r'^2 \cdot f_2(g, r_0)\,d\phi, \tag{25}$$

where f_1 and f_2 are given by

$$f_1(\nabla^2 g, \nabla g, g, r_0)(\phi) = r_0\frac{\partial}{\partial n}\left(\frac{\partial g}{\partial r}\right) + \frac{\partial g}{\partial \phi}\frac{r_0 r_0'}{\sqrt{r_0^2+r_0'^2}^3}$$

$$+\frac{\partial g}{\partial r}\frac{2r_0^3 + 4r_0'^2 r_0 - r_0^2 r_0''}{\sqrt{r_0^2+r_0'^2}^3} + g\frac{2r_0'^4 + r_0^3 r_0'' - 2r_0 r_0'^2 r_0'' - r_0^2 r_0'^2}{\sqrt{r_0^2+r_0'^2}^5}$$

$$\text{and} \qquad f_2(g, r_0)(\phi) = \frac{r_0^2 g}{\sqrt{r_0^2+r_0'^2}^3}.$$

Remark 19: Here, only a H^1-estimate

$$\nabla^2 J_2(r_0)[r;r] \;\geq\; c_0\|r\|_{H^1}^2, \text{ with some } c_0 > 0 \tag{26}$$

is possible. For the verification of such an estimate a Riccati equation technique may be used.

Theorem 3 *For $\Omega_0 \in C_p^2[0,2\pi]$ and $g \in C^3$ the condition $\left[\frac{\partial g}{\partial n} + g\kappa\right]|_0 \equiv 0$ and estimate (26) are sufficient for optimality.*

Proof: Similar to the volume case we have to estimate $R_2(r) = d^2J_2(r_\nu)[r,r] - d^2J_2(r_0)[r,r]$. From (25) it follows

$$|R_2(r)| \leq \int_0^{2\pi} r^2|f_1^\nu(\phi) - f_1^0(\phi)| + r'^2|f_2^\nu(\phi) - f_2^0(\phi)|\,d\phi,$$

where $f_1^\nu(\phi) = f_1(\nabla^2 g, \nabla g, g, r_\nu)(\phi)$ and $f_2^\nu(\phi) = f_2(g, r_\nu)(\phi)$, respectively.
Moreover, with $g \in C^3$ and (25) we get
(because of $\quad |f_1^\nu(\phi) - f_1^0(\phi)| \leq |\frac{r_\nu^2}{\sqrt{r_\nu^2+r_\nu'^2}}\frac{\partial^2 g}{\partial r^2}|_\nu - \frac{r_0^2}{\sqrt{r_0^2+r_0'^2}}\frac{\partial^2 g}{\partial r^2}|_0| + \quad \ldots +$

$$+|\frac{2r_\nu^3+4r'^2_\nu-r_\nu^2 r"_\nu}{\sqrt{r_\nu^2+r'^2_\nu}^3}\frac{\partial g}{\partial r}|_\nu - \frac{2r_0^3+4r'^2_0-r_0^2 r"_0}{\sqrt{r_0^2+r'^2_0}^3}\frac{\partial g}{\partial r}|_0| + \quad \ldots \quad \text{and so on})$$

$$R_2(r)| \leq \int_0^{2\pi} r^2\left\{c_1|r|+c_2|r'|+c_3|r"|\right\}+r'^2 c_4|r|\,d\phi,$$

with $\quad c_i = c_i(g, r_0, \eta),\ i = 1(1)4.$

$$\leq \tilde{c}(g, r_0, \eta)\cdot\|r\|_{C^2}\cdot\|r\|_{H^1}^2, \text{ for } \|r\|_{C^2} < \eta.$$

Summarizing up, we are able to estimate (for sufficiently small $\eta > 0$)

$$J_2(r_0+r)-J_2(r_0) \geq \frac{c_0}{2}\|r\|_{H^1}^2, \text{ for } \|r\|_{C^2} < \eta. \qquad \square$$

Remark 20: If $g < 0$ holds somewhere on Γ_0, Ω_0 cannot be optimal.

The similarity of the sufficient conditions can be seen by the following transformation of $\nabla^2 J_2(\gamma_0)[\vec{d};\vec{d}]$. We use

$$\vec{d} = \langle\vec{n},\vec{d}\rangle\vec{n}+\langle\vec{\tau},\vec{d}\rangle\vec{\tau} = d_n\vec{n}+d_\tau\vec{\tau}, \Rightarrow$$

$$\dot{d}_n = \frac{d}{dt}\langle\vec{n},\vec{d}\rangle = \kappa\sqrt{\dot{y}^2+\dot{x}^2}d_\tau+\langle\vec{n},\frac{d}{dt}\vec{d}\rangle, \quad \dot{d}_\tau = \frac{d}{dt}\langle\vec{\tau},\vec{d}\rangle = -\kappa\sqrt{\dot{y}^2+\dot{x}^2}d_n+\langle\vec{\tau},\frac{d}{dt}\vec{d}\rangle$$

and obtain

$$\begin{aligned}
\nabla^2 J_2(\gamma_0)[\vec{d};\vec{d}] &= \int_0^T \langle\nabla^2 g_0\,\vec{d},\vec{d}\rangle\sqrt{\dot{x}_0^2+\dot{y}_0^2}+2\langle\nabla g_0,\vec{d}\rangle\cdot\langle\vec{\tau},\frac{d}{dt}\vec{d}\rangle+g_0\cdot\frac{(\dot{x}_0\dot{d}_y-\dot{y}_0\dot{d}_x)^2}{\sqrt{\dot{x}_0^2+\dot{y}_0^2}^3}\,dt \\
&= \int_0^T \left(d_n^2\frac{\partial^2 g_0}{\partial n^2}+2d_n d_\tau\frac{\partial^2 g_0}{\partial n\partial\tau}+d_\tau^2\frac{\partial^2 g_0}{\partial\tau^2}\right)\sqrt{\dot{x}_0^2+\dot{y}_0^2}+\frac{g_0(\dot{d}_n-\kappa_0\sqrt{\dot{x}_0^2+\dot{y}_0^2}d_\tau)^2}{\sqrt{\dot{x}_0^2+\dot{y}_0^2}} \\
&\quad +2(\dot{d}_\tau+\kappa_0\sqrt{\dot{x}_0^2+\dot{y}_0^2}d_n)(d_n\frac{\partial g_0}{\partial n}+d_tau\frac{\partial g_0}{\partial\tau})\,dt \\
&= \int_0^T I_1+I_2+I_3\,dt.
\end{aligned}$$

Here we introduced

$$I_1 = d_n^2\left(\frac{\partial^2 g_0}{\partial n^2}+2\kappa_0\frac{\partial g_0}{\partial n}\right)\sqrt{\dot{x}_0^2+\dot{y}_0^2}+\dot{d}_n^2\frac{g_0}{\sqrt{\dot{x}_0^2+\dot{y}_0^2}},$$

$$I_2 = d_\tau^2\left(\frac{\partial^2 g_0}{\partial\tau^2}+\kappa_o^2 g_0\right)\sqrt{\dot{x}_0^2+\dot{y}_0^2}+2\dot{d}_\tau d_\tau\frac{\partial g_0}{\partial\tau},$$

$$I_3 = 2d_n d_\tau \left(\frac{\partial^2 g_0}{\partial n \partial \tau} + \kappa_0 \frac{\partial g_0}{\partial \tau}\right)\sqrt{\dot{x}_0^2 + \dot{y}_0^2} + 2\dot{d}_\tau d_n \frac{\partial g_0}{\partial n} - 2d_n \dot{d}_\tau \kappa_0 g_0.$$

By using the necessary optimality condition $\kappa_0 g_0 + \dfrac{\partial g_0}{\partial n} \equiv 0$ on Γ_0 , we immediately get

$$\int_0^T I_2\, dt = 0 \quad \text{and} \quad \int_0^T I_3\, dt = 0,$$

because of

$$\int_0^T 2\dot{d}_\tau\, d_\tau \frac{\partial g_0}{\partial \tau}\, dt = -\int_0^T d_\tau^2 \left(\frac{\partial^2 g_0}{\partial \tau^2} - \kappa_0 \frac{\partial g_0}{\partial n}\right)\sqrt{\dot{x}_0^2 + \dot{y}_0^2}, dt,$$

for the second part and the third part vanishes by

$$2\dot{d}_\tau d_n \frac{\partial g_0}{\partial n} - 2d_n \dot{d}_\tau \kappa_0 g_0 = 2\frac{d}{dt}[d_\tau d_n]\frac{\partial g_0}{\partial n} \qquad \text{and}$$

$$\left(\frac{\partial^2 g_0}{\partial n \partial \tau} + \kappa_0 \frac{\partial g_0}{\partial \tau}\right)\sqrt{\dot{x}_0^2 + \dot{y}_0^2} - \frac{d}{dt}\frac{\partial g_0}{\partial n} \equiv 0.$$

Hence, we arrive at (for a domain Ω_0, satisfying the necessary condition)

$$\nabla^2 J_2(\gamma_0)[\vec{d};\vec{d}] = \int_0^T \left[d_n^2\left(\frac{\partial^2 g_0}{\partial n^2} + 2\kappa_0 \frac{\partial g_0}{\partial n}\right) + (\dot{d}_n)^2 \frac{g_0}{\dot{x}_0^2 + \dot{y}_0^2}\right]\sqrt{\dot{x}_0^2 + \dot{y}_0^2}\, dt. \tag{27}$$

The same can be directly obtained from (20).

Remark 21: The equivalence between (27) and (25) is also obvious for starshaped domains. Moreover, coercivity holds simultaneously.

7 The Dido problem

As an illustrating example we want to apply the foregoing investigations to the Dido problem of maximizing the volume (area) of a domain subject to a given lenght of the perimeter. There are two elementary proofs known for the optimality of the circle (see, for example [17]) . One of them is mainly based on investigations of Zenodorus in the ancient greece. The second proof was developed by Steiner in the 19th century. Moreover, several formulations of the problem are given in the calculus of variation (cf. [11]). If we restrict our considerations to starshaped domains only, the problem seems to become

$$(\mathrm{P}) \begin{cases} J_1(r_0) &= \int_{\Omega_0} -1\, dx = \int_0^{2\pi} -\frac{1}{2} r_0^2(\phi) d\phi \to \inf, \\ \text{subject to} & \\ J_2(r_0) &= \int_{\Gamma_0} 1\, dS_\Gamma = \int_0^{2\pi} \sqrt{r_0^2(\phi) + r_0'^2(\phi)} d\phi = l_0. \end{cases}$$

However, the problem is invariant with respect to parallel shifting. Hence, for the investigation of **sufficient** condition we additionally fix the baricentre, for convenience at the origin, which "forbids" the parallel shifting and does not influence the original problem otherwise. We arrive at the following modified problem

$$
(\mathrm{PM})\left\{\begin{array}{rcl}
J_1(r_0) & = & \int\limits_{\Omega_0} -1\,dx = \int\limits_0^{2\pi} -\frac{1}{2}r_0^2(\phi)d\phi \;\rightarrow\; \inf, \\
\text{subject} & \text{to} & \\
J_2(r_0) & = & \int\limits_{\Gamma_0} 1\,dS_\Gamma - l_0 = \int\limits_0^{2\pi} \sqrt{r_0^2(\phi) + r'^2_0(\phi)}d\phi - l_0 = 0, \\
J_3(r_0) & = & \int\limits_{\Omega_0} x_1\,dx = \int\limits_0^{2\pi} \cos\phi \int\limits_0^{r_0(\phi)} \rho^2 d\rho d\phi = 0, \\
J_4(r_0) & = & \int\limits_{\Omega_0} x_2\,dx = \int\limits_0^{2\pi} \sin\phi \int\limits_0^{r_0(\phi)} \rho^2 d\rho d\phi = 0.
\end{array}\right.
$$

Whereas the discussion of necessary conditions is known from calculus of variation, we repeat it in terms of shape funtionals. We define the Lagrangian

$$L(r_0;\lambda) = J_1(r_0) - \sum_{k=2}^{4} \lambda_k J_k(r_0), \qquad \text{and obtain for } r_0 \in C_p^2$$

$$
\begin{aligned}
dL(r_0;\lambda)[r_1] &= \int\limits_0^{2\pi} -r_0(\phi)r_1(\phi)[1 + \lambda_3 r_0(\phi)\cos\phi + \lambda_4 r_0(\phi)\sin\phi] - \lambda_2 \frac{r_0 r_1 + r_0' r_1'}{\sqrt{r_0^2 + r'^2_0}}(\phi)\,d\phi \\
&= \int\limits_0^{2\pi} -r_0(\phi)r_1(\phi)\left(1 + \lambda_2\cdot\kappa_0(\phi) + \lambda_3\cos\phi r_0(\phi) + \lambda_4 \sin\phi r_0(\phi)\right)d\phi \stackrel{!}{=} 0,
\end{aligned}
$$

$$\Rightarrow \qquad 1 + \lambda_2\cdot\kappa_0(\phi) + \lambda_3\cos\phi r_0(\phi) + \lambda_4\sin\phi r_0(\phi) = 0, \quad \phi\in[0,2\pi].$$

With $\lambda_3^0 = \lambda_4^0 = 0$ and according to our constraints, we get

$$\kappa_0 \equiv \text{ const.} \neq 0 \;\Rightarrow\; r_0(\phi) \equiv r_0,\; \lambda_2^0 = -\kappa_0^{-1} = -r_0 = -\frac{l_0}{2\pi}.$$

Remark 22: The assertion $\lambda_3^0 = \lambda_4^0 = 0$ makes sense, because the optimal value function is obviuosly constant with respect to a variation of the value of the second and third constraint. Moreover, a vanishing Lagrange multiplier of the objective (i.e., $\lambda_1 = 0$) implying $\lambda_2 = 0$ or $\kappa_0 \equiv 0$. Therefore, regularity of the Lagrangian can be assumed.

Remark 23: The additional constraints are formally not needed for the necessary condition. Also for Problem (P) we obtain

$$\kappa_0 \equiv \text{ const.} \neq 0 \;\text{ and }\; \lambda_2^0 = -\kappa_0^{-1} = -\frac{l_0}{2\pi}.$$

However, we cannot conclude uniquely $r_0(\phi) \equiv r_0$, because all "shifted" circle with centre at $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2)^T$ satisfies the necessary condition
(for $\varepsilon_1^2 + \varepsilon_2^2 < r_0^2 \Rightarrow r_\varepsilon(\phi) = \varepsilon_1 \cos\phi + \varepsilon_2 \sin\phi + \sqrt{r_0^2 - \varepsilon_1^2 \sin^2\phi - \varepsilon_2^2 \cos^2\phi - \varepsilon_1\varepsilon_2 \sin 2\phi}$) .

For the validity of a sufficient second order condition we need

$$\nabla^2 L(r_0, \lambda_0)[r; r] \;\geq\; c_0 \|r\|_{H^1}^2,$$

for all r from the tangent cone T_c^0 at Ω_0 of the constraints. Due to the regularity, the tangent cone coincides with the linearizing cone, i.e., according to the derivatives of J_k,

$$T_c^0 = T_c(\Omega_0) = \left\{ r \in C^2 \mid \int_0^{2\pi} r(\phi) d\phi = 0,\; \int_0^{2\pi} r(\phi)\cos\phi d\phi = 0,\; \int_0^{2\pi} r(\phi)\sin\phi d\phi = 0 \right\}.$$

Lemma 6 *It holds*

$$\nabla^2 L(r_0, \lambda_0)[r; r] \;\geq\; \frac{3}{5}\|r\|_{H^1}^2,$$

for all $r \in T_c^0$, ensuring that a sufficient second order condition is satisfied for the circle.

Proof: An easy calculation yields

$$\nabla^2 L(r_0, \lambda^0)[r; r] = \int_0^{2\pi} r'^2(\phi) - r^2(\phi)\, d\phi.$$

Moreover, the system of trigonometric functions $\{1,\ \cos n\phi,\ \sin n\phi,\ n \geq 1\}$ is complete in C^2 and a orthonormal basis in H^1, hence,

$$\|r\|_{H^1}^2 = \int_0^{2\pi} r'^2(\phi) + r^2(\phi)\, d\phi = \mu_0^2(r) + (1 + n^2) \sum_{n=1}^{\infty} \mu_n^2(r) + \nu_n^2(r).$$

The Fourier-coefficients of r are given as usual

$$\nu_n(r) = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} r(\phi)\sin n\phi d\phi, \quad \mu_n(r) = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} r(\phi)\cos n\phi d\phi, \quad \mu_0(r) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} r(\phi) d\phi.$$

Furthermore, the tangent cone is contained in the closure of the linear hull of $\{\cos n\phi,\ \sin n\phi,\ n \geq 2\}$. Therefore, we are able to estimate as follows for $r \in T_c^0$

$$\int_0^{2\pi} r'^2(\phi) - r^2(\phi)\, d\phi = (n^2 - 1) \sum_{n=2}^{\infty} \mu_n^2(r) + \nu_n^2(r) \geq \frac{3(n^2+1)}{5} \sum_{n=2}^{\infty} \mu_n^2(r) + \nu_n^2(r) = \frac{3}{5}\|r\|_{H^1}^2.$$

Hence, we have the desired coercivity of $\nabla^2 L(r_0, \lambda^0)[r; r]$ □

Remark 24: From calculus of variation the validity of

$$\nabla^2 L(r_0, \lambda^0)[r; r] \geq 0, \ \forall r \in T^0_{c2} = \{r \in C^2 \mid \int_0^{2\pi} r(\phi)d\phi = 0\}$$

is known. However, this is directly clear from the discussion above. Moreover, the functions $r_1(\phi) = \cos\phi$ and $r_2(\phi) = \sin\phi$ are associated with the "linearized directions of parallel shifting" at Ω_0 with respect to x_1 and x_2, respectively.

Remark 25: Sufficient conditions for shape functionals only are not to much important, because some of the results are obviously or intuitively clear. Nevertheless, it can be a first step for the study of more interesting shape optimization problems. For example, it seems to be possible to combine the presented technique with BIE- or potential methods ([3],[9], [10]) for the computation of shape derivatives for elliptic equations ([13],[14],[8],[4]), also related to investigations of Fuji ([5],[6],[7],[1]). This will be discussed in a forthcoming paper.

References

[1] S. Belov and N. Fuji: Symmetry and sufficient condition of optimality in a domain optimization problem, Control and Cybernetics vol. **26** (1997), No. 1, 45-56.

[2] Bögel, K. and M. Tasche: Analysis in normierten Räumen, Akademie-Verlag, Berlin, 1974.

[3] Colton, D. and R. Kress: Integral equation methods in scattering theory, Krieger, Malabar, 1992.

[4] Eppler, K.: Optimal shape design for elliptic equations via BIE-methods. preprint TU Chemnitz (1998)

[5] Fuji, N.: Necessary conditions for a domain optimization problem in elliptic boundary value problems, SIAM J. Control and Optimzation vol. **24** (1986), No. 3, 346-360.

[6] Fuji, N.: Second order necessary conditions in a domain optimization problem, Journal of Optimization Theory and Applications vol. **65** (1990), No. 2, 223-245.

[7] Fuji, N.: Sufficient conditions for optimality in shape optimizations, Control and Cybernetics vol. **23** (1994), No. 3, 393-406.

[8] Fuji, N. and Y. Goto: A potential method for shape optimization problems, Control and Cybernetics vol. **23** (1994), No. 3, 383-392.

[9] Guenther, N. M.: Die Potentialtheorie und ihre Anwendungen auf Grundaufgaben der mathematischen Physik, Teubner, Leipzig, 1957.

[10] Hackbusch, W.: Integralgleichungen, Teubner, Stuttgart, 1989.

[11] Ioffe, A. D. and V. M. Tichomirow: Theorie der Extremalaufgaben, Deutscher Verlag der Wissenschaften, Berlin 1979.

[12] Mazja, W.: Einbettungssätze für Sobolewsche Räume I, Teubner, Leipzig, 1979.

[13] Potthast, R.: Fréchet differentiability of boundary integral operators in inverse acoustic scattering, Inverse Problems vol. **10** (1994), 431-447.

[14] Potthast, R.: Fréchet Differenzierbarkeit von Randintegraloperatoren und Randwertproblemen zur Helmholtzgleichung und den zeitharmonischen Maxwellgleichungen. Dissertation, Göttingen, 1994.

[15] Pironneau, O.: Optimal Shape Design for Elliptic Systems, Springer, New York, 1983.

[16] Sokolowski, J. and J.-P. Zolesio: Introduction to Shape Optimization, Springer, Berlin, 1992.

[17] V. M. Tichomirow: Stories about Maxima and Minima, Mathematical World **1**, American Mathematical Society, 1990.