TECHNISCHE UNIVERSITÄT CHEMNITZ

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A. Böttcher

Preprint 98-8

Fakultät für Mathematik

ON THE CORONA THEOREM FOR ALMOST PERIODIC FUNCTIONS

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Let $AP^+_\Sigma(\mathbf{R}^n)$ denote the Banach algebra of all continuous almost periodic functions on $\mathbf{R}^n$ whose Bohr-Fourier spectrum is contained in an additive semi-group $\Sigma \subset [0,\infty)^n$. We show that the maximal ideal space of $AP^+_\Sigma(\mathbf{R}^n)$ may have a nonempty corona and we characterize all Σ for which the corona is empty. Analogous results are established for algebras of almost periodic functions with absolutely convergent Fourier series.

1. Introduction

Recent work on the factorization of almost periodic matrix functions (and thus on the solution of convolution integral equations over finite intervals) has resulted in a revival of the interest in corona theorems for almost periodic functions; see [3], [4], [5], [6], [17], [18]. The general problem is as follows: given analytic almost periodic functions $f_1, \ldots, f_m$ in the upper half-plane $\mathbf{C}_+$ such that

$$\inf_{z\in\mathbf{C}_+} \sum_{j=1}^{m} |f_j(z)| > 0,$$

are there analytic almost periodic functions $g_1, \ldots, g_m$ such that

$$f_1(z)g_1(z) + \cdots + f_m(z)g_m(z) = 1 \quad \text{for all} \quad z \in \mathbf{C}_+ \, ?$$

The answer to this problem is yes, and as far as the author knows, it was Xia [21] who was the first to state this explicitly.

Now suppose that, in addition, the Bohr-Fourier spectra of $f_1, \ldots, f_m$ are all contained in some additive semi-subgroup Σ of $[0,\infty)$. Does the above problem have a solution $g_1, \ldots, g_m$ such that the Bohr-Fourier spectra of $g_1, \ldots, g_m$ are also contained in Σ ? The results along these lines the author is aware of all concern the case where Σ is the intersection of some additive group $G \subset \mathbf{R}$ with $[0,\infty)$, and in this case the answer to the question is again yes (Rodman and Spitkovsky [18]). We will show that in the case of arbitrary semi-groups Σ the answer may nevertheless be no. This happens, for instance, if $\Sigma \subset [0,\infty)$ is given by

$$\Sigma = \left\{k + l\sqrt{2} : (k,l) \in \mathbf{Z}^2_+\right\}.$$

Moreover, we will characterize all Σ for which the answer is in the affirmative.

The proofs of [18] and [21] make use of Carleson's corona theorem [7] in its full strength. This theorem says that for every $m \geq 1$ and every $\varepsilon > 0$ there exists a constant $C(m, \varepsilon) < \infty$ such that if $f_1, \ldots, f_m$ are bounded analytic functions in $\mathbf{C}_+$ satisfying

$$\sup_{z \in \mathbf{C}_+} |f_j(z)| \leq 1, \quad \inf_{z \in \mathbf{C}_+} \sum_{j=1}^{m} |f_j(z)| \geq \varepsilon,$$

then there exist bounded analytic functions $g_1, \ldots, g_m$ in $\mathbf{C}_+$ such that

$$\sup_{z \in \mathbf{C}_+} |g_j(z)| \leq C(m, \varepsilon), \quad \sum_{j=1}^{m} f_j(z) g_j(z) = 1 \quad \text{for} \quad z \in \mathbf{C}_+$$

(see also [8] and [15]).

We here proceed in a different way. We compute the maximal ideal spaces of the relevant algebras and then look whether $\mathbf{C}_+$ is a dense subset of the maximal ideal space. Note that the maximal ideal spaces of Banach algebras of almost periodic functions are in principle known since Arens and Singer's 1956 paper [2] and that they are much simpler than the maximal ideal space of H^∞. Moreover, the approach pursued here does not have any recourse to Carleson's corona theorem and it allows us to establish corona theorems for almost periodic functions on $\mathbf{R}^n$.

The paper is organized as follows. Section 2 contains the preliminaries, in Section 3 we determine the maximal ideal spaces, and Section 4 is devoted to the problem whether $\mathbf{C}_+^n$ is dense in the maximal ideal space. Section 5 contains a few remarks on the role of corona theorems in connection with the factorization of almost periodic matrix functions. Part of the results of this paper are not at all new, but several things are made explicit and the paper is a reasonably self-contained exposition of corona theorems for almost periodic functions on $\mathbf{R}^n$ without too many accessories from abstract harmonic analysis.

2. Basic definitions

For $\lambda \in \mathbf{R}^n$, define $e_\lambda : \mathbf{R}^n \to \mathbf{C}$ by $e_\lambda(x) = e^{i(\lambda, x)}$ where $(\lambda, x) = \lambda_1 x_1 + \ldots + \lambda_n x_n$. We denote by $AP^0(\mathbf{R}^n)$ the set of all almost periodic polynomials, that is, the set of all functions of the form $\Sigma_{\lambda \in H} a_\lambda e_\lambda$ where $a_\lambda \in \mathbf{C}$ and H is a finite subset of $\mathbf{R}^n$. Let $AP(\mathbf{R}^n)$ stand for the closure of $AP^0(\mathbf{R}^n)$ in $L^\infty(\mathbf{R}^n)$. For every $a \in AP(\mathbf{R}^n)$, the Bohr mean-value

$$M(a) := \lim_{T \to \infty} \frac{1}{(2T)^n} \int_{[-T,T]^n} a(x)\, dx,$$

exists and is finite. The set

$$\Omega(a) := \left\{ \lambda \in \mathbf{R}^n : M(a e_{-\lambda}) \neq 0 \right\}$$

is called the Bohr-Fourier spectrum of a. The set $\Omega(a)$ is at most countable and the series

$$\sum_{\lambda \in \Omega(a)} M(a e_{-\lambda}) e_\lambda$$

is referred to as the Fourier series of a. We let $APW(\mathbf{R}^n)$ denote the set of all $a \in AP(\mathbf{R}^n)$ with absolutely convergent Fourier series:

$$a \in APW(\mathbf{R}^n) \Longleftrightarrow \|a\|_W := \sum_{\lambda \in \Omega(a)} |M(ae_{-\lambda})| < \infty.$$

Note that $AP(\mathbf{R}^n)$ is a C^*-subalgebra of $L^\infty(\mathbf{R}^n)$ and that $APW(\mathbf{R}^n)$ is a Banach algebra with pointwise operations and the norm $\|\cdot\|_W$.

Given any subset Σ of $\mathbf{R}^n$, we put

$$AP^0_\Sigma(\mathbf{R}^n) := \Big\{a \in AP^0(\mathbf{R}^n) : \Omega(a) \subset \Sigma\Big\}.$$

Let $AP_\Sigma(\mathbf{R}^n)$ and $APW_\Sigma(\mathbf{R}^n)$ be the closures of $AP^0_\Sigma(\mathbf{R}^n)$ in $AP(\mathbf{R}^n)$ and $APW(\mathbf{R}^n)$, respectively. Using the Bochner-Fejér operators (see, e.g., [16, Chap. 1, Sec. 2.3]), one can show that

$$AP_\Sigma(\mathbf{R}^n) = \Big\{a \in AP(\mathbf{R}^n) : \Omega(a) \subset \Sigma\Big\},$$
$$APW_\Sigma(\mathbf{R}^n) = \Big\{a \in APW(\mathbf{R}^n) : \Omega(a) \subset \Sigma\Big\}.$$

The sets $AP_\Sigma(\mathbf{R}^n)$ and $APW_\Sigma(\mathbf{R}^n)$ are closed subalgebras of $AP(\mathbf{R}^n)$ and $APW(\mathbf{R}^n)$, respectively, if and only if Σ is an additive semi-group, i.e., if and only if $\lambda, \mu \in \Sigma$ implies that $\lambda + \mu \in \Sigma$.

Let $\mathbf{C}_+ := \{x + iy : x \in \mathbf{R},\, y \in (0, \infty)\}$. If $\Sigma \subset [0, \infty)^n$, then every function a in $AP_\Sigma(\mathbf{R}^n)$ can be extended to an analytic function in $\mathbf{C}^n_+$ via the Poisson integral,

$$a(z) := \frac{1}{\pi^n} \int_{\mathbf{R}^n} \left(\prod_{j=1}^{n} \frac{y_j}{(x_j - t_j)^2 + y_j^2} \right) a(t_1, \ldots, t_n)\, dt_1, \ldots dt_n,$$

where $z = (x_1 + iy_1, \ldots, x_n + iy_n) \in \mathbf{C}^n_+$. Obviously,

$$\sup_{z \in \mathbf{C}^n_+} |a(z)| \le \|a\|_\infty \quad (:= \sup_{t \in \mathbf{R}^n} |a(t)|). \tag{1}$$

Throughout what follows we suppose that Σ *is an additive semi-group contained in* $[0, \infty)^n$ *and that* $0 \in \Sigma$. To emphasize this convention, we henceforth write

$$AP^+_\Sigma(\mathbf{R}^n) := AP_\Sigma(\mathbf{R}^n), \quad APW^+_\Sigma(\mathbf{R}^n) := APW_\Sigma(\mathbf{R}^n).$$

Clearly, $AP^+_\Sigma(\mathbf{R}^n)$ and $APW^+_\Sigma(\mathbf{R}^n)$ are unital commutative Banach algebras (with the norms $\|\cdot\|_\infty$ and $\|\cdot\|_W$, respectively). We tacitly identify functions in $AP^+_\Sigma(\mathbf{R}^n)$ and $APW^+_\Sigma(\mathbf{R}^n)$ with their analytic extension into $\mathbf{C}^n_+$.

3. Maximal ideal spaces

Let $G \subset \mathbf{R}^n$ be the smallest additive group containing the semi-group Σ,

$$G = \Sigma - \Sigma := \{\lambda - \mu : \lambda, \mu \in \Sigma\},$$

and denote by G_B the Bohr compactification of G. Thus, G_B is the set of all maps χ of G into the complex unit circle $\mathbf{T}$ such that $\chi(\lambda+\mu)=\chi(\lambda)\chi(\mu)$ for all $\lambda,\mu\in G$.

The maximal ideal spaces of the algebras $AP_G(\mathbf{R}^n)$ and $APW_G(\mathbf{R}^n)$ can be identified with G_B in the following sense: φ is a nonzero multiplicative linear functional if and only if there is a $\chi\in G_B$ such that $\varphi=\varphi_\chi$ where

$$\varphi_\chi\Big(\sum_\lambda a_\lambda e_\lambda\Big):=\sum_\lambda a_\lambda\chi(\lambda)\ \text{ for }\ \sum_\lambda a_\lambda e_\lambda\in AP^0_G(\mathbf{R}^n)$$

(see [9], [10], [16], for example). In short:

$$M\big(AP_G(\mathbf{R}^n)\big)=M\big(APW_G(\mathbf{R}^n)\big)=G_B.$$

We now return to the semi-group $\Sigma\subset[0,\infty)^n$. We denote by $\overline{Y}_\Sigma$ the set of all maps $\theta:\Sigma\to[0,\infty]$ such that

$$\theta(0)=0\ \text{ and }\ \theta(\lambda+\mu)=\theta(\lambda)+\theta(\mu)\ \text{ for all }\ \lambda,\mu\in\Sigma. \tag{2}$$

Here, of course, $\nu+\infty:=\infty$ for all $\nu\in[0,\infty]$. Given $\theta\in\overline{Y}_\Sigma$, we put

$$\Sigma_\theta:=\{\lambda\in\Sigma:\theta(\lambda)<\infty\}, \tag{3}$$

we denote by G^θ the smallest additive subgroup of $\mathbf{R}^n$ containing Σ_θ, and we let G^θ_B stand for the Bohr compactification of G^θ:

$$G^\theta:=\Sigma_\theta-\Sigma_\theta,\quad G^\theta_B:=(G^\theta)_B. \tag{4}$$

Theorem 3.1 (Arens and Singer). *The maximal ideal spaces of both $AP^+_\Sigma(\mathbf{R}^n)$ and $APW^+_\Sigma(\mathbf{R}^n)$ can be identified with*

$$M_\Sigma:=\bigcup_{\theta\in\overline{Y}_\Sigma}\big(G^\theta_B\times\{\theta\}\big)$$

in the following sense: φ is a nonzero multiplicative linear functional if and only if there are $\theta\in\overline{Y}_\Sigma$ and $\chi\in G^\theta_B$ such that $\varphi=\varphi_{\chi,\theta}$ where

$$\varphi_{\chi,\theta}\Big(\sum_{\lambda\in\Sigma}a_\lambda e_\lambda\Big):=\sum_{\lambda\in\Sigma_\theta}a_\lambda\chi(\lambda)e^{-\theta(\lambda)} \tag{5}$$

for $\sum_{\lambda\in\Sigma}a_\lambda e_\lambda\in AP^0_\Sigma(\mathbf{R}^n)$.

This result is contained in Theorem 4.1 of [2]. For the readers convenience, we give the proof below. We first make a few remarks and consider some illustrative examples.

Every $\chi\in G^\theta_B$ can be extended (not necessarily in a unique way) to a $\tilde{\chi}\in G_B$; see, e.g., [11, Lemma 24.4]. Thus, Theorem 3.1 implies that

$$M_\Sigma\subset G_B\times\overline{Y}_\Sigma$$

in the following sense: every nonzero multiplicative linear functional φ is of the form $\varphi = \varphi_{\chi,\theta}$ where $\chi \in G_B$, $\theta \in \overline{Y}_\Sigma$, and

$$\varphi_{\chi,\theta}\Big(\sum_{\lambda\in\Sigma} a_\lambda e_\lambda\Big) := \sum_{\lambda\in\Sigma} a_\lambda \chi(\lambda) e^{-\theta(\lambda)} \text{ for } \sum_{\lambda\in\Sigma} a_\lambda e_\lambda \in AP^0_\Sigma(\mathbf{R}^n).$$

Notice that the values of χ on $\Sigma\setminus\Sigma_\theta$ are irrelevant because $e^{-\theta(\lambda)} = e^{-\infty} = 0$ for $\lambda \in \Sigma\setminus\Sigma_\theta$.

Let Y_Σ be the set of all $\theta \in \overline{Y}_\Sigma$ which do not assume the value ∞. Thus, Y_Σ is the set of all maps $\theta : \Sigma \to [0,\infty)$ satisfying (2). We denote both the $\theta \in \overline{Y}_\Sigma$ given by $\theta(\lambda) := \infty$ for all $\lambda \in \Sigma \setminus \{0\}$ and the $\varphi \in M_\Sigma$ defined by $\varphi(\sum a_\lambda e_\lambda) := a_0$ simply by ∞. If $\theta \in Y_\Sigma$, then $\Sigma_\theta = \Sigma$ and $G^\theta = G$. Hence, in case $\overline{Y}_\Sigma = Y_\Sigma \cup \{\infty\}$, we have

$$M_\Sigma = (G_B \times Y_\Sigma) \cup \{\infty\}. \tag{6}$$

Example 3.2. If H is an additive subgroup of $\mathbf{R}^n$ and

$$\Sigma := H \cap \Big(\{0\} \cup (0,\infty)^n\Big),$$

then $\overline{Y}_\Sigma = Y_\Sigma \cup \{\infty\}$ and therefore (6) holds. Indeed, suppose $\theta \in \overline{Y}_\Sigma$ and $\theta(\mu) < \infty$ for some $\mu = (\mu_1,\ldots,\mu_n) \in \Sigma \setminus \{0\}$. Let $\lambda = (\lambda_1,\ldots,\lambda_n)$ be an arbitrary point of Σ. Since $\mu_j > 0$ for all j, there is a natural number k such that $k\mu_j > \lambda_j$ for all j. As $k\mu - \lambda \in H$ and $k\mu - \lambda \in (0,\infty)^n$, it follows that $k\mu - \lambda \in \Sigma$. Therefore $\theta(\lambda) + \theta(k\mu - \lambda) = k\theta(\mu)$, and because $\theta(\mu) < \infty$, we see that $\theta(\lambda) < \infty$.

Taking $n = 1$, we obtain in particular that if H is an additive subgroup of $\mathbf{R}$ and $\Sigma = H \cap [0,\infty)$, then $\overline{Y}_\Sigma = Y_\Sigma \cup \{\infty\}$ and M_Σ is of the form (6). ■

Example 3.3. It is well known that $\theta \in Y_{[0,\infty)}$ if and only if there is a $y \in [0,\infty)$ such that

$$\theta(\lambda) = \lambda y \text{ for } \lambda \in [0,\infty)$$

(this is an 1880 result of Darboux; see [1, p. 45]). Using this result, it is easy to show that $\theta \in Y_{[0,\infty)^m}$ if and only if there exists a $y \in [0,\infty)^m$ such that

$$\theta(\lambda) = (\lambda, y) \text{ for } \lambda \in [0,\infty)^m.$$

Thus, we can identify $Y_{[0,\infty)^m}$ and $[0,\infty)^m$ in a natural way.

Let now $n = 1$ and $\Sigma = [0,\infty)$. Then, by Example 3.2 and by what was said in the previous paragraph, $\overline{Y}_\Sigma = [0,\infty) \cup \{\infty\}$ and hence,

$$M_{[0,\infty)} = \Big(\mathbf{R}_B \times [0,\infty)\Big) \cup \{\infty\}.$$

We may think of $\mathbf{R}_B \times [0,\infty)$ as a "disk" whose center was deleted and may interprete $\{\infty\}$ as the center of this "disk". ■

Example 3.4. Let $n = 2$ and $\Sigma = [0,\infty)^2$. Employing the argument of Example 3.2 it is not difficult to show that

$$\overline{Y}_{[0,\infty)^2} = Y_{[0,\infty)^2} \cup \Big(Y_{[0,\infty)} \times \{\infty\}\Big) \cup \Big(\{\infty\} \times Y_{[0,\infty)}\Big) \cup \{\infty\}$$

where $Y_{[0,\infty)} \times \{\infty\}$ is the set of all θ of the form

$$\theta(\lambda_1,\lambda_2) = \begin{cases} \theta_1(\lambda_1) & \text{if } \lambda_2 = 0 \\ \infty & \text{if } \lambda_2 > 0 \end{cases}$$

with some $\theta_1 \in Y_{[0,\infty)}$ and where $\{\infty\} \times Y_{[0,\infty)}$ is defined similarly. From Example 3.3 we therefore deduce that $\overline{Y}_{[0,\infty)^2}$ can be identified with $[0,\infty]^2$ in the following sense: $\theta \in \overline{Y}_{[0,\infty)^2}$ if and only if there is a $(y_1,y_2) \in [0,\infty)^2$ such that

$$\theta(\lambda_1,\lambda_2) = \lambda_1 y_1 + \lambda_2 y_2$$

or if there is a $y_1 \in [0,\infty)$ such that

$$\theta(\lambda_1,\lambda_2) = \lambda_1 y_1 + \lambda_2 \infty := \begin{cases} \lambda_1 y_1 & \text{for } \lambda_2 = 0 \\ \infty & \text{for } \lambda_2 > 0 \end{cases}$$

or if there is a $y_2 \in [0,\infty)$ such that

$$\theta(\lambda_1,\lambda_2) = \lambda_1 \infty + \lambda_2 y_2 := \begin{cases} \lambda_2 y_2 & \text{for } \lambda_1 = 0 \\ \infty & \text{for } \lambda_1 > 0 \end{cases}$$

or if $\theta = \infty$, that is,

$$\theta(\lambda_1,\lambda_2) = \lambda_1 \infty + \lambda_2 \infty := \infty.$$

Accordingly, Theorem 3.1 gives

$$M_{[0,\infty)^2} = \Big((\mathbf{R}^2)_B \times [0,\infty)^2\Big) \cup \Big([0,\infty) \times \mathbf{R}_B\Big) \cup \Big(\mathbf{R}_B \times [0,\infty)\Big) \cup \{\infty\}. \qquad (7)$$

Let $\mathbf{T} := \{z \in \mathbf{C} : |z| = 1\}$, $\overline{\mathbf{D}} := \{z \in \mathbf{C} : |z| \le 1\}$, $\overline{\mathbf{D}}_* := \overline{\mathbf{D}} \setminus \{0\}$. In a sense, (7) is the analogue of the decomposition

$$\begin{aligned} \overline{\mathbf{D}} \times \overline{\mathbf{D}} &= \Big(\overline{\mathbf{D}}_* \cup \{0\}\Big) \times \Big(\overline{\mathbf{D}}_* \cup \{0\}\Big) \\ &= \Big(\overline{\mathbf{D}}_* \times \overline{\mathbf{D}}_*\Big) \cup \Big(\overline{\mathbf{D}}_* \times \{0\}\Big) \cup \Big(\{0\} \times \overline{\mathbf{D}}_*\Big) \cup \Big\{(0,0)\Big\} \\ &\cong \Big(\mathbf{T}^2 \times (0,1]^2\Big) \cup \Big(\mathbf{T} \times (0,1]\Big) \cup \Big((0,1] \times \mathbf{T}\Big) \cup \Big\{(0,0)\Big\}. \blacksquare \end{aligned}$$

Example 3.5. Let $\mathbf{Z}_+ := \{0,1,2,\ldots\}$ and let $\mathbf{Q}_+$ stand for the nonnegative rational numbers. It is easily seen that if $\Sigma = \mathbf{Z}_+^m$ or $\Sigma = \mathbf{Q}_+^m$, then

$$Y_\Sigma = [0,\infty)^m$$

in the sense that $\theta \in Y_\Sigma$ if and only if there is a $y \in [0,\infty)^m$ such that $\theta(\lambda) = (\lambda, y)$ for all $\lambda \in \Sigma$. The set $\overline{Y}_\Sigma$ can be identified with $[0,\infty]^m$ as in Example 3.4. $\blacksquare$

Example 3.6. Let $n = 1$ and $\Sigma = \{k + l\sqrt{2} : (k,l) \in \mathbf{Z}_+^2\}$. If $\theta \in Y_\Sigma$ then

$$\theta(k + l\sqrt{2}) = k\theta(1) + l\theta(\sqrt{2}) =: k\eta_1 + l\eta_2 \qquad (8)$$

with $(\eta_1, \eta_2) \in [0, \infty)^2$. Conversely, for every $(\eta_1, \eta_2) \in [0, \infty)^2$ the map θ given by (8) belongs to Y_Σ. The inclusion $Y_\Sigma \subset [0, \infty)$ would mean that there is a $y \in [0, \infty)$ such that

$$k\eta_1 + l\eta_2 = (k + l\sqrt{2})y \text{ for all } (k, l) \in \mathbf{Z}_+^2.$$

This is impossible for $\eta_2 \neq \sqrt{2}\eta_1$, and hence Y_Σ contains $[0, \infty)$ properly. Consequently, $\overline{Y}_\Sigma$ is all the more much bigger than $[0, \infty]$.

Note that for every $\eta_1, \eta_2 \in [0, \infty)$ the maps θ_1 and θ_2 given by

$$\theta_1(k + l\sqrt{2}) := \begin{cases} k\eta_1 & \text{for } l = 0 \\ \infty & \text{for } l \geq 1 \end{cases}, \quad \theta_2(k + l\sqrt{2}) := \begin{cases} l\eta_2 & \text{for } k = 0 \\ \infty & \text{for } k \geq 1 \end{cases}$$

belong to $\overline{Y}_\Sigma \setminus Y_\Sigma$. ■

Proof of Theorem 3.1. Put $A_0 := AP_\Sigma^+(\mathbf{R}^n)$ and $A_1 := APW_\Sigma^+(\mathbf{R}^n)$. It is obvious that $\varphi_{\chi,\theta}$ is a nonzero multiplicative linear functional on A_1. To prove that $\varphi_{\chi,\theta}$ is a multiplicative linear functional on A_0, it suffices to prove that $\varphi_{\chi,\theta}$ is bounded. The boundedness of $\varphi_{\chi,\theta}$ will follow once we have shown that

$$|\varphi_{\chi,\theta}(a)| \leq 2\|a\|_\infty \text{ for all } a \in AP_\Sigma^0(\mathbf{R}^n). \tag{9}$$

So let $a = \sum_{j=1}^m a_{\lambda_j} e_{\lambda_j} \in AP_\Sigma^0(\mathbf{R}^n)$. Put

$$a^N = \Big(\sum_{j=1}^m a_{\lambda_j} e_{\lambda_j}\Big)^N =: \sum_{\lambda_k \in F_N} a_{\lambda_k}^{(N)} e_{\lambda_k}.$$

It is easily seen that the number $|F_N|$ of elements in F_N is at most $(N+1)^{m-1}$. Hence $|F_N|^{1/N} \to 1$ as $N \to \infty$. Since $\varphi_{\chi,\theta}$ has norm 1 on A_1, we get

$$\begin{aligned} |\varphi_{\chi,\theta}(a)| &= |\varphi_{\chi,\theta}(a^N)|^{1/N} \leq \|a^N\|_W^{1/N} = \Big(\sum_{\lambda_k \in F_N} |a_{\lambda_k}^{(N)}|\Big)^{1/N} \\ &\leq |F_N|^{1/(2N)} \Big(\sum_{\lambda_k \in F_N} |a_{\lambda_k}^{(N)}|^2\Big)^{1/(2N)} \leq 2\Big(\sum_{\lambda_k \in F_N} |a_{\lambda_k}^{(N)}|^2\Big)^{1/(2N)} \end{aligned}$$

if only N is large enough. Parseval's equality says that

$$\sum_{\lambda_k \in F_N} |a_{\lambda_k}^{(N)}|^2 = \int_{(\mathbf{R}^n)_B} |a^N(\chi)|^2 d\mu(\chi)$$

where $d\mu$ is the normalized Haar measure on $(\mathbf{R}^n)_B$. Hence,

$$|\varphi_{\chi,\theta}(a)| \leq 2\Big(\int_{(\mathbf{R}^n)_B} |a(\chi)|^{2N} d\mu(\chi)\Big)^{1/(2N)} \leq 2\|a\|_\infty,$$

which is (9).

Conversely, let φ be a nonzero multiplicative linear functional on A_0 or A_1. Since $|\varphi(e_\lambda)| \leq \|e_\lambda\| = 1$, it follows that $|\varphi(e_\lambda)| = e^{-\theta(\lambda)}$ with $\theta \in \overline{Y}_\Sigma$. Define Σ_θ and G^θ by

(3) and (4). The map θ admits a unique extension $\tilde{\theta} : G^\theta \to \mathbf{R}$ from Σ_θ to G^θ such that $\tilde{\theta}(\lambda + \mu) = \tilde{\theta}(\lambda) + \tilde{\theta}(\mu)$ for all $\lambda, \mu \in G^\theta$:

$$\tilde{\theta}(\lambda - \mu) = \theta(\lambda) - \theta(\mu) \;\text{ for }\; \lambda, \mu \in \Sigma_\theta$$

(note that θ is finite on Σ_θ). There is also a unique extension of the map

$$\{e_\lambda\}_{\lambda \in \Sigma_\theta} \to \mathbf{C} \setminus \{0\},\; e_\lambda \mapsto \varphi(e_\lambda)$$

to a map $\tilde{\varphi} : \{e_\lambda\}_{\lambda \in G^\theta} \to \mathbf{C} \setminus \{0\}$ such that

$$\tilde{\varphi}(e_{\lambda+\mu}) = \tilde{\varphi}(e_\lambda)\tilde{\varphi}(e_\mu) \;\text{ for all }\; \lambda, \mu \in G^\theta,$$

namely

$$\tilde{\varphi}(e_{\lambda-\mu}) = \varphi(e_\lambda)\big(\varphi(e_\mu)\big)^{-1} \;\text{ for }\; \lambda, \mu \in \Sigma_\theta$$

(recall that $|\varphi(e_\mu)| > 0$ for $\mu \in \Sigma_\theta$). For $\lambda \in G^\theta$, put

$$\chi(\lambda) := e^{\tilde{\theta}(\lambda)}\tilde{\varphi}(e_\lambda).$$

Then $|\chi(\lambda)| = 1$ and $\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)$ for all $\lambda, \mu \in G^\theta$, whence $\chi \in G^\theta_B$. Since

$$\varphi(e_\lambda) = \begin{cases} \chi(\lambda)e^{-\theta(\lambda)} & \text{for } \;\lambda \in \Sigma_\theta \\ e^{-\infty} = 0 & \text{for } \;\lambda \in \Sigma \setminus \Sigma_\theta \end{cases}$$

and φ is a multiplicative linear functional, it results that φ is of the form (5). ■

4. The corona theorem

Let Σ be an additive semi-subgroup of $[0, \infty)^n$ and suppose $0 \in \Sigma$. The map

$$\tau : \mathbf{R}^n \times (0, \infty)^n \to \mathbf{C}^n_+,\; (x, y) \mapsto (x_1 + iy_1, \ldots, x_n + iy_n)$$

identifies $\mathbf{R}^n \times (0, \infty)^n$ and $\mathbf{C}^n_+$. For $x \in \mathbf{R}^n$ and $y \in (0, \infty)^n$, we define

$$\varphi_{x,y} : AP^0_\Sigma(\mathbf{R}^n) \to \mathbf{C},\; \sum a_\lambda e_\lambda \mapsto \sum a_\lambda e^{i(\lambda,x)}e^{-(\lambda,y)}. \tag{10}$$

Since

$$\varphi_{x,y}\Big(\sum a_\lambda e_\lambda\Big) = \sum a_\lambda e^{i(\lambda, x+iy)}, \tag{11}$$

we see that $\varphi_{x,y}(a)$ is the value of the analytic extension of a at $x + iy \in \mathbf{C}^n_+$. From (10) we get

$$\Big|\varphi_{x,y}\Big(\sum a_\lambda e_\lambda\Big)\Big| \le \sum |a_\lambda| = \|a\|_W,$$

which shows that $\varphi_{x,y}$ extends to a nonzero multiplicative functional on $APW^+_\Sigma(\mathbf{R}^n)$. Taking into account (1) and (11), we conclude that $\varphi_{x,y}$ also extends to a nonzero multplicative functional on $AP^+_\Sigma(\mathbf{R}^n)$. Thus, we have a map

$$\sigma : \mathbf{R}^n \times (0, \infty)^n \to M_\Sigma,\; (x, y) \mapsto \varphi_{x,y}.$$

In general, σ is not injective.

Example 4.1. Let $\Sigma = \mathbf{Z}_+ := \{0, 1, 2, \ldots\}$. We have $\varphi_{x_1,y_1} = \varphi_{x_2,y_2}$ if and only if

$$e^{i\lambda x_1}e^{-\lambda y_1} = e^{i\lambda x_2}e^{-\lambda y_2} \text{ for all } \lambda \in \mathbf{Z}_+,$$

which happens if and only if $y_1 = y_2$ and $x_1 - x_2 \in 2\pi\mathbf{Z}$. Thus, σ is not injective. On the other hand, if $\Sigma = [0, \infty)$, then σ is clearly injective. ∎

The map $\sigma\tau^{-1} : \mathbf{C}^n_+ \to M_\Sigma$ sends each point of $\mathbf{C}^n_+$ to a maximal ideal of $AP^+_\Sigma(\mathbf{R}^n)$ and $APW^+_\Sigma(\mathbf{R}^n)$. If σ is injective, this is an embedding of $\mathbf{C}^n_+$ into the maximal ideal space M_Σ. In case σ is not injective, we think of $\mathbf{C}^n_+$ as being contained in M_Σ in "rolled up" form. In what follows, when saying that $\mathbf{C}^n_+$ is dense in M_Σ we always mean that $\sigma\tau^{-1}(\mathbf{C}^n)$ is dense in M_Σ (in the Gelfand topology).

We begin with a standard result, which relates the problem formulated in Section 1 to the density of $\mathbf{C}^n_+$ in M_Σ. The proof is also standard (see, e.g., [15]) and is only given for the reader's convenience.

Proposition 4.2. *The following are equivalent:*

(i) *for every* $f_1, \ldots, f_m \in AP^+_\Sigma(\mathbf{R}^n)$ *(resp.* $APW^+_\Sigma(\mathbf{R}^n)$*) satisfying*

$$\inf_{z \in \mathbf{C}^n_+} \sum_{j=1}^{m} |f_j(z)| > 0 \tag{12}$$

there exist $g_1, \ldots, g_m \in AP^+_\Sigma(\mathbf{R}^n)$ *(resp.* $APW^+_\Sigma(\mathbf{R}^n)$*) such that*

$$f_1 g_1 + \ldots f_m g_m = 1. \tag{13}$$

(ii) $\mathbf{C}^n_+$ *is dense in* M_Σ.

Proof. (i) $\Rightarrow$ (ii). Let $\varphi \in M_\Sigma$ and suppose φ is not in the closure of $\sigma\tau^{-1}(\mathbf{C}^n_+)$. Then there exist $h_1, \ldots, h_m \in AP^0_\Sigma(\mathbf{R}^n)$ and $\varepsilon > 0$ such that for each $(x, y) \in \mathbf{R}^n \times (0, \infty)^n$ at least one of the inequalities $|\varphi_{x,y}(h_j) - \varphi(h_j)| \geq \varepsilon$ is satisfied. Put $f_k := h_k - \varphi(h_k)$. Then $f_k \in AP^0_\Sigma(\mathbf{R}^n)$ and for each $(x, y) \in \mathbf{R}^n \times (0, \infty)^n$ there is a j such that $|f_j(x + iy)| = |\varphi_{x,y}(h_j) - \varphi(h_j)| \geq \varepsilon$. Hence, by (i), we can find $g_1, \ldots, g_m \in AP^+_\Sigma(\mathbf{R}^n)$ (resp. $APW^+_\Sigma(\mathbf{R}^n)$) such that $\sum f_k g_k = 1$. It follows that $\sum \varphi(f_k)\varphi(g_k) = 1$, which is impossible because $\varphi(f_k) = 0$ for all k.

(ii) $\Rightarrow$ (i). Let A stand for $AP^+_\Sigma(\mathbf{R}^n)$ or $APW^+_\Sigma(\mathbf{R}^n)$. Assume there are $f_1, \ldots, f_m \in A$ satisfying (12) but that there are no $g_1, \ldots, g_m \in A$ such that (13) holds. Then the set $\{\sum f_j g_j : g_j \in A\}$ is a proper ideal of A and therefore contained in some maximal ideal. Consequently, there exists a $\varphi \in M_\Sigma$ such that $\varphi(f_j) = 0$ for all $j = 1, \ldots, m$. Since $\sigma\tau^{-1}(\mathbf{C}^n_+)$ is dense in M_Σ, there are $(x_n, y_n) \in \mathbf{R}^n \times (0, \infty)^n$ such that $|\varphi(f_j) - \varphi_{x_n,y_n}(f_j)| < 1/n$ for all j. It results that $|f_j(x_n + iy_n)| = |\varphi_{x_n,y_n}(f_j)| < 1/n$ for all j, which contradicts (12). ∎

For $\alpha \in [0,\infty)$, we define $\alpha + \infty := \infty$, $\alpha \cdot \infty := \infty$ if $\alpha \neq 0$ and $\alpha \cdot \infty := 0$ if $\alpha = 0$. We write $\overline{Y}_\Sigma \subset [0,\infty]^n$ if every $\theta \in \overline{Y}_\Sigma$ is of the form $\theta(\lambda) = (\lambda, y)$ for some $y \in [0,\infty]^n$. Note that if, for example, $n = 4$ and $y = (y_1, \infty, y_3, \infty)$, then $\theta(\lambda) = (\lambda, y)$ is given by

$$\begin{aligned}\theta(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= \alpha_1 y_1 + \alpha_2\infty + \alpha_3 y_3 + \alpha_4\infty \\ &:= \begin{cases} \infty & \text{if } \alpha_2 > 0 \text{ or } \alpha_4 > 0, \\ \alpha_1 y_1 + \alpha_3 y_3 & \text{if } \qquad \alpha_2 = \alpha_4 = 0. \end{cases}\end{aligned}$$

Here is our main result.

Theorem 4.3. *The set $\mathbf{C}_+^n$ is dense in M_Σ if and only if $\overline{Y}_\Sigma \subset [0,\infty]^n$.*

Before giving the proof, we discuss a few examples.

Example 4.4. Let $\mathbf{Z}_+ := \{0,1,2,\ldots\}$ and let $\mathbf{Q}_+$ be the set of the nonnegative rational numbers. If

$$\Sigma = [0,\infty)^n \text{ or } \Sigma = \mathbf{Q}_+^n \text{ or } \Sigma = \mathbf{Z}_+^n, \tag{14}$$

then $\overline{Y}_\Sigma = [0,\infty]^n$ (recall Examples 3.3 to 3.5). Thus, in the cases (14) we deduce from Theorem 4.3 that $\mathbf{C}_+^n$ is dense in M_Σ. For $\Sigma = [0,\infty)$, this was already proved by Xia [21], for $\Sigma = \mathbf{Q}_+$ and $\Sigma = \mathbf{Z}_+$, the density of $\mathbf{C}_+$ in M_Σ was established by Rodman and Spitkovsky [18]. ∎

Example 4.5. Let $n = 2$ and $\Sigma = \{(k,k) : k \in \mathbf{Z}_+\}$. Clearly, $\theta \in \overline{Y}_\Sigma$ if and only if there is a $y \in [0,\infty)$ such that

$$\theta(k,k) = k\theta(1,1) = ky \text{ for all } k \in \mathbf{Z}_+.$$

We can write

$$\theta(k,k) = k(y/2) + k(y/2) = \big((k,k),(y/2,y/2)\big)$$

and accordingly,

$$\overline{Y}_\Sigma = \{(y_1,y_2) \in [0,\infty]^2 : y_1 = y_2\}. \tag{15}$$

We can also write

$$\theta(k,k) = k(y/3) + k(2y/3) = \big((k,k),(y/3,2y/3)\big)$$

and thus,

$$\overline{Y}_\Sigma = \{(y_1,y_2) \in [0,\infty]^2 : 2y_1 = y_2\}.$$

Evidently, there infinitely many other possibilities of representing $\overline{Y}_\Sigma$. In any case, $\overline{Y}_\Sigma$ is a subset of $[0,\infty]^2$ and hence $\mathbf{C}_+^2$ is dense in M_Σ by virtue of Theorem 4.3.

In the situation considered here, the density of $\mathbf{C}_+^2$ in M_Σ can be easily understood. Indeed, we have $\varphi \in \sigma\tau^{-1}(\mathbf{C}_+^2)$ if and only if $\varphi = \varphi_{(x_1,x_2),(y_1,y_2)}$ with $(x_1,x_2) \in \mathbf{R}^2$ and $(y_1,y_2) \in (0,\infty)^2$. Because

$$\varphi_{(x_1,x_2),(y_1,y_2)}(e_{(k,k)}) = e^{ik(x_1+x_2)}e^{-k(y_1+y_2)},$$

we see that

$$\sigma\tau^{-1}(\mathbf{C}_+^2) = \{\psi_{x,y} : x \in [0, 2\pi), y \in (0, \infty)\}$$

where $\psi_{x,y}(e_{(k,k)}) := e^{ikx}e^{-ky}$. Consequently, $\sigma\tau^{-1}(\mathbf{C}_+^2)$ may be identified with $\mathbf{T} \times (0, \infty)$. Taking the representation (15) of $\overline{Y}_\Sigma$, we can identify $\overline{Y}_\Sigma$ with $[0, \infty] = [0, \infty) \cup \{\infty\}$. Since

$$G := \Sigma - \Sigma = \{k(1,1)\}_{k \in \mathbf{Z}} \cong \mathbf{Z}$$

and $\mathbf{Z}_B \cong \mathbf{T}$, we deduce from Theorem 3.1 that

$$M_\Sigma \cong \Big(\mathbf{T} \times [0, \infty)\Big) \cup \{\infty\}. \tag{16}$$

Clearly, one expects that $\mathbf{T} \times (0, \infty)$ is dense in (16). ∎

Example 4.6. Let $\Sigma = \{k + l\sqrt{2} : (k, l) \in \mathbf{Z}_+^2\}$ be as in Example 3.6. Since $\overline{Y}_\Sigma$ is not contained in $[0, \infty]$, Theorem 4.3 implies that $\mathbf{C}_+$ is not dense in M_Σ.

Let us again try to understand the case at hand in a direct way. By Theorem 3.1,

$$G_B \times Y_\Sigma \subset M_\Sigma.$$

Since G is dense in G_B and $G \subset \mathbf{R}$, one can show that $\mathbf{C}_+ \cong \mathbf{R} \times (0, \infty)$ is dense in $G_B \times [0, \infty)$, but as Y_Σ is much bigger than $[0, \infty)$, one cannot expect that $G_B \times [0, \infty)$ is dense in $G_B \times Y_\Sigma$.

To be more specific, notice first that in the present situation

$$G := \Sigma - \Sigma = \{k + l\sqrt{2} : (k, l) \in \mathbf{Z}^2\}.$$

Hence $G_B = \mathbf{T}^2$, where $(e^{ix_1}, e^{ix_2}) \in \mathbf{T}^2$ is identified with the character

$$\chi_{x_1,x_2}(k + l\sqrt{2}) := e^{ikx_1}e^{ilx_2}.$$

Thus, we can write

$$G_B \times Y_\Sigma \cong \mathbf{T}^2 \times Y_\Sigma.$$

Why is $\mathbf{C}_+ \cong \mathbf{R} \times (0, \infty)$ not dense in $\mathbf{T}^2 \times Y_\Sigma$? One might think this is due to the fact that $\mathbf{R}$ is not dense in $\mathbf{T}^2$. However, $\mathbf{R}$ is dense in $\mathbf{T}^2$. Indeed, the density of $\mathbf{R}$ in $\mathbf{T}^2$ is equivalent to the following: given $(e^{ix_1}, e^{ix_2}) \in \mathbf{T}^2$, $\varepsilon > 0$, and rationally independent $k_j + l_j\sqrt{2} \in G$ $(j = 1, \ldots, m)$, there exists an $x \in \mathbf{R}$ such that

$$\left|e^{ik_jx_1}e^{il_jx_2} - e^{i(k_j+l_j\sqrt{2})x}\right| < \varepsilon \text{ for all } j.$$

But by Kronecker's theorem (see, e.g., [12, Chapter 2]), the image of the map

$$\mathbf{R} \to \mathbf{T}^m, \quad x \mapsto \left(e^{i(k_1+l_1\sqrt{2})x}, \ldots, e^{i(k_m+l_m\sqrt{2})x}\right)$$

is dense in $\mathbf{T}^m$ and hence comes as closely as desired to the point

$$\left(e^{i(k_1x_1+l_1x_2)}, \ldots, e^{i(k_mx_1+l_mx_2)}\right).$$

Thus, $\mathbf{R}$ is dense in $\mathbf{T}^2$. In contrast to this, the density of $[0,\infty)$ in Y_Σ is equivalent to the following: given $(\eta_1,\eta_2) \in [0,\infty)^2$, $\varepsilon > 0$, and $k_j + l_j\sqrt{2} \in G$ $(j = 1,\ldots,m)$, there exists a $y \in \mathbf{R}$ such that

$$\left|e^{-(k_j\eta_1 + l_j\eta_2)} - e^{-(k_j+l_j\sqrt{2})y}\right| < \varepsilon \text{ for all } j$$

This can be shown to be impossible in general. In summary, $\mathbf{C}_+$ is not dense in M_Σ because $[0,\infty)$ is not dense in Y_Σ. ■

Proof of Theorem 4.3. Suppose $\overline{Y_\Sigma} \subset [0,\infty]^n$ and pick $(\chi,\theta) \in G^\theta \times \overline{Y_\Sigma}$ (Theorem 3.1). We must show that if we are given $f_1,\ldots,f_m \in AP^0_\Sigma(\mathbf{R}^n)$ and $\varepsilon > 0$, then there is a point $(x,y) \in \mathbf{R}^n \times [0,\infty)^n$ such that

$$|\varphi_{\chi,\theta}(f_j) - \varphi_{x,y}(f_j)| < \varepsilon \text{ for } j = 1,\ldots,m. \tag{17}$$

Write $f_j = \sum_\lambda f^{(j)}_\lambda e_\lambda$.

Let Σ_θ, G^θ be given by (3), (4). For $x \in G^\theta$, define $\chi_x \in G^\theta_B$ by $\chi_x(\lambda) := e^{i(\lambda,x)}$ $(\lambda \in G^\theta)$. Since G^θ is dense in G^θ_B, there is an $x \in G^\theta \subset \mathbf{R}^n$ such that

$$|\varphi_{\chi,\theta}(f_j) - \varphi_{\chi_x,\theta}(f_j)| \le \sum_{\lambda\in\Sigma_\theta} |f^{(j)}_\lambda||\chi(\lambda) - \chi_x(\lambda)| < \frac{\varepsilon}{2}. \tag{18}$$

We write the points $\lambda \in \Sigma$ in the form $\lambda = (\alpha_1,\ldots,\alpha_n)$. Since $\overline{Y_\Sigma} \subset [0,\infty]^n$, there is a $y^0 = (y^0_1,\ldots,y^0_n) \in [0,\infty]^n$ such that $\theta(\lambda) = (\lambda, y^0)$. Let $K := \{k \in \{1,\ldots,n\} : y^0_k = \infty\}$. Clearly,

$$\Sigma_\theta = \{(\alpha_1,\ldots,\alpha_n) : \alpha_k = 0 \text{ for all } k \in K\}. \tag{19}$$

For every $y \in (0,\infty)^n$, we have

$$\varphi_{\chi_x,\theta}(f_j) = \sum_{\lambda\in\Sigma_\theta} f^{(j)}_\lambda e^{i(\lambda,x)} e^{-(\lambda,y^0)},$$

$$\varphi_{x,y}(f_j) = \sum_{\lambda\in\Sigma_\theta} f^{(j)}_\lambda e^{i(\lambda,x)} e^{-(\lambda,y)} + \sum_{\lambda\in\Sigma\setminus\Sigma_\theta} f^{(j)}_\lambda e^{i(\lambda,x)} e^{-(\lambda,y)}.$$

Choosing $y = (y_1,\ldots,y_n) \in (0,\infty)^n$ so that y_k is sufficiently large if $y^0_k = \infty$, y_k is sufficiently small if $y^0_k = 0$, and $y_k = y^0_k$ if $y^0_k \in (0,\infty)$, we can by virtue of (19) guarantee that

$$\sum_{\lambda\in\Sigma\setminus\Sigma_\theta} |f^{(j)}_\lambda| e^{-(\lambda,y)} < \frac{\varepsilon}{4}, \quad \sum_{\lambda\in\Sigma_\theta} |f^{(j)}_\lambda||e^{-(\lambda,y^0)} - e^{-(\lambda,y)}| < \frac{\varepsilon}{4}.$$

Hence,

$$|\varphi_{\chi_x,\theta}(f_j) - \varphi_{x,y}(f_j)| < \varepsilon/2. \tag{20}$$

Adding (19) and (20) we arrive at (18).

Conversely, suppose now that $\sigma\tau^{-1}(\mathbf{C}^n_+)$ is dense in M_Σ. Define χ_0 on $G := \Sigma - \Sigma$ by $\chi_0(\lambda) = 1$ for all $\lambda \in G$. Pick any $\theta \in \overline{Y_\Sigma}$. By Theorem 3.1, $\varphi_{\chi_0,\theta} \in M_\Sigma$. The density of $\sigma\tau^{-1}(\mathbf{C}^n_+)$ in M_Σ implies that if we are given any $\lambda_1,\ldots,\lambda_m \in \Sigma$ and any $\varepsilon > 0$, we can find $(x,y) \in \mathbf{R}^n \times (0,\infty)^n$ such that

$$|\varphi_{\chi_0,\theta}(e_{\lambda_j}) - \varphi_{x,y}(e_{\lambda_j})| < \varepsilon \text{ for } j = 1,\ldots,m.$$

Equivalently,

$$\left|e^{-\theta(\lambda_j)} - e^{i(\lambda_j,x)}e^{-(\lambda_j,y)}\right| < \varepsilon \text{ for } j = 1, \ldots, m.$$

Passage to real and imaginary parts gives

$$\left|e^{-\theta(\lambda_j)} - \cos(\lambda_j, x)e^{-(\lambda_j,y)}\right| < \varepsilon, \quad \left|\sin(\lambda_j, x)e^{-(\lambda_j,y)}\right| < \varepsilon,$$

and adding the squares of these inequalities we obtain

$$e^{-2\theta(\lambda_j)} - 2\cos(\lambda_j, x)e^{-\theta(\lambda_j)}e^{-(\lambda_j,y)} + e^{-2(\lambda_j,y)} < 2\varepsilon^2.$$

Since $-\cos(\lambda_j, x) \geq -1$, it results that

$$\left|e^{-\theta(\lambda_j)} - e^{-(\lambda_j,y)}\right| < \sqrt{2}\,\varepsilon \text{ for } j = 1, \ldots, m.$$

Suppose $n = 1$. We have proved that for every $\lambda_1, \lambda_2 \in \Sigma$ and every natural number N there exists a $y_N = y_N(\lambda_1, \lambda_2) \in (0, \infty)$ such that

$$\left|e^{-\theta(\lambda_j)} - e^{-\lambda_j y_N}\right| < \frac{1}{N} \text{ for } j = 1, 2.$$

Hence $\lambda_j y_N \to \theta(\lambda_j)$ as $N \to \infty$. Assume Σ contains a nonzero point λ_1. Then

$$y_N \to \frac{\theta(\lambda_1)}{\lambda_1} =: y \in [0, \infty] \text{ as } N \to \infty$$

and therefore $\theta(\lambda_2) = \lambda_2 y$. As $\lambda_2 \in \Sigma$ was arbitrary, we arrive at the conclusion that $\theta \in [0, \infty]$. If $\Sigma = \{0\}$, then $\overline{Y_\Sigma}$ is the singleton consisting of the zero map and hence $\overline{Y_\Sigma} \subset [0, \infty]$. At this point the proof is complete for $n = 1$.

Now let $n = 2$. Suppose first that there are $\lambda_1 = (\alpha_1, \beta_1)$ and $\lambda_2 = (\alpha_2, \beta_2)$ in Σ such that

$$\det\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} \neq 0. \tag{21}$$

We know that if $\lambda_3 = (\alpha_3, \beta_3)$ is an arbitrary point in Σ and N is any natural number, then there are $y_N^1, y_N^2 \in (0, \infty)$ such that

$$\left|e^{-\theta(\lambda_j)} - e^{-\alpha_j y_N^1 - \beta_j y_N^2}\right| < \frac{1}{N} \text{ for } j = 1, 2, 3,$$

whence

$$\alpha_j y_N^1 + \beta_j y_N^2 \to \theta(\lambda_j) \text{ as } N \to \infty \ (j = 1, 2, 3). \tag{22}$$

Taking into account (21), we get

$$\begin{pmatrix} y_N^1 \\ y_N^2 \end{pmatrix} \to \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}^{-1} \begin{pmatrix} \theta(\lambda_1) \\ \theta(\lambda_2) \end{pmatrix} \text{ as } N \to \infty,$$

that is, there are y^1 and y^2 in $[0, \infty]$ such that $y_N^1 \to y^1$ and $y_N^2 \to y^2$ as $N \to \infty$. From (22) for $j = 3$ we obtain

$$\theta(\alpha_3, \beta_3) = \alpha_3 y^1 + \beta_3 y^2,$$

and as $(\alpha_3, \beta_3) \in \Sigma$ is arbitrary, we see that $\theta \in [0, \infty]^2$.

If there are no (α_1, β_1), $(\alpha_2, \beta_2) \in \Sigma$ satisfying (21), then

$$\Sigma = \{\nu(\alpha_0, \beta_0)\}_{\nu \in \Delta}$$

with some semi-group $\Delta \subset [0, \infty)$. By what was proved for $n = 1$, there is a $y \in [0, \infty]$ such that $\theta(\nu\alpha_0, \nu\beta_0) = \nu y$ for all $\nu \in \Delta$. A little thought reveals that there is a (not necessarily unique) $(\xi, \eta) \in [0, \infty]^2$ such that $y = \alpha_0\xi + \beta_0\eta$. Consequently,

$$\theta(\nu\alpha_0, \nu\beta_0) = \nu y = \nu\alpha_0 \cdot \xi + \nu\beta_0 \cdot \eta$$

and thus, $\theta \in [0, \infty]^2$. This completes the proof for $n = 2$.

It is clear that the above reasoning can be extended to $n = 3, 4, \ldots$. ■

Here is a case in which the corona $M_\Sigma \setminus \mathbf{C}_+^n$ is empty.

Theorem 4.7. *If $H \subset \mathbf{R}^n$ is an additive group and $\Sigma = H \cap [0, \infty)^n$, then $\mathbf{C}_+^n$ is dense in the maximal ideal space M_Σ.*

This theorem was established by Rodman and Spitkovsky [18] for $n = 1$ by having recourse to the argument of Xia [21]. In particular, this proof uses the Carleson corona theorem for $H^\infty(\mathbf{R})$ in its full strength. The following proof is based on Theorem 4.3 and the observation that in the case at hand $\overline{Y_\Sigma}$ is contained in $[0, \infty]^n$.

Proof of Theorem 4.7. We first prove that $Y_\Sigma \subset [0, \infty)^n$. The smallest subgroup of $\mathbf{R}^n$ which contains Σ is $G = \Sigma - \Sigma$. Obviously, $G \subset H$. Since

$$\Sigma \subset G \cap [0, \infty)^n \subset H \cap [0, \infty)^n = \Sigma,$$

it follows that $\Sigma = G \cap [0, \infty)^n$. Let $\theta \in Y_\Sigma$. We extend θ from Σ to all of G by defining $\theta(\lambda - \mu) := \theta(\lambda) - \theta(\mu)$.

Fix $\lambda_0 = (\lambda_0^{(1)}, \ldots, \lambda_0^{(n)}) \in \Sigma$. If $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(n)}) \in \Sigma$ and $\lambda^{(j)} \leq \lambda_0^{(j)}$ for all j, then $\lambda_0 - \lambda \in G$ and $\lambda_0 - \lambda \in [0, \infty)^n$. Hence $\lambda_0 - \lambda \in \Sigma$, and therefore

$$0 \leq \theta(\lambda) \leq \theta(\lambda) + \theta(\lambda_0 - \lambda) = \theta(\lambda_0).$$

Consequently, θ is bounded on $\{\lambda \in \Sigma : \lambda^{(j)} \leq \lambda_0^{(j)} \text{ for all } j\}$. This easily implies that θ is also bounded on $\{\lambda \in G : |\lambda^{(j)}| \leq \lambda_0^{(j)} \text{ for all } j\}$, which in turn shows that $\theta : G \to \mathbf{R}$ is continuous at the origin (recall that θ satisfies (2) on G). Thus,

$$\theta(\lambda_N) \to 0 \text{ as } \lambda_N \in G \text{ and } \lambda_N \to 0. \tag{23}$$

Let $\lambda_1, \ldots, \lambda_m, \lambda \in \Sigma$ and suppose

$$\lambda = c_1\lambda_1 + \ldots + c_m\lambda_m \quad (c_j \in \mathbf{R}). \tag{24}$$

By a theorem of Dirichlet (see, e.g., [12, p. 11]), for every natural number N there are a natural number $k_N \le N^m$ and integers $l_j^{(N)}$ such that

$$\left| c_j - \frac{l_j^{(N)}}{k_N} \right| < \frac{1}{N k_N} \quad \text{for } j = 1, \ldots, m.$$

Hence,

$$\left| k_N \lambda - \sum_{j=1}^{m} l_j^{(N)} \lambda_j \right| = \left| \sum_{j=1}^{m} (k_N c_j - l_j^{(N)}) \lambda_j \right| \le \frac{1}{N} \sum_{j=1}^{m} |\lambda_j|,$$

and this approaches zero as N goes to infinity. From (23) we therefore deduce that

$$k_N \theta(\lambda) - \sum_{j=1}^{m} l_j^{(N)} \theta(\lambda_j) \to 0 \text{ as } N \to \infty,$$

whence

$$\theta(\lambda) = c_1 \theta(\lambda_1) + \ldots + c_m \theta(\lambda_m). \tag{25}$$

Let first $n = 1$ and let $\lambda_0 \in \Sigma \setminus \{0\}$. Every $\lambda \in \Sigma$ can be written as $\lambda = c\lambda_0$ with $c \in [0, \infty)$, and from the implication (24) $\Rightarrow$ (25) we get

$$\theta(\lambda) = c\theta(\lambda_0) = \lambda(\theta(\lambda_0)/\lambda_0) =: \lambda y.$$

Thus, $Y_\Sigma \subset [0, \infty)$.

Now suppose $n = 2$ and Σ contains two linearly independent elements $\lambda_1 = (\alpha_1, \beta_1)$, $\lambda_2 = (\alpha_2, \beta_2)$. There are $y_1, y_2 \in \mathbf{R}$ such that

$$\theta(\lambda_1) = \alpha_1 y_1 + \beta_1 y_2, \quad \theta(\lambda_2) = \alpha_2 y_1 + \beta_2 y_2.$$

Every $\lambda = (\alpha, \beta) \in \Sigma$ is of the form $\lambda = c_1\lambda_1 + c_2\lambda_2$, and the implication (24) $\Rightarrow$ (25) gives

$$\begin{aligned} \theta(\alpha, \beta) &= \theta(\lambda) = c_1\theta(\lambda_1) + c_2\theta(\lambda_2) \\ &= c_1(\alpha_1 y_1 + \beta_1 y_2) + c_2(\alpha_2 y_1 + \beta_2 y_2) \\ &= (c_1\alpha_1 + c_2\alpha_2)y_1 + (c_1\beta_1 + c_2\beta_2)y_2 \\ &= \alpha y_1 + \beta y_2. \end{aligned}$$

Since $\theta(\alpha, \beta) \ge 0$ for all $(\alpha, \beta) \in \Sigma$, it results that $y_1, y_2 \in [0, \infty)$, and hence we have proved that $Y_\Sigma \subset [0, \infty)^2$. If Σ does not contain linearly independent elements, the inclusion $Y_\Sigma \subset [0, \infty)^2$ follws from the $n = 1$ case.

It is obvious that the above argument extends to $n = 3, 4, \ldots$ and gives that Y_Σ is contained in $[0, \infty)^n$.

Let now $\theta \in \overline{Y_\Sigma}$ and define Σ_θ by (3). If Σ_θ contains a point of $(0, \infty)^n$, then the argument employed in Example 3.2 shows that $\Sigma_\theta = \Sigma$ and hence $\theta \in Y_\Sigma$. Thus, let us assume that $\Sigma_\theta \cap (0, \infty)^n = \emptyset$. This implies that Σ_θ is completely contained in one of the $(n-1)$-dimensional "faces"

$$F_j := \{(\lambda_1, \ldots, \lambda_n) \in [0, \infty)^n : \lambda_j = 0\} \quad (j = 1, \ldots, n).$$

Put $K_j := \{(\lambda_1, \ldots, \lambda_n) \in \mathbf{R}^n : \lambda_j = 0\}$, $H_j := H \cap K_j$, $\Sigma_j := H_j \cap F_j$. If Σ_θ coincides with Σ_j, we obtain from what was already proved that $\theta(\lambda) = (\lambda, y)$ with

$$y = (y_1, \ldots, y_{j-1}, \infty, y_{j+1}, \ldots, y_n), \quad y_k < \infty \text{ for } k \neq j.$$

Otherwise we can compress the problem to one of the $(n-2)$-dimensional "faces" of F_j. Continuing in this way we arrive at the conlusion that $\theta \in [0, \infty]^n$.

Theorem 4.3 finally gives the density of $\mathbf{C}_+^n$ in M_Σ. ■

Remark 4.8. The converse of Theorem 4.7 is not true: if $n = 1$ and

$$\Sigma := \mathbf{Z}_+ \setminus \{1\} = \{0, 2, 3, 4, \ldots\},$$

then obviously $\overline{Y_\Sigma} = [0, \infty]$ and hence $\mathbf{C}_+$ is dense in M_Σ, but there is no additive subgroup H of $\mathbf{R}$ such that $\Sigma = H \cap [0, \infty)$. ■

5. The Portuguese transformation

For $f \in APW(\mathbf{R})$ and $\lambda \in (0, \infty)$, consider the matrix function

$$\Gamma_\lambda(f) := \begin{pmatrix} e_\lambda & 0 \\ f & e_{-\lambda} \end{pmatrix}.$$

Karlovich and Spitkovsky [13], [14] showed that some central problems of Wiener-Hopf theory can be reduced to finding a so-called APW factorization of $\Gamma_\lambda(f)$, i.e., to representing $\Gamma_\lambda(f)$ in the form

$$\Gamma_\lambda(f) = K_+ \begin{pmatrix} e_\sigma & 0 \\ 0 & e_{-\sigma} \end{pmatrix} K_-$$

where $\sigma \in \mathbf{R}$ and $K_+^{\pm 1}$ and $K_-^{\pm 1}$ are 2×2 matrix functions with entries in

$$APW^+ := APW_{[0,\infty)}(\mathbf{R}) \quad \text{and} \quad APW^- := APW_{(-\infty,0]}(\mathbf{R}),$$

respectively. One can show (see [14]) that $\Gamma_\lambda(f)$ admits an APW factorization if and only if $\Gamma_\lambda(P_\lambda f)$ has such a factorization, where

$$P_\lambda f := \sum_{\mu \in (-\lambda, \lambda)} M(f e_{-\mu}) e_\mu.$$

The idea of [4] is to construct functions $u, v, g_1, g_2 \in APW^+$ such that

$$\begin{pmatrix} u & v \\ g_1 & g_2 \end{pmatrix} \begin{pmatrix} e_\lambda & 0 \\ f & e_{-\lambda} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} e_\nu & 0 \\ h & e_{-\nu} \end{pmatrix},$$

and, in addition,

$$u g_2 - v g_1 \in \mathbf{C} \setminus \{0\}, \quad \nu < \lambda.$$

Because $u g_2 - v g_1$ is a nonzero constant, the entries of

$$\begin{pmatrix} u & v \\ g_1 & g_2 \end{pmatrix}^{-1} = \frac{1}{u g_2 - v g_1} \begin{pmatrix} g_2 & -v \\ -g_1 & u \end{pmatrix}$$

belong to APW^+, and hence $\Gamma_\lambda(f)$ has an APW factorization if and only if $\Gamma_\nu(h)$ has an APW factorization. Since $\nu < \lambda$ and h may be replaced by $P_\nu h$, there is some hope that finding an APW factorization of $\Gamma_\nu(h)$ is simpler than the same problem for $\Gamma_\lambda(f)$. As shown in [4], [5], [6], [17], [18], there are indeed surprisingly many cases in which this strategy works. The above approach was first employed by Spitkovsky and Tishin [19], [20] to factorize $\Gamma_\lambda(f)$ for certain classes of trinomials f. However, the actual impact of this method for APW factorization was realized only in [4] and the subsequent papers [5], [6], [17], [18]. The Bastos, Karlovich, Spitkovsky, Tishin paper [4] received an essential impetus from joint work of its first three authors in Lisbon, and therefore I henceforth call the passage from $\Gamma_\lambda(f)$ to $\Gamma_\nu(h)$ (or $\Gamma_\nu(P_\nu h)$) the Portuguese transformation.

Suppose $\Omega(f) \subset (-\lambda, \lambda)$ contains a minimal element $-\nu < 0$ and suppose we can find $g_1, g_2 \in APW^+$ such that

$$e_{\lambda+\nu} g_1 + e_\nu f g_2 = 1. \tag{26}$$

Letting $u := -e_\nu f$ and $v := e_{\lambda+\nu}$, we get

$$\begin{pmatrix} -e_\nu f & e_{\lambda+\nu} \\ g_1 & g_2 \end{pmatrix} \begin{pmatrix} e_\lambda & 0 \\ f & e_{-\lambda} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} e_\nu & 0 \\ g_2 e_{-\lambda} & e_{-\nu} \end{pmatrix}$$

and

$$u g_2 - v g_1 = -e_\nu f g_2 - e_{\lambda+\nu} g_1 = -1 \in \mathbf{C} \setminus \{0\}.$$

Thus, the Portuguese transformation replaces the matrix function $\Gamma_\lambda(f)$ by $\Gamma_\nu(g_2 e_{-\lambda})$ and thus by $\Gamma_\nu(P_\nu\ (g_2(e_{-\lambda}))$.

Clearly, (26) is a corona problem. Putting $e_{\lambda+\mu} =: e_\mu$ and $e_\nu f =: f_2$, we can rewrite (26) in the form

$$e_\mu g_1 + f_2 g_2 = 1. \tag{27}$$

Passage from $\Gamma_\lambda(f)$ to $\Gamma_\nu(g_2 e_{-\lambda})$ is advantageous because $\nu < \lambda$. However, this transformation is not helpful in case it destroys some good structure owned by f. Thus, suppose we are given an additive semi-group $\Sigma \subset [0, \infty)$ such that $\nu + \Omega(f) \subset \Sigma$ and $0 \in \Sigma$. We then want to have a solution g_2 of (26) such that $\Omega(P_\nu(g_2 e_{-\lambda}))$ is also contained in Σ. In the language of problem (27), this means that we have

$$f_2 \in APW_\Sigma^+(\mathbf{R}) \text{ and } \{0\} \subset \Omega(f_2) \subset (0, \mu) \tag{28}$$

and that we are looking for a solution of (27) such that $\Omega(g_2) \cap [0, \mu) \subset \Sigma$ or, equivalently, $P_\mu g_2 \in APW_\Sigma^+(\mathbf{R})$.

Since

$$|e_\mu(x + iy)| = e^{-\mu y}, \quad |f_2(x + iy)| \to |M(f_2)| \text{ as } y \to +\infty, \tag{29}$$

and since $|M(f_2)| \neq 0$ due to the inclusion $\{0\} \subset \Omega(f_2)$, Proposition 4.2 implies that if, in addition, $\mu \in \Sigma$, then (27), (28) has a solution $g_1, g_2 \in APW_\Sigma^+(\mathbf{R})$ provided $\mathbf{C}^+$ is dense in the maximal ideal space M_Σ. We know from Section 4 that $\mathbf{C}_+$ need not be dense in M_Σ. In this light, the following result is quite remarkable.

Theorem 5.1. *The problem* (27), (28) *always has a solution* $g_1, g_2 \in APW^+$ *such that* $P_\mu g_2 \in APW_\Sigma^+(\mathbf{R})$.

Proof (after Rodman and Spitkovsky). Suppose first that $f_2 \in AP^0_\Sigma(\mathbf{R})$ and write

$$f_2 = \sum_{j=0}^{m} a_j e_{\gamma_j} \quad \text{where} \quad 0 = \gamma_0 < \gamma_1 < \ldots < \gamma_m < \mu.$$

For $k = (k_1, \ldots, k_m) \in \mathbf{Z}^m_+$, put

$$c_k = \frac{(k_1 + \ldots + k_m)!}{k_1! \ldots k_m!} \left(-\frac{a_1}{a_0}\right)^{k_1} \ldots \left(-\frac{a_m}{a_0}\right)^{k_m}$$

(note that $a_0 = M(f_2) \neq 0$). Further, let $\gamma = (\gamma_1, \ldots, \gamma_m)$ and set

$$g_2 := \sum_{(k,\gamma)<\mu} c_k \frac{1}{a_0} e_{(k,\gamma)},$$

$$g_1 := \sum_{j=1}^{m} \sum_{\mu-\gamma_j \leq (k,\gamma) < \mu} c_k \left(-\frac{a_j}{a_0}\right) e_{(k,\gamma)+\gamma_j-\mu}.$$

A direct computation (see [17]) shows that $e_\mu g_1 + f_2 g_2 = 1$. Since $\gamma_1, \ldots, \gamma_m \in \Sigma$, it follows that $g_2 \in AP^0_\Sigma(\mathbf{R})$.

Now let $f_2 \in APW^+_\Sigma(\mathbf{R})$ be arbitrary and determine $g_1, g_2 \in APW^+$ so that $e_\mu g_1 + f_2 g_2 = 1$ (which is possible because of (29) and the density of $\mathbf{C}_+$ in $M_{[0,\infty)}$). Choose $f^{(m)} \in AP^0_\Sigma(\mathbf{R})$ so that $\|f_2 - f^{(m)}\|_W \to 0$. Then

$$e_\mu g_1 + f^{(m)} g_2 = 1 + (f^{(m)} - f_2) g_2 =: \psi_m,$$

and as $\|\psi_m - 1\|_W \to 0$ as $m \to \infty$, it results that $\psi_m^{-1} \in APW^+$ for all sufficiently large m. Obviously,

$$e_\mu g_1 \psi_m^{-1} + f^{(m)} g_2 \psi_m^{-1} = 1. \tag{30}$$

In [18], it is shown that the general solution $\tilde{g}_1, \tilde{g}_2 \in APW^+$ of problem (27), (28) is given by

$$\tilde{g}_1 = g_1 + \varphi f_2, \quad \tilde{g}_2 = g_2 - \varphi e_\mu$$

where $g_1, g_2 \in APW^+$ is any particular solution and φ is an arbitrary function in APW^+. Hence, from what was proved in the first paragraph of this proof and from (30) we deduce that there are $\varphi_m \in APW^+$ such that

$$g_2 \psi_m^{-1} - \varphi_m e_\mu \in AP^0_\Sigma(\mathbf{R}).$$

This implies that $P_\mu(g_2 \psi_m^{-1}) \in AP^0_\Sigma(\mathbf{R})$. Because $\|g_2 \psi_m^{-1} - g_2\|_W \to 0$ as $m \to \infty$, we finally see that $P_\mu g_2 \in APW^+_\Sigma(\mathbf{R})$. ■

In the case where $\Sigma = G \cap [0, \infty)$ for some additive group $G \subset \mathbf{R}$, Theorem 5.1 is in Rodman and Spitkovsky's paper [18]. They also realized that the $\Sigma = G \cap [0, \infty)$ version of the theorem is insufficient for the purposes of AP factorization: for example, if $\Sigma = \{0\} \cup [\rho, \infty)$ for some $\rho > 0$, one wants to know that $P_\mu g_2$ belongs to $APW^+_\Sigma(\mathbf{R})$ together with f_2. Therefore, Rodman and Spitkovsky [18] repeatedly employed the argument of the proof of Theorem 5.1. Thus, although they did not state this theorem explicitly, one can nevertheless say that they already had it.

References

[1] J. Aczél: *Vorlesungen über Funktionalgleichungen und ihre Anwendungen.* Birkhäuser Verlag, Basel and Stuttgart 1961.

[2] R. Arens and I. Singer: Generalized analytic functions. *Trans. Amer. Math. Soc.* **81** (1956), 379-393.

[3] J.A. Ball, Yu.I. Karlovich, L. Rodman, I.M. Spitkovsky: Sarason interpolation and Toeplitz corona for almost periodic matrix functions. Preprint 1997.

[4] M.A. Bastos, Yu.I. Karlovich, I.M. Spitkovsky, P.M. Tishin: On a new algorithm for almost periodic factorization. Preprint 12/96, Inst. Sup. Técnico, Lisbon 1996.

[5] M.A. Bastos, Yu.I. Karlovich, A.F. dos Santos, P.M. Tishin: The corona theorem and the existence of canonical factorization of triangular AP-matrix functions. Preprint 8/97, Inst. Sup. Técnico, Lisbon 1997.

[6] M.A. Bastos, Yu.I. Karlovich, A.F. dos Santos, P.M. Tishin: The corona theorem and the factorization of triangular AP-matrix functions – criteria and explicit formulas. Preprint 11/97, Inst. Sup. Técnico, Lisbon 1997.

[7] L. Carleson: Interpolation by bounded analytic functions and the corona problem. *Ann. of Math.* **76** (1962), 547-559.

[8] J. Garnett: *Bounded Analytic Functions.* Academic Press, New York 1981 [Russian transl.: Mir, Moscow 1984].

[9] I.M. Gelfand, D.A. Raikow, G.E. Schilow: *Kommutative normierte Algebren.* Deutscher Verlag d. Wiss., Berlin 1964 [Russian original: Fizmatgiz, Moscow 1960].

[10] I. Gohberg and I.A. Feldman: *Convolution Equations and Projection Methods for Their Solution.* Amer. Math. Soc. Transl. Math. Monographs, Vol. **41**, Providence, R.I., 1974 [Russian original: Nauka, Moscow 1971; German transl.: Akademie-Verlag, Berlin 1974].

[11] E. Hewitt and K.A. Ross: *Abstract Harmonic Analysis* I. Springer-Verlag, Berlin, Göttingen, Heidelberg 1963.

[12] E. Hlawka, J. Schoissengeier, R. Taschner: *Geometric and Analytic Number Theory.* Springer-Verlag, Berlin, Heidelberg, New York 1991.

[13] Yu.I. Karlovich: On the Haseman problem. *Demonstratio Mathematica* **26** (1993), 581-595.

[14] Yu.I. Karlovich and I.M. Spitkovsky: Factorization of almost periodic matrix-functions and the Noether theory for certain classes of equations of convolution type. *Math. of the USSR, Izvestiya* **34** (1990), 281-316 [Russian original: *Izvestiya, Ser. Matem.*, **53** (1989), 276-308].

[15] P. Koosis: *Introduction to H^p Spaces (With an Appendix on Wolff's Proof of the Corona Theorem).* Cambridge Univ. Press, Cambridge 1980 [Russian transl.: Mir, Moscow 1984].

[16] A.A. Pankov: *Bounded and Almost Periodic Solutions of Nonlinear Differential Operator Equations.* Kluwer, Dordrecht, Boston, London 1990 [Russian original: Naukova Dumka, Kiev 1985].

[17] D. Quint, L. Rodman, I.M. Spitkovsky: New cases of almost periodic factorization of triangular matrix functions. *Michigan Math. J.* **45** (1998)

[18] L. Rodman and I.M. Spitkovsky: Almost periodic factorization and corona theorem. *Indiana Univ. Math. J.* (to appear).

[19] I.M. Spitkovsky and P.M. Tishin: Factorization of new classes of almost periodic matrix functions. *Reports of the Extended Sessions of the I.N. Vekua Inst. for Applied Math.* **3** (1989), 170-173 [Russian].

[20] P.M. Tishin: *Factorization and Formulas for the Partial Indices of Certain Classes of Matrix Functions.* Ph.D. Thesis, Odessa State Univ. 1990 [Russian].

[21] J. Xia: Conditional expectations and the corona problem of ergodic Hardy spaces. *J. Funct. Analysis* **64** (1985), 251-274.

Fakultät für Mathematik
TU Chemnitz
D - 09107 Chemnitz
Germany

aboettch@mathematik.tu-chemnitz.de

MSC 1991: Primary 46 J 15
Secondary 30 H 05, 42 A 75, 43 A 60, 47 A 68