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A NEW THEOREM OF THE ALTERNATIVE-A LOCATION APPROACH

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Abstract—By means of a location approach there is given a new method to obtain theroems of the alternative. This consists in associating a location problem to the considered original system of linear inequalities. There is introduced a dual optimization problem and strong duality is established. From the dual problem there is derived a second system of equations and inequalities and for both systems the duality and solvability properties allow to conclude a general theorem of the alternative. The assigned location problem includes weighting factors and powers of the norms. These weighting factors and exponents may be considered as parameters appearing in the theorem of the alternative, which therefore represents a whole class of theorems of the alternative. The presented theorem of the alternative may be considered as a generalization of the well-known Gale's theorem of the alternative for linear inequalities.

Keywords: theorem of the alternative, location problem, duality

1 Introduction

Theorems of the alternative play an important role in mathematics and especially in optimization, also in the historical sense. To confirm this let us recall the classical theorems of the alternative due to Gordan (1873), Farkas (1902), Stiemke (1915) and Motzkin (1936). Especially Farkas' result and its generalizations have been applied in optimization to derive optimality conditions and duality results.

For extensive presentations quoting a lot of different theorems of the alternative we refer to the books by O.L. Mangasarian (1969) and M.J. Panik (1993).

The purpose of the present paper is to establish a new theorem of the alternative using a new location approach as for the first time introduced by Wanka (1996, a, b.).

This method can be characterized by associating a location problem to a given system of equations or inequalities. Then there is constructed a dual optimization problem to that location problem. From the dual problem there is derived an adjoint or dual system of equations and inequalities. Strong duality then allows to conclude a theorem of the alternative for the original and the dual system, i.e. the assertion that exactly one of the systems is solvable, but never both.

Different from the approach due to Wanka (1996 b) the assigned multi facility location problem includes weighting factors and powers of the norms describing the distances between the location variables. Therefore these parameters (weighting factors and exponents) also appear within the adjoint system and in the formulation of the theorem of the alternative. This means that we have the interesting fact and new viewpoint of a theorem of the alternative with parameters. Choosing for the parameters special numbers we obtain also special theorems of the alternative. As an example there is given the case with squared norms and constant weighting parameters. The paper finishes with the remark that the introduced theorem of the alternative may be considered as a generalization of the well-known Gale's theorem of the alternative for linear inequalities.

2 Location formulation of linear inequalities

We have to consider systems of inequalities in $\mathbb{R}^n$ and therefore we agree concerning to the partial ordering in $\mathbb{R}^n$ the denotations

$$x \geqq y \text{ iff } x_i \geqq y_i, \quad i = 1, \ldots, n,$$

$$x > y \text{ iff } x_i > y_i, \quad i = 1, \ldots, n,$$

for $x = (x_1, \ldots, x_n)^T$, $y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$.

We consider a system of linear inequalities

$$Ax \leqq b, \tag{1}$$

where A is a $(m+1) \times n$ matrix, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^{m+1}$. We represent the matrix A in the form

$$A = \begin{pmatrix} a_0^T \\ a_1^T \\ \vdots \\ a_m^T \end{pmatrix}$$

with rows a_i^T and $a_i \in \mathbb{R}^n$, $i = 0, 1, \ldots, m$. The system (1) can be written in the form

$$\langle a_i, x \rangle := a_i^T x \leqq b_i, \; i = 0, 1, \ldots, m. \tag{2}$$

As announced we assign to (1) and (2), respectively, the following location problem

$$(P) \qquad \inf_{\substack{\langle a_i, x_i \rangle \leqq b_i, \\ i = 0, 1, \ldots, m}} \sum_{i=1}^{m} \lambda_i \|x_i - x_0\|^{n_i}$$

with weighting factors $\lambda_i > 0$ and numbers $n_i \geqq 1$, $i = 1, \ldots, m$.

Let us spend some words to the interpretation of the problem (P). If we are looking at the objective function then we recognize that we are concerned with a multifacility location problem. Indeed, let us set the exponents $n_i = 1, i = 1, \ldots, m$, then (P) turns out to be a Weber problem in $\mathbb{R}^n$, where the locations of the facilities $x_0, x_1, \ldots, x_m$ to be determined are restricted to regions described by the inqualities $a_i^T x_i \leqq b_i$, i.e. they represent half-spaces of $\mathbb{R}^n$.

We have here the situation that there are no fixed facilities contrary to the classical Weber problem (also called Fermat - Weber problem) consisting in minimizing the weighted sum of distances between a new point x (sought facility) and m given fixed facilities $x_1, \ldots, x_m$ (demand points, locations of given customers). The objective function to be minimized than reads as $\sum_{i=1}^{m} \lambda_i \|x_i - x\|$, where the numbers $\lambda_1, \ldots, \lambda_m$ are weighting factors. In the classical Weber problem as norm is taken the Euclidean norm, but also other norms (ℓ_p, ℓ_∞ norm etc.) play an important role in theory and practice of location problems. As norm in (P) may be chosen any norm, too, even there may be different norms depending on i.

In terms of location theory one could interpret $x_1, \ldots, x_m$ as the locations of m customers or demand points, which are not fixed but have potential locations in given half-spaces. And x_0 represents the location of a new facility (like x within the above mentioned Weber problem) that has the task to serve the clients.

The obvious connection between the system (1) and the location problem (P) may be expressed in the following proposition.

Proposition 1 *The minimum of the location problem (P) is equal to zero if and only if the inequality system (1) has a solution.*

3 The dual location problem

We now assign a dual problem to the location problem (P). This will enable us to formulate a dual or adjoint system to (1). The construction of the dual problem will be done on the base of the Fenchel-Rockafellar duality concept. This approach works as follows (cf. Ekeland and Temam (1976)). The objective function $f(x)$ of the optimization problem $\inf_{x \in X} f(x)$ has to be perturbed introducing a perturbation variable (parameter) $\varphi \in Y$. The perturbation space Y is an appropriate linear space (e.g. a normed one). If the perturbed objective function is denoted by $\Phi(x, \varphi)$ with $\Phi(x, 0) = f(x)$ then the perturbed optimization problem reads as $\inf_{x \in X} \Phi(x, \varphi)$.

We recall that the conjugate function g^* to a function g is defined by $g^*(x^*) = \sup_{x \in X} \{\langle x^*, x\rangle - g(x)\}$ where $x^* \in X^*$ (dual space to X) and $\langle x^*, x\rangle$ denotes the linear continuous functional x^* taken at the point $x \in X$. Because we are working in $X = \mathbb{R}^n$ we have $\langle x^*, x\rangle = x^{*T}x$, where $x, x^* \in \mathbb{R}^n$, (cf. (2)).

With the conjugate function $\Phi^*(x^*, \varphi^*)$ a perturbed dual problem is defined by $\sup_{p^* \in Y^*} \{-\Phi^*(x^*, p^*)\}$. By Y^* the topological dual space of linear continuous functionals to Y is denoted. Here $x^* \in X^*$ plays the role of the perturbation variable for the dual problem. Setting the dual perturbation variable $x^* = 0$ one obtains a dual programming problem to the original primal problem $\inf_{x \in X} f(x)$ and it holds the so-called weak duality relation $\sup_{p^* \in Y^*} \{-\Phi^*(0, p^*)\} \leqq \inf_{x \in X} f(x)$. If instead of this inequality there is an equation one says that strong duality is fulfilled. In general there is weak duality but under suitable additional assumptions strong duality holds.

Accordingly, we define a perturbation function

$$\Phi(x_0, x_1, \ldots, x_m, \varphi, \gamma) = \begin{cases} \sum\limits_{i=1}^{m} \lambda_i \|x_i + \varphi_i - x_0\|^{n_i} \\ \text{for } \langle a_i, x_i\rangle - b_i \leqq \gamma_i,\ i = 0, 1, \ldots, m, \\ \infty \text{ otherwise,} \end{cases} \tag{3}$$

with the perturbation variables $\varphi = (\varphi_1, \ldots, \varphi_m)$, $\varphi_i \in \mathbb{R}^n$, $i = 1, \ldots, m$, $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_m)^T \in \mathbb{R}^{m+1}$.

So we have the perturbed location problem

$$\inf_{x_i \in \mathbb{R}^n, i=0,\ldots,m} \Phi(x_0, x_1, \ldots, x_m, \varphi, \gamma). \tag{4}$$

For $\varphi_i = (0, \ldots, 0)^T$, $i = 1, \ldots, m$ and $\gamma_i = 0$, $i = 0, \ldots, m$, arises the original problem (P).

The perturbed location problem (4) admits an interesting geometrical interpretation. Namely, by (3) the perturbation variables $\varphi_i, i = 1, \ldots, m$, are introduced in such a way that they cause a displacement of the facilities or demand points x_i to $x_i + \varphi_i$. The perturbation variables $\gamma_i, i = 0, \ldots, m$, however, effect parallel translations of the boundaries of the given halfspaces as admissible regions for the locations x_i.

A perturbed dual optimization problem may be generated according to the Fenchel-Rockafellar concept as explained above in the form

$$\sup_{\substack{\varphi_i^* \in \mathbb{R}^n,\ i=1,\ldots,m,\\ \gamma_j^* \in \mathbb{R},\ j=0,\ldots,m}} \{-\Phi^*(x_0^*, \ldots, x_m^*, \varphi^*, \gamma^*)\} \tag{5}$$

with Φ^* the conjugate function to Φ and $\varphi^* = (\varphi_1^*, \ldots, \varphi_m^*)$, $\gamma^* = (\gamma_0^*, \ldots, \gamma_m^*)^T$. For the problem (5) the variables $x_0^*\ x_1^*, \ldots, x_m^* \in \mathbb{R}^n$ are the perturbation variables.
Choosing $x_i^* = (0, \ldots, 0)^T$, $i = 0, \ldots, m$, we get the dual problem (P^*) to (P).
Regarding to the definition of the conjugate we obtain

$$\begin{aligned}&\Phi^*(x_0^*, \ldots, x_m^*, \varphi^*, \gamma^*)\\ &= \sup\left\{\sum_{i=0}^m \langle x_i^*, x_i\rangle + \sum_{i=1}^m \langle \varphi_i^*, \varphi_i\rangle + \langle \gamma^*, \gamma\rangle - \Phi(x_0, \ldots, x_m, \varphi, \gamma)\right\},\end{aligned}$$

where the supremum is taken over $x_i \in \mathbb{R}^n$, $i = 0, \ldots, m$, $\varphi_i \in \mathbb{R}^n$, $i = 1, \ldots, m$, and $\gamma \in \mathbb{R}^{m+1}$. With (3) we have

$$\begin{aligned}&\Phi^*(x_0^*, \ldots, x_m^*, \varphi^*, \gamma^*)\\ &= \sup_{\substack{x_i \in \mathbb{R}^n,\ i=0,\ldots,m,\\ \varphi_i \in \mathbb{R}^n,\ i=1,\ldots,m,\\ \gamma \in \mathbb{R}^{m+1},\\ \langle a_i, x_i\rangle \leqq b_i + \gamma_i,\ i=0,\ldots,m}} \left\{\sum_{i=0}^m \langle x_i^*, x_i\rangle + \sum_{i=1}^m \langle \varphi_i^*, \varphi_i\rangle + \langle \gamma^*, \gamma\rangle - \sum_{i=1}^m \lambda_i \|x_i + \varphi_i - x_0\|^{n_i}\right\}\end{aligned}$$

We introduce new variables $y = (y_1, \ldots, y_m)$, $y_i \in \mathbb{R}^n$, $i = 1, \ldots, m$, and $z = (z_0, \ldots, z_m)^T \in \mathbb{R}^{m+1}$ substituting φ and γ by

$$\begin{aligned} y_i &= x_i + \varphi_i - x_0, && i = 1, \ldots, m,\\ z_i &= \gamma_i + b_i - \langle a_i, x_i\rangle, && i = 0, \ldots, m.\end{aligned}$$

This implies

$$\begin{aligned}
&\Phi^*(x_0^*,\dots,x_m^*,\varphi^*,\gamma^*)\\
&= \sup_{\substack{z_i \geqq 0,\, x_i\in\mathbb{R}^n,\, i=0,\dots,m,\\ y_i\in\mathbb{R}^n,\, i=1,\dots,m}} \Big\{ \sum_{i=0}^m \langle x_i^*, x_i\rangle + \sum_{i=1}^m \langle \varphi_i^*, x_0 + y_i - x_i\rangle \\
&\qquad + \sum_{i=0}^m \gamma_i^*\,(z_i - b_i + \langle a_i, x_i\rangle) - \sum_{i=1}^m \lambda_i \|y_i\|^{n_i}\Big\}\\
&= \sup_{\substack{z_i \geqq 0,\, x_i\in\mathbb{R}^n,\, i=0,\dots,m,\\ y_i\in\mathbb{R}^n,\, i=1,\dots,m}} \Big\{ \sum_{i=1}^m \Big[\lambda_i\Big(\Big\langle \tfrac{1}{\lambda_i}\varphi_i^*, y_i\Big\rangle - \|y_i\|^{n_i}\Big)\Big] + \langle x_0^*, x_0\rangle + \sum_{i=1}^m \langle \varphi_i^*, x_0\rangle \\
&\qquad + \sum_{i=1}^m \langle x_i^* - \varphi_i^* + \gamma_i^* a_i, x_i\rangle + \langle \gamma^*, z - b\rangle + \langle \gamma_0^* a_0, x_0\rangle \Big\}
\end{aligned}$$

with $b = (b_0,\dots,b_m)^T \in \mathbb{R}^{m+1}$. Therefore holds

$$\begin{aligned}
&\Phi^*(x_0^*,\dots,x_m^*,\varphi^*,\gamma^*)\\
&= \sum_{i=1}^m \lambda_i \sup_{y_i\in\mathbb{R}^n} \Big\{\Big\langle \tfrac{1}{\lambda_i}\varphi_i^*, y_i\Big\rangle - \|y_i\|^{n_i}\Big\} + \sum_{i=1}^m \sup_{x_i\in\mathbb{R}^n} \langle x_i^* - \varphi_i^* + \gamma_i^* a_i, x_i\rangle\\
&+ \sup_{x_0\in\mathbb{R}^n} \langle x_0^* + \gamma_0^* a_0 + \sum_{i=1}^m \varphi_i^*, x_0\rangle + \sup_{z\geqq 0}\langle \gamma^*, z\rangle - \langle \gamma^*, b\rangle.
\end{aligned}$$

Computing the different suprema we get

$$\sup_{y_i\in\mathbb{R}^n} \left\{\left\langle \frac{1}{\lambda_i}\varphi_i^*, y_i\right\rangle - \|y_i\|^{n_i}\right\} = (n_i - 1)\left\|\frac{1}{n_i\lambda_i}\varphi_i^*\right\|_*^{\frac{n_i}{n_i-1}}$$

for $n_i > 1$. This follows from $f^*(u^*) = \frac{n-1}{n}\|u^*\|_*^{\frac{n}{n-1}}$ for $f(u) = \frac{1}{n}\|u\|^n$, $n > 1$ (cf. Ekeland and Temam (1976)). For $n_i = 1$ holds (cf. also Ekeland and Temam (1976))

$$\sup_{y_i\in\mathbb{R}^n} \left\{\left\langle \frac{1}{\lambda_i}\varphi_i^*, y_i\right\rangle - \|y_i\|\right\} = \begin{cases} 0 \text{ for } \|\varphi_i^*\|_* \leqq \lambda_i, \\ \infty \text{ otherwise.}\end{cases}$$

By $\|\cdot\|_*$ there is denoted the dual norm to the given norm $\|\cdot\|$.

Moreover there is

$$\sup_{x_i\in\mathbb{R}^n} \langle x_i^* - \varphi_i^* + \gamma_i^* a_i, x_i\rangle = \begin{cases} 0 \text{ for } x_i^* - \varphi_i^* + \gamma_i^* a_i = 0, \\ \infty \text{ otherwise,}\end{cases}$$

$$\sup_{x_0\in\mathbb{R}^n} \left\langle x_0^* + \gamma_0^* a_0 + \sum_{i=1}^m \varphi_i^*, x_0\right\rangle = \begin{cases} 0 \text{ for } x_0^* + \gamma_0^* a_0 + \sum_{i=1}^m \varphi_i^* = 0, \\ \infty \text{ otherwise,}\end{cases}$$

$$\sup_{z\geqq 0} \langle \gamma^*, z\rangle = \begin{cases} 0 \text{ for } \gamma^* \leqq 0 \\ \infty \text{ otherwise.}\end{cases}$$

This yields with (5) the perturbed dual problem

$$\sup_{\substack{\gamma^* \leqq 0,\ \|\varphi_i^*\|_* \leqq \lambda_i \text{ for } n_i = 1,\\ \varphi_i^* - \gamma_i^* a_i = x_i^*,\ i = 1,\dots,m,\\ -\sum_{i=1}^m \varphi_i^* - \gamma_0^* a_0 = x_0^*}} \left\{ \langle \gamma^*, b\rangle + \sum_{\substack{i=1\\ i:n_i>1}}^m \lambda_i(1-n_i) \left\| \frac{1}{n_i\lambda_i}\varphi_i^* \right\|_*^{\frac{n_i}{n_i-1}} \right\}.$$

Substituting $p_i^* := \frac{1}{n_i\lambda_i}\varphi_i^*$, $i = 1,\dots,m$, and setting the perturbation variables $x_0^* = x_1^* = \dots = x_m^* = 0$ we obtain the dual problem to (P)

$$\sup_{\substack{\gamma^* \leqq 0,\ \|p_i^*\|_* \leqq 1 \text{ for } n_i = 1,\\ n_i\lambda_i p_i^* - \gamma_i^* a_i = 0,\ i = 1,\dots,m,\\ -\sum_{i=1}^m n_i\lambda_i p_i^* - \gamma_0^* a_0 = 0}} \left\{ \langle \gamma^*, b\rangle + \sum_{\substack{i=1\\ i:n_i>1}}^m \lambda_i(1-n_i) \|p_i^*\|_*^{\frac{n_i}{n_i-1}} \right\}.$$

The variables p_i^* may be eliminated in the constraints and the emerging equation $\sum_{i=0}^m \gamma_i^* a_i = 0$ may be described by $A^T\gamma^* = 0$. Therefore the dual problem (P^*) to (P) reads as

$$(P^*) \qquad \sup_{\substack{\gamma^* \leqq 0,\ |\gamma_i^*| \leqq \frac{\lambda_i}{\|a_i\|_*} \text{ for } n_i = 1,\\ A^T\gamma^* = 0}} \left\{ \langle \gamma^*, b\rangle + \sum_{\substack{i=1\\ i:n_i>1}}^m \lambda_i(1-n_i) \left\| \frac{\gamma_i^*}{n_i\lambda_i} a_i \right\|_*^{\frac{n_i}{n_i-1}} \right\}.$$

For the primal problem (P) a regularity condition of the following form is satisfied: there exist variables $(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_m)$ with $\langle a_i, \bar{x}_i\rangle < b_i$, $i = 0,\dots,m$, i.e. the inequalities representing the constraints are strictly fulfilled.This fact implies the so-called stability of (P) and therefore the strong duality between (P) and (P^*) according to the Fenchel-Rockafellar duality theory (Ekeland and Temam, 1976). Moreover this means that the dual problem (P^*) has a solution allowing to write "max" instead of "sup" within the formulation of(P^*). Writing for the infimum (minimum) of (P) and the supremum (maximum) of (P^*), respectively, inf (P) (min (P)) and sup (P^*) (max (P^*)) we may state the following strong duality Proposition.

Proposition 2 *For (P) and the dual problem (P^*) applies*

$$inf\ (P) = \ max\ (P^*).$$

4 The theorem of the alternative

As mentioned before the existence of a solution to $Ax \leqq b$ is equivalent to $\min(P) = 0$. Therefore it is obvious that $\inf(P) > 0$ implies that $Ax \leqq b$ has no solution. But what about $\inf(P) = 0$? The answer is given by the following Proposition.

Proposition 3 *From* $inf\ (P) = 0$ *follows* $\min(P) = 0$, *i.e.* (P) *is solvable.*

Proof: Let $(x_{0l}, x_{1l}, \ldots, x_{ml})$, $l = 1, 2, \ldots$, be an infimal sequence to (P) on the assumption that $\inf(P) = 0$.
Because of $\sum\limits_{i=1}^{m} \lambda_i \|x_{il} - x_{0l}\|^{n_i} \to 0$ for $l \to \infty$, we have also $\|x_{il} - x_{0l}\| \to 0$ for $l \to \infty$ and moreover there is $\langle a_i, x_{il}\rangle \leqq b_i$, $i = 0, \ldots, m$. By Wanka (1996 b) it has been verified (cf. formula (7) p. 242) that these both conditions imply the existence of the limit $(\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_m)$ of an appropriate subsequence of the above infimal sequence with the property $\bar{x}_i = \bar{x}_0, i = 1, \ldots, m$, and fulfilling the side conditions of (P) $\langle a_i, \bar{x}_i\rangle \leqq b_i, i = 0, \ldots, m$. Therefore $(\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_m)$ is feasible for (P) and the objective function value is equal to zero

$$\sum_{i=1}^{m} \lambda_i \|\bar{x}_i - \bar{x}_0\|^{n_i} = \sum_{i=1}^{m} \lambda_i \|\bar{x}_0 - \bar{x}_0\|^{n_i} = 0,$$

i.e. $\min(P) = 0$. ■

Using the strong duality of Proposition 2 and the assertion of Proposition 3 we may point out the following general new theorem of the alternative.

Theorem *For a given* $(m+1) \times n$ *matrix* A *and a given vector* $b \in \mathbb{R}^{m+1}$, *either*

$I \quad Ax \leqq b \quad$ *has a solution* $\quad x \in \mathbb{R}^n$

or

$$II \quad A^T\gamma^* = 0, \quad \gamma^* \leqq 0, \quad \langle \gamma^*, b\rangle > \sum_{\substack{i=1 \\ i:\, n_i > 1}}^{m} \lambda_i (n_i - 1) \left\| \frac{\gamma_i^*}{n_i \lambda_i} a_i \right\|_*^{\frac{n_i}{n_i - 1}},$$

$\gamma_i^* \geq -\lambda_i \|a_i\|_*^{-1}$ *for* i *with* $n_i = 1$ *has a solution* $\gamma^* \in \mathbb{R}^{m+1}$,
but never both.

Proof: Let us suppose $Ax \leqq b$ to be solvable. Then there is $\min(P) = 0$. Proposition 2 says $\max(P^*) = 0$, too. From the formulation of (P^*) follows

$$\langle \gamma^*, b\rangle + \sum_{\substack{i=1 \\ i:\, n_i > 1}}^{m} \lambda_i (1 - n_i) \left\| \frac{\gamma_i^*}{n_i \lambda_i} a_i \right\|_*^{\frac{n_i}{n_i - 1}} \leqq 0$$

for all $\gamma^* \in \mathbb{R}^{m+1}$ with $\gamma^* \leqq 0$, $A^T\gamma^* = 0$ and $\gamma_i^* \geqq -\lambda_i \|a_i\|_*^{-1}$ for i with $n_i = 1$,i.e. the system II has no solution.

Now let $Ax \leqq b$ be unsolvable. From Proposition 3 follows $\inf(P) > 0$ and with Proposition 2 $\max(P^*) > 0$. This yields the existence of an element $\gamma^* \in \mathbb{R}^{m+1}$ with the properties

$$\gamma^* \leqq 0,\ A^T\gamma^* = 0,\ \gamma_i^* \geqq \lambda_i \|a_i\|_*^{-1} \text{ for } i \text{ with } n_i = 1$$

$$\text{and } \langle \gamma^*, b\rangle + \sum_{\substack{i=1 \\ i:n_i>1}}^{m} \lambda_i(1-n_i) \left\| \frac{\gamma_i^*}{n_i\lambda_i} a_i \right\|_*^{\frac{n_i}{n_i-1}} > 0,$$

i.e. the system II has a solution. ∎

In this theorem of the alternative there are some parameters λ_i and n_i, $1 \leqq i \leqq m$. Choosing special values for these parameters one obtains special theorems of the alternative. Therefore this Theorem gives even a whole class of such assertions.
An interesting case emerges if we choose $\lambda_i = \frac{1}{4}$ and $n_i = 2$, $i = 1, \ldots, m$.
Then we get from the Theorem the following assertion with linear and quadratic inequalities.

Corollary *For a given $(m+1) \times n$ matrix A and a given vector $b \in \mathbb{R}^{m+1}$, either*

I $\quad Ax \leqq b \quad$ has a solution $\quad x \in \mathbb{R}^n$

or

II $\quad A^T\gamma^* = 0, \quad \gamma^* \leqq 0, \quad \langle \gamma^*, b\rangle > \sum_{i=1}^{m} \|\gamma_i^* a_i\|_*^2 \quad$ has a solution $\gamma^* \in \mathbb{R}^{m+1}$,

but never both.

It is possible to normalize the system $Ax \leqq b$ in the sense that $\|a_i\|_* = 1$, $i = 1, \ldots, m$, what of course also changes the coordinates of b. Then the quadratic inequality of the system II in the Corollary takes the simpler form $\langle \gamma^*, b\rangle > \sum_{i=1}^{m} \gamma_i^{*2}$. This can be done also for the Theorem. Finally, let us refer to Gale's theorem of the alternative for linear inequalities (cf. Gale (1960), Mangasarian (1969)).

Proposition *For a given $m \times n$ matrix A and a given vector $b \in \mathbb{R}^m$, either*

I $\quad Ax \leqq b \quad$ has a solution $\quad x \in \mathbb{R}^n$

or

II $\quad A^T\gamma^* = 0, \quad \gamma^* \leqq 0, \quad \langle \gamma^*, b\rangle = 1 \quad$ has a solution $\gamma^* \in \mathbb{R}^{m+1}$,

but never both.

Obviously, we can replace $\langle \gamma^*, b\rangle = 1$ by $\langle \gamma^*, b\rangle > 0$. Comparing this result with our Theorem (or the Corollary) we see that the Theorem (or the Corollary) may be considered as a generalization of Gale's theorem of the alternative. We have namely (e.g. according to the Corollary) the result that in the case of the unsolvability of $Ax \leqq b$ there exists a solution of $A^T\gamma^* = 0$, $\gamma^* \leqq 0$ with not only fulfilling $\langle \gamma^*, b\rangle > 0$ as said by Gale's theorem, but even satisfying $\langle \gamma^*, b\rangle > \sum_{i=1}^{m} \|a_i\|_*^2 \gamma_i^2 > 0$.

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