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# THE INTERFACE CRACK PROBLEM FOR ANISOTROPIC BODIES

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**Abstract** The two-dimensional interface crack problem is investigated for anisotropic bodies in the Comninou formulation. It is established that, as in the isotropic case, properly incorporating contact zones at the crack tips avoids contradictions connected with the oscillating asymptotic behaviour of physical and mechanical characteristics leading to the overlapping of material. Applying the special integral representation formulae for the displacement field the problem in question is reduced to the scalar singular integral equation with the index equal to $-1$. The analysis of this equation is given. The comparison with the results of previous authors shows that the integral equations corresponding to the interface crack problems in the anisotropic and isotropic cases are actually the same from the point of view of the theoretical and numerical analysis.

## 1 Introduction

This paper is concerned with two-dimensional mathematical problems of solid-solid interaction. We consider in $\mathbb{R}^2$ two elastic half-planes $x_2 > 0$ and $x_2 < 0$ filled up by different anisotropic materials which are coupled via the $x_1$ axis. In addition, this solid structure contains an interface crack (cut) of finite length. The conventional formulation of the problem, which assumes that the two faces of the crack are traction free and outside of the crack the rigid contact conditions (continuity of the displacements and stresses) are given, leads to contradictions. This is connected with the oscillating asymptotic behaviour of solutions near the tips of the crack which exibits overlapping of material (for details see [7], [12], [3], [4], [5], and references therein).

M.Comninou in [3], [4] considered an alternative version of the above mixed interface problem in isotropic case. The basic idea of the new approach is that the crack faces are always in contact at the very tips of the crack. Mathematically it means that the crack (cut) interval is devided into three subintervals: two of them are contact intervals, one-sided small vicinities of the tips of the crack, and the third one (the middle interval) corresponds to the open traction free part of the crack. On these contact intervals special type transmission conditions are given which describe that the normal components of the displacement and the stress vectors are continuous on the contact interval, while the tangent components of the stress vector vanish on the same contact interval. The Comninou formulation of the interface crack problem is free of the above contradictions. Moreover, we note that the interface crack problem in Comninou formulation is uniquely solvable (see [15]). This approach has been considered and studied by a lot of authors. An exhaustive information concerning theoretical and numerical results for the isotropic case can be found in [3], [4], [5], [6], [8], [9], [10], [15].

As to the general anisotropic case, to the authors' best knowlege, the above problem has not been treated systematically in the scientific literature. In the present paper we apply special potential type intregral representation formulae for the displacement vector (obtained in [1], [2]) and investigate the above interface crack problem in the Comninou formulation. We reduce the problem to the scalar singular integral equation of index $-1$ and give the corresponding analysis. The equation in anisotropic case is quite similar to the equation obtained by Gautesen and Dundurs in [9], [10] for the isotropic case. Therefore, applying the same approach as in [8], [9], and [10] one can solve explicitly the singular integral equation corresponding to the anisotropic case.

# 2 Formulation of the Problem

Denote by $S^{(0)}$ the upper half-plane ($x_2 > 0$) and by $S^{(1)}$ the lower half-plane ($x_2 < 0$). We assume that the domains $S^{(j)}$ are filled up by anisotropic materials with elastic constants $A_{11}^{(j)}$, $A_{12}^{(j)}$, $A_{13}^{(j)}$, $A_{22}^{(j)}$, $A_{23}^{(j)}$, and $A_{33}^{(j)}$, $j = 0, 1$. The common boundary ($x_1$-axis) of the above two half-planes will be referred to as *the contact line* $l$. In what follows we will use the superscript $(j)$ with the physical characteristics corresponding to the domain $S^{(j)}$. Sometimes we will omit the superscript $(j)$ when this causes no confusion.

The system of equations of elastostatics in the anisotropic case, sans body forces, reads as ([11])

$$\begin{aligned} A_{11}\frac{\partial^2 u_1}{\partial x_1^2} + 2A_{13}\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + A_{33}\frac{\partial^2 u_1}{\partial x_2^2} + A_{13}\frac{\partial^2 u_2}{\partial x_1^2} + (A_{12} + A_{33})\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + A_{23}\frac{\partial^2 u_2}{\partial x_2^2} = 0, \\ A_{13}\frac{\partial^2 u_1}{\partial x_1^2} + (A_{12} + A_{33})\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + A_{23}\frac{\partial^2 u_1}{\partial x_2^2} + A_{33}\frac{\partial^2 u_2}{\partial x_1^2} + 2A_{23}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + A_{22}\frac{\partial^2 u_2}{\partial x_2^2} = 0, \end{aligned} \quad (2.1)$$

where $u = (u_1, u_2)^\top$ is the displacement vector, and $x = (x_1, x_2) \in S^{(j)} \subset I\!R^2$. Here and in what follows $\top$ denotes transposition.

The stress components $\sigma_{x_1}$, $\sigma_{x_2}$, $\tau_{x_1 x_2}$, and the strain components $\varepsilon_{x_1}$, $\varepsilon_{x_2}$, $\varepsilon_{x_1 x_2}$ are related by Hook's law

$$\begin{aligned} \sigma_{x_1} &= A_{11}\varepsilon_{x_1} + A_{12}\varepsilon_{x_2} + A_{13}\varepsilon_{x_1 x_2}, \\ \sigma_{x_2} &= A_{12}\varepsilon_{x_1} + A_{22}\varepsilon_{x_2} + A_{23}\varepsilon_{x_1 x_2}, \\ \tau_{x_1 x_2} &= A_{13}\varepsilon_{x_1} + A_{23}\varepsilon_{x_2} + A_{33}\varepsilon_{x_1 x_2}, \end{aligned} \quad (2.2)$$

where

$$\varepsilon_{x_1} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{x_2} = \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_{x_1 x_2} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}. \quad (2.3)$$

The positive definiteness of the potential energy implies that ([11])

$$\begin{aligned} &A_{11}>0, \quad A_{22}>0, \quad A_{33}>0, \\ &A_{11}A_{22}-A_{12}^2>0, \quad A_{11}A_{33}-A_{13}^2>0, \\ &A_{22}A_{33}-A_{23}^2>0, \end{aligned} \qquad \Delta=\det\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}>0. \tag{2.4}$$

In the isotropic case

$$A_{11}=A_{22}=\lambda+2\mu, \quad A_{12}=\lambda, \quad A_{13}=A_{23}=0, \quad A_{33}=\mu, \quad \Delta=4\mu^2(\lambda+\mu),$$

where $\lambda$ and $\mu$ are the Lamé constants.

As it has been indicated in the introduction, we follow the mathematical model of M.Comninou developed in ([3], [4]) for the interface crack problem and assume that we have an interface crack (cut) along the segment $[-L, L]$ on the contact line. According to the Comninou approach outside of the cut ($|x_1|>L$) we have the perfect bond (the rigid contact conditions) between the two materials. Moreover, the crack interval $(-L, L)$ is devided into the three subintervals $(-L,-a)$, $(-a, b)$, and $(b, L)$, where the different boundary and contact conditions are given describing that the middle (open) part $(-a, b)$ of the crack is free of tractions, while the two remaining intervals represent small one-sided vicinites of the tips of the crack and are the contact zones with special contact conditions (see below the formulation of problem $(C)$). We observe that in this model the constants $a$ and $b$ are unknowns and they must be determined in the course of solution.

In addition, we provide that the following conditions

$$\sigma_{x_2}^{\infty}=T, \quad \tau_{x_1x_2}^{\infty}=S, \tag{2.5}$$

where $T$ and $S$ are given real constant numbers, are fulfilled at infinity.

*Problem* $(C)$. Find regular solutions $u^{(j)}$ to the system (2.1) in $S^{(j)}$ $(j=0,1)$ satisfying the following interface and boundary conditions on the contact line $l$

$$\text{i)}\ \ [u^{(0)}]^+=[u^{(1)}]^-, \quad [\sigma_{x_2}^{(0)}]^+=[\sigma_{x_2}^{(1)}]^-, \quad [\tau_{x_1x_2}^{(0)}]^+=[\tau_{x_1x_2}^{(1)}]^-, \qquad x_2=0,\ |x_1|>L, \tag{2.6}$$

$$\text{ii)}\ \ [u_2^{(0)}]^+=[u_2^{(1)}]^-, \quad [\sigma_{x_2}^{(0)}]^+=[\sigma_{x_2}^{(1)}]^-, \quad [\tau_{x_1x_2}^{(0)}]^+=[\tau_{x_1x_2}^{(1)}]^-=0, \qquad x_2=0, \quad x_1\in(-L,-a)\cup(b,L), \tag{2.7}$$

$$\text{iii)}\ \ [\sigma_{x_2}^{(0)}]^+=[\sigma_{x_2}^{(1)}]^-=0, \quad [\tau_{x_1x_2}^{(0)}]^+=[\tau_{x_1x_2}^{(1)}]^-=0, \quad x_2=0,\ x_1\in(-a,b), \tag{2.8}$$

where $a$ and $b$ are unknown real numbers $(0<a<L,\ 0<b<L)$, and where the symbols $[\cdot]^+$ and $[\cdot]^-$ denote limits on $l$ from $S^{(0)}$ and $S^{(1)}$, respectively.

By a regular solution to the system (2.1) is understood a two-dimensional vector $u^{(j)}=(u_1^{(j)}, u_2^{(j)})^\top$ such that:

a) $u^{(j)}\in \mathrm{C}(\overline{S^{(j)}})\cap \mathrm{C}^2(S^{(j)})$,

b) the corresponding stress components $\sigma_{x_1}^{(j)}$, $\sigma_{x_2}^{(j)}$, and $\tau_{x_1x_2}^{(j)}$ (see (2.2)) are continuously extendible on the whole $x_1$ axis except the points $\{-L;\ -a;\ b;\ L\}$ in the vicinity of which they have integrable singularities,

c) for sufficiently large $|x| = (x_1^2 + x_2^2)^{1/2}$

$$u^{(j)}(x) - u_\infty^{(j)}(x) = O(1), \tag{2.9}$$

$$\frac{\partial}{\partial x_k}\left[u^{(j)}(x) - u_\infty^{(j)}(x)\right] = O(|x|^{-2}),\ k = 1, 2, \tag{2.10}$$

where

$$u_\infty^{(j)}(x) = \frac{1}{A_{22}^{(j)}A_{33}^{(j)} - (A_{23}^{(j)})^2}\begin{bmatrix} A_{22}^{(j)}S - A_{23}^{(j)}T \\ A_{33}^{(j)}T - A_{23}^{(j)}S \end{bmatrix} x_2. \tag{2.11}$$

# 3 Preliminary material

In this section we collect some auxiliary material connected with the potential method in the two-dimensional elasticity theory of anisotropic bodies.

**3.1.** First let us note that the conditions (2.2) and (2.4) imply

$$\begin{aligned} \varepsilon_{x_1} &= a_{11}\sigma_{x_1} + a_{12}\sigma_{x_2} + a_{13}\tau_{x_1x_2}, \\ \varepsilon_{x_2} &= a_{12}\sigma_{x_1} + a_{22}\sigma_{x_2} + a_{23}\tau_{x_1x_2}, \\ \varepsilon_{x_1x_2} &= a_{13}\sigma_{x_1} + a_{23}\sigma_{x_2} + a_{33}\tau_{x_1x_2}, \end{aligned} \tag{3.1}$$

where $a_{11},\ a_{12}, \cdots, a_{33}$ are the entries of the matrix $[a_{kj}]_{3\times3}$ inverse to $[A_{kj}]_{3\times3}$ and satisfy the conditions similar to (2.4).

In the isotropic case we have

$$a_{11} = a_{22} = \frac{\lambda + 2\mu}{4\mu(\lambda+\mu)},\ \ a_{12} = -\frac{\lambda}{4\mu(\lambda+\mu)},\ \ a_{13} = a_{23} = 0,\ \ a_{33} = \frac{1}{\mu}.$$

We recall that the so-called characteristic equation of the system (2.1) (see [11])

$$a_{11}\alpha^4 - 2a_{13}\alpha^3 + (2a_{12} + a_{33})\alpha^2 - 2a_{23}\alpha + a_{22} = 0 \tag{3.2}$$

possesses only the complex roots

$$\alpha_k = a_k + \mathrm{i}b_k, \quad \bar\alpha_k = a_k - \mathrm{i}b_k, \quad (b_k > 0), \quad k = 1, 2. \tag{3.3}$$

In the isotropic case $\alpha_1 = \alpha_2 = \mathrm{i}$.

**3.2.** The fundamental matrix of solutions to the system (2.1) has been constructed by M.Basheleishvili in [1]

$$\Gamma(z,t) = \mathrm{Im}\sum_{k=1}^{2}\begin{bmatrix} A_k & B_k \\ B_k & C_k \end{bmatrix}\ln\sigma_k, \tag{3.4}$$

where

$$\begin{aligned} A_k &= -\frac{2}{\Delta a_{11}}(A_{22}\alpha_k^2 + 2A_{23}\alpha_k + A_{33})d_k, \\ B_k &= \frac{2}{\Delta a_{11}}(A_{23}\alpha_k^2 + (A_{12} + A_{33})\alpha_k + A_{13})d_k, \\ C_k &= -\frac{2}{\Delta a_{11}}(A_{33}\alpha_k^2 + 2A_{13}\alpha_k + A_{11})d_k, \\ d_1^{-1} &= (\alpha_1 - \bar\alpha_1)(\alpha_1 - \alpha_2)(\alpha_1 - \bar\alpha_2), \quad d_2^{-1} = (\alpha_2 - \alpha_1)(\alpha_2 - \bar\alpha_1)(\alpha_2 - \bar\alpha_2); \end{aligned} \tag{3.5}$$

here $z = x_1 + \mathrm{i}x_2$ and $t = y_1 + \mathrm{i}y_2$ are arbitrary points in the plane, and

$$\sigma_k = z_k - t_k, \quad z_k = x_1 + \alpha_k x_2, \quad t_k = y_1 + \alpha_k y_2.$$

In the sequel we will use the notation $v(z)$ for the function $v(x_1, x_2)$ of the variables $x_1$ and $x_2$.

The coefficients $A_k, B_k$, and $C_k$ satisfy the equations

$$A_k C_k - B_k^2 = 0, \quad k = 1, 2. \tag{3.6}$$

Next, we set (see [1])

$$C = 2\mathrm{i} \sum_{k=1}^{2} d_k, \quad A = 2\mathrm{i} \sum_{k=1}^{2} \alpha_k d_k, \quad B = 2\mathrm{i} \sum_{k=1}^{2} \alpha_k^2 d_k. \tag{3.7}$$

It is easy to show that

$$C > 0, \quad B > 0, \quad BC - A^2 > 0. \tag{3.8}$$

From (3.7) it follows that

$$\begin{aligned} d_k &= -\frac{\mathrm{i}}{2}(C s_k - A r_k), \quad \alpha_k d_k = -\frac{\mathrm{i}}{2}(A p_k - C q_k), \\ \alpha_k^2 d_k &= -\frac{\mathrm{i}}{2}(B p_k - A q_k), \quad A p_k - C q_k = A s_k - B r_k, \end{aligned} \tag{3.9}$$

where

$$r_k = \frac{(-1)^k}{\alpha_1 - \alpha_2}, \quad s_k = \frac{(-1)^k \alpha_1 \alpha_2}{\alpha_k(\alpha_1 - \alpha_2)}, \quad p_k = -\alpha_k r_k, \quad q_k = -\alpha_k s_k. \tag{3.10}$$

These equations yield

$$\begin{aligned} &\begin{bmatrix} p_k & q_k \\ r_k & s_k \end{bmatrix} = \frac{2\mathrm{i} d_k}{BC - A^2} \begin{bmatrix} \alpha_k^2 & -\alpha_k \\ -\alpha_k & 1 \end{bmatrix} \begin{bmatrix} C & A \\ A & B \end{bmatrix}, \\ &\sum_{k=1}^{2} \begin{bmatrix} p_k & q_k \\ r_k & s_k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =: E. \end{aligned} \tag{3.11}$$

In what follows we shall also use the abridged notation

$$\mathcal{P}_{(k)} := \begin{bmatrix} p_k & q_k \\ r_k & s_k \end{bmatrix}. \tag{3.12}$$

The stress vector $T(\partial_z, n)u(z)$, acting on an arc element with the unit normal vector $n = (n_1, n_2)$, and the corresponding stress operator $T(\partial_z, n)$ are calculated by the formulae

$$
\begin{aligned}
&[T(\partial_z, n)u(z)]_1 = \sigma_{x_1} n_1(z) + \tau_{x_1x_2} n_2(z),\\
&[T(\partial_z, n)u(z)]_2 = \tau_{x_1x_2} n_1(z) + \sigma_{x_2} n_2(z),\\
&T(\partial_z, n) = [T_{kj}(\partial_z, n)]_{2\times 2},\\
&T_{11}(\partial_z, n) = n_1(z)\left(A_{11}\frac{\partial}{\partial x_1} + A_{13}\frac{\partial}{\partial x_2}\right) + n_2(z)\left(A_{13}\frac{\partial}{\partial x_1} + A_{33}\frac{\partial}{\partial x_2}\right),\\
&T_{12}(\partial_z, n) = n_1(z)\left(A_{13}\frac{\partial}{\partial x_1} + A_{12}\frac{\partial}{\partial x_2}\right) + n_2(z)\left(A_{33}\frac{\partial}{\partial x_1} + A_{23}\frac{\partial}{\partial x_2}\right),\\
&T_{21}(\partial_z, n) = n_1(z)\left(A_{13}\frac{\partial}{\partial x_1} + A_{33}\frac{\partial}{\partial x_2}\right) + n_2(z)\left(A_{12}\frac{\partial}{\partial x_1} + A_{23}\frac{\partial}{\partial x_2}\right),\\
&T_{22}(\partial_z, n) = n_1(z)\left(A_{33}\frac{\partial}{\partial x_1} + A_{23}\frac{\partial}{\partial x_2}\right) + n_2(z)\left(A_{23}\frac{\partial}{\partial x_1} + A_{22}\frac{\partial}{\partial x_2}\right).
\end{aligned}
\tag{3.13}
$$

Taking into account (3.6) we get

$$
[T(\partial_t, n)\Gamma(z,t)]^\top = \operatorname{Im}\sum_{k=1}^{2}\begin{bmatrix} N_k & M_k \\ L_k & R_k \end{bmatrix}\frac{\partial \ln \sigma_k}{\partial s(t)},
\tag{3.14}
$$

where

$$
\begin{aligned}
&M_k = -\frac{2}{a_{11}}(a_{11}\alpha_k^2 - a_{13}\alpha_k + a_{12})d_k, \quad R_k = -\frac{2}{a_{11}\alpha_k}(a_{12}\alpha_k^2 - a_{23}\alpha_k + a_{22})d_k,\\
&N_k = -\alpha_k M_k, \quad L_k = -\alpha_k R_k, \quad \frac{\partial}{\partial s(t)} = n_1(t)\frac{\partial}{\partial y_2} - n_2(t)\frac{\partial}{\partial y_1}.
\end{aligned}
\tag{3.15}
$$

We have also the following relationship between the above coefficients

$$
\begin{bmatrix} N_k & L_k \\ M_k & R_k \end{bmatrix} = \begin{bmatrix} p_k & q_k \\ r_k & s_k \end{bmatrix} + 2\omega d_k \begin{bmatrix} \alpha_k & \alpha_k^2 \\ -1 & -\alpha_k \end{bmatrix},
\tag{3.16}
$$

where

$$
\omega = b_1 b_2 - a_1 a_2 + \frac{a_{12}}{a_{11}}.
\tag{3.17}
$$

The equations (3.9) imply

$$
\begin{bmatrix} N_k & L_k \\ M_k & R_k \end{bmatrix} = \begin{bmatrix} p_k & q_k \\ r_k & s_k \end{bmatrix}\begin{bmatrix} 1 - \mathrm{i}\omega A & -\mathrm{i}\omega B \\ \mathrm{i}\omega C & 1 + \mathrm{i}\omega A \end{bmatrix}.
\tag{3.18}
$$

Further, since $\alpha_k$ represents the root of the equation (3.2), with the help of the Vieta's theorem we see that

$$
A_k = \frac{a_{11}}{2d_k}M_k^2, \quad B_k = \frac{a_{11}}{2d_k}M_k R_k, \quad C_k = \frac{a_{11}}{2d_k}R_k^2.
\tag{3.19}
$$

Whence due to (3.11) and (3.16)

$$\begin{bmatrix} A_k & B_k \\ B_k & C_k \end{bmatrix} = \begin{bmatrix} N_k & M_k \\ L_k & R_k \end{bmatrix} \left\{ \frac{\mathrm{i}a_{11}}{BC - A^2} \begin{bmatrix} C & A \\ A & B \end{bmatrix} + \omega a_{11} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}. \tag{3.20}$$

Therefore, from (3.18) and (3.20) we derive

$$\sum_{k=1}^{2} \begin{bmatrix} N_k & L_k \\ M_k & R_k \end{bmatrix} = \begin{bmatrix} 1 - \mathrm{i}\omega A & -\mathrm{i}\omega B \\ \mathrm{i}\omega C & 1 + \mathrm{i}\omega A \end{bmatrix}, \tag{3.21}$$

$$\sum_{k=1}^{2} \begin{bmatrix} A_k & B_k \\ B_k & C_k \end{bmatrix} = \frac{\mathrm{i}m}{BC - A^2} \begin{bmatrix} C & A \\ A & B \end{bmatrix}, \tag{3.22}$$

where

$$m = a_{11}(1 - \omega^2(BC - A^2)). \tag{3.23}$$

We note that the constants $\omega$ and $m$ introduced above are positive [1]

$$\omega > 0, \quad m > 0. \tag{3.24}$$

Furthermore, let us introduce the so-called pseudo-stress operator which will be essentially amployed in the next sections

$$N(\partial_z, n)u(z) = T(\partial_z, n)u(z) + \kappa_N \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial u(z)}{\partial s(z)}, \tag{3.25}$$

where

$$\kappa_N = \omega(BC - A^2)m^{-1} > 0. \tag{3.26}$$

By the direct calculations we get

$$[N(\partial_t, n)\Gamma(z,t)]^\top = \mathrm{Im} \sum_{k=1}^{2} \begin{bmatrix} E_k & F_k \\ G_k & H_k \end{bmatrix} \frac{\partial \ln \sigma_k}{\partial s(t)}. \tag{3.27}$$

Here the coefficients $E_k, F_k, G_k$, and $H_k$, due to (3.20), can be written as

$$\begin{bmatrix} E_k & F_k \\ G_k & H_k \end{bmatrix} = \frac{a_{11}}{m} \begin{bmatrix} N_k & M_k \\ L_k & R_k \end{bmatrix} \begin{bmatrix} 1 + \mathrm{i}\omega A & -\mathrm{i}\omega C \\ \mathrm{i}\omega B & 1 - \mathrm{i}\omega A \end{bmatrix},$$

whence

$$\mathcal{E}_{(k)} := \begin{bmatrix} E_k & F_k \\ G_k & H_k \end{bmatrix} = -\frac{\mathrm{i}}{m} \begin{bmatrix} A_k & B_k \\ B_k & C_k \end{bmatrix} \begin{bmatrix} B & -A \\ -A & C \end{bmatrix}. \tag{3.28}$$

Notice that

$$\mathcal{E}_{(1)} + \mathcal{E}_{(2)} = E, \tag{3.29}$$

where $E$ is the $2 \times 2$ identity matrix.

We note that the columns of the matrices (3.14) and (3.27), considered as vector functions of the variable $z$ (i.e., of the variables $x_1$ and $x_2$), solve the system (2.1) for $z \neq t$. Moreover, by equations (3.21) we can show that the entries of the matrix (3.14) have singularities of type $|z-t|^{-1}$ and on arbitrary $\mathrm{C}^{1+\alpha}$-smooth curve (with $\alpha > 0$) they represent singular kernels in variables $z$ and $t$. Making use of (3.29) it can also be shown that the entries of the matrix (3.27) on a $\mathrm{C}^{1+\alpha}$-smooth curve have integrable singularities of type $|z-t|^{-1+\alpha}$.

For our purposes we need also the so-called hypersingular kernels constructed by the fundamental solution (3.4). The direct calculations lead to the formulae

$$T(\partial_z, n)[T(\partial_t, n)\Gamma(z,t)]^\top = -a_{11}^{-1}\,\mathrm{Re}\sum_{k=1}^{2}\mathcal{P}_{(k)}\begin{bmatrix} B & -A \\ -A & C \end{bmatrix}\frac{\partial^2 \ln \sigma_k}{\partial s(z)\partial s(t)}, \tag{3.30}$$

$$T(\partial_z, n)[N(\partial_t, n)\Gamma(z,t)]^\top = \mathrm{Im}\sum_{k=1}^{2}\begin{bmatrix} E'_k & F'_k \\ G'_k & H'_k \end{bmatrix}\frac{\partial^2 \ln \sigma_k}{\partial s(z)\partial s(t)}, \tag{3.31}$$

where

$$\mathcal{E}'_{(k)} := \begin{bmatrix} E'_k & F'_k \\ G'_k & H'_k \end{bmatrix} = -\frac{\mathrm{i}}{m}\begin{bmatrix} N_k & L_k \\ M_k & R_k \end{bmatrix}\begin{bmatrix} B & -A \\ -A & C \end{bmatrix}, \quad k = 1, 2. \tag{3.32}$$

These matrices can be also represented as follows

$$\mathcal{E}'_{(k)} = \begin{bmatrix} p_k & q_k \\ r_k & s_k \end{bmatrix}\left\{\kappa_N\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \frac{\mathrm{i}}{m}\begin{bmatrix} B & -A \\ -A & C \end{bmatrix}\right\}, \quad k = 1, 2. \tag{3.33}$$

The formulae (3.32) and (3.33) yield

$$E'_k = -\alpha_k G'_k, \quad F'_k = -\alpha_k H'_k,$$

$$\begin{bmatrix} E_k & F_k \\ G_k & H_k \end{bmatrix} = \left\{\omega a_{11}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \frac{\mathrm{i}a_{11}}{BC - A^2}\begin{bmatrix} C & A \\ A & B \end{bmatrix}\right\}\begin{bmatrix} E'_k & F'_k \\ G'_k & H'_k \end{bmatrix},$$

$$\sum_{k=1}^{2}\begin{bmatrix} E'_k & F'_k \\ G'_k & H'_k \end{bmatrix} = \kappa_N\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \frac{\mathrm{i}}{m}\begin{bmatrix} B & -A \\ -A & C \end{bmatrix}, \quad k = 1, 2. \tag{3.34}$$

We remark that the coefficients $A$, $B$, $C$, $\omega$, $m$, and $\kappa_N$ defined by (3.7), (3.17), (3.23), and (3.26), respectively, in the isotropic case read as

$$A = 0, \quad B = C = \frac{1}{2}, \quad \omega = 2\frac{\kappa - 1}{\kappa + 1}, \quad m = \frac{\kappa}{2\mu(\kappa + 1)}, \quad \kappa_N = \frac{\mu(\kappa - 1)}{\kappa}, \tag{3.35}$$

where $\kappa = (\lambda + 3\mu)(\lambda + \mu)^{-1}$.

**3.3.** Let $l_0$ be a simple, closed, nonselfintersecting, $C^{1+\alpha}$-smooth curve which surrounds a bounded domain $\Omega^+$. Further, let $\Omega^- = I\!R^2 \backslash \overline{\Omega^+}$, $\overline{\Omega^+} = \Omega^+ \cup l_0$, $l_0 = \partial\Omega^\pm$. Throughout this paper by $n(z)$ $(z \in l_0)$ we denote the outward unit normal vector with respect to $\Omega^+$.

We introduce the following generalized potentials

$$V(g)(z) := \frac{1}{\pi} \int_{l_0} \Gamma(z,t)\, g(t)\, ds, \tag{3.36}$$

$$W(g)(z) := \frac{1}{\pi} \int_{l_0} [T(\partial_t, n)\Gamma(z,t)]^\top g(t)\, ds, \tag{3.37}$$

$$\tilde{W}(g)(z) := \frac{1}{\pi} \int_{l_0} [N(\partial_t, n)\Gamma(z,t)]^\top g(t)\, ds, \tag{3.38}$$

where $g = (g_1, g_2)^\top$ is a real-valued density; $V$, $W$ and $\widetilde{W}$ will be referred to as a single-layer potential, a double-layer potential of the first kind and a double-layer potential of the second kind, respectively. Clearly, the potentials (3.36)-(3.38) are solutions of the system (2.1) in $\Omega^+$ and $\Omega^-$ for arbitrary integrable density $g$. Properties of the above potentials in the space of Hölder-continuous vector functions have been investigated in [1].

**THEOREM 3.1** [1] *Let $l_0 \in C^{2+\alpha}$, $0 < \alpha < 1$, and $0 < \beta < \alpha$. Then*

i) *the single-layer potential $V(g)$ is continuous in the whole plane if $g \in C(l_0)$ and $V(g) \in C^{k+1+\beta}(\overline{\Omega^\pm})$ if $g \in C^{k+\beta}(l_0)$, with $k = 0, 1, 2$;*

ii) *the double-layer potentials $W(g)$ and $\widetilde{W}(g)$ belong to the spaces $C^{k+\beta}(\overline{\Omega^\pm})$ if $g \in C^{k+\beta}(l_0)$ with $k = 0, 1, 2$; moreover, for arbitrary $g \in C^\beta(l_0)$ and $t_0 \in l_0$ there hold the following jump formulae*

$$[W(g)(t_0)]^\pm = \pm g(t_0) + \frac{1}{\pi} \int_{l_0} [T(\partial_t, n(t))\Gamma(t_0, t)]^\top g(t) ds, \tag{3.39}$$

$$[\widetilde{W}(g)(t_0)]^\pm = \pm g(t_0) + \frac{1}{\pi} \int_{l_0} [N(\partial_t, n(t))\Gamma(t_0, t)]^\top g(t) ds; \tag{3.40}$$

iii) *for arbitrary $g \in C^\beta(l_0)$ and $t_0 \in l_0$ there hold the jump relations*

$$[T(\partial_{t_0}, n)V(g)(t_0)]^\pm = \mp g(t_0) + \frac{1}{\pi} \int_{l_0} [T(\partial_{t_0}, n(t_0))\Gamma(t_0, t)]\, g(t) ds, \tag{3.41}$$

$$[N(\partial_{t_0}, n)V(g)(t_0)]^\pm = \mp g(t_0) + \frac{1}{\pi} \int_{l_0} [N(\partial_{t_0}, n(t_0))\Gamma(t_0, t)]\, g(t) ds; \tag{3.42}$$

iv) *for arbitrary $g \in C^{1+\beta}(l_0)$ the vectors $T(\partial_z, n)W(g)(z)$ and $T(\partial_z, n)\widetilde{W}(g)(z)$ are continuously extendible on $l_0$ from $\Omega^\pm$ and for arbitrary $t_0 \in l_0$ there hold*

$$[T(\partial_{t_0}, n)W(g)(t_0)]^\pm = \frac{1}{\pi a_{11}} \mathrm{Re} \sum_{k=1}^{2} \mathcal{P}_{(k)} \begin{bmatrix} B & -A \\ -A & C \end{bmatrix} \int_{l_0} \frac{\partial \ln \sigma_k}{\partial s(t_0)} \frac{\partial g(t)}{\partial s(t)} ds, \tag{3.43}$$

$$[T(\partial_{t_0}, n)\widetilde{W}(g)(t_0)]^\pm = \mp \kappa_N \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial g(t_0)}{\partial s(t_0)} - \frac{1}{\pi} \mathrm{Im} \sum_{k=1}^{2} \mathcal{E}'_{(k)} \int_{l_0} \frac{\partial \ln \sigma_k}{\partial s(t_0)} \frac{\partial g(t)}{\partial s(t)} ds. \tag{3.44}$$

We note that the integral operators in (3.39) and (3.41) have singular kernels, while the integral operators in (3.40) and (3.42) have weakly singular kernels (cf. [13], [14], [1]).

**3.4.** Finally, we recall the Green formulae in anisotropic elasticity (see, e.g., [1]). Let $u \in \mathrm{C}^1(\overline{\Omega^+}) \cap \mathrm{C}^2(\Omega^+)$ represents a solution to the system (2.1) in $\Omega^+$. Then the following equations hold

$$\int_{\Omega^+} T(u,u)d\sigma = \int_{l_0} [u]^+[T(\partial_t, n)u]^+ ds, \tag{3.45}$$

$$\int_{\Omega^+} N(u,u)d\sigma = \int_{l_0} [u]^+[N(\partial_t, n)u]^+ ds, \tag{3.46}$$

where $T(u,u)$ and $N(u,u)$ are positive definite quadratic forms and read as

$$\begin{aligned} T(u,u) &= A_{11}\varepsilon_{x_1}^2 + 2A_{12}\varepsilon_{x_1}\varepsilon_{x_2} + A_{22}\varepsilon_{x_2}^2 + 2A_{13}\varepsilon_{x_1}\varepsilon_{x_1x_2} + \\ &\quad + 2A_{23}\varepsilon_{x_2}\varepsilon_{x_1x_2} + A_{33}\varepsilon_{x_1x_2}^2, \end{aligned} \tag{3.47}$$

$$\begin{aligned} N(u,u) &= A_{11}\varepsilon_{x_1}^2 + 2(A_{12}+\kappa_N)\varepsilon_{x_1}\varepsilon_{x_2} + A_{22}\varepsilon_{x_2}^2 + \\ &\quad + 2A_{13}\varepsilon_{x_1}\varepsilon_{x_1x_2} + 2A_{23}\varepsilon_{x_2}\varepsilon_{x_1x_2} + (A_{33} - \frac{1}{2}\kappa_N)\varepsilon_{x_1x_2}^2 + \frac{1}{2}\kappa_N\omega_{x_1x_2}^2, \end{aligned} \tag{3.48}$$

where $\varepsilon_{x_1}$, $\varepsilon_{x_2}$, and $\varepsilon_{x_1x_2}$ are given by (2.3), and

$$\omega_{x_1x_2} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}.$$

It can be easily seen that the equatoin $T(u,u)=0$ implies $u(z) = (e_1 - e_3x_2,\ e_2 + e_3x_1)$, while the equation $N(u,u) = 0$ yields $u(z) = (e_1, e_2)$, where $e_1$, $e_2$, and $e_3$ are arbitrary real constants.

In the case of the domain $\Omega^-$ the formulae (3.45) and (3.46) (with the sign "–" in the left-hand side) remain also valid if, in addition, the vector $u$ meets the conditions

$$u(z) = O(1), \quad \frac{\partial u(z)}{\partial x_k} = O(|z|^{-2}),\ k = 1,2, \tag{3.49}$$

at infinity.

**3.5.** For the interface crack problem formulated above the following uniqueness theorem holds.

**THEOREM 3.2** *Let a pair $(u^{(0)}, u^{(1)})$ of vector functions $u^{(0)} = (u_1^{(0)}, u_2^{(0)})^\top$ and $u^{(1)} = (u_1^{(1)}, u_2^{(1)})^\top$ be a regular solution to the homogeneous version of Problem $(C)$ (i.e., $T=0$ and $S=0$ ).*

*Then $u^{(0)} = (e_1, e_2)^\top$ in $S^{(0)}$ and $u^{(1)} = (e_1, e_2)^\top$ in $S^{(1)}$ where $e_1$ and $e_2$ are arbitrary real constants.*

*Proof.* We start with the remark that $u^{(0)}$ and $u^{(1)}$ satisfy conditions (3.49) at infinity, since in the case under consideration $u_\infty^{(j)}(x) = 0$, $j = 0,1$ (see (2.9) and (2.10)). We proceed as follows. Denote by $K_R$ the circle centered at the origin and radius $R$, and let $l_R^+$ and $l_R^-$ be the corresponding upper and lower semicircles. We put $S_R^{(0)} = S^{(0)} \cap K_R$ and $S_R^{(1)} = S^{(1)} \cap K_R$. Further, due to formula (3.45) and conditions (2.6)-(2.8), we have

$$\int_{S_R^{(0)}} T^{(0)}(u^{(0)}, u^{(0)}) d\sigma + \int_{S_R^{(1)}} T^{(1)}(u^{(1)}, u^{(1)}) d\sigma =$$
$$= \int_{l_R^+} [u^{(0)}][T^{(0)}(\partial_t, n) u^{(0)}] ds + \int_{l_R^-} [u^{(1)}][T^{(1)}(\partial_t, n) u^{(1)}] ds, \tag{3.50}$$

where $n$ is the exterior unit normal vector to $l_R = l_R^+ \cup l_R^-$. Taking into account the behaviour of the vectors $u^{(j)}$ at infinity, the positive definiteness of the forms $T^{(0)}(u^{(0)}, u^{(0)})$ and $T^{(1)}(u^{(1)}, u^{(1)})$, and passing to the limit in (3.50) as $R \to +\infty$, we conclude that the right-hand side integrals in (3.50) vanish and, therefore,

$$T^{(j)}(u^{(j)}, u^{(j)}) = 0, \quad j = 0, 1.$$

Whence

$$u^{(j)}(z) = (e_1^{(j)} - e_3^{(j)} x_2,\ e_2^{(j)} + e_3^{(j)} x_1), \quad j = 0, 1,$$

with arbitrary constants $e_k^{(j)}$, $k = 1, 2, 3$, and $j = 0, 1$. The behaviour of the displacement fields at infinity implies that $e_3^{(j)} = 0$, $j = 0, 1$, while from the contact conditions it follows that $e_k^{(0)} = e_k^{(1)}$, $k = 1, 2$. ∎

**REMARK 3.3** *We note that, if, in addition, the displacement fields in $S^{(0)}$ and $S^{(1)}$ vanish at invinity, then the homogeneous Problem $(C)$ possesses only the trivial solution.*

# 4 The Basic Interface Problem

In this section we present the explicit solution of the basic interface problem for the piecewise homogeneous anisotropic elastic plane $\overline{S^{(0)}} \cup \overline{S^{(1)}}$ introduced above (see Section 2). This problem can be formulated as follows [1]: Find regular solutions $u^{(j)}$ to the system (2.1) in the domains $S^{(j)}$ $(j = 0, 1)$ satisfying on the interface $x_2 = 0$ (i.e., on the line $l$) the transmission conditions

$$\begin{aligned} &[u^{(0)}]^+ - [u^{(1)}]^- = f(x_1), \\ &[\tau_{x_1 x_2}^{(0)}]^+ - [\tau_{x_1 x_2}^{(1)}]^- = \varphi_1(x_1), \\ &[\sigma_{x_2}^{(0)}]^+ - [\sigma_{x_2}^{(1)}]^- = \varphi_2(x_1), \quad -\infty < x_1 < +\infty, \end{aligned} \tag{4.1}$$

where $f = (f_1, f_2)^\top \in \mathrm{C}^{1+\alpha}(l)$ and $\varphi = (\varphi_1, \varphi_2)^\top \in \mathrm{C}^{\alpha}(l)$ are given vector functions with the following asymptotics at infinity

$$f(x_1) = p_0 + \frac{q_0}{|x_1|^{\varepsilon}}, \quad \varphi(x_1) = \frac{r_0}{|x_1|^{1+\eta}}; \tag{4.2}$$

here $p_0,\ q_0,\ r_0,\ \varepsilon > 0$, and $\eta > 0$ are real constants.

In addition, we assume that the resultant vector vanishes, i.e.,

$$\int\limits_{-\infty}^{+\infty} \varphi(x_1)\,dx_1 = 0, \tag{4.3}$$

and that the stress components vanish at infinity as well.

The solution of the above basic interface problem is then representable in the form (for details see [1], [2])

$$\begin{aligned} u^{(j)}(z) &= \frac{\kappa_N^{(1)}-\kappa_N^{(0)}}{\Delta_0}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} K + \frac{1}{\pi}\mathrm{Im}\sum_{k=1}^{2}\mathcal{E}_{(k)}^{(j)}\Bigg\{(X^{(j)}+\mathrm{i}Y^{(j)})\int\limits_{-\infty}^{+\infty}\frac{f(t)dt}{t-z_{kj}}+ \\ &+ (\tilde{X}^{(j)}+\mathrm{i}\tilde{Y}^{(j)})\int\limits_{-\infty}^{+\infty}\varphi(t)\ln(t-z_{kj})dt\Bigg\},\ j=0,1, \end{aligned} \tag{4.4}$$

where $K$ is an arbitrary real constant, $\mathcal{E}_{(k)}^{(j)}$ is given by (3.28), and $z_{kj} = x_1 + \alpha_k^{(j)} x_2$; moreover,

$$\begin{aligned} \Delta_0 &= 2\kappa_N^{(0)}\kappa_N^{(1)} + \frac{B^{(0)}C^{(0)}-(A^{(0)})^2}{m^{(0)}a_{11}^{(0)}} + \frac{B^{(1)}C^{(1)}-(A^{(1)})^2}{m^{(1)}a_{11}^{(1)}} + \\ &\quad + \frac{B^{(1)}C^{(0)}+B^{(0)}C^{(1)}-2A^{(0)}A^{(1)}}{m^{(0)}m^{(1)}} > 0; \\ X^{(0)}+\mathrm{i}Y^{(0)} &= \frac{1}{\Delta_0}\Bigg\{\left(\frac{B^{(1)}C^{(1)}-(A^{(1)})^2}{m^{(1)}a_{11}^{(1)}}+\kappa_N^{(0)}\kappa_N^{(1)}\right)E+ \\ &\quad + \frac{1}{m^{(0)}m^{(1)}}\begin{bmatrix} B^{(1)}C^{(0)}-A^{(0)}A^{(1)} & A^{(0)}C^{(1)}-A^{(1)}C^{(0)} \\ A^{(0)}B^{(1)}-A^{(1)}B^{(0)} & B^{(0)}C^{(1)}-A^{(0)}A^{(1)} \end{bmatrix} - \\ &\quad - \mathrm{i}\left(\frac{\kappa_N^{(1)}}{m^{(0)}}\begin{bmatrix} A^{(0)} & -C^{(0)} \\ B^{(0)} & -A^{(0)} \end{bmatrix} + \frac{\kappa_N^{(0)}}{m^{(1)}}\begin{bmatrix} A^{(1)} & -C^{(1)} \\ B^{(1)} & -A^{(1)} \end{bmatrix}\right)\Bigg\}, \\ \tilde{X}^{(0)}+\mathrm{i}\tilde{Y}^{(0)} &= \frac{1}{\Delta_0}\Bigg\{(\kappa_N^{(0)}-\kappa_N^{(1)})\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \\ &\quad + \mathrm{i}\left(\frac{1}{m^{(1)}}\begin{bmatrix} C^{(1)} & A^{(1)} \\ A^{(1)} & B^{(1)} \end{bmatrix} + \frac{1}{m^{(0)}}\begin{bmatrix} C^{(0)} & A^{(0)} \\ A^{(0)} & B^{(0)} \end{bmatrix}\right)\Bigg\}; \end{aligned} \tag{4.5}$$

the constant matrices $X^{(1)}+\mathrm{i}Y^{(1)}$ and $\tilde{X}^{(1)}+\mathrm{i}\tilde{Y}^{(1)}$ are obtained from the above formulae for $X^{(0)}+\mathrm{i}Y^{(0)}$ and $\tilde{X}^{(0)}+\mathrm{i}\tilde{Y}^{(0)}$ by the interchange of the superscripts (0) and (1). Note that

$$X^{(0)}+X^{(1)} = E, \quad Y^{(0)} = Y^{(1)}, \quad \tilde{X}^{(0)}+\tilde{X}^{(1)} = 0, \quad \tilde{Y}^{(0)} = \tilde{Y}^{(1)}. \tag{4.6}$$

Making use of the relationships between the coefficients established in Subsection 3.2, from (4.4) we derive the following representation formulae for the stress components $\sigma_{x_2}^{(j)}$ and $\tau_{x_1x_2}^{(j)}$

$$\begin{bmatrix} \tau_{x_1x_2}^{(j)} \\ \sigma_{x_2}^{(j)} \end{bmatrix} = -\frac{1}{\pi}\mathrm{Im}\sum_{k=1}^{2}\mathcal{E}_{(k)}^{\prime(j)}\Bigg\{(X^{(j)}+\mathrm{i}Y^{(j)})\int\limits_{-\infty}^{+\infty}\frac{f'(t)dt}{t-z_{kj}} - (\tilde{X}^{(j)}+\mathrm{i}\tilde{Y}^{(j)})\int\limits_{-\infty}^{+\infty}\frac{\varphi(t)dt}{t-z_{kj}}\Bigg\}, \tag{4.7}$$

where $j=0,1$, and $\mathcal{E}'^{(j)}_{(k)}$ is given by (3.32).

# 5 Reduction of the Mixed Interface Problem to the Integral Equation

In this section we shall essentially use the representation formulae (4.4) and (4.7) together with the equations (3.28), (3.29), (3.32), (3.33), (3.34), (4.5), and (4.6), to investigate the mixed interface problem (2.6)-(2.8).

Taking into account the interface conditions (2.6)-(2.8) we easily conclude that $[\sigma^{(0)}_{x_2}]^+ - [\sigma^{(1)}_{x_2}]^- = 0$ and $[\tau^{(0)}_{x_1x_2}]^+ - [\tau^{(1)}_{x_1x_2}]^- = 0$ on the whole contact line $l$. Therefore, due to formula (4.4), we look for the displacement fields in the domains $S^{(j)}$ in the form

$$u^{(j)}(z) = u^{(j)}_\infty(z) + \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{2}\mathcal{E}^{(j)}_{(k)}(X^{(j)}+\mathrm{i}Y^{(j)})\begin{bmatrix}\int\limits_{-L}^{L}\frac{u_1^{(0)}(t)-u_1^{(1)}(t)}{t-z_{kj}}dt\\ \int\limits_{-a}^{b}\frac{u_2^{(0)}(t)-u_2^{(1)}(t)}{t-z_{kj}}dt\end{bmatrix}, \tag{5.1}$$

where $z\in S^{(j)}$, $j=0,1$.

In (5.1) the first summand $u^{(j)}_\infty(z)$, given by (2.11), represents the solution of (2.1) which satisfies conditions (2.5) at infinity. Here and throughout this section we use the notations $u^{(0)}(t) := [u^{(0)}(t,0)]^+$ and $u^{(1)}(t) := [u^{(1)}(t,0)]^-$ for $-\infty < t < +\infty$.

Clearly, the difference $u_1^{(0)}(t) - u_1^{(1)}(t)$ is unknown in the interval $(-L,L)$, while the difference $u_2^{(0)}(t)-u_2^{(1)}(t)$ is unknown in the interval $(-a,b)$.

It is evident that the conditions

$$u_1^{(0)}(\pm L) - u_1^{(1)}(\pm L) = 0, \tag{5.2}$$

$$u_2^{(0)}(-a) - u_2^{(1)}(-a) = 0, \quad u_2^{(0)}(b) - u_2^{(1)}(b) = 0, \tag{5.3}$$

are sufficient for the above displacement vectors $u^{(j)}$ to be continuously extendible on the whole contact line $l$ (see [14]).

Further, let

$$u_1^{(0)}(x_1) - u_1^{(1)}(x_1) = -\int\limits_{-L}^{x_1} B_1(t)dt, \tag{5.4}$$

$$u_2^{(0)}(x_1) - u_2^{(1)}(x_1) = -\int\limits_{-a}^{x_1} B_2(t)dt. \tag{5.5}$$

These equations yield

$$B(x_1) = -\frac{d}{dx_1}(u^{(0)}(x_1) - u^{(1)}(x_1)), \tag{5.6}$$

where $B(x_1) = (B_1(x_1), B_2(x_1))^\top$ is the so-called dislocation vector.

We require that

$$\int\limits_{-L}^{L} B_1(t)dt = 0, \tag{5.7}$$

$$\int\limits_{-a}^{b} B_2(t)dt = 0, \tag{5.8}$$

which guarantee the conditions (5.2) and (5.3).

Moreover, we provide (cf. [3], [4])

$$B_2(-a) = B_2(b) = 0, \tag{5.9}$$

which shows that the normal displacements $u_2^{(0)}$ and $u_2^{(1)}$ have a "smooth contact" at the points $-a$ and $b$ of the contact line $l$. From (5.1) it follows that the stress components can be represented in terms of the dislocation vector

$$\begin{bmatrix} \tau_{x_1x_2}^{(j)} \\ \sigma_{x_2}^{(j)} \end{bmatrix} = \begin{bmatrix} S \\ T \end{bmatrix} + \frac{1}{\pi}\mathrm{Im}\sum_{k=1}^{2} \mathcal{E}'^{(j)}_{(k)}(X^{(j)} + \mathrm{i}Y^{(j)}) \begin{bmatrix} \int\limits_{-L}^{L} \frac{B_1(t)dt}{t-z_{kj}} \\ \int\limits_{-a}^{b} \frac{B_2(t)dt}{t-z_{kj}} \end{bmatrix}. \tag{5.10}$$

The limits of the displacement vectors (5.1) on the contact line are expressed by the formulae

$$u^{(j)}(x_1) = \frac{1}{\pi}Y^{(j)} \begin{bmatrix} \int\limits_{-L}^{L} \frac{u_1^{(0)}(t)-u_1^{(1)}(t)}{t-x_1}dt \\ \int\limits_{-a}^{b} \frac{u_2^{(0)}(t)-u_2^{(1)}(t)}{t-x_1}dt \end{bmatrix}, \quad |x_1| > L, \;\; j = 0, 1.$$

Since $Y^{(0)} = Y^{(1)}$, due to (4.6), we have $u^{(0)}(x_1) = u^{(1)}(x_1)$ for $|x_1| > L$, and, therefore, we can check that the first equation in (2.6) is automatically satisfied.

Next, applying the formula (5.10) we arrive at the equation on $l$

$$\begin{aligned} \begin{bmatrix} \tau_{x_1x_2}^{(0)} \\ \sigma_{x_2}^{(0)} \end{bmatrix}^{+} - \begin{bmatrix} \tau_{x_1x_2}^{(1)} \\ \sigma_{x_2}^{(1)} \end{bmatrix}^{-} &= \frac{1}{\pi}\mathrm{Im}\sum_{k=1}^{2} \Big(\mathcal{E}'^{(0)}_{(k)}(X^{(0)} + \mathrm{i}Y^{(0)}) - \\ &\quad -\mathcal{E}'^{(1)}_{(k)}(X^{(1)} + \mathrm{i}Y^{(1)})\Big) \begin{bmatrix} \int\limits_{-L}^{L} \frac{B_1(t)dt}{t-x_1} \\ \int\limits_{-a}^{b} \frac{B_2(t)dt}{t-x_1} \end{bmatrix}, \quad |x_1| > L. \end{aligned}$$

With the help of the equality

$$\mathrm{Im}\sum_{k=1}^{2} \Big(\mathcal{E}'^{(0)}_{(k)}(X^{(0)} + \mathrm{i}Y^{(0)}) - \mathcal{E}'^{(1)}_{(k)}(X^{(1)} + \mathrm{i}Y^{(1)})\Big) = 0, \tag{5.11}$$

which follows from the fomulae (3.32), (3.33), (3.34), (4.5), and (4.6), we can easily show that the stress components (5.10) automatically satisfy the second and the third equations in (2.6).

In that case when $x_1 \in (-L,-a)\cup(b,L)$ we have

$$u^{(j)}(x_1) = (-1)^j X^{(j)} \begin{bmatrix} u_1^{(0)}(x_1) - u_1^{(1)}(x_1) \\ 0 \end{bmatrix} + \frac{Y^{(j)}}{\pi} \begin{bmatrix} \int\limits_{-L}^{L} \frac{u_1^{(0)}(t)-u_1^{(1)}(t)}{t-x_1}dt \\ \int\limits_{-a}^{b} \frac{u_2^{(0)}(t)-u_2^{(1)}(t)}{t-x_1}dt \end{bmatrix}.$$

Whence it follows that

$$u^{(0)}(x_1) - u^{(1)}(x_1) = (X^{(0)} + X^{(1)}) \begin{bmatrix} u_1^{(0)}(x_1) - u_1^{(1)}(x_1) \\ 0 \end{bmatrix} = \begin{bmatrix} u_1^{(0)}(x_1) - u_1^{(1)}(x_1) \\ 0 \end{bmatrix}.$$

Therefore, $u_2^{(0)}(x_1) = u_2^{(1)}(x_1)$ for $x_1 \in (-L,-a)\cup(b,L)$, and the first equation in (2.7) is also fulfilled.

Further, we derive the following relation

$$\begin{aligned}
&\begin{bmatrix} \tau_{x_1x_2}^{(0)} \\ \sigma_{x_2}^{(0)} \end{bmatrix}^+ - \begin{bmatrix} \tau_{x_1x_2}^{(1)} \\ \sigma_{x_2}^{(1)} \end{bmatrix}^- = \\
&= \mathrm{Re} \sum_{k=1}^{2} \left( \mathcal{E}'^{(0)}_{(k)}(X^{(0)} + \mathrm{i}Y^{(0)}) + \mathcal{E}'^{(1)}_{(k)}(X^{(1)} + \mathrm{i}Y^{(1)}) \right) \begin{bmatrix} B_1(x_1) \\ 0 \end{bmatrix} + \\
&+ \frac{1}{\pi} \mathrm{Im} \sum_{k=1}^{2} \left( \mathcal{E}'^{(0)}_{(k)}(X^{(0)} + \mathrm{i}Y^{(0)}) - \mathcal{E}'^{(1)}_{(k)}(X^{(1)} + \mathrm{i}Y^{(1)}) \right) \begin{bmatrix} \int\limits_{-L}^{L} \frac{B_1(t)dt}{t-x_1} \\ \int\limits_{-a}^{b} \frac{B_2(t)dt}{t-x_1} \end{bmatrix}, \\
&x_1 \in (-L,-a)\cap(b,L).
\end{aligned}$$

Now the equations (5.11) and the equality

$$\mathrm{Re} \sum_{k=1}^{2} \left( \mathcal{E}'^{(0)}_{(k)}(X^{(0)} + \mathrm{i}Y^{(0)}) + \mathcal{E}'^{(1)}_{(k)}(X^{(1)} + \mathrm{i}Y^{(1)}) \right) = 0 \tag{5.12}$$

imply that the second condition in (2.7) and the equation $[\tau_{x_1x_2}^{(0)}]^+ = [\tau_{x_1x_2}^{(1)}]^-$ are also fulfilled.

Finally, from (5.10) we deduce for $x_1 \in (-a,b)$

$$\begin{aligned}
&\begin{bmatrix} \tau_{x_1x_2}^{(0)} \\ \sigma_{x_2}^{(0)} \end{bmatrix}^+ - \begin{bmatrix} \tau_{x_1x_2}^{(1)} \\ \sigma_{x_2}^{(1)} \end{bmatrix}^- = \\
&= \mathrm{Re} \sum_{k=1}^{2} \left( \mathcal{E}'^{(0)}_{(k)}(X^{(0)} + \mathrm{i}Y^{(0)}) + \mathcal{E}'^{(1)}_{(k)}(X^{(1)} + \mathrm{i}Y^{(1)}) \right) \begin{bmatrix} B_1(x_1) \\ B_2(x_1) \end{bmatrix} + \\
&+ \frac{1}{\pi} \mathrm{Im} \sum_{k=1}^{2} \left( \mathcal{E}'^{(0)}_{(k)}(X^{(0)} + \mathrm{i}Y^{(0)}) - \mathcal{E}'^{(1)}_{(k)}(X^{(1)} + \mathrm{i}Y^{(1)}) \right) \begin{bmatrix} \int\limits_{-L}^{L} \frac{B_1(t)dt}{t-x_1} \\ \int\limits_{-a}^{b} \frac{B_2(t)dt}{t-x_1} \end{bmatrix}.
\end{aligned}$$

Whence one can easily show that $\sigma_{x_2}^{(0)} = \sigma_{x_2}^{(1)}$ and $\tau_{x_1x_2}^{(0)} = \tau_{x_1x_2}^{(1)}$ for $x_1 \in (-a,b)$. Therefore, the conditions (2.6)-(2.8) will be completely fulfilled if

$$\left[\sigma_{x_2}^{(0)}\right]^+ = 0, \quad x_1 \in (-a,b), \tag{5.13}$$

$$\left[\tau_{x_1x_2}^{(0)}\right]^+ = 0, \quad x_1 \in (-L,L). \tag{5.14}$$

These boundary conditions together with the formula (5.10) lead to the system of integral equations

$$QB_1(x_1) + \frac{R_{21}}{\pi}\int_{-L}^{L}\frac{B_1(t)dt}{t-x_1} + \frac{R_{22}}{\pi}\int_{-a}^{b}\frac{B_2(t)dt}{t-x_1} = T, \quad x_1 \in (-a,b),$$

$$Q(H(x_1-b) - H(x_1+a))B_2(x_1) + \frac{R_{11}}{\pi}\int_{-L}^{L}\frac{B_1(t)dt}{t-x_1} +$$

$$+\frac{R_{12}}{\pi}\int_{-a}^{b}\frac{B_2(t)dt}{t-x_1} = S, \quad x_1 \in (-L,L), \tag{5.15}$$

where $H$ stands for the Heviside step function, and

$$Q = \frac{1}{\Delta_0}\left(\kappa_N^{(1)}\frac{B^{(0)}C^{(0)} - (A^{(0)})^2}{m^{(0)}a_{11}^{(0)}} - \kappa_N^{(0)}\frac{B^{(1)}C^{(1)} - (A^{(1)})^2}{m^{(1)}a_{11}^{(1)}}\right),$$

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \frac{1}{\Delta_0}\left(\frac{B^{(1)}C^{(1)} - (A^{(1)})^2}{m^{(0)}m^{(1)}a_{11}^{(1)}}\begin{bmatrix} B^{(0)} & -A^{(0)} \\ -A^{(0)} & C^{(0)} \end{bmatrix} + \right.$$

$$\left. + \frac{B^{(0)}C^{(0)} - (A^{(0)})^2}{m^{(0)}m^{(1)}a_{11}^{(0)}}\begin{bmatrix} B^{(1)} & -A^{(1)} \\ -A^{(1)} & C^{(1)} \end{bmatrix}\right).$$

Remark that

$$R_{11} > 0, \quad R_{22} > 0, \quad R_{12} = R_{21}. \tag{5.16}$$

Thus, we have to find the unknown functions $B_1(x_1)$ and $B_2(x_1)$ satisfying the system of integral equations (5.15) and conditions (5.7)-(5.9).

It is easy to check that in the isotropic case (cf. [3], [4])

$$R_{12} = R_{21} = 0, \quad R_{11} = R_{22} = \frac{2\mu_1(1+\alpha_*)}{(\kappa_1+1)(1-\beta_*^2)} = \frac{2\mu_0(1-\alpha_*)}{(\kappa_0+1)(1-\beta_*^2)} = C_*,$$

$$Q = \beta_* C_*, \quad \beta_* = \frac{\mu_0(\kappa_1-1) - \mu_1(\kappa_0-1)}{\mu_0(\kappa_1+1) + \mu_1(\kappa_0+1)}, \quad \alpha_* = \frac{\mu_0(\kappa_1+1) - \mu_1(\kappa_0+1)}{\mu_0(\kappa_1+1) + \mu_1(\kappa_0+1)}.$$

Therefore, in that case, the system (5.15) reads as

$$\beta_* B_1(x_1) + \frac{1}{\pi}\int_{-a}^{b}\frac{B_2(t)dt}{t-x_1} = \frac{T}{C_*}, \quad x_1 \in (-a,b),$$

$$\beta_*(H(x_1-b) - H(x_1+a))B_2(x_1) + \frac{1}{\pi}\int_{-L}^{L}\frac{B_1(t)dt}{t-x_1} = \frac{S}{C_*}, \quad x_1 \in (-L,L). \tag{5.17}$$

This system coincides with the equations obtained in [6] for the mixed interface problem in question in the isotropic case (i.e., when the domains $S^{(0)}$ and $S^{(1)}$ in the problem (2.6)-(2.8) are isotropic half-planes with distinct Lamé constants). The same system has been investigated in [10] where the explicit solution is also constructed (see also [8], [9]).

We follow to [10] and make in (5.15) the change of independent variables

$$x_1 = L\frac{s-\gamma}{1-s\gamma}, \quad t = L\frac{r-\gamma}{1-r\gamma}, \quad \gamma = \frac{L(a-b)}{L^2-ab+\sqrt{(L^2-a^2)(L^2-b^2)}}. \tag{5.18}$$

Next we introduce the new unknown functions $A_1$ and $A_2$ rather than $B_1$ and $B_2$

$$R_{11}B_1(x_1) = (1-s\gamma)^2 A_1(s), \quad R_{22}B_2(x_1) = (1-s\gamma)^2 A_2(s). \tag{5.19}$$

After these manipulations the equations (5.15) are recast into the system

$$\begin{aligned}
&\frac{Q}{R_{11}}A_1(s) + \frac{R_{21}}{\pi R_{11}}\int_{-1}^{1}\frac{(1-r\gamma)A_1(r)}{(1-s\gamma)(r-s)}dr + \frac{1}{\pi}\int_{-c}^{c}\frac{(1-r\gamma)A_2(r)}{(1-s\gamma)(r-s)}dr = \\
&\qquad = \frac{T}{(1-s\gamma)^2}, \quad |s|<c, \\
&-\frac{Q}{R_{22}}H(c^2-s^2)A_2(s) + \frac{1}{\pi}\int_{-1}^{1}\frac{(1-r\gamma)A_1(r)}{(1-s\gamma)(r-s)}dr + \\
&\qquad + \frac{R_{12}}{\pi R_{22}}\int_{-c}^{c}\frac{(1-r\gamma)A_2(r)}{(1-s\gamma)(r-s)}dr = \frac{S}{(1-s\gamma)^2}, \quad |s|<1,
\end{aligned} \tag{5.20}$$

where

$$c = \frac{a\sqrt{L^2-b^2}+b\sqrt{L^2-a^2}}{L(\sqrt{L^2-b^2}+\sqrt{L^2-a^2})} < 1.$$

The conditions (5.7)-(5.9) now read as

$$\int_{-1}^{1} A_1(r)dr = 0, \tag{5.21}$$

$$\int_{-c}^{c} A_2(r)dr = 0, \tag{5.22}$$

$$A_2(-c) = A_2(c) = 0. \tag{5.23}$$

Applying the equality

$$\int_{-a}^{a}\frac{(1-r\gamma)A(r)}{(1-s\gamma)(r-s)}dr = \int_{-a}^{a}\frac{A(r)dr}{r-s} - \frac{\gamma}{1-s\gamma}\int_{-a}^{a}A(r)dr,$$

and bearing in mind that $A_1$ and $A_2$ are subjected to the conditions (5.21)-(5.23), from the system (5.20) we infer

$$\frac{Q}{R_{11}}A_1(s) + \frac{R_{21}}{\pi R_{11}}\int_{-1}^{1}\frac{A_1(r)dr}{r-s} + \frac{1}{\pi}\int_{-c}^{c}\frac{A_2(r)dr}{r-s} = \frac{T}{(1-s\gamma)^2}, \quad |s|<c,$$

$$-\frac{Q}{R_{22}}H(c^2-s^2)\,A_2(s) + \frac{1}{\pi}\int_{-1}^{1}\frac{A_1(r)dr}{r-s} + \frac{R_{12}}{\pi R_{22}}\int_{-c}^{c}\frac{A_2(r)dr}{r-s} = \frac{S}{(1-s\gamma)^2}, \quad |s|<1. \quad (5.24)$$

Clearly, the unknown functions $A_1$ and $A_2$ have to meet the conditions (5.21)-(5.23). Now we show that this problem can be reduced to a single integral equation (cf. [10]). To this end let us rewrite the second equation in (5.24) as follows

$$\frac{1}{\pi}\int_{-1}^{1}\frac{A_1(r)dr}{r-s} = \frac{Q}{R_{22}}H(c^2-s^2)\;A_2(s) - \frac{R_{12}}{\pi R_{22}}\int_{-c}^{c}\frac{A_2(r)dr}{r-s} + \frac{S}{(1-s\gamma)^2}, \quad |s|<1. \quad (5.25)$$

Next, we shall apply the general theory of singular integral equations on arcs developed in [14] (Chapter 5). According to this theory, if we look for a solution $A_1$ to the equation (5.25) in the class of functions which are unbounded at the both ends $\pm 1$ and satisfy the condition (5.21), we can invert the left-hand side operator in (5.25). As a result we obtain

$$\begin{aligned} A_1(s) \;\;=\;\; & -\frac{X(s)}{\pi}\int_{-1}^{1}\left(\frac{Q}{R_{22}}H(c^2-r^2)A_2(r) - \frac{R_{12}}{\pi R_{22}}\int_{-c}^{c}\frac{A_2(t)dt}{t-r} + \right. \\ & \left. + \frac{S}{(1-r\gamma)^2}\right)\frac{dr}{X(r)(r-s)}, \quad |s|<1, \end{aligned} \quad (5.26)$$

where $X(s) = (1-s^2)^{-1/2}$. On account the additional conditions (5.22) and (5.23) we can simplify (5.26)

$$A_1(s) = -\frac{R_{12}}{R_{22}}H(c^2-s^2)A_2(s) - \frac{QX(s)}{\pi R_{22}}\int_{-c}^{c}\frac{A_2(r)dr}{X(r)(r-s)} + SX(s)\frac{d}{d\gamma}\left(\frac{\sqrt{1-\gamma^2}}{1-s\gamma}\right). \quad (5.27)$$

Substitution of this expression for $A_1$ into the first equation of the system (5.24) implies

$$\frac{1}{\pi}\int_{-c}^{c}\frac{A_2(r)}{r-s}(\sqrt{1-s^2} - \nu^2\sqrt{1-r^2})dr = \frac{d}{d\gamma}\left(\frac{\gamma T_*\sqrt{1-s^2} - \delta S\sqrt{1-\gamma^2}}{1-s\gamma}\right), \quad (5.28)$$

where $|s|<c$ and

$$\nu^2 = \frac{Q^2}{R_{11}R_{22}-R_{12}^2} > 0, \quad T_* = \frac{R_{22}(R_{11}T - R_{12}S)}{R_{11}R_{22}-R_{12}^2}, \quad \delta = \frac{QR_{22}}{R_{11}R_{22}-R_{12}^2}.$$

Thus, we have obtained the single integral equation (5.28) together with the conditions (5.22)-(5.23) to find the unknown $A_2$ in the class of functions bounded at the both ends $\pm c$.

Let us single out the so-called characteristic (dominant singular) part of the equation (5.28)

$$\frac{1}{\pi}\int\limits_{-c}^{c}\frac{A_2(r)\,dr}{r-s}+\frac{1}{\pi}\int\limits_{-c}^{c}K(s,r)A_2(r)\,dr=\Phi(s),\quad -c<s<c, \tag{5.29}$$

where

$$K(s,r)=\frac{\nu^2}{1-\nu^2}\frac{r+s}{\sqrt{1-s^2}(\sqrt{1-s^2}+\sqrt{1-r^2})},$$
$$\Phi(s)=\frac{1}{1-\nu^2}\frac{1}{\sqrt{1-s^2}}\frac{d}{d\gamma}\left(\frac{\gamma T_*\sqrt{1-s^2}-\delta S\sqrt{1-\gamma^2}}{1-s\gamma}\right).$$

Obviously, $K(s,r)\in \mathrm{C}^{1/2}([-c,c]\times[-c,c])$ and $\Phi\in \mathrm{C}^{1/2}([-c,c])$, i.e., the kernel $K$ and the right-hand side function $\Phi$ are Hölder-continuous functions with the exponent $1/2$.

We look for solutions of the equation (5.29) in the class of functions which are bounded at the both ends $-c$ and $c$. Due to [14], we denote this class by $h(-c,c)$. We emphasize that, in addition, the solution has to satisfy the contitions (5.22) and (5.23). It is evident that (see [14]), if equation (5.29) possesses a solution, then this solution automatically satisfies the conditions (5.23) since the function $\Phi$ is Hölder-continuous. Thus, we have to look for a solution to the equation (5.29) in the class $h(-c,c)$ with the only additional condition (5.22).

It can be easily seen that the index of the equation (5.29) in the class $h(-c,c)$ is equal to $-1$. Furthermore, applying the uniqueness Theorem 3.2 and Remark 3.3 we can show that the homogeneous version of equation (5.29) has no nontrivial solutions in the class $h(-c,c)$. Therefore, the homogeneous equation

$$\frac{1}{\pi}\int\limits_{-c}^{c}\frac{\sigma(r)}{r-s}(\sqrt{1-s^2}-\nu^{-2}\sqrt{1-r^2})dr=0, \tag{5.30}$$

which is adjoint to equation (5.29), has only one linearly independent solution in the class of functions unbounded at the both ends $-c$ and $c$. We denote this nontrivial solution by $\sigma_0(s)$. Due to the Noether's theorems the condition

$$\int\limits_{-c}^{c}\Phi(r)\,\sigma_0(r)\,dr=0 \tag{5.31}$$

is then necessary and sufficient for the equation (5.29) (i.e., (5.28)) to be solvable (cf. [14]). The unknown parameters $c$ and $\gamma$ (i.e., $a$ and $b$) are to be defined by the conditions (5.22) and (5.31).

A direct comparison shows that the equations (5.28), (5.22), and (5.23) are quite similar to the equations obtained by Gautesen and Dundurs (see equations (2.18), (2.13), and (2.19) in [10]). In fact, the above integral equation (5.28) and the equation (2.18) in [10] coincide term by term within the constant coefficients involved in these equations. Therefore, the analysis given in [10] can also be applied word for word to our case to construct the explicit solution of the equation (5.28) and to define the unknown parameters $a$ and $b$.

The above results show that the theoretical and numerical analysis of the singular integral equations, corresponding to the Comninou formulation of the interface crack problem, are quite similar in the isotropic and in the general anisotropic cases.

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