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NON-LOCAL APPROACH IN MATHEMATICAL PROBLEMS OF FLUID-STRUCTURE INTERACTION

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Abstract Three-dimensional mathematical problems of interaction between elastic and scalar oscillation fields are investigated. An elastic field is to be defined in a bounded inhomogeneous anisotropic body occupying the domain $\overline{\Omega}_1 \subset I\!R^3$ while a physical (acoustic) scalar field is to be defined in the exterior domain $\overline{\Omega}_2 = I\!R^3 \setminus \Omega_1$ which is filled up also by an anisotropic (fluid) medium. These two fields satisfy the governing equations of steady state oscillations in the corresponding domains together with special kinematic and dynamic transmission conditions on the interface $\partial\Omega_1$. The problems are studied by the so-called non-local approach, which is the coupling of the boundary integral equation method (in the unbounded domain) and the functional-variational method (in the bounded domain). The uniqueness and existence theorems are proved and the regularity of solutions are established with the help of the corresponding Steklov-Poincaré type operators and on the basis of the Gårding inequality and the Lax-Milgram theorem. In particular, it is shown that the physical fluid-solid acoustic interaction problem is solvable for arbitrary values of the frequency parameter.

Introduction

This paper is concerned with three-dimensional mathematical problems of fluid-structure interaction and their generalizations. The physical fluid-solid acoustic interaction problem leading to the generalized mathematical model considered in the present paper can be described as follows. We consider an acoustic wave propagation in an anisotropic fluid domain of infinite extent where a bounded inhomogeneous anisotropic elastic structure is immersed. The elastic body is coupled to the fluid via the smooth interface where the so-called *kinematic and dynamic boundary conditions* are given: the normal velocity of the fluid on one

side of the boundary must match the normal velocity of the solid on the other side and the resultant force acting on all parts of the interface has to vanish. We suppose that a time-harmonic acoustic wave is incident upon the elastic target and the problem is to determine the scattered acoustic pressure in the fluid domain and the displacement field in the elastic structure.

The fluid-structure interaction problem, described above, has been investigated by a lot of authors. An exhaustive information concerning theoretical and numerical results, for the case when the both interacting media are isotropic, can be found in [1], [2], [5], [3], [4], [15], [16], [18], [19], [21], [25], [26], [33], [34], [37], [51], [53], [54], [55], [56], [57].

The case, when the elastic body under consideration is homogeneous and anisotropic, and the fluid is isotropic, has been considered in [48]. In [29] and [30] the interface problems of the interaction between a four-dimensional thermoelastic field and a scalar oscillation field have been studied in general homogeneous anisotropic case.

Mainly, in the above cited papers the direct and the indirect boundary integral equations methods and the theory of pseudodifferential equations on manifolds are the principal tools to investigate the interface problems in question. We note that by the same methods a wide class of three-dimensional solid-solid interaction problems has been investigated in [35], [46], [27], [31], [32] (see also the references therein).

By these methods the original fluid-solid interaction problems are reduced to the corresponding system of boundary pseudodifferential equations (PsDE) on the interface. The analysis of these PsDEs (uniqueness, index problem and existence), besides the radiation conditions for the scalar acoustic field, in addition, needs special, the so-called *generalized Sommerfeld-Kupradze type radiation conditions* for elastic (thermoelastic) field (for details see [45] [48], [29], [30]). These latter conditions are too complicated in anisotropic case. Moreover, the principal thing is that they are not involved in the initial formulation of the original interface problem. Therefore, it is very desirable to get rid of these additional radiation conditions. This is done in the present paper for general anisotropic case. We apply the so-called non-local approach which, in fact, is the coupling of the boundary integral equations methods and the functional-variational methods. In isotropic case this approach has been used for similar problems in [25], [26] (see also references therein).

The basic idea of the non-local approach is the reduction of the original interface problem to an equivalent problem for a bounded domain (occupied by an elastic body). To this end, firstly, with the help of a special integral representation formula we derive *the generalized Steklov-Poincaré relations* between the Dirichlet and the Neumann type data for the exterior scalar (acoustic pressure) field. Secondly, applying the Green identities for the elastic field in a bounded domain and taking into account the transmission conditions and the Steklov-Poincaré formulae on the interface, we derive the equivalent weak functional-variational formulation in a bounded domain for the original generalized fluid-structure interaction problem. In this formulation the field of elastic displacements and the Dirichlet (or Neumann) datum for the scalar (acoustic pressure) field are involved as the anknowns. Thirdly, we show that for the sesquilinear form, involved in the weak formulation, the Gårding inequality holds and on the basis of the Lax-Milgram theorem we prove various variants of the existence and uniqueness theorems.

On the basis of the functional-variational approach developed in [36], we established the necessary and sufficient conditions of solvability for the general interface problem and proved

that, in particular, the physical fluid-solid acoustic interaction problem is always solvable for arbitrary values of the frequency parameter.

Finally, we note that by the approach developed in the present paper the above interaction problems can also be investigated (with corresponding obvious modification) when the interface is a non-smooth polyhedral (or, in general, Lipschitz) surface. In this case we have to apply results obtained in [39], [63], [8], [14], [42], [49], [22], [50].

1 Classical Formulation of Problem

1.1. Elastic field. Let $\Omega_1 = \Omega^+ \subset I\!R^3$ be a bounded domain ($\operatorname{diam}\Omega_1 < +\infty$) with a smooth, connected, nonselfintersecting boundary $S = \partial\Omega_1$; $\Omega_2 = \Omega^- = I\!R^3\setminus\overline{\Omega}_1$, $\overline{\Omega}_1 = \Omega_1 \cup S$. Later we shall state the smoothness conditions required for S more precisely.

The domain $\overline{\Omega}_1$ is supposed to be filled up by an inhomogeneous anisotropic material with the elastic coefficients

$$c_{kjpq}(x) = c_{pqkj}(x) = c_{jkpq}(x), \quad k,j,p,q = 1,2,3, \tag{1.1}$$

and the density $\rho_1(x) > 0$. For simplicity, throughout the paper we assume that $c_{kjpq}, \rho_1 \in C^1(\overline{\Omega}_1)$.

The stress tensor $\{\tau_{kj}\}$ and the strain tensor $\{\varepsilon_{kj}\}$ are related by Hook's law

$$\tau_{kj} = c_{kjpq}(x)\varepsilon_{pq}, \quad \varepsilon_{kj} = 2^{-1}(D_k u_j + D_j u_k), \tag{1.2}$$

where $u = (u_1, u_2, u_3)^\top$ is the displacement vector and $D_p = D_{x_p} = \partial/\partial x_p$, $p = 1,2,3$; here and in what follows the summation over repeated indices is meant from 1 to 3, unless otherwise stated; the superscript $\top$ denotes transposition.

As usual, the quadratic form corresponding to the potential energy is supposed to be positive definite in the symmetric real variables $\varepsilon_{kj} = \varepsilon_{jk}$ (see, e.g., [20])

$$E(u,u) = \tau_{kj}\varepsilon_{kj} = c_{kjpq}(x)\varepsilon_{kj}\varepsilon_{pq} \geq \delta_1\varepsilon_{kj}\varepsilon_{kj}, \quad \delta_1 = \text{const} > 0, \quad x \in \overline{\Omega}_1.$$

Due to the symmetry relations (1.1) we also have

$$E(u,\overline{u}) = \tau_{kj}\overline{\varepsilon_{kj}} = c_{kjpq}(x)\varepsilon_{kj}\overline{\varepsilon_{pq}} \geq \delta_1[\varepsilon'_{kj}\varepsilon'_{kj} + \varepsilon''_{kj}\varepsilon''_{kj}], \quad x \in \overline{\Omega}_1, \tag{1.3}$$

where an overbar denotes complex conjugation, and where $\varepsilon_{kj} = \varepsilon'_{kj} + \mathrm{i}\varepsilon''_{kj}$ are complex variables (complex strain tensor) corresponding to the complex vector $u = u' + \mathrm{i}u''$; $\mathrm{i} = \sqrt{-1}$.

The elastic field in $\Omega^+ = \Omega_1$ is described by the following steady state oscillation equations (see [35])

$$D_j[c_{kjpq}(x)D_q u_p(x)] + \rho_1(x)\omega^2 u_k(x) = F_k(x), \quad x \in \Omega_1, \quad k = 1,2,3. \tag{1.4}$$

These equations can be written in the matrix form as

$$A_\omega(x,D)u(x) = F(x), \quad x \in \Omega_1, \tag{1.5}$$

where $F = (F_1, F_2, F_3)^\top$ is a given (complex-valued) body force, $D = \nabla = (D_1, D_2, D_3)$, u is the complex-valued amplitude of displacements, and

$$\begin{aligned} &A_\omega(x, D) := [\ (A_\omega(x, D))_{kp}]_{3\times 3}\,, \\ &(A_\omega(x, D))_{kp} = (A(x, D))_{kp} + \delta_{kp}\rho_1(x)\omega^2, \\ &(A(x, D))_{kp} = D_j(c_{kjpq}(x)D_q); \end{aligned} \tag{1.6}$$

here $\omega > 0$ is the frequency parameter and δ_{kp} is the Kronecker's delta.

Equalities (1.1), (1.2), (1.6) and inequality (1.3) imply that $A_\omega(x, D)$ is a strongly elliptic, formally self-adjoint matrix differential operator in $\overline{\Omega}_1$ (see [20], Theorem 5.11).

By $T(D_x, n)u(x)$ we denote the stress vector acting on a surface element with the unit normal vector $n = (n_1, n_2, n_3)$:

$$[T(D_x, n(x))u(x)]_k = \tau_{kj}n_j(x) = c_{kjpq}(x)n_j(x)D_q u_p(x), \quad k = 1, 2, 3. \tag{1.7}$$

We note that throughout the paper we will use the following notations (when this causes no confusion):

a) if all elements of a vector $v = (v_1, \cdots, v_m)^\top$ (matrix $N = [N_{kj}]_{m\times n}$) belong to one and the same space X, we will write $v \in X$ ($N \in X$) instead of $v \in [X]^m$ ($N \in [X]_{m\times n}$);

b) if $K : X_1 \times \cdots \times X_m \to Y_1 \times \cdots \times Y_n$ and $X_1 = \cdots = X_m$, $Y_1 = \cdots = Y_n$, we will write $K : X \to Y$ instead of $K : [X]^m \to [Y]^n$;

c) if a and b are complex vectors of $\mathbb{C}^m$, then by $a \cdot b := \sum_{k=1}^m a_k\overline{b}_k$ we denote the scalar product of vectors a and b, while $\langle a\ ,\ b\rangle := \sum_{k=1}^m a_k b_k$;

d) by $||u; X||$ we denote the norm of the element u in the space X.

1.2. Scalar field. We assume that the exterior domain Ω_2 is filled up by a homogeneous anisotropic (fluid) medium with the constant density ρ_2. Moreover, let $a_{pq} = a_{qp}$ be real constants defining a positive definite matrix, i.e.,

$$\widetilde{a} := [a_{pq}]_{3\times 3} \quad \text{and} \quad \widetilde{a}\zeta\cdot\zeta = a_{pq}\zeta_q\overline{\zeta}_p \geq \delta_2|\zeta|^2, \quad \delta_2 = \text{const} > 0,$$

for arbitrary $\zeta \in \mathbb{C}^3$.

Further, let some physical process (say the propagation of acoustic waves) in Ω_2 be described by a complex-valued scalar function (scalar field) $w(x)$ being a solution of the homogeneous "wave equation" (generalized Helmholtz equation)

$$\mathcal{A}_\omega(D)w(x) := a(D)w(x) + \rho_2\omega^2 w(x) = 0, \quad x \in \Omega^- = \Omega_2, \tag{1.8}$$

where $a(D) = a_{pq}D_pD_q$, and where ω is again the frequency parameter.

To prove the uniqueness and existence theorems of solutions to the exterior boundary value problems (BVPs) we require some special decay conditions and structural restrictions at infinity which are similar to the Sommerfeld radiation conditions for solutions to the Helmholtz equation (cf. [62], [6]).

To derive the Sommerfeld type radiation condition at infinity for the scalar field defined by the equation (1.8), let us introduce the characteristic function corresponding to the operator $\mathcal{A}_\omega(D)$ (cf. [60], [61], [29], [30]):

$$\Phi_{\mathcal{A}}(\xi, \omega) := -\mathcal{A}_\omega(-\mathrm{i}\xi) = a_{pq}\xi_p\xi_q - \rho_2\omega^2, \quad \xi \in \mathbb{R}^3.$$

Denote by S_ω the characteristic surface (ellipsoid) given by the equation

$$\Phi_{\mathcal{A}}(\xi,\omega) = \widetilde{a}\xi\cdot\xi - \rho_2\omega^2 = 0, \quad \xi\in I\!R^3.$$

It is evident that for an arbitrary vector $\eta\in I\!R^3$ with $|\eta|=1$ there exists only one point $\xi(\eta)\in S_\omega$ such that the outward unit normal vector $n(\xi(\eta))$ to S_ω at the point $\xi(\eta)$ has the same direction as η, i.e., $n(\xi(\eta))=\eta$.

Note that $\xi(-\eta)=-\xi(\eta)\in S_\omega$ and $n(-\xi(\eta))=-\eta$.

It can be easily verified that

$$\xi(\eta) = \omega\sqrt{\rho_2}\,(\widetilde{a}^{-1}\eta\cdot\eta)^{-1/2}\;\widetilde{a}^{-1}\eta, \tag{1.9}$$

since

$$n(\xi) = \frac{\nabla_\xi\Phi_{\mathcal{A}}(\xi,\omega)}{|\nabla_\xi\Phi_{\mathcal{A}}(\xi,\omega)|} = \frac{\widetilde{a}\xi}{|\widetilde{a}\xi|};$$

here $\widetilde{a}^{-1}$ is the matrix inverse to $\widetilde{a}$.

Now we are in the position to define the class $\mathrm{Som}_r(\Omega^-)$ of complex-valued functions satisfying the generalized Sommerfeld type radiation conditions (see [60], [29], [30]).

A function w belongs to $\mathrm{Som}_r(\Omega^-)$, where either $r=1$ or $r=2$, if $w\in C^1(\Omega^-)$ and for sufficiently large $|x|$

$$w(x) = O(|x|^{-1}), \quad \frac{\partial w(x)}{\partial x_k} + \mathrm{i}(-1)^r\xi_k(\eta)w(x) = O(|x|^{-2}), \quad k=1,2,3, \tag{1.10}$$

where $\xi(\eta)\in S_\omega$ corresponds to the vector $\eta = x/|x|$ (i.e., $\xi(\eta)$ is defined by (1.9) with $\eta = x/|x|$).

Obviously, the conditions (1.10) are equivalent to the classical Sommerfeld radiation conditions for the Helmholtz equation if $a(D)$ is the Laplace operator (see, for example, [62], [6]). In the sequel elements of the class $\mathrm{Som}_r(\Omega^-)$ will also be referred to as radiating functions.

1.3. Classical formulation of the basic interface problem. Now we can formulate (in the classical setting) the basic mathematical transmission problem corresponding to the interaction of the fields introduced above.

Problem $(P^{(\omega,r)})$. Find a regular vector $u\in C^2(\Omega_1)\cap C^1(\overline{\Omega}_1)$ and a regular radiating function $w\in C^2(\Omega_2)\cap C^1(\overline{\Omega}_2)\cap \mathrm{Som}_r(\Omega_2)$ which are solutions of equations (1.5) and (1.8), respectively, and satisfy the following coupling conditions on $S=\partial\Omega^\pm$:

$$[u(x)\cdot n(x)]^+ = d_1\,[\partial_n w(x)]^- + f_0(x); \tag{1.11}$$

$$[T(D,n)u(x)]^+ = d_2\,[w(x)]^- n(x) + f(x), \quad f=(f_1,f_2,f_3)^\top, \tag{1.12}$$

where $T(D,n)u$ is the stress vector given by formula (1.7), $\partial_n w = a_{pq}n_pD_qw$ is the so-called co-normal derivative; here and throughout this paper $n(x)$ denotes the unit outward normal vector to S at the point $x\in S$; the symbols $[\cdot]^\pm$ denote limits on S from $\Omega^\pm$, f_j $(j=0,...,3)$ are given functions on S; d_1 and d_2 are given complex constants satisfying the conditions $d_1d_2\neq 0$ and $\mathrm{Im}[\overline{d_1}d_2]=0$.

We remark that the fluid-solid acoustic interaction problem is the particular case of the problem $(P^{(\omega,r)})$. In fact, in that case $w(x) = p^{sc}(x)$ is the scattered acoustic pressure and

$$d_1 = [\rho_2\omega^2]^{-1}, \quad d_2 = -1, \quad f_0(x) = f_0^{inc}(x) = [\rho_2\omega^2]^{-1}\partial_n p^{inc}(x),$$
$$f(x) = f^{inc}(x) = -p^{inc}(x)\, n(x), \tag{1.13}$$

where p^{inc} is the given incident field, e.g.,

$$p^{inc}(x) = \exp\{\mathrm{i}\, d \cdot x\} \quad \text{with} \quad d = \omega\sqrt{\rho_2}\,(\widetilde{a}^{-1}\eta' \cdot \eta')^{-1/2}\,\widetilde{a}^{-1}\eta', \tag{1.14}$$

where η' is an arbitrary unit vector. Clearly, the p^{inc} defined by formula (1.13) solves the equation (1.8) but it does not represent a radiating function (cf. [25], [37]). The particular case, when the elastic characteristics c_{kjpq} are concstants and $a_{pq} = \delta_{pq}$ (i.e. (1.8) is the Helmholtz equation), the problem $(P^{(\omega,r)})$ has been studied by the indirect boundary integral equation method in [48]. When, in addition, the elastic body is homogeneous and isotropic, i.e.

$$A_\omega(D)u := \mu\Delta u + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u + \rho_2\omega^2 u,$$

where λ and μ are the Lamé constants, the problem $(P^{(\omega,r)})$ has been considered in [25] by using the non-local approach (see also [37]).

In what follows we will develope this approach in the general (inhomogeneous) anisotropic case.

To this end we need some auxiliary material which is outlined in the next subsections.

1.4. Integral representation formulae and potentials. It is well known that the fundamental functions (solutions) of the class $\mathrm{Som}_r(I\!R^3\backslash\{0\})$ for the operator $\mathcal{A}_\omega(D)$ reads (cf., e.g., [43], [30])

$$\gamma_{\mathcal{A}}(x,\omega,r) = -\frac{\exp\{\mathrm{i}(-1)^{r+1}\omega\sqrt{\rho_2}(\widetilde{a}^{-1}x\cdot x)^{1/2}\}}{4\pi|\widetilde{a}|^{1/2}(\widetilde{a}^{-1}x\cdot x)^{1/2}}, \quad r = 1,2, \tag{1.15}$$

where $|\widetilde{a}| = \det\widetilde{a}$.

The principal dominant homogeneous singular part of these fundamental solutions has the form

$$\gamma_a(x) = -[4\pi|\widetilde{a}|^{1/2}(\widetilde{a}^{-1}x\cdot x)^{1/2}]^{-1}, \quad x \in I\!R^3\backslash\{0\}.$$

We emphasize that $\gamma_a(x)$ represents the fundamental function for the operator $a(D)$ (see (1.8)). Now we list here some results obtained in [30].

LEMMA 1.1 *The fundamental solution $\gamma_{\mathcal{A}}$ defined by* (1.15) *satisfies the conditions*:

i) $\gamma_{\mathcal{A}}(\cdot,\omega,r) \in C^\infty(I\!R^3\backslash\{0\}) \cap \mathrm{Som}_r(I\!R^3\backslash\{0\})$;

ii) *in a neighbourhood of the origin*

$$|D_x^\beta\gamma_{\mathcal{A}}(x,\omega,r) - D_x^\beta\gamma_a(x)| \le c|x|^{1-|\beta|}, \quad c = \mathrm{const} > 0,$$

where $\beta = (\beta_1,\beta_2,\beta_3)$ is an arbitrary multi-index and $|\beta| = \beta_1+\beta_2+\beta_3$;

iii) *for sufficiently large $|x|$*

$$\gamma_{\mathcal{A}}(x-y,\omega,r) = -\frac{|\widetilde{a}\xi|}{4\pi\omega\,(\rho_2|\widetilde{a}|)^{1/2}}\frac{\exp\{\mathrm{i}(-1)^{r+1}(\xi\cdot(x-y))\}}{|x|} + O(|x|^{-2}), \tag{1.16}$$

where y varies in a bounded subset of $\mathbb{R}^3$ and $\xi = \xi(\eta) \in S_\omega$ corresponds to the direction $\eta = x/|x|$ (see (1.9)); the asymptotic formula (1.16) can be differentiated any times with respect to x and y.

LEMMA 1.2 (Analogue of Rellich's lemma). *Let $w \in \mathrm{Som}_r(\Omega^-)$ be a solution of (1.8) in Ω^- and let*

$$\lim_{R\to+\infty} \mathrm{Im} \left\{ \int_{\Sigma_R} \overline{w(x)}\, \partial_n w(x)\, d\Sigma_R \right\} = 0, \tag{1.17}$$

where Σ_R is the sphere centered at the origin and radius R, ∂_n denotes the co-normal differentiation.

Then $w(x) = 0$ in Ω^-.

In [30] it is shown that for functions of the class $\mathrm{Som}_r(\Omega^-)$ the condition (1.17) is equivalent to the following one

$$\lim_{R\to+\infty} \int_{\Sigma_R} |w(x)|^2 (\widetilde{a}^{-1}\eta\cdot\eta)^{-1/2} d\Sigma_R = 0, \quad \eta = x/|x|,$$

which is very similar to the condition in the classical Rellich's lemma for the Helmholtz equation (see, e.g., [6]).

Note that, if w is a solution of the homogeneous equation (1.8), which is an elliptic equation with constant coefficients, then w is an analytic function of the real variable x in the domain $\Omega^- = \Omega_2$. Moreover, if, in addition, $w \in C^1(\overline{\Omega}_2) \cap \mathrm{Som}_r(\Omega_2)$, then the following integral representation formula holds (cf. [60])

$$\begin{aligned} &\int_S \gamma_{\mathcal{A}}(x-y,\omega,r)[\partial_n w(y)]^- dS_y - \int_S [\partial_{n(y)}\gamma_{\mathcal{A}}(y-x,\omega,r)][w(y)]^- dS_y = \\ &\begin{cases} w(x) & \text{for} \quad x \in \Omega^-, \\ 0 & \text{for} \quad x \in \Omega^+, \end{cases} \end{aligned} \tag{1.18}$$

where n is again the outward unit normal vector to S.

Due to Lemma 1.1, item iii) and the formula (1.18) we have for sufficiently large $|x|$

$$w(x) = |x|^{-1} b(\xi) \exp\{\mathrm{i}(-1)^{1+r}\xi\cdot x\} + O(|x|^{-2}), \tag{1.19}$$

where $\xi = \xi(\eta) \in S_\omega$ corresponds to $\eta = x/|x|$ and where

$$\begin{aligned} b(\xi) \;=\; & -|\widetilde{a}\xi|[4\pi\omega(\rho_2|\widetilde{a}|)^{1/2}]^{-1} \left\{ \int_S [\partial_n w(y)]^- \exp\{\mathrm{i}(-1)^r \xi\cdot y\}\, dS_y - \right. \\ & \left. \mathrm{i}(-1)^r \int_S a_{pq}\xi_p n_q(y)[w(y)]^- \exp\{\mathrm{i}(-1)^r\xi\cdot y\}\, dS_y \right\}, \end{aligned} \tag{1.20}$$

which is the so-called far-field pattern of the radiating function $w(x)$ given by (1.18) (cf. [7]).

The representations (1.18)-(1.20) play a crucial role also in the inverse scattering problems (for details see [7] and references therein).

Let us introduce the single- and double-layer potentials corresponding to the differential operator $\mathcal{A}_\omega(D)$:

$$\begin{aligned} V(g)(x) &= \int\limits_S \gamma_{\mathcal{A}}(x-y,\omega,r)\, g(y)\, dS_y, \quad x \in I\!R^3 \backslash S, \\ W(g)(x) &= \int\limits_S [\partial_{n(y)} \gamma_{\mathcal{A}}(y-x,\omega,r)]\, g(y)\, dS_y, \quad x \in I\!R^3 \backslash S, \end{aligned} \tag{1.21}$$

where g is a scalar function.

In what follows we essentialy will use the following properties of these potentials (for details see [30]).

LEMMA 1.3 *Let $S \in C^{k+1+\alpha'}$ with integer $k \geq 0$ and $0 < \alpha < \alpha' \leq 1$, $0 \leq m \leq k$.*

Then

i) *the operators*

$$\begin{aligned} V &: C^{m+\alpha}(S) \to C^{m+1+\alpha}(\overline{\Omega^+}), \\ &: C^{m+\alpha}(S) \to C^{m+1+\alpha}(\overline{\Omega^-}) \cap \mathrm{Som}_r(\Omega^-), \\ W &: C^{m+\alpha}(S) \to C^{m+\alpha}(\overline{\Omega^+}), \\ &: C^{m+\alpha}(S) \to C^{m+\alpha}(\overline{\Omega^-}) \cap \mathrm{Som}_r(\Omega^-), \end{aligned}$$

are bounded;

ii) *for arbitrary $g \in C^{k+\alpha}(S)$ and $z \in S$ the following jump relations hold on S :*

$$[V(g)(z)]^{\pm} = \int\limits_S \gamma_{\mathcal{A}}(z-y,\omega,r)\, g(y)\, dS_y =: \mathcal{H}\, g(z), \quad m \geq 0, \tag{1.22}$$

$$\begin{aligned} [\partial_{n(z)} V(g)(z)]^{\pm} &= \mp 2^{-1} g(z) + \int\limits_S [\partial_{n(z)} \gamma_{\mathcal{A}}(z-y,\omega,r)]\, g(y)\, dS_y \\ &=: [\mp 2^{-1} I + \mathcal{K}^{(1)}]\, g(z), \quad m \geq 0, \end{aligned} \tag{1.23}$$

$$\begin{aligned} [W(g)(z)]^{\pm} &= \pm 2^{-1} g(z) + \int\limits_S [\partial_{n(y)} \gamma_{\mathcal{A}}(y-z,\omega,r)]\, g(y)\, dS_y \\ &=: [\pm 2^{-1} I + \mathcal{K}^{(2)}]\, g(z), \quad m \geq 0, \end{aligned} \tag{1.24}$$

$$[\partial_{n(z)} W(g)(z)]^{+} = [\partial_{n(z)} W(g)(z)]^{-} =: \mathcal{L}\, g(z), \quad m \geq 1, \tag{1.25}$$

where I stands for the identical operator;

iii) *the operators*

$$\begin{aligned} \mathcal{H} &: C^{m+\alpha}(S) \to C^{m+1+\alpha}(S), \\ \mathcal{K}^{(1)},\ \mathcal{K}^{(2)} &: C^{m+\alpha}(S) \to C^{m+\alpha}(S), \\ \mathcal{L} &: C^{m+1+\alpha}(S) \to C^{m+\alpha}(S), \end{aligned}$$

are bounded; *moreover, if* $S \in C^{1+\alpha'}$, *then* $\mathcal{H}$ *has a weakly singular kernel-function of type* $O(|x-y|^{-1})$, *while* $\mathcal{K}^{(1)}$ *and* $\mathcal{K}^{(2)}$ *have weakly singular kernel-functions of type* $O(|x-y|^{-2+\alpha'})$ *on* S; $\mathcal{L}$ *is a singular integro-differential operator*;

iv) *the operators* $\mathcal{H}$, $\mp 2^{-1}I + \mathcal{K}^{(1)}$, $\pm 2^{-1}I + \mathcal{K}^{(2)}$ *and* $\mathcal{L}$ *are elliptic pseudodifferential operators of order* -1, 0, 0, *and* 1, *respectively, with the index zero*;

v) *the principal homogeneous symbols of the above operators read*:

$$\sigma(\mathcal{H})(x,\widetilde{\xi}) = -(2\pi)^{-1} \int_{-\infty}^{+\infty} \frac{d\,\xi_3}{a_{pq}\,(B\xi)_p\,B(\xi)_q} < 0,$$

$$\sigma(\pm 2^{-1}I + \mathcal{K}^{(1)})(x,\widetilde{\xi}) = \sigma(\pm 2^{-1}I + \mathcal{K}^{(2)})(x,\widetilde{\xi}) = \pm\frac{1}{2},$$

$$\sigma(\mathcal{L})(x,\widetilde{\xi}) = -[4\sigma(\mathcal{H})(x,\widetilde{\xi})]^{-1} > 0,$$

$$\widetilde{\xi} = (\xi_1,\xi_2) \in I\!R^2\backslash\{0\}, \quad x \in S, \quad \xi = (\xi_1,\xi_2,\xi_3) \in I\!R^3;$$

here

$$B(x) = \begin{bmatrix} l_1(x) & m_1(x) & n_1(x) \\ l_2(x) & m_2(x) & n_2(x) \\ l_3(x) & m_3(x) & n_3(x) \end{bmatrix}$$

with $\det B(x) = 1$, *where* $n(x)$ *is the exterior (with respect to* Ω^+*) unit normal vector, while* $l(x) = (l_1(x), l_2(x), l_3(x))$ *and* $m(x) = (m_1(x), m_2(x), m_3(x))$ *are orthogonal unit vectors in the tangent plane at the point* $x \in S$;

vi) *the following operator equations*

$$\mathcal{H}\mathcal{K}^{(1)} = \mathcal{K}^{(2)}\mathcal{H}, \quad \mathcal{L}\mathcal{K}^{(2)} = \mathcal{K}^{(1)}\mathcal{L},$$
$$\mathcal{H}\mathcal{L} = -4^{-1}I + [\mathcal{K}^{(2)}]^2, \quad \mathcal{L}\mathcal{H} = -4^{-1}I + [\mathcal{K}^{(1)}]^2,$$

hold in appropriate functional spaces (cf. [7], pp. 41-42).

LEMMA 1.4 *Let* $g \in C^{1+\alpha}(S)$ *with* $S \in C^{2+\alpha'}$, $0 < \alpha < \alpha' < 1$, *and*

$$w(x) = W(g)(x) + p_0 V(g)(x), \quad x \in \Omega^- = \Omega_2,$$

where $p_0 = p_1 + \mathrm{i}p_2$ *is a complex number with* $p_2 \neq 0$.

If w *vanishes in* Ω^-, *then* $g = 0$ *on* S.

We can apply results and arguments of [10], [9] [31], [46], [12], [13], [32] to formulate the above lemmata in the Sobolev W_p^s, Bessel-potential H_p^s and Bessov $B_{p,q}^s$ spaces. Namely, the following propositions hold.

LEMMA 1.5 *Let* $s \in I\!R$, $1 < p < \infty$, $1 \leq q \leq \infty$ *and* $S \in C^\infty$. *The operators* V, W, $\mathcal{H}$, $\mathcal{K}^{(1)}$, $\mathcal{K}^{(2)}$, *and* $\mathcal{L}$ *can be extended by continuity to the following bounded operators:*

$$\begin{aligned}
V &: B_{p,p}^s(S) \to H_p^{s+1+1/p}(\Omega^+) \cap C^\infty(\Omega^+) \quad [B_{p,p}^s(S) \to H_{p,loc}^{s+1+1/p}(\Omega^-) \cap C^\infty(\Omega^-)], \\
&: B_{p,q}^s(S) \to B_{p,q}^{s+1+1/p}(\Omega^+) \cap C^\infty(\Omega^+) \quad [B_{p,q}^s(S) \to B_{p,q,loc}^{s+1+1/p}(\Omega^-) \cap C^\infty(\Omega^-)], \\
W &: B_{p,p}^s(S) \to H_p^{s+1/p}(\Omega^+) \cap C^\infty(\Omega^+) \quad [B_{p,p}^s(S) \to H_{p,loc}^{s+1/p}(\Omega^-) \cap C^\infty(\Omega^-)], \\
&: B_{p,q}^s(S) \to B_{p,q}^{s+1/p}(\Omega^+) \cap C^\infty(\Omega^+) \quad [B_{p,q}^s(S) \to B_{p,q,loc}^{s+1/p}(\Omega^-) \cap C^\infty(\Omega^-)],
\end{aligned}$$

$$\begin{aligned}
\mathcal{H} &: H^s_p(S) \to H^{s+1}_p(S) \quad [B^s_{p,q}(S) \to B^{s+1}_{p,q}(S)],\\
\mathcal{K}^{(j)} &: H^s_p(S) \to H^s_p(S) \qquad [B^s_{p,q}(S) \to B^s_{p,q}(S)], \quad j = 1,2,\\
\mathcal{L} &: H^{s+1}_p(S) \to H^s_p(S) \quad [B^{s+1}_{p,q}(S) \to B^s_{p,q}(S)].
\end{aligned}$$

For these extended operators formulae (1.22)-(1.25) *remain valid in the corresponding spaces. Representation* (1.18) *holds for* $w \in W^1_{p,loc}(\Omega^-)$ *if, in addition,* $\mathcal{A}_\omega(D)w(x) = 0$ *in* Ω^-.

The proof of this lemma is verbatim the proofs of Theorem 2 and Theorem 4 in [13].

We only note here that $H^s_2 = W^s_2 = B^s_{2,2}$, $W^t_p = B^t_{p,p}$ and $H^k_p = W^k_p$ for any $s \in \mathbb{R}$, for any positive and non-integer t and for any non-negative integer k (see, e.g., [58], [59]).

For $w \in W^1_{p,loc}(\Omega^-)$ by the trace theorem (see, e.g., [59], Theorem 3.3.3) we have $[w]^-_S \in B^{1-1/p}_{p,p}(S) = W^{1-1/p}_p(S)$, $1 < p < \infty$. If, in addition, $\mathcal{A}_\omega(D)w(x) = 0$ in Ω^-, using the corresponding Green's formula, we define also $[\partial_n w]^-_S$ as a functional in $B^{-1/p}_{p,p}(S)$ by the following duality relation (cf. [9], [13])

$$\langle [\partial_n w]^- , [v]^- \rangle_S := \int\limits_{\Omega^-} [\rho_2 \omega^2 \, w\, v \; - \; a_{pq} D_p w \, D_q v] \, dx; \tag{1.26}$$

here v is an arbitrary element of the space $W^1_{p',comp}(\Omega^-)$ (i.e., $v \in W^1_{p',loc}(\Omega^-)$ and has a compact support), $1/p + 1/p' = 1$, and, obviously, $[v]^-_S \in B^{1/p}_{p',p'}(S)$.

Clearly,

$$\langle f \, , \, g \rangle_S \; = \int\limits_S f \, g \, dS$$

for the smooth functions f and g.

1.5. The Exterior Dirichlet and Neumann BVPs. Throughout this subsection we assume

$$S = \partial\Omega^- \in C^{k+1+\alpha'}, \;\; \text{where} \;\; k \geq 1 \;\; \text{is a positive integer and} \;\; 0 < \alpha' \leq 1. \tag{1.27}$$

Further, let $Y(\Omega^-)$ be one of the following four functional classes:

a) $C^{k+\alpha}(\overline{\Omega^-})$, b) $W^1_{p,loc}(\Omega^-)$, c) $C(\overline{\Omega^-})$, d) $C^1(\overline{\Omega^-})$,

where $0 < \alpha < \alpha'$.

Now we formulate the exterior Dirichlet and Neumann BVPs for the equation (1.8).

Find a complex-valued scalar function $w \in Y(\Omega^-) \cap \mathrm{Som}_r(\Omega^-)$ satisfying the equation (1.8) in $\Omega^- = \Omega_2$ and the following boundary conditions on S:

$$\text{Dirichlet Problem } (D^{(\omega,r)})^- : \quad [w]^- = \varphi \qquad \text{(in the cases a), b), c))}, \tag{1.28}$$

$$\text{Neumann Problem } (N^{(\omega,r)})^- : \quad [\partial_n w]^- = \psi \qquad \text{(in the cases a), b), d))}. \tag{1.29}$$

Clearly, the boundary data in (1.28) and (1.29) have to meet the following natural compatibility conditions corresponding to the above four cases for the functional class $Y(\Omega^-)$:

a) $\varphi \in C^{k+\alpha}(S)$ and $\psi \in C^{k-1+\alpha}(S)$;

b) $\varphi \in B^{1-1/p}_{p,p}(S) \quad \text{and} \quad \psi \in B^{-1/p}_{p,p}(S)$;

c) $\varphi \in C(S)$;

d) $\psi \in C(S)$.

Note that, in the cases a), c), and d), the above boundary conditions can be concidered in the classical sense, while, in the case b), the conditions (1.28) and (1.29) are to be understood in the usual trace sense and in the functional trace sense described above, respectively.

LEMMA 1.6 *The homogeneous problems* $(D^{(\omega,r)})^-$ *and* $(N^{(\omega,r)})^-$ $[\varphi = 0, \quad \psi = 0]$ *in the class of regular functions* $C^1(\overline{\Omega}_2) \cap C^\infty(\Omega_2) \cap \mathrm{Som}_r(\Omega_2)$ *admit only the trivial solution.*

Proof. Let $B(O,R)$ be the ball centered at the origin and radius R. We assume $\overline{\Omega}_1 \subset B(O,R)$ and denote $\Omega^-_R = \Omega_2 \cap B(O,R)$. Clearly, $\partial\Omega^-_R = S \cup \Sigma_R$, where the Σ_R is as in Lemma 1.2.

Further, let $w \in C^1(\overline{\Omega}_2) \cap C^\infty(\Omega_2) \cap \mathrm{Som}_r(\Omega_2)$ be an arbitrary regular solution to the homogeneous problem either $(D^{(\omega,r)})^-$ or $(N^{(\omega,r)})^-$.

Since w is a regular function in the bounded domain Ω^-_R, by applying the divergence theorem for w and $\overline{w}$, we obtain

$$\int\limits_{\Omega^-_R} \{[\mathcal{A}_\omega(D)w(x)]\,\overline{w(x)} \,-\, w(x)\,[\mathcal{A}_\omega(D)\overline{w(x)}\}\,dx = \int\limits_{\partial\Omega^-_R} \{[\partial_{n(x)}w(x)]\,\overline{w(x)} \,-\, w(x)\,[\overline{\partial_{n(x)}w(x)}\}\,dS. \tag{1.30}$$

Whence it follows that

$$\mathrm{Im} \int\limits_{\Sigma_R} [\partial_{n(x)}w(x)]\,\overline{w(x)}\,d\Sigma_R = 0, \tag{1.31}$$

since the integral over S vanishes due to the homogeneous boundary condition.

Now Lemma 1.2 completes the proof. ∎

REMARK 1.7 *If we apply definition* (1.26) *it is easy to show that Lemma* 1.6 *remains also valid in the class* $W^1_{2,loc}(\Omega_2) \cap C^\infty(\Omega_2) \cap \mathrm{Som}_r(\Omega_2)$*, i.e., the homogeneous problems* $(D^{(\omega,r)})^-$ *and* $(N^{(\omega,r)})^-$ *admit only the trivial solutions in this class.*

In fact, let $\chi_R \in C^\infty(I\!R^3)$, $\chi_R(x) = 1$ for $|x| \le R$, $\chi_R(x) = 0$ for $|x| \ge 2R$, and denote $v(x) = \overline{w(x)}\chi_R(x) \in W^1_{2,comp}(\Omega_2)$, where $w \in W^1_{2,loc}(\Omega_2)$ is a radiating solution of the homogeneous Dirichlet [Neumann] problem. The definition (1.26) and the homogeneous boundary condition then imply

$$\langle [\partial_n w]^-, [\overline{v}]^- \rangle_S = \int\limits_{\Omega^-_{2R}} [\rho_2\omega^2 w(x)\,\overline{w(x)}\,\chi_R(x) - a_{pq}\,D_p w(x)\,D_q(\overline{w(x)}\chi_R(x))]\,dx = 0.$$

Hence, using the decomposition $\Omega_{2R}^- = \Omega_R^- \cup \{B(O,2R)\backslash B(O,R)\}$, we get

$$\int\limits_{\Omega_R^-} [\rho_2\omega^2|w(x)|^2 \,-\, a_{pq}\, D_p w(x)\, D_q\overline{w(x)}]\, dx \,+ \int\limits_{B(O,2R)\backslash B(O,R)} [\rho_2\omega^2 w(x) \,+$$
$$a_{pq}\, D_p D_q w(x)]\overline{w(x)}\,\chi_R(x)\, dx \,+ \int\limits_{\Sigma_R} [a_{pq}\, n_q(x)\, D_p w(x)]\,\overline{w(x)}]\, d\Sigma_R \,=\, 0.$$

Since $\mathcal{A}_\omega(D)w(x) = 0$ in $B(O,2R)\backslash B(O,R)$ and $\tilde{a} = [a_{pq}]_{3\times 3}$ is a positive definite matrix, we arrive again to the equation (1.31) which due to Lemma 1.2 yields $w(x) = 0$ in Ω_2. ∎

Before we go over to the question of existence of solutions to the nonhomogeneous problems $(D^{(\omega,r)})^-$ and $(N^{(\omega,r)})^-$ we present some properties of the operators

$$\mathcal{N}\, g \,:=\, \left[(-2^{-1}\, I + \mathcal{K}^{(2)}) \,+\, p_0\, \mathcal{H}\right]\, g \tag{1.32}$$

and

$$\mathcal{M}\, g \,:=\, \left[\mathcal{L} \,+\, p_0\, (2^{-1}\, I + \mathcal{K}^{(1)})\right]\, g, \tag{1.33}$$

where I, $\mathcal{K}^{(1)}$, $\mathcal{K}^{(2)}$, $\mathcal{H}$, $\mathcal{L}$, and p_0 are as in Lemmata 1.3 and 1.4.

LEMMA 1.8 *Let* (1.27) *be fulfiled and* $0 < \alpha < \alpha'$, $0 \le m \le k$.
Then the operators

$$\mathcal{N} \,:\, C^{m+1+\alpha}(S) \to C^{m+1+\alpha}(S) \tag{1.34}$$

and

$$\mathcal{M} \,:\, C^{m+1+\alpha}(S) \to C^{m+\alpha}(S) \tag{1.35}$$

are isomorphisms.

Proof. Due to Lemma 1.3, items iii) and iv), it remains only to prove that the null-spaces of the operators (1.34) and (1.35) are trivial. To this end we consider the homogeneous equations $\mathcal{N}\, g \,=\, 0$ and $\mathcal{M}\, h \,=\, 0$. Let $g \in C^{m+1+\alpha}(S)$ and $h \in C^{m+1+\alpha}(S)$ be arbitrary solutions of these equations and construct the following regular functions in Ω_2:

$$w_1(x) = W(g)(x) + p_0 V(g)(x) \quad \text{and} \quad w_2(x) = W(h)(x) + p_0 V(h)(x).$$

Obviously, w_1, $w_2 \in C^{m+1+\alpha}(\overline{\Omega}_2) \cap \mathrm{Som}_r(\Omega_2)$, and $[w_1(x)]^- = \mathcal{N}\, g(x) = 0$ and $[\partial_n w_2(x)]^- = \mathcal{M}\, h(x) = 0$ on S. Therefore, by Lemmata 1.6 and 1.4 we conclude that $g = 0$ and $h = 0$, which completes the proof. ∎

COROLLARY 1.9 *Let* $S \in C^\infty$ *and* $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$.
Then the operators

$$\begin{aligned} \mathcal{N} \quad &:\quad H_p^s(S) \to H_p^s(S) \qquad [B_{p,q}^s(S) \to B_{p,q}^s(S)], \\ \mathcal{M} \quad &:\quad H_p^{s+1}(S) \to H_p^s(S) \quad [B_{p,q}^{s+1}(S) \to B_{p,q}^s(S)], \end{aligned}$$

are isomorphisms.

Proof. It easily follows from Lemma 1.8 and the ellipticity of the operators $\mathcal{N}$ and $\mathcal{M}$ (see Lemma 1.3, items iv) and v)), since the null-spaces of the above operators are invariant with respect to s, p and q. ∎

COROLLARY 1.10 *Let $S \in C^{2+\alpha'}$.*

Then the operators

$$\mathcal{N} \; : \; W_2^{1/2}(S) \to W_2^{1/2}(S) \quad \text{and} \quad \mathcal{M} \; : \; W_2^{1/2}(S) \to W_2^{-1/2}(S) \tag{1.36}$$

are isomorphisms. Moreover,

$$\mathcal{N}^{-1} \; : \; W_2^{1/2}(S) \to W_2^{1/2}(S) \tag{1.37}$$

is a singular integral operator on S of normal type, while

$$\mathcal{M}^{-1} \; : \; W_2^{-1/2}(S) \to W_2^{1/2}(S)$$

is a smoothing integral operator on S with the kernel-function of type $O(|x-y|^{-1})$.

Proof. Invertibility of the operators (1.36) follows from Lemma 1.5 and Remark 1.7. The second part of the Lemma can be shown as follows.

It is well known that the operator

$$\mathcal{R}g(x) \; = \; \frac{1}{2\pi} \int\limits_S \frac{g(y)}{|x-y|}\, dS, \quad x \in S, \tag{1.38}$$

is an equivalent lifting pseudodifferential operator of order -1 on S (see, e.g., [28], [47]). Its principal homogeneous symbol is $|\xi|^{-1}$. Therefore the equations

$$\mathcal{M}\, h \; = \; \Psi \quad \text{and} \quad \mathcal{R}\mathcal{M}\, h \; = \; \mathcal{R}\, \Psi,$$

with given $\Psi \in W_2^{-1/2}(S)$ and unknown $h \in W_2^{1/2}(S)$, are uniquely solvable and equivalent. Note that , $\mathcal{R}\, \Psi \in W_2^{1/2}(S)$. But, since $\mathcal{R}\mathcal{M}$ is an elliptic invertible singular integral operator, we conclude that its inverse $(\mathcal{R}\mathcal{M})^{-1}$ is also a singular integral operator of normal type (see [40], Chapter 14; [35], Chapter 4). Whence $(\mathcal{R}\mathcal{M})^{-1}\mathcal{R} = \mathcal{M}^{-1}$ is an integral operator on S with the kernel function of type $O(|x-y|^{-1})$ (see, e.g., [17]).

Applying again results of [35], [40], and taking into account the ellipticity and invertibility of the singular integral operator $\mathcal{N}$, we conclude that $\mathcal{N}^{-1}$ is also a singular integral operator of normal type (see (1.37)). ∎

Now we are in the position to prove the existence theorems for the nonhomogeneous problems $(D^{(\omega,r)})^-$ and $(N^{(\omega,r)})^-$.

LEMMA 1.11 *Let conditions* (1.27) *be fulfiled and $\varphi \in C^{m+1+\alpha}(S)$ with $0 \le m \le k$ and $0 < \alpha < \alpha'$.*

Then the Dirichlet problem $(D^{(\omega,r)})^-$ is uniquely solvable in the class $C^{m+1+\alpha}(\overline{\Omega}_2) \cap \mathrm{Som}_r(\Omega_2)$ and the solution is representable in the form of linear combination of double- and single-layer potentials

$$w(x) = W(g)(x) + p_0 V(g)(x), \quad x \in \Omega_2, \quad p_0 = p_1 + \mathrm{i}p_2, \quad p_2 \neq 0, \tag{1.39}$$

where the density $g \in C^{m+1+\alpha}(S)$ is the unique solution of the singular (Fredholm) integral equation

$$\mathcal{N}g(x) \; = \; \varphi(x), \quad x \in S. \tag{1.40}$$

Proof. It follows immediately from Lemmata 1.4, 1.6, 1.8 and 1.3. ∎

REMARK 1.12 *From Lemmata* 1.8 *and* 1.11 *it also follows that the radiating solution* $w \in C^{m+1+\alpha}(\overline{\Omega}_2) \cap \mathrm{Som}_r(\Omega_2)$ *of the problem* $(D^{(\omega,r)})^-$ *with the Dirichlet datum* (1.28) *can be represented as*

$$w(x) = [W + p_0 V]\,(\mathcal{N}^{-1}\varphi)(x), \quad x \in \Omega_2, \tag{1.41}$$

where $[w]^- =: \varphi \in C^{m+1+\alpha}(S)$ $0 \le m \le k$. *Applying Lemma* 1.5, *Corollary* 1.9 *and Corollary* 1.10 *we can obtain the same representation formula* (1.41) *for the radiating solution* $w \in W^1_{p,loc}(\Omega_2) \cap \mathrm{Som}_r(\Omega_2)$ *of the problem* $(D^{(\omega,r)})^-$ *with* $[w]^- =: \varphi \in B^{1-1/p}_{p,p}(S)$. *In particular, if* $S \in C^{2+\alpha'}$ *and* $p = 2$, *then* $\varphi \in W_2^{1/2}(S)$ *and the formula* (1.41) *represents the unique radiating solution of the problem* $(D^{(\omega,r)})^-$ *of the class* $W^1_{2,loc}(\Omega_2)$, *i.e., the mapping*

$$\begin{aligned} &w := [W + p_0 V]\,(\mathcal{N}^{-1}) \;:\; W_2^{1/2}(S) \to \\ &\Big\{ w \in W^1_{2,loc}(\Omega_2) \cap \mathrm{Som}_r(\Omega_2) \;:\; \mathcal{A}_\omega(D) w \;=\; 0 \;\text{ in }\; \Omega_2 \Big\} \end{aligned} \tag{1.42}$$

is an isomorphism.

LEMMA 1.13 *Let conditions* (1.27) *be fulfilled and* $\psi \in C^{m+\alpha}(S)$ *with* $0 \le m \le k$ *and* $0 < \alpha < \alpha'$.

Then the Neumann problem $(N^{(\omega,r)})^-$ *is uniquely solvable in the class* $C^{m+1+\alpha}(\overline{\Omega}_2) \cap \mathrm{Som}_r(\Omega_2)$ *and the solution is representable in the form of* (1.39) *where the density* $g \in C^{m+1+\alpha}(S)$ *is the unique solution of the singular integro-differential equation*

$$\mathcal{M} g(x) \;=\; \psi(x), \quad x \in S.$$

Proof. It follows immediately again from Lemmata 1.4, 1.6, 1.8 and 1.3. ∎

REMARK 1.14 *Note that, due to the Lemmata* 1.13 *and* 1.8, *it is evident that the radiating solution* $w \in C^{m+1+\alpha}(\overline{\Omega}_2) \cap \mathrm{Som}_r(\Omega_2)$ *of the problem* $(N^{(\omega,r)})^-$ *can be represented as*

$$w(x) = [W + p_0 V]\,(\mathcal{M}^{-1}\psi)(x), \quad x \in \Omega_2, \tag{1.43}$$

where $[\partial_n w]^- =: \psi \in C^{m+\alpha}(S)$ $0 \le m \le k$. *Again, with the help of Lemma* 1.5, *Corollary* 1.9 *and Corollary* 1.10 *we can obtain the same representation formula* (1.43) *for the radiating solution* $w \in W^1_{p,loc}(\Omega_2) \cap \mathrm{Som}_r(\Omega_2)$ *of the problem* $(N^{(\omega,r)})^-$ *with the boundary condition* $[\partial_n w]^- = \psi \in B^{-1/p}_{p,p}(S)$. *Again, as in Remark* 1.12, *if* $S \in C^{2+\alpha'}$ *and* $p = 2$, *then* $\psi \in W_2^{-1/2}(S)$ *and the formula* (1.43) *represents the unique radiating solution to the problem* $(N^{(\omega,r)})^-$ *of the class* $W^1_{2,loc}(\Omega_2)$, *i.e., the mapping*

$$\begin{aligned} &w := [W + p_0 V]\,(\mathcal{M}^{-1}) \;:\; W_2^{-1/2}(S) \to \\ &\Big\{ w \in W^1_{2,loc}(\Omega_2) \cap \mathrm{Som}_r(\Omega_2) \;:\; \mathcal{A}_\omega(D) w \;=\; 0 \;\text{ in }\; \Omega_2 \Big\} \end{aligned} \tag{1.44}$$

is an isomorphism.

REMARK 1.15 *From the above results it follows that the Dirichlet and the Neumann data for an arbitrary radiating solution of the equation* (1.8) *are related on* S *by the following Steklov-Poincaré type equations* (*see* (1.41), (1.42), (1.43), (1.44)) :

$$[\partial_n w]^- \;=\; \mathcal{M}\mathcal{N}^{-1}[w]^-, \quad [w]^- \;=\; \mathcal{N}\mathcal{M}^{-1}[\partial_n w]^-,$$

i.e.,

$$\mathcal{M}^{-1}[\partial_n w]^- \;=\; \mathcal{N}^{-1}[w]^-,$$

where either $[w]^- \in C^{m+1+\alpha}(S)$ *and* $[\partial_n w]^- \in C^{m+\alpha}(S)$ $(0 \le m \le k$, *see* (1.27)) *or* $[w]^- \in B^{1-1/p}_{p,p}(S)$ *and* $[\partial_n w]^- \in B^{-1/p}_{p,p}(S)$. *In particular, we can assume that* $[w]^- \in W_2^{1/2}(S)$ *and* $[\partial_n w]^- \in W_2^{-1/2}(S)$ *with* $S \in C^{2+\alpha'}$ (*cf.* [6], *Section* 3).

Moreover, we can easily prove that, if

$$\mathcal{M}^{-1}\psi \;=\; \mathcal{N}^{-1}\varphi \tag{1.45}$$

for some functions $\varphi \in C^{m+1+\alpha}(S)$ $[B^{1-1/p}_{p,p}(S)]$ *and* $\psi \in C^{m+\alpha}(S)$ $[B^{-1/p}_{p,p}(S)]$, *then the pair* (φ, ψ) *is a Cauchy datum for some radiating solution* $w \in C^{m+1+\alpha}(\overline{\Omega}_2)$ $[W^1_{p,loc}(\Omega_2)]$ *of the equation* (1.8), *i.e.,* $[w]^- = \varphi$ *and* $[\partial_n w]^- = \psi$. *Obviously, this solution can be explicitly written either by the formula* (1.41) *or* (1.43), *which are the same due to the equation* (1.45). *Therefore, for any function* $g \in C^{m+1+\alpha}(S)$ $[B^{1-1/p}_{p,p}(S)]$ *the pair* $(\mathcal{N}g, \mathcal{M}g)$ *is a Cauchy datum* $([w]^-, [\partial_n w]^-)$ *of the radiating solution* $w \in C^{m+1+\alpha}(\overline{\Omega}_2)$ $[W^1_{p,loc}(\Omega_2)]$ *given by formula* (1.39).

REMARK 1.16 *Applying the arguments similar to* [23], *Chapter* 3, *and* [6], [7], *Section* 3, *we can prove that the Dirichlet problem is uniquely solvable in the class* $C(\overline{\Omega}_2) \cap \mathrm{Som}_r(\Omega_2)$ *and the Neumann problem in the class* $C^1(\overline{\Omega}_2) \cap \mathrm{Som}_r(\Omega_2)$ *if* $S \in C^{2+\alpha'}$. *In this case, the solution of the Dirichlet problem can be represented again by formula* (1.39), *where* $g \in C(S)$ *solves the equation* (1.40), *and where* $\varphi \in C(S)$. *Note that* $\mathcal{N} : C(S) \to C(S)$ *is an isomorphism as well.*

The solution to the Neumann problem can be represented as

$$w(x) = V(g)(x) + p_0 W(\mathcal{R}^2 g)(x), \quad x \in \Omega_2, \tag{1.46}$$

where the boundary integral operator $\mathcal{R}$ *is given by* (1.38), *and* $g \in C(S)$ *solves the integral equation*

$$\widetilde{\mathcal{M}} g \;:=\; (2^{-1} I + \mathcal{K}^{(1)} + p_0 \mathcal{L}\mathcal{R}^2) g = \psi, \tag{1.47}$$

where $\psi \in C(S)$ *and* $\widetilde{\mathcal{M}} : C(S) \to C(S)$ *is an isomorphism* (*for details see* [52], [6] *and* [7], *Section* 3). *Note that the operator* $\mathcal{K}^{(1)} + p_0 \mathcal{L}\mathcal{R}^2$ *has a weakly singular kernel-function of type* $O(|x-y|^{-1})$ *on* S.

Note that, if one looks for the solution to the Dirichlet problem in the same form (1.46), *then one arrives to the pseudodifferential equation* (*of order* -1) *for the unknown density* g *on* S

$$\widetilde{\mathcal{N}} g \;:=\; (\mathcal{H} + p_0(-2^{-1} I + \mathcal{K}^{(2)})\mathcal{R}^2) g = \varphi. \tag{1.48}$$

Moreover, it is evident that the integral (pseudodifferential) operators (see Lemma 1.5 *and Remark* 1.7)

$$\widetilde{\mathcal{N}} \ : \ B^{-1/p}_{p,p}(S) \to B^{1-1/p}_{p,p}(S),$$
$$\widetilde{\mathcal{M}} \ : \ B^{-1/p}_{p,p}(S) \to B^{-1/p}_{p,p}(S),$$

where $1 < p < \infty$, *are also isomorphisms* (*cf. Corollary* 1.9).

2 Non-local Formulation of the Interaction Problem. Uniqueness Theorem

2.1. Green formulae for vector field. Let $u, v \in [C^2(\overline{\Omega}_1)]^3$ be three-dimensional vector-functions and $S \in C^{2+\alpha'}$. Then we can write the Green's formula:

$$\int\limits_{\Omega_1} A(x,D)u \cdot v \, dx = \int\limits_S [Tu]^+ \cdot [v]^+ dS - \int\limits_{\Omega^+} E(u,\overline{v}) \, dx, \tag{2.1}$$

where Au and Tu are given by (1.6) and (1.7), respectively, and

$$E(u,\overline{v}) = 4^{-1} c_{kjpq}(x) \, (D_k u_j + D_j u_k)(D_p \overline{v}_q + D_q \overline{v}_p) = c_{kjpq}(x) \, D_k u_j D_p \overline{v}_q. \tag{2.2}$$

If u is a solution of the equation (1.4), then (2.1) implies:

$$\int\limits_{\Omega_1} \{E(u,\overline{v}) - \rho_1 \omega^2 \, u \cdot v\} \, dx = \int\limits_S [Tu]^+ \cdot [v]^+ dS - \int\limits_{\Omega_1} F \cdot v \, dx,$$

where $F \in C^\alpha(\Omega_1)$ is a given vector-function (body force). The last formula can be extended by continuity for the vector functions $u \in [W^1_p(\Omega_1)]^3$, $v \in [W^1_{p'}(\Omega_1)]^3$, and $F \in [L_p(\Omega_1)]^3$ (i.e., $A_\omega(x,D)u \in [L_p(\Omega_1)]^3$) (cf. [9], [12]):

$$\int\limits_{\Omega_1} \{E(u,\overline{v}) - \rho_1 \omega^2 \, u \cdot v\} \, dx = \langle [Tu]^+, [\overline{v}]^+ \rangle_S \, - \int\limits_{\Omega_1} F \cdot v \, dx, \tag{2.3}$$

where $\langle \cdot, \cdot \rangle_S$ is the duality between the spaces $[B^{-1/p}_{p,p}(S)]^3$ and $[B^{1/p}_{p',p'}(S)]^3$ which coincides to the usual $[L_2(S)]^3$ scalar product for regular vector-functions, i.e., if f and g are continuous vector-functions then

$$\langle f, \overline{g} \rangle_S = \int\limits_S f \cdot g \, dS = \int\limits_S \sum_{k=1}^3 f_k \, \overline{g}_k \, dS.$$

Note that, in fact, the equation (2.3) is the definition of the stress vector-functional $[Tu]^+$ on the boundary S for elements of the space $[W^1_p(\Omega_1)]^3$, $1 < p < \infty$, for which $A_\omega(x,D)u \in [L_p(\Omega_1)]^3$ (see Subsection 1.4).

2.2. Weak formulation of the interaction problem. In Subsection 1.3 we have formulated the fluid-structure interaction problem in the classical setting. Now we formulate the same problem in the weak setting. We look for a vector-function $u \in [W^1_2(\Omega_1)]^3$ and a scalar function $w \in W^1_{2,loc}(\Omega_2) \cap \mathrm{Som}_r(\Omega_2)$ which satisfy respectivelly the differential

equations (1.5) and (1.8) in the distributional sense (with $F \in [L_2(\Omega_1)]^3$), and the conditions (1.11) and (1.12) in the sense of the space $W_2^{-1/2}(S)$, with $f_k, f_0 \in W_2^{-1/2}(S),\ k = 1,2,3$.

By virtue of Remarks 1.12, 1.14 and 1.15, the sought for scalar function $w \in W^1_{2,loc}(\Omega_2) \cap \mathrm{Som}_r(\Omega_2)$ is uniquely representable in the form (1.39) with some density $g \in W_2^{1/2}(S)$. Moreover,

$$\mathcal{N}g = [w]^- \in W_2^{1/2}(S) \ \text{ and } \ \mathcal{M}g = [\partial_n w]^- \in W_2^{-1/2}(S).$$

Therefore, we can recast the interface conditions (1.11) and (1.12) on S as follows

$$[u \cdot n]^+ = d_1 \mathcal{M}g + f_0, \tag{2.4}$$

$$[Tu]^+ = d_2\, n(x)\, \mathcal{N}g + f. \tag{2.5}$$

Further, substitution of $[Tu]^+$ from (2.5) into (2.3) and multiplication of the equation (2.4) by $\overline{h} \in W_2^{1/2}(S)$ in the duality sense, lead us to the relations

$$\int\limits_{\Omega_1} \{E(u,\overline{v}) - \rho_1\omega^2\, u \cdot v\}\, dx - d_2 \langle \mathcal{N}g, [\overline{v}]^+ \cdot n\rangle_S = \langle f, [\overline{v}]^+\rangle_S \ - \int\limits_{\Omega_1} F \cdot v\ dx, \tag{2.6}$$

$$\langle [u \cdot n]^+, \overline{h}\rangle_S \ - \ d_1 \langle \mathcal{M}g, \overline{h}\rangle_S = \langle f_0, \overline{h}\rangle_S, \tag{2.7}$$

where, as above, $\langle \cdot, \cdot \rangle_S$ denotes the duality between the spaces $W_2^{-1/2}(S)$ and $W_2^{1/2}(S)$. Note that the third summand in (2.6) and the first summand in (2.7) are the usual $L_2(S)$ scalar products, since all functions involved in these terms are $W_2^{1/2}(S)$ functions. Multiplying (2.7) by $(-\overline{d}_1)$ and adding termwise to (2.6) we arrive to the equation:

$$\mathcal{B}(u, g; v, h) = \mathcal{F}(v, h), \tag{2.8}$$

where

$$\begin{aligned} \mathcal{B}(u,g;v,h) \ &= \ \int\limits_{\Omega_1} E(u,\overline{v})\, dx - \omega^2 \int\limits_{\Omega_1} \rho_1\ u \cdot v\ dx - d_2 \int\limits_S \mathcal{N}g\ ([\overline{v}]^+ \cdot n)\, dS - \\ &\quad -\overline{d}_1 \int\limits_S [u \cdot n]^+ \overline{h}\, dS + |d_1|^2 \langle \mathcal{M}g, \overline{h}\rangle_S, \end{aligned} \tag{2.9}$$

$$\mathcal{F}(v,h) \ = \ \langle f, [\overline{v}]^+\rangle_S - \overline{d}_1 \langle f_0, \overline{h}\rangle_S - \int\limits_{\Omega_1} F \cdot v\ dx. \tag{2.10}$$

Now we can formulate the so-called non-local setting of the basic interaction problem.

Problem ($\widetilde{\mathcal{P}}^{(\omega,r)}$): Find a pair $(u, g) \in [W_2^1(\Omega_1)]^3 \times W_2^{1/2}(S)$ satisfying the aquation (2.8) for arbitrary $(v, h) \in [W_2^1(\Omega_1)]^3 \times W_2^{1/2}(S)$, where the sesquilinear form $\mathcal{B}$ and the functional $\mathcal{F}$ are defined by (2.9) and (2.10), respectively, with

$$F = (F_1, F_2, F_3)^\top \in [L^2(\Omega_1)]^3, \ \ f = (f_1, f_2, f_3)^\top, \ \ f_k, f_0 \in W_2^{-1/2}(S).$$

If (u, g) is a solution of the problem ($\widetilde{\mathcal{P}}^{(\omega,r)}$), then u describes the elastic field of displacements in Ω_1, while the scalar field w is constructed with the help of the g by formula (1.39).

REMARK 2.1 *If instead of* (1.39) *we apply the representation* (1.46)*, then due to Remark* 1.16 *we have*

$$[w]^- = \widetilde{\mathcal{N}} g = [\mathcal{H} + p_0(-2^{-1} I + \mathcal{K}^{(2)})\mathcal{R}^2]\, g \in W_2^{1/2}(S) \tag{2.11}$$

and

$$[\partial_n w]^- = \widetilde{\mathcal{M}} g = [2^{-1} I + \mathcal{K}^{(1)} + p_0 \mathcal{L}\mathcal{R}^2] g \in W_2^{-1/2}(S),$$

where $w \in W^1_{2,loc}(\Omega_2)$*. Obviously, in this case* $g \in W_2^{-1/2}(S)$*. Clearly, all the above arguments and formulae, including the formulation of problem* $(\widetilde{\mathcal{P}}^{(\omega,r)})$*, remain valid with* $\widetilde{\mathcal{N}}$ *and* $\widetilde{\mathcal{M}}$ *in the place of* $\mathcal{N}$ *and* $\mathcal{M}$*, respectively.*

Let us show that in regular classes the both above formulations of the interaction problem are equivalent.

LEMMA 2.2 *If* $(u, g) \in [C^2(\Omega_1) \cap C^1(\overline{\Omega}_1)]^3 \times C^{1+\alpha}(S)]$ *is a solution of the problem* $(\widetilde{\mathcal{P}}^{(\omega,r)})$*, with* $F_k \in C(\overline{\Omega}_1)$*,* $f_k, f_0 \in C(S)$*, then* u *and* w*, where* w *is given by* (1.39) *as a linear combination of single- and double-layer potentials with the density* g*, represent a classical solution to the problem* $(P^{(\omega,r)})$ *and vice versa.*

Proof. Let (u, g) be some solution of problem $(\widetilde{\mathcal{P}}^{(\omega,r)})$ with the smoothness properties indicated in Lemma 2.2. Then all the terms, involved in the duality brackets in formulae (2.8)-(2.10), are, actually, usual L_2 scalar products (see Lemma 1.8).

Let us first take $v \in C_0^\infty(\Omega_1)$ (i.e. v is C^∞-regular and has a compact support in Ω_1) and $h = 0$. Then (2.8) implies

$$\int\limits_{\Omega_1} \{E(u, \overline{v}) - \rho_1(x)\omega^2\, u \cdot v\}\, dx = -\int\limits_{\Omega_1} F \cdot v\, dx, \tag{2.12}$$

whence it follows that u is a weak solution to the equation (1.5). Further, the integration by parts then gives

$$\int\limits_{\Omega_1} [A_\omega(x, D)u - F] \cdot v\, dx = 0, \tag{2.13}$$

which shows that u is a classical solution to (1.5) in Ω_1, i.e., $A_\omega(x, D)u(x) = F(x)$, $x \in \Omega_1$. The scalar field $w(x)$ represented by formula (1.39) is also classical radiating solution of the equation (1.8) in Ω_2, since by assumption $g \in C^{1+\alpha}(S)$ (see Lemma 1.3).

It remains to verify that u and w meet the interface conditions (1.11) and (1.12) in the classical sense.

To this end, let us take an arbitrary $v \in C^1(\overline{\Omega}_1)$ and $h \in C^1(S)$. Applying again the integration by parts in (2.8) we get

$$-\int\limits_{\Omega_1} A_\omega(x, D)u \cdot v\, dx + \int\limits_S [T(D, n)u]^+ \cdot [v]^+\, dS - d_2 \int\limits_S (\mathcal{N} g) n \cdot [v]^+\, dS -$$

$$-\overline{d}_1 \int\limits_S [u \cdot n]^+ \overline{h}\, dS + |d_1|^2 \int\limits_S (\mathcal{M} g)\overline{h}\, dS = \int\limits_S f \cdot [v]^+ dS - \int\limits_{\Omega_1} F \cdot v\, dx - \overline{d}_1 \int\limits_S f_0 \overline{h}\, dS,$$

i.e.,

$$\int_S \{[T(D,n)u]^+ - d_2(\mathcal{N}g)n - f\} \cdot [v]^+ \, dS - \overline{d}_1 \int_S \{[u \cdot n]^+ - d_1 \mathcal{M}g - f_0\} \, \overline{h} \, dS = 0$$

due to (2.13).

Obviously, these equations are equivalent to the conditions (1.11) and (1.12) since $\mathcal{N}g = [w]^-$ and $\mathcal{M}g = [\partial_n w]^-$, and $[v]^+$ and h are arbitrary C^1-smooth functions on S. ∎

2.3. Uniqueness theorem. Jones modes and Jones frequencies. Let us consider the homogeneous version of the problem $(\widetilde{\mathcal{P}}^{(\omega,r)})$ (i.e., $F = 0$, $f = 0$, $f_0 = 0$) and let $(u, g) \in [W_2^1(\Omega_1)]^3 \times W_2^{1/2}(S)$ be its arbitrary solution. Then

$$\mathcal{B}(u, g; v, h) = 0 \quad \text{for} \quad \forall (v, h) \in [W_2^1(\Omega_1)]^3 \times W_2^{1/2}(S), \tag{2.14}$$

whence (see (2.9))

$$\int_{\Omega_1} \{E(u, \overline{v}) - \rho_1 \omega^2 u \cdot v\} \, dx = d_2 \int_S (\mathcal{N}g) n \cdot [v]^+ dS \ \text{ for } \ \forall v \in [W_2^1(\Omega^+)]^3, \tag{2.15}$$

and

$$\langle [u \cdot n]^+, \overline{h} \rangle_S = d_1 \langle \mathcal{M}g, \overline{h} \rangle_S \ \text{ for } \ \forall h \in W_2^{1/2}(S). \tag{2.16}$$

From now on we provide that $S \in C^{2+\alpha'}$. Due to formula (2.3) (with $F = 0$) the equation (2.15) shows that

$$[Tu]^+ = d_2 \, n(\mathcal{N}g) \ \text{ on } \ S, \tag{2.17}$$

while (2.16) yields

$$[u \cdot n]^+ = d_1 \mathcal{M}g \ \text{ on } \ S. \tag{2.18}$$

The both equations are understood in the sense of the functional space $W_2^{-1/2}(S)$.

But, since $g \in W_2^{1/2}(S)$, by Corollary 1.9 we conclude that $[Tu]^+ \in W_2^{1/2}(S)$. Making use of the interior and boundary regularity results for solutions of elliptic BVPs (see e.g., [20]) we infer that $u \in [W_2^{3/2}(\Omega_1) \cap C^\infty(\Omega_1)]^3$. Therefore, by the trace theorem, then $[u \cdot n]^+ \in W_2^1(S)$ and the equation (2.18) implies $g \in W_2^{3/2}(S)$, due to Corollary 1.9. In turn, the last embedding shows that the corresponding scalar function w, given by (1.39), belongs to the space $W_{2,loc}^2(\Omega_2) \cap C^\infty(\Omega_2) \cap \mathrm{Som}_r(\Omega_2)$. Applying again (2.17) and taking into account that $\mathcal{N}g \in W_2^{3/2}(S)$, we get $u \in [W_2^2(\Omega_1)]^3$. Note that, if $S \in C^\infty$, then we can continue the above arguments to prove that the both fields u and w are C^∞-smoth in $\overline{\Omega}_1$ and $\overline{\Omega}_2$, respectively.

Thus, we have established that (2.14) yields

$$u \in [W_2^2(\Omega_1) \cap C^\infty(\Omega_1)]^3, \quad w \in W_{2,loc}^2(\Omega_2) \cap C^\infty(\Omega_2) \cap \mathrm{Som}_r(\Omega_2). \tag{2.19}$$

Consequently, the interface conditions (2.17) and (2.18) can be writen as

$$[Tu]^+ = d_2 \, [w]^- \, n, \tag{2.20}$$

$$[u \cdot n]^+ = d_1 \, [\partial_n w]^-, \tag{2.21}$$

which are understood in the usual sense (note that by Sobolev embedding theorem $w \in C(\overline{\Omega}_2)$ and $u \in C(\overline{\Omega}_1)$). Moreover, the field equations (1.5) (with $F = 0$) and (1.8) are fulfilled in the classical sense.

Let $B(O,R)$, Σ_R and Ω_R^- be the same as in the proof of Lemma 1.6. Then, by virtue of (2.19), we can apply formula (1.30) for w and formula (2.1) for u (with $v = u$) to obtain

$$\int\limits_{\Sigma_R} \{[\partial_n w][\overline{w}] - [w][\overline{\partial_n w}]\}\, d\Sigma_R - \int\limits_S \{[\partial_n w]^-[\overline{w}]^- - [w]^-[\overline{\partial_n w}]^-\}\, dS = 0,$$

$$\int\limits_S \{[Tu]^+ \cdot [u]^+ - [u]^+ \cdot [Tu]^+\}\, dS = 0.$$

In accordance with the equations (2.20) and (2.21) we get

$$[Tu]^+ \cdot [u]^+ = d_2\overline{d_1}[w]^-[\overline{\partial_n w}]^-.$$

Since $\mathrm{Im}\,(\overline{d}_1\, d_2) = 0$, from the above integral equalities it follows that

$$\mathrm{Im} \int\limits_{\Sigma_R} [\partial_n w][\overline{w}]\, d\Sigma_R = 0.$$

Hence by Lemma 1.2 $w = 0$ in Ω_2. Therefore, for the elastic field u we obtain the following eigenvalue problem

$$\begin{aligned} &A_\omega(x,D)u = 0 \ \text{ in } \ \Omega_1, \\ &[u \cdot n]^+ = 0 \ \text{ and } \ [Tu]^+ = 0 \ \text{ on } \ S. \end{aligned} \tag{2.22}$$

We denote by $J(\Omega_1)$ the set of values of the frequency parameter $\omega > 0$ for which the problem (2.22) admits a nontrivial solution.

Such solutions are called *Jones modes*, while the corresponding values of ω are called *Jones eigenfrequencies* (cf. [37]).

Clearly, $J(\Omega_1)$ is at most denumerable [44]. For arbitrary $\omega \in J(\Omega_1)$ there exist only finitely many linearly independent Jones modes. In general, $J(\Omega_1) \neq \emptyset$ (see [33], [35]), however, there exist domains for which $J(\Omega_1) = \emptyset$ ([24], [35], [29]).

Combining the above results we arrive to the following proposition (cf. [48], Theorem 4.1).

LEMMA 2.3 *Let*

$$\omega \notin J(\Omega_1), \quad \mathrm{Im}(\overline{d}_1\, d_2) = 0, \quad d_1 d_2 \neq 0. \tag{2.23}$$

Then the homogeneous problem $(\widetilde{P}^{(\omega,r)})$ $(f = 0,\ f_0 = 0,\ F = 0)$ *has only the trivial solution.*

3 Existence of Weak Solutions

3.1. Properties of the sesquilinear form $\mathcal{B}$ and the anti-linear functional $\mathcal{F}$. To apply the well-known functional-variational methods to the equation (2.8) we have to study some properties of the sesquilinear form $\mathcal{B}$ and the anti-linear functional $\mathcal{F}$. In particular, we will establish the boundedness of $\mathcal{B}$ and $\mathcal{F}$, and the Gårding type inequality for $\mathcal{B}$, in appropriate function spaces (see, e.g., [36], Chapter 2, Section 9, [20], [49].) To this end, we recall the well-known Ehrling inequalities (see, e.g., [36], Chapter 1, §16, Theorem 16.3; [41], Chapter 3, §7, Theorem 3.16; [38], Chapter 14, §3). Let $u \in W_2^1(\Omega^+)$. Then for arbitrary $\varepsilon > 0$ there exist positive constants $c_1'(\varepsilon)$ and $c_2'(\varepsilon)$ such that

$$\begin{aligned} ||u\ ;\ W_2^{s_2}(\Omega_1)|| \le \varepsilon ||u\ ;\ W_2^{s_1}(\Omega_1)|| + c_1'(\varepsilon)||u\ ;\ W_2^{s_3}(\Omega_1)||, \\ 1 \ge s_1 > s_2 > s_3 \ge 0, \\ ||[u]^+\ ;\ W_2^{t_2}(S)|| \le \varepsilon ||u\ ;\ W_2^{t_1}(\Omega_1)|| + c_2'(\varepsilon)||u\ ;\ W_2^{t_3}(\Omega_1)||, \\ 0 \le t_3 < t_2 + \frac{1}{2} < t_1 \le 1, \quad t_2 \ge 0, \end{aligned} \tag{3.1}$$

where $[u]^+$ is the trace of u on $S = \partial\Omega_1$. Here and in what follows we again assume $S \in C^{2+\alpha'}$ with $0 < \alpha' \le 1$.

The following assertion holds.

LEMMA 3.1 *Let*

$$F \in [L_2(\Omega_1)]^3, \quad f \in [W_2^{-1/2}(S)]^3, \quad f_0 \in W_2^{-1/2}(S). \tag{3.2}$$

Then the anti-linear functional $\mathcal{F}(\cdot,\cdot)$ and the sesquilinear form $\mathcal{B}(\cdot,\cdot\ ;\ \cdot,\cdot)$ defined by (2.10) *and* (2.9)*, respectively, are bounded on $[W_2^1(\Omega_1)]^3 \times W_2^{1/2}(S)$.*

Moreover, there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$\begin{aligned} \mathrm{Re}\,[\mathcal{B}(u,g;u,g)] \ge c_1 \left\{ ||u\ ;\ W_2^1(\Omega_1)||^2 + ||g\ ;\ W_2^{1/2}(S)||^2 \right\} - \\ c_2 \left\{ ||u\ ;\ L_2(\Omega_1)||^2 + ||g\ ;\ L_2(S)||^2 \right\} \end{aligned} \tag{3.3}$$

for arbitrary $u \in [W_2^1(\Omega_1)]^3$ and $g \in W_2^{1/2}(S)$.

Proof. Let us denote

$$[W_2^{s_1}(\Omega_1)]^3 \times W_2^{s_2}(S) = \mathcal{V}^{s_1,s_2},$$

and for arbitrary $(u,g) \in \mathcal{V}^{s_1,s_2}$ put

$$||(u,g)\ ;\ \mathcal{V}^{s_1,s_2}||^2 = ||u\ ;\ W_2^{s_1}(\Omega_1)||^2 + ||g\ ;\ W_2^{s_2}(S)||^2.$$

Due to the Schwarz inequality and the trace theorem, from (2.10) for arbitrary $(v,h) \in \mathcal{V}^{1,1/2}$ we have

$$\begin{aligned} |\mathcal{F}(v,h)| \le ||f\ ;\ W_2^{-1/2}(S)||\ ||[v]^+\ ;\ W_2^{1/2}(S)|| + ||F\ ;\ L_2(\Omega_1)||\ ||v\ ;\ L_2(\Omega_1)|| + \\ |d_1|\ ||f_0\ ;\ W_2^{-1/2}(S)||\ ||h\ ;\ W_2^{1/2}(S)|| \le \left\{ ||F\ ;\ L_2(\Omega_1)|| + ||f\ ;\ W_2^{-1/2}(S)|| + \right. \\ \left. |d_1|\ ||f_0\ ;\ W_2^{-1/2}(S)|| \right\} \left\{ ||v\ ;\ W_2^1(\Omega_1)|| + ||h\ ;\ W_2^{1/2}(S)|| \right\}, \end{aligned}$$

i.e.,

$$|\mathcal{F}(v,h)| \;\le\; c\,||(v,h)\,;\,\mathcal{V}^{1,1/2}||, \quad c=\text{const}>0,$$

which shows that the functional $\mathcal{F}$ is bounded and continuous in the space $\mathcal{V}^{1,1/2}$.

Further, we establish the boundedness (continuity) of the sesquilinear form $\mathcal{B}$.

Applying the equality (2.2), the continuity of the functions c_{kjpq} and ρ_1, Corollary 1.10 and the trace lemma, we easily get from (2.9) for arbitrary $(u,g),\ (v,h)\in\mathcal{V}^{1,1/2}$:

$$\begin{aligned}
&|\mathcal{B}(u,g;v,h)|\le b_1\,||u;W_2^1(\Omega_1)||\,||v\,;\,W_2^1(\Omega_1)||+\\
&|d_2|\,||\mathcal{N}g;L_2(S)||\,||[v]^+\,;\,L_2(S)||+|d_1|\,||[u]^+\,;\,L_2(S)||\,||h;L^2(S)||+\\
&|d_1|^2\,||\mathcal{M}g\,;\,W_2^{-1/2}(S)||\,||h\,;\,L^2(S)||\le b_2||(u,g)\,;\,\mathcal{V}^{1,1/2}||\,||(v,h)\,;\,\mathcal{V}^{1,1/2}||, \qquad (3.4)
\end{aligned}$$

where b_1 and b_2 are appropriate positive constants.

To prove the Gårding inequality (3.3), we proceed as follows.

First, we note that by virtue of the inequality (1.3) and the so-called Korn's second inequality (see, e.g., [20]) we derive

$$\begin{aligned}
\int\limits_{\Omega_1} E(u,\overline{u})\,dx \;&=\; \int\limits_{\Omega_1} c_{kjpq}(x)\,D_k u_j\,\overline{D_p u_q}\,dx \;\ge \delta_1 \int\limits_{\Omega_1}\sum_{k,j=1}^{3}|D_k u_j+D_j u_k|^2\,dx \;\ge\\
&b_3\,||u;W_2^1(\Omega_1)||^2 \;-\; b_4\,||u;L_2(\Omega_1)||^2, \qquad (3.5)
\end{aligned}$$

where b_3 and b_4 are positive constants. Clearly,

$$\omega^2\int\limits_{\Omega_1}\rho_1(x)\,u(x)\cdot u(x)\,dx \;\le b_5\,||u;L_2(\Omega_1)||^2 \qquad (3.6)$$

with $b_5=\omega^2\sup\rho_1(x)<+\infty$.

By Corollary 1.10 together with the Schwarz and Ehrling inequalities (see (3.1)) we conclude

$$\begin{aligned}
&\left|\int\limits_S \mathcal{N}g\,[\overline{u}]^+\cdot n)\,dS\right| \;\le b_6\,||g;L_2(S)||\,||[u]^+;L_2(S)||\le\\
&b_6\,||g;L_2(S)||\,\left\{\varepsilon\,||u;W_2^1(\Omega_1)||+c_2'(\varepsilon)\,||u;L_2(\Omega_1)||\right\},\quad b_6=\text{const}>0, \qquad (3.7)\\
&\left|\int\limits_S [u\cdot n]^+\overline{g}\,dS\right| \;\le ||[u]^+;L_2(S)||\,||g;L_2(S)||\le\\
&||g;L_2(S)||\left\{\varepsilon\,||u;W_2^1(\Omega_1)||+c_2'(\varepsilon)\,||u;L_2(\Omega_1)||\right\}. \qquad (3.8)
\end{aligned}$$

By Lemma 1.3, item v) we have that the principal homogeneous (of order 1) symbol of the pseudodifferential operator $\mathcal{M}$ is positive. Therefore, by standard arguments we can easily show (cf. [9], [11], Chapter 11, §2) that there exist positive constants b_7 and b_8 such that

$$\text{Re}\,\{\langle\mathcal{M}g,\overline{g}\rangle_S\}\ge b_7\,||g;W_2^{1/2}(S)||^2-b_8\,||g;L_2(S)||^2. \qquad (3.9)$$

Now from (2.9) and (3.5)-(3.9) it follows that

$$\mathrm{Re}\{\mathcal{B}(u,g;u,g)\} \geq \int\limits_{\Omega_1} E(u,\overline{u})\,dx + |d_1^2|\mathrm{Re}\{\langle \mathcal{M}g,\overline{g}\rangle_S\} -$$

$$\omega^2 \int\limits_{\Omega_1} \rho_1\,|u|^2\,dx - \left| d_2 \int\limits_S \mathcal{N}g\,([\overline{u}]^+\cdot n)\,dS \right| - \left| \overline{d}_1 \int\limits_S [u\cdot n]^+\overline{g}\,dS \right| \geq$$

$$[b_3 - \tfrac{\varepsilon}{2}(b_6|d_2| + |d_1|)]\;||u;W_2^1(\Omega_1)||^2 + b_7|d_1^2|\;||g;W_2^{1/2}(S)||^2 -$$

$$[b_4 + b_5 + \tfrac{1}{2}c_2'(\varepsilon)(|d_1| + b_6|d_2|)]\;||u;L_2(\Omega_1)||^2 -$$

$$[b_8|d_1|^2 + \tfrac{1}{2}(\varepsilon + c_2'(\varepsilon))(b_6|d_2| + |d_1|)]\;||g;L_2(S)||^2.$$

This inequality completes the proof since we can take $\varepsilon > 0$ sufficiently small. Clearly, the inequality (3.3) then holds with

$$c_1 = \min\left\{ b_3 - \frac{\varepsilon}{2}(b_6|d_2| + |d_1|),\quad b_7|d_1^2| \right\} > 0,$$

and

$$c_2 = \max\left\{ b_4 + b_5 + \frac{1}{2}c_2'(\varepsilon)(|d_1| + b_6|d_2|),\quad b_8|d_1|^2 + \frac{1}{2}(\varepsilon + c_2'(\varepsilon))(b_6|d_2| + |d_1|) \right\} > 0.$$

■

3.2. Existence theorem. Now we are ready to formulate the existence theorem for the problem $(\widetilde{\mathcal{P}}^{(\omega,r)})$.

THEOREM 3.2 *Let $S \in C^{2+\alpha'}$ with $0 < \alpha' \leq 1$ and F, f and f_0 satisfy conditions* (3.2). *Moreover, let the conditions* (2.23) *be fulfilled.*

Then the non-local problem $(\widetilde{\mathcal{P}}^{(\omega,r)})$ is uniquely solvable.

Proof. It immediately follows by standard arguments (see, e.g., [49], [36], [20]) from the well-known Lax-Milgram theorem and Lemma 3.1. ■

REMARK 3.3 *The existence Theorem* 3.2 *implies that the scalar field w represented by formula* (1.39) *with $g \in W_2^{1/2}(S)$ belongs to the class $W^1_{2,loc}(\Omega_2) \cap \mathrm{Som}_r(\Omega_2)$, while the elastic-field $u \in W_2^1(\Omega_1)$. Applying the boundary regularity results and the ellipticity of the pseudodifferential operator $\mathcal{M}$ we easily conclude that:*

i) *if $S \in C^{2+\alpha'}$, $f_0 \in W_2^{1/2}(S)$, $f \in W_2^{1/2}(S)$ and $F \in W_2^1(\Omega_1)$, then $u \in W_2^2(\Omega_1)$ and $g \in W_2^{3/2}(S)$, i.e. $w \in W^2_{2,loc}(\Omega_2) \cap \mathrm{Som}_r(\Omega_2)$;*

ii) *if $S \in C^{2+\alpha'}$, $f \in C^\alpha(S)$, $f_0 \in C^\alpha(S)$ and $F \in C^\alpha(\overline{\Omega}_1)$, then $u \in C^{1+\alpha}(\overline{\Omega}_1)$ and $g \in C^{1+\alpha}(S)$, i.e. $w \in C^{1+\alpha}(\overline{\Omega}_2) \cap \mathrm{Som}_r(\Omega_2)$.*

iii) *if $S \in C^\infty$, $F \in C^\infty(\overline{\Omega}_1)$, $f \in C^\infty(S)$ and $f_0 \in C^\infty(S)$, then $u \in C^\infty(\overline{\Omega}_1)$ and $g \in C^\infty(S)$, and, consequently, $w \in C^\infty(\overline{\Omega}_2) \cap \mathrm{Som}_r(\Omega_2)$.*

Let ω be a Jones eigenfrequency for the domain Ω_1 and let us consider the fluid-solid acoustic interface problem (in classical setting):

$$A_\omega(x,D)u(x) = 0 \quad \text{in} \quad \Omega_1, \quad u \in C^1(\overline{\Omega}_1) \cap C^\infty(\Omega_1), \tag{3.10}$$

$$\mathcal{A}_\omega(D)w(x) = 0 \quad \text{in} \quad \Omega_2, \quad w \in C^1(\overline{\Omega}_2) \cap C^\infty(\Omega_2), \tag{3.11}$$

$$[Tu]^+ = d_2[w]^-\,n + f^{inc} \quad \text{on} \quad S, \tag{3.12}$$

$$[u\cdot n]^+ = d_1\,[\partial_n w]^- + f_0^{inc} \quad \text{on} \quad S, \tag{3.13}$$

where $d_1,\ d_2,\ f^{inc} \in C^\infty(S)$ and $f_0^{inc} \in C^\infty(S)$ are given by (1.13) and (1.14). Clearly, the non-local formulation of the problem (3.10)-(3.13) reads as the equation (2.8) where $\mathcal{B}$ and $\mathcal{F}$ are represented by (2.9) and (2.10) with $F = 0$, and $f = f^{inc}$ and $f_0 = f_0^{inc}$, respectively. We denote this problem by $(\widetilde{\mathcal{P}}_{inc}^{(\omega,r)})$.

Denote by $X_\omega(\Omega_1)$ the null-space of the corresponding homogeneous problem (2.22), i.e., the space of Jones modes. Obviously, if the interface problem is solvable, a solution of the non-homogeneous problem is then defined within the summand $(u_0, w_0)^\top$ where $u_0 \in X_\omega(\Omega_1)$ and $w_0 = 0$. Let us show that this nonhomogeneous problem (3.10)-(3.13) is always solvable.

We note that, if (u, w) is a solution to the above non-local problem $(\widetilde{\mathcal{P}}_{inc}^{(\omega,r)})$, then, actually, it is a classical solution due to the regularity results indicated in Remark 3.3. Therefore, in this case the classical and the non-local weak formulations are equivalent.

To establish the necessary and sufficient conditions for the problem (3.10)-(3.13) to be solvable, we have to formulate the so-called "adjoint" interface problem.

To this end, we note that the problem (3.10)-(3.13) can be equivalently rewritten as (see Remarks 1.12, 1.14 and 1.15):

$$A_\omega(x, D)u(x) = 0 \quad \text{in} \quad \Omega_1, \tag{3.14}$$

$$[Tu]^+ - d_1^{-1}d_2\{\mathcal{N}\mathcal{M}^{-1}[u \cdot n]^+\}n = f^{inc} - d_1^{-1}d_2\{\mathcal{N}\mathcal{M}^{-1}f_0^{inc}\}n, \quad \text{on} \quad S \tag{3.15}$$

$$w(x) = W(g)(x) + p_0 V(g)(x), \quad x \in \Omega_2, \quad p_0 = p_1 + \mathrm{i}p_2, \quad p_2 \neq 0, \tag{3.16}$$

$$g = d_1^{-1}\mathcal{M}^{-1}([u \cdot n]^+ - f_0^{inc}), \tag{3.17}$$

where $\mathcal{N}$ and $\mathcal{M}$ are given by (1.32) and (1.33).

Thus, the original interface problem is completely splited into the two problems; the first one is the boundary value problem (3.14)-(3.15) for the elastic field u in Ω_1, and the second one is the construction of the scalar field w in Ω_2 by formula (3.16), where g solves the equation (3.17). Clearly, the w is uniquely defined by the boundary function $[u \cdot n]^+$. So we need only to solve the problem (3.14)-(3.15) and construct the inverse operator $\mathcal{M}^{-1}$.

It is evident that the problems (3.14)-(3.15) and (3.10)-(3.13) are simultaneously solvable or unsolvable.

Next, we denote by $\mathcal{P}^*$ the operator adjoint to $\mathcal{P}$ with respect to the "real" scalar product, which for complex-valued regular vector-functions f and g reads as follows

$$\int\limits_S \langle f, g\rangle \, dS = \int\limits_S \sum_{k=1}^{3} f_k g_k \, dS = \langle f, g\rangle_S,$$

where $\langle a, b\rangle = \sum_{k=1}^3 a_k b_k$ for $a, b \in \mathbb{C}^3$.

Therefore, we have by definition $\langle \mathcal{P}f, g\rangle_S = \langle f, \mathcal{P}^* g\rangle_S$.

Now, let us construct the "adjoint" boundary value problem to the problem (3.14)-(3.15) with respect to the Green formula:

$$\begin{aligned}\int\limits_{\Omega_1} \{\langle A_\omega u, v\rangle - \langle u, A_\omega v\rangle\} \, dx = \int\limits_S \{\langle [Tu]^+, [v]^+\rangle - \langle [u]^+, [Tv]^+\rangle\} \, dS = \\ \int\limits_S \{\langle [Tu]^+ - d_1^{-1}d_2[\mathcal{N}\mathcal{M}^{-1}[u \cdot n]^+]n \ , \ [v]^+\rangle - \\ \langle [u]^+ \ , \ [Tv]^+ - d_1^{-1}d_2[(\mathcal{N}\mathcal{M}^{-1})^*[v \cdot n]^+]n\rangle\} \, dS.\end{aligned} \tag{3.18}$$

Taking into account that $\mathcal{L} = \mathcal{L}^*$, $\mathcal{H} = \mathcal{H}^*$ and $\mathcal{K}^{(1)} = (\mathcal{K}^{(2)})^*$, and applying Lemma 1.3, itemvi), we deduce that $\mathcal{M}^*\mathcal{N} = \mathcal{N}^*\mathcal{M}$, i.e. $\mathcal{N}\mathcal{M}^{-1} = (\mathcal{N}\mathcal{M}^{-1})^*$. Therefore, (3.18) implies

$$\int\limits_{\Omega_1} \{\langle A_\omega u, v\rangle - \langle u, A_\omega v\rangle\}\, dx = \int\limits_S \{\langle [Tu]^+ - d_1^{-1} d_2[\mathcal{N}\mathcal{M}^{-1}[u\cdot n]^+]n\ ,\ [v]^+\rangle - \langle [u]^+\ ,\ [Tv]^+ - d_1^{-1} d_2[\mathcal{N}\mathcal{M}^{-1}[v\cdot n]^+]n\rangle\}\, dS,$$

which shows that the sought for homogeneous adjoint boundary value problem can be formulated as follows (cf. [36], Chapter 2)

$$\begin{aligned} &A_\omega(x,D)v(x) = 0 \quad \text{in} \quad \Omega_1,\\ &[Tv]^+ - d_1^{-1} d_2[\mathcal{N}\mathcal{M}^{-1}[v\cdot n]^+]n = 0 \ \text{ on } \ S. \end{aligned} \tag{3.19}$$

Moreover, the problem (3.14)-(3.15) is solvable if and only if (see, e.g., [36], Chapter2)

$$\int\limits_S \langle f^{inc} - d_1^{-1} d_2[\mathcal{N}\mathcal{M}^{-1} f_0^{inc}]n\ ,\ [v]^+\rangle\, dS = 0, \tag{3.20}$$

where v is an arbitrary solution of the adjoint problem (3.19).

Let us prove that (3.19) coincides with the homogeneous problem of Jones modes (2.22). Clearly, we need only to show that (3.19) yields $[v\cdot n]^+ = 0$ on S. We denote

$$d_1^{-1}\mathcal{M}^{-1}[v\cdot n]^+ =: g, \tag{3.21}$$

and construct a scalar function $w(x)$ by formula (3.16) with g given by (3.21) . Then the boundary condition in (3.19) is equivalent to the conditions on S:

$$[Tv]^+ = d_2\ (\mathcal{N}g)\ n, \quad [v\cdot n]^+ = d_1\ \mathcal{M}g,$$

i.e.,

$$[Tv]^+ = d_2[w]^- n, \quad [v\cdot n]^+ = d_1\ [\partial_n w]^-.$$

Now the arguments of Subsection 2.3 show that $w(x) = 0$ in Ω_2. Whence $[v\cdot n]^+ = 0$ on S, and, therefore, (3.19) is equivalent to the eigenvalue problem of Jones modes:

$$\begin{aligned} &A_\omega(x,D)v(x) = 0 \quad \text{in} \quad \Omega_1\\ &[Tv]^+ = 0 \quad \text{and} \quad [v\cdot n]^+ = 0 \quad \text{on} \quad S. \end{aligned}$$

The condition (3.20) is then converted into the equation

$$\int\limits_S \langle f^{inc}, [v]^+\rangle\, dS = 0, \quad \text{for all} \quad v \in X_\omega(\Omega_1), \tag{3.22}$$

which represents the necessary and sufficient condition for the problem (3.14)-(3.15) (and, consequently, for the problem (3.10)-(3.13)) to be solvable.

Since in the fluid-solid acoustic interaction problem $f^{inc} = -p^{inc}(x)n(x)$ due to (1.13), where $p^{inc}(x)$ is a scalar function, we have

$$\langle f^{inc}, [v]^+\rangle = -p^{inc}(x)\langle n, [v]^+\rangle = -p^{inc}(x)[v\cdot n]^+ = 0$$

for an arbitrary Jones mode v. From this fact it follows that (3.22) is satisfied and the problem (3.10)-(3.13) is always solvable (cf. [29], [30]).

Thus, the fluid-solid acoustic interaction problem is solvable for arbitrary value of the frequency parameter ω (in the both classical and non-local settings). If $\omega \notin J(\Omega_1)$ then the problem is uniquely solvable, and, if $\omega \in J(\Omega_1)$, then solutions are defined modulo $(u_0, 0)$, where $u_0 \in X_\omega(\Omega_1)$ is an arbitrary Jones mode.

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