Multiplication operators and its ill-posedness properties

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Abstract

This paper deals with the characterization of multiplication operators, especially with its behaviour in the ill-posed case. We want to classify the different types and degrees of ill-posedness. We give some connections between this classification and regularization methods.

1 Introduction

Multiplication operators occur in various fields of mathematics. The most typical application is usually the case of coefficient operators in ordinary and partial differential equations. Here it is often necessary to solve multiplication equations to reconstruct or identify parameters multiplied with certain other functions. Another topic where we can find multiplication operators are convolution equations of the first kind. If we apply the Fourier transform to such an equation we obtain a multiplication equation.

Why do we not simply divide the equation by the factor and solve the equation this way? This is true, whenever the factor is bounded away from zero. However, in the other case, this can not be done without leaving the function space. Why do we not want to leave the function space? At first, every change of the function space comes together with a loss of smoothness of the solution. This can not always be accepted, above all if the lower smoothness can not be measured. Secondly, the equation is ill-posed, whenever the multiplication function has zeros and the operator maps a function space into itself. In a comparable case, some (compact) integral equations of the first kind can be made well-posed, when the function spaces are changed. Nevertheless, these operators has to be considered as ill-posed operators. So it makes sense for multiplication operators, too.

2 Preliminaries

In the beginning we want to give some definitions and summarize some well-known facts. At first we define the object "multiplication operator".

Definition 1 *Let X, Y and Φ three linear normed function spaces over the same set Ω. Then for every $\varphi \in \Phi$ the operator*

$$\mathcal{M} : X \to Y \tag{2.1}$$

with

$$[\mathcal{M}x](t) := \varphi(t) \cdot x(t), \quad t \in \Omega \tag{2.2}$$

is called a multiplication operator *for all $x \in X$, for which the product is defined.*

It depends on the situation and the type of the function spaces, whether the product has to be taken pointwise, almost everywhere or in a generalized sense. In typical cases the spaces X, Y and Φ are Banach spaces, X and Y are often Hilbert spaces.

It is well-known that the *spectrum* $\sigma(A)$ of a linear operator A is defined as the set of all points λ of the complex plain, where the operator $A - \lambda I$ is not continuously invertible. The most important point of the spectrum concerning the ill-posedness of the operator is the point 0.

Now we concentrate on self-adjoint linear operators $\mathcal{M}$ between Hilbert spaces $X =: H_1$ and $Y =: H_2$. For such operators the following holds:
The spectrum is a subset of the real axis and the values

$$m := \inf_{\|x\|_{H_1}} (Ax, x) \tag{2.3}$$

and

$$M := \sup_{\|x\|_{H_1}} (Ax, x) \tag{2.4}$$

belong to the spectrum whenever they are finite. Further, $\sigma(A)$ is a closed subset of the interval $[m, M]$ and the operator is bounded whenever m and M are finite.

If the operator is not self-adjoint but bounded, we may substitute it by its Gauss-transformed operator A^*A or the operator $(A^*A)^{\frac{1}{2}}$. Here A^* is the adjoint of the operator A. The ill-posedness properties of the operator are not influenced by this transformation. Therefore we may assume, that the bounded operator A is self-adjoint and non-negative in general.

For such operators we divide the spectrum into two parts, the point and the continuous spectrum. Unfortunately, these definitions are not uniformly chosen in the literature. We will use the following definition.

Definition 2 *Obviously it holds*

$$R(A-\lambda I) \subseteq \overline{R(A-\lambda I)} \subseteq H_2 \quad \forall \lambda \in \mathbb{C}. \tag{2.5}$$

Then we say, that λ is an element of the continuous spectrum $\sigma_c(A)$, *if*

$$R(A-\lambda I) \neq \overline{R(A-\lambda I)}, \tag{2.6}$$

that means the range is not closed. Furthermore, λ belongs to the point spectrum $\sigma_p(A)$, *if*

$$\overline{R(A-\lambda I)} \neq H_2, \tag{2.7}$$

that means the range is not dense in the Hilbert space H_2.

Another expression we need is the definition of the generalized inverse. For this we consider the set of all solutions x of the minimum problem

$$\|Ax-y\|_{H_2} = \inf\{\|Au-y\|_{H_2} : u \in H_1\}. \tag{2.8}$$

Every solution of this problem we call *least-squares-solution*. As it is known, the infimum is a minimum, whenever $y \in R(A)+R(A)^{\perp}$ with $R(A)$ being the range of the operator A. The manifold of the least-squares-solutions can be expressed by $x_0 + N(A)$, where x_0 is any element from this manifold and $N(A)$ is the null space of A. Then we denote by $x^{\dagger}$ the element with the least norm in this manifold (minimum norm least squares solution).

Definition 3 *We call the operator*

$$A^{\dagger} : R(A)+R(A)^{\perp} \subset H_2 \to H_1, \tag{2.9}$$

defined by

$$A^{\dagger}y := x^{\dagger}, \quad y \in R(A)+R(A)^{\perp} \tag{2.10}$$

the generalized inverse *of the operator A.*

3 Properties of one-dimensional multiplication operators

In the following we want to consider the case that Ω is a bounded subset of $\mathbb{R}^1$. Additionally, we assume that $H_1 = H_2 = L^2(0,1)$ is the Hilbert space of all quadratically integrable functions over the interval $(0,1)$. Then the operator

$$\mathcal{M} : L^2(0,1) \to L^2(0,1) \tag{3.1}$$

of the multiplication with the measurable function $\varphi \in \Phi$ is defined by

$$[\mathcal{M}x](t) := \varphi(t) \cdot x(t) \tag{3.2}$$

for almost every $t \in (0,1)$.

To guarantee the boundedness of the operator $\mathcal{M}$ it is sufficient and necessary, that φ is essentially bounded, that means $\varphi \in L^\infty(0,1)$. As already mentioned in the preliminaries we may assume, that φ is a real-valued function and $\varphi(t) \geq 0$ almost everywhere in $(0,1)$. Otherwise the operator

$$[\mathcal{M}^*\mathcal{M}x](t) = [\mathcal{M}\mathcal{M}^*x](t) = |\varphi(t)|^2 x(t) \tag{3.3}$$

or the operator

$$[(\mathcal{M}^*\mathcal{M})^{\frac{1}{2}}x](t) = |\varphi(t)|x(t) \tag{3.4}$$

can be used.

Now we want to characterize the spectrum of the multiplication operator. To do this we need the following definition first.

Definition 4 *Let G_φ the union of all open sets $G \subset \mathbb{R}$ with* $\text{meas}(\varphi^{-1}(G)) = 0$. *Then we define by*

$$R_e(\varphi) := \mathcal{C}G_\varphi \quad (\textit{set-theoretical complement with respect to } \mathbb{R}) \tag{3.5}$$

the essential range *of φ (cf. [8]).*

Now the following lemma can be shown (For the lemma without a proof compare [8] again.):

Lemma 5 *The spectrum $\sigma(\mathcal{M})$ of the multiplication operator $\mathcal{M}$ can be expressed by*

$$\sigma(\mathcal{M}) = R_e(\varphi). \tag{3.6}$$

Proof: We have the set $G_\varphi = \cup G : \text{meas}(\varphi^{-1}(G)) = 0$. Now let $\{D_i\}$ a countable base of the topology of $\mathbb{R}$. Then obviously $G_\varphi = \cup D_i : (\text{meas}(\varphi^{-1}(D_i)) = 0)$ and therefore $\text{meas}(\varphi^{-1}(G_\varphi)) = 0$ is fulfilled.

Hence we have

G_φ is the largest open (with respect to $\mathbb{R}$) set G fulfilling $\text{meas}(\varphi(G)) = 0$.

Analogously, for the complement (the essential range) holds

$F_\varphi := \mathcal{C}G_\varphi = R_e(\varphi)$ is the smallest closed set containing the value $\varphi(t)$ for almost every $t \in (0,1)$.

(a) Let $\lambda \in G_\varphi$. Since G_φ is a open set, it exists a open ball (=interval) $U = B_\varepsilon(\lambda)$ in G_φ with $\lambda \in U$. For this set we have $\text{meas}(\varphi^{-1}(U)) = 0$ or $\text{meas}\{t : \varphi(t) \in U\} = 0$. Translating U in the zero, we obtain $\text{meas}\{t : \varphi(t) - \lambda \in U_0\} = 0$, with $U_0 := B_\varepsilon(0)$. Following the definition of the essential infimum, we see that

$$\inf_{t\in(0,1)} \text{ess}\{|\varphi(t) - \lambda|\} \geq \varepsilon \tag{3.7}$$

holds including

$$\frac{1}{|\varphi(t) - \lambda|} \in L^\infty(0,1) \tag{3.8}$$

and

$$\frac{1}{\varphi(t)-\lambda} \in L^\infty(0,1). \tag{3.9}$$

That means, for any $y \in L^2(0,1)$ the function $\frac{y}{\varphi-\lambda}$ is in $L^2(0,1)$, too. Hence, $(\mathcal{M}-\lambda I)^{-1}$ exists and is continuous. Consequently, λ is an element of the resolvent set $\varrho(\mathcal{M})$.

(b) Now let $\lambda \in \varrho(\mathcal{M})$. Then $\frac{1}{\varphi(t)-\lambda}$ has to be in $L^\infty(0,1)$. By definition, this is valid, if and only if an $\varepsilon > 0$ exists such that $\inf\operatorname{ess}|\varphi(t)-\lambda| \geq \varepsilon$. In analogy to (a) it follows

$$\exists U_0 := B_\varepsilon(0) : \operatorname{meas}\{t : \varphi(t)-\lambda \in U_0\} = 0, \tag{3.10}$$

$$\exists U = B_\varepsilon(\lambda) : \operatorname{meas}\{t : \varphi(t) \in U\} = 0 \tag{3.11}$$

and from this the existence of a open set containing λ with

$$\operatorname{meas}(\varphi^{-1}(U)) = 0, \tag{3.12}$$

and finally $\lambda \in G_\varphi$. ■

Remark 6 It is in general not useful to restrict the considerations to real values of λ, since the spectrum is always a subset of the complex plain. (Real operators may have complex spectral values, of course.) Nevertheless, in our case it makes sense, because we have a real spectrum and the function φ is only defined for real values. The complex case works analogously, but we can not extend our function to a set in the complex plain by zero, since then the function is only defined on a null set.

In the following we want to classify the continuous and the point spectrum of the multiplication operator. If λ belongs to the point spectrum, that means it is an eigenvalue, the equation

$$(\mathcal{M}-\lambda I)x = (\varphi(t)-\lambda)x(t) = 0 \quad \text{a.e. in } L^2(0,1) \tag{3.13}$$

has to be true for any $x \in L^2(0,1)$. Obviously λ can only be an eigenvalue, if the set

$$K_\lambda := \varphi^{-1}(\{\lambda\}) \tag{3.14}$$

has a positive measure. K_λ is uniquely defined except on a null set. All eigenvalues have an infinite multiplicity and each closed orthogonal system of functions in $L^2(0,1)$ with support in K_λ is a closed system of eigenfunctions. For all $\lambda \in \sigma(\mathcal{M})$ with $\operatorname{meas}(K_\lambda) = 0$ we find $\lambda \in \sigma_c(\mathcal{M})$. Now we have to answer the question, which eigenvalues are also in the continuous spectrum. For this we define the set Ω_λ by

$$\Omega_\lambda := \operatorname{supp}(\varphi(t)-\lambda). \tag{3.15}$$

Here supp denotes the support in the sense of distributions (i.e. in our case the local non-vanishing by measure). Now we consider the operator $\mathcal{M}_\lambda$ as the restriction of $\mathcal{M}$ to $L^2(\Omega_\lambda)$. Obviously, λ is not an eigenvalue of $\mathcal{M}_\lambda$. Now, if λ is in the spectrum of $\mathcal{M}_\lambda$, then it is an element of the continuous spectrum. This includes, that the range $R(\mathcal{M} - \lambda I)$ is not closed. We see, that the range $R(\mathcal{M} - \lambda I)$ is also not closed, since it consists of all functions continued by zero in $[0,1] \setminus \Omega_\lambda$. Therefore, these elements of the point spectrum are in the continuous spectrum, too.

To illustrate our considerations we give a short example.

Example 7
(a) *Let* $\varphi(t) = t$. *Then* $\sigma(\mathcal{M}) = R_e(\varphi) = [0,1]$. *The point spectrum* $\sigma_p(\mathcal{M})$ *is empty, hence* $\sigma_c(\mathcal{M}) = [0,1]$.
(b) *Let* $\varphi(t) = \begin{cases} 0, & t \le \frac{1}{2} \\ 1, & t > \frac{1}{2} \end{cases}$. *Then* $\sigma(\mathcal{M}) = R_e(\varphi) = \{0,1\}$, $\sigma_p(\mathcal{M}) = \{0,1\}$ *and* $\sigma_c(\mathcal{M})$ *is empty.*
(c) *Let* $\varphi(t) = \begin{cases} 2t, & t \le \frac{1}{2} \\ 1, & t > \frac{1}{2} \end{cases}$*.Then* $\sigma(\mathcal{M}) = R_e(\varphi) = [0,1]$, $\sigma_p(\mathcal{M}) = \{0\}$ *and* $\sigma_c(\mathcal{M}) = [0,1]$.

Similar to the compact operators it is possible to make a spectral decomposition. In general every self-adjoint operator A in a Hilbert space H can be expressed by the Stieltjes integral

$$Ax = \int_{m-0}^{M} \lambda \, dE_\lambda x. \tag{3.16}$$

Here $\{E_\lambda\}$ is called the *spectral family* of the operator. This is (cf. [9]) a family of self-adjoint projectors $E_\lambda : H \to H$ with $E_\mu E_\lambda = E_{\min \mu,\lambda}$ (this is equivalent to $E_\mu \ge E_\lambda$ for $\mu \ge \lambda$), E_λ is right-continuous with respect to λ and $E_{m-0} = O$ as well as $E_M = I$. Note that the right-continuity is substituted by many authors by the left-continuity. Then the other conditions has to be adapted.

At first we want to consider the operator of the multiplication with the independent variable, i.e.

$$[\tilde{\mathcal{M}}x](t) := tx(t), \quad t \in (0,1). \tag{3.17}$$

For this operator the spectral family $\tilde{E}_\lambda$ is declared by

$$[\tilde{E}_\lambda] = \begin{cases} x(t), & t \le \lambda \\ 0, & t > \lambda \end{cases}. \tag{3.18}$$

Namely, for $x, y \in H$ we have due to $m = 0$ and $M = 1$

$$\tilde{\mathcal{M}}x = \int_0^1 \lambda \, d\tilde{E}_\lambda x = \int_0^1 \lambda \, d(x(t)H(\lambda - t)) = \int_0^1 \lambda x(t) \, dH(\lambda - t) = tx(t) \tag{3.19}$$

with the Heaviside function H.
Note that $m = \inf\operatorname{ess}_{(0,1)} \varphi(t)$ and $M = \sup\operatorname{ess}_{(0,1)} \varphi(t)$.

Now we want to consider the operator of the multiplication with any function from $L^\infty(0,1)$. There are two possible ways:
At first, we determine the spectral family of the operator $[\mathcal{M}x](t) = \varphi(t)x(t)$. According to [9] we obtain here

$$[E_\lambda] = \begin{cases} x(t), & \varphi(t) \leq \lambda \\ 0, & \varphi(t) > \lambda \end{cases}. \tag{3.20}$$

On the other hand, a representation of the operator $\mathcal{M}$ as a function of the operator $\tilde{\mathcal{M}}$ is possible. Namely, for any self-adjoint operator A with the spectral family $\{E_\lambda\}$ the operator $u(A)$ with a function u can be defined as

$$[u(A)]x := \int_{m-0}^{M} u(\lambda)\, dE_\lambda x \quad \forall x \in H. \tag{3.21}$$

In the case, that u is summable with respect to all the functions of finite variation $(E_\lambda x, y)$, $x, y \in H$ (the functions $(E_\lambda x, x) = \|E_\lambda x\|^2$ suffice), that means the integral on the right-hand side exists, the function $u(A)$ is uniquely defined.

For the multiplication operator we obtain analogously to the $\tilde{\mathcal{M}}$ case

$$[\varphi(\mathcal{M})]x = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda x = \varphi(t)x(t). \tag{3.22}$$

The integrals exist, since we have $\varphi \in L^\infty(0,1)$, and we obtain $\mathcal{M} = \varphi(\tilde{\mathcal{M}})$.

Now we want to consider the solvability of a multiplication equation. For this we want to determine the generalized inverse of the operator. First, we consider the resolvent R_{λ_0} of the operator $\mathcal{M}$ in the point λ_0. It is known (cf. [9]), that the resolvent of any linear operator A can be expressed by

$$R_{\lambda_0} x = (A - \lambda_0 I)^{-1} x = \int_m^M \frac{1}{\lambda - \lambda_0}\, dE_\lambda x. \tag{3.23}$$

This resolvent can be derived for all elements λ_0 from the resolvent set $\varrho(A)$.

Now the domain of the generalized inverse is given by $R(\mathcal{M}) + R(\mathcal{M})^\perp$. The solution x of the equation $\mathcal{M}x = y$ is uniquely determined whenever the null space of $\mathcal{M}$ contains only the zero element. Else it can written as $x = x_0 + N(\mathcal{M})$, where x_0 is any least squares solution of the equation. The element with the least norm from this manifold is the minimum norm least squares solution $x^\dagger$, which defines the generalized inverse $\mathcal{M}^\dagger$ by

$$\mathcal{M}^\dagger y := x^\dagger \tag{3.24}$$

for every $y \in R(\mathcal{M}) + R(\mathcal{M})^\perp$. We obtain due to $R(\mathcal{M})^\perp = N(\mathcal{M}^*) = N(\mathcal{M})$ for all these $y = y_1 + y_2$ a least squares solution, if $y_1 \in R(\mathcal{M})$ and $y_2 \in N(\mathcal{M})$. How we can find an expression for $R(\mathcal{M})$ and $N(\mathcal{M})$? For the null space we have already seen, that λ is an eigenvalue if and only if $\mathrm{meas}([0,1] \setminus \mathrm{supp}\, \varphi) = 0$. The null space is the subspace of all eigenfunctions belonging to the eigenvalue

0. As we have already mentioned, this is the set of all functions y_2 with $y_2(t) = 0$ almost everywhere on the support of φ:

$$N(\mathcal{M}) = \{x \in L^2(0,1) : x(t) = 0 \text{ for a.e. } t \in \operatorname{supp}\varphi\}. \tag{3.25}$$

For any $y_1 \in R(\mathcal{M})$ any function x with

$$x = \begin{cases} \frac{y_1(t)}{\varphi(t)}, & t \in \operatorname{supp}\varphi \\ \text{ambiguous}, & t \notin \operatorname{supp}\varphi \end{cases} \tag{3.26}$$

solves the equation $\varphi(t)x(t) = y_1(t)$. From this it follows immediately, that the generalized inverse $\mathcal{M}^\dagger$ for $y \in R(\mathcal{M})$ can be expressed by

$$\mathcal{M}^\dagger y = \begin{cases} \frac{y_1(t)}{\varphi(t)}, & t \in \operatorname{supp}\varphi \\ 0, & t \notin \operatorname{supp}\varphi \end{cases}. \tag{3.27}$$

Since for $y \in R(\mathcal{M})^\perp$ the generalized inverse is given by $A^\dagger y = 0$, (3.27) holds for all $y \in R(\mathcal{M}) + R(\mathcal{M})^\perp$. Now the range $R(\mathcal{M})$ of the multiplication operator can easily be characterized. Namely, we have

$$R(\mathcal{M}) = \{y \in L^2(0,1) : \frac{y}{\varphi} \in L^2(\operatorname{supp}\varphi),\ y = 0 \text{ a.e. in } [0,1] \setminus \operatorname{supp}\varphi\}. \tag{3.28}$$

The range is closed if and only if the continuous spectrum is empty.

Now we want to characterize the generalized inverse a bit more. It is known (cf. [9]), that the generalized inverse can be expressed by the resolvent (3.23) for $\lambda_0 = 0$. This also works in the case, where 0 belongs to the spectrum, but the integral exists. Therefore, we have for any linear self-adjoint operator A

$$A^\dagger = \int_{m+0}^{M} \frac{1}{\lambda}\, dE_\lambda x. \tag{3.29}$$

From this identity we can derive a condition according to Picard's condition for compact operators. Namely, we find:
An element y belongs to the range of an operator A in a Hilbert space H whenever the formally defined generalized inverse from (3.29) is bounded. This includes

$$\|A^\dagger y\|_H^2 = \int_m^M \frac{1}{\lambda^2} d\, \|E_\lambda y\|_H^2 < \infty. \tag{3.30}$$

This condition can be seen as a generalization of Picard's condition. For compact operators this coincides with the classical condition. For the multiplication operator $\mathcal{M}$ (in the injective case, i.e. $\lambda = 0$ is not an eigenvalue) this means, that

$$\int_0^1 \frac{(y(t))^2}{(\varphi(t))^2} < \infty. \tag{3.31}$$

This is equivalent with the range characterization (3.28) in the injective case.

4 Multiplication operators and ill-posedness

In this section we consider one-dimensional multiplication operators again. How it it possible to characterize such operators in a similar way as compact operators? For such operators the *degree of ill-posedness* is introduced by the decay rate of the singular values of the operators. In the case of non-negative self-adjoint operators these correspond to the eigenvalues. Therefore, a compact linear self-adjoint operator, given by

$$Ax = \sum_{i=1}^{\infty} \lambda_i (x, x_i) x_i \tag{4.1}$$

with the eigenvalues λ_i and the eigenfunctions x_i has the degree of ill-posedness ν, if

$$\lambda_i \sim \frac{1}{i^\nu} \tag{4.2}$$

can be found. Obviously, the operator A^α, $\alpha > 0$ has due to

$$A^\alpha x = \sum_{i=1}^{\infty} \lambda_i^\alpha (x, x_i) x_i \tag{4.3}$$

the degree of ill-posedness $\alpha \cdot \nu$.

Hence, it seems practically to introduce the degree of ill-posedness for multiplication operators in such a way, that any operator

$$\mathcal{M}x = \int_0^M \lambda \, dE_\lambda x \tag{4.4}$$

with degree of ill-posedness ν causes a degree $\alpha\nu$ for the operator

$$\mathcal{M}^\alpha x = \int_0^M \lambda^\alpha \, dE_\lambda x. \tag{4.5}$$

Without loss of generality we may set the value $\nu = 1$ for the operator $\tilde{\mathcal{M}}x = tx$. Then for all potential function $\varphi(t) = t^\alpha$, $\alpha > 0$ we conclude a degree $\nu = \alpha$, of course. How we can generalize this to all $L^\infty(0,1)$-functions? At first, we have to assume, that the operator is injective. This is equivalent to $\mathrm{meas}\{[0,1] \setminus \mathrm{supp}\,\varphi\} = 0$. In the other case, the solution is not identifiable from the data (compare the expression (3.26)), so we can restrict the interval to $\mathrm{supp}\,\varphi$. Now in the injective case the set, where φ vanishes, has measure 0. For such functions φ we introduce the following.

Definition 8 *Let $\varphi \in L^\infty(0,1)$. We introduce the distribution function $p(\lambda)$ by*

$$p(\lambda) := \mathrm{meas}(\varphi^{-1}([0,\lambda])). \tag{4.6}$$

Then we define the increasing rearrangement *of φ as the function*

$$\tilde{\varphi}(s) := \sup\{\lambda : p(\lambda) \le s\}. \tag{4.7}$$

Here the expression φ^{-1} is the complete pre-image of the function φ.

Now we want to give some properties of the rearranged function $\tilde{\varphi}$. This function is in general the inverse function of p, whenever p is not constant in an interval. Else $\tilde{\varphi}$ has jumps. Furthermore, it is a monotonous increasing function. It is continuous from the right and is bounded, since φ is bounded. Due to the rearrangement, $\tilde{\varphi}$ has at most one zero, namely for $t = 0$, whenever the set, where φ vanishes, has measure zero. In this case of an injective multiplication operator we are able to give the following definition.

Definition 9 *Let $\mathcal{M}$ be the operator of the multiplication with a function φ, $\varphi \in L^\infty(0,1)$. Then we define the number $\nu = \nu(\mathcal{M})$ to be the* degree of ill-posedness *of this multiplication operator if we have*

$$\tilde{\varphi}(t) \sim t^\nu \tag{4.8}$$

in a sufficiently small environment of $t = 0$.

Remark 10 The proportionalilty in (4.8) is given in the following form: There exist two positive constants $\underline{c}$ and $\overline{c}$ such that

$$\underline{c}t^\nu \leq \tilde{\varphi}(t) \leq \overline{c}t^\nu. \tag{4.9}$$

It is possible to generalize this definition of the degree of ill-posedness in a similar way as in connection with compact operators. At first, we may define the degree ν of ill-posedness to be the value with

$$0 < \lim_{t\to 0} \frac{\tilde{\varphi}(t)}{t^\nu} < \infty. \tag{4.10}$$

It is clear that such a ν is uniquely defined if it exists. Furthermore, this definition contains the original definition. Secondly, we may generalize this again. Now we define ν by

$$\nu := \sup\{\mu : \lim_{t\to 0} \frac{\tilde{\varphi}(t)}{t^\mu} = 0\}. \tag{4.11}$$

Note that this value need also not exist. This coincides with the appropriate case for compact operators (cf. [5]).

We will show now, that the degree of ill-posedness is connected with the order of the zeros of the function φ. First we have to define the expression "zero" of a $L^\infty(0,1)$-function.

Definition 11 *If for a function $\varphi(t) \in L^\infty(0,1)$ the value*

$$\alpha_0 := \sup\{\alpha : \inf_{B_\varepsilon(t_0)} \mathrm{ess} \left|\frac{\varphi(t)}{(t-t_0)^\alpha}\right| = 0\}. \tag{4.12}$$

exists and is positive for a point $t_0 \in (0,1)$, then we call t_0 a zero *of φ with the order α_0.*

The essential infimum has to be taken over all balls $B_\varepsilon(t_0)$ with sufficiently small radius ε. Using the previous definition we can give the following lemma.

Lemma 12 *If $\varphi \in L^\infty(0,1)$ has only a finite number of zeros, then the degree of ill-posedness of the associated multiplication operator is not greater than the maximum of all the orders of the zeros of φ.*

For a proof we refer to [2]. There it can also be found a counterexample, that the converse assertion of the lemma is not true. That means, the degree of ill-posedness may be smaller than the maximal order of the zeros.

In the following we demonstrate that our degree ν really characterizes the ill-posedness, i.e. it grows with increasing ill-posedness. First we want to give a classification of the multiplication operators, namely a semi-ordering.

Definition 13 *We say, that two multiplication operator A_1 and A_2 are related with respect to the semi-ordering "$\prec$" (A_1 is "smaller" than A_2), i.e.*

$$A_1 \prec A_2, \tag{4.13}$$

if a constant $c > 0$ exists, such that

$$\|A_1 h\|_{L^2(0,1)} \leq c\|A_2 h\|_{L^2(0,1)} \quad \forall h \in L^2(0,1). \tag{4.14}$$

Note, that there may be operators A_1 and A_2 with $A_1 \neq A_2$ fulfilling $A_1 \prec A_2$ and $A_2 \prec A_1$. This rather contradicts the definition of a semi-ordering, but in this case the operators are spectral equivalent and we want to identify these operators.

It is possible to give a connection between this semi-ordering and the degree of ill-posedness.

Lemma 14 *If the multiplication operators A_1 and A_2 are ordered with respect to the semi-ordering (4.14), i.e. $A_1 \prec A_2$, then for the corresponding degrees of ill-posedness we obtain $\nu_1 \geq \nu_2$.*

Proof: Let A_i the operators of multiplication with the function φ_i, $i = 1, 2$. From $A_1 \prec A_2$ it follows

$$\|\varphi_1 h\|_{L^2(0,1)} \leq c\|\varphi_2 h\|_{L^2(0,1)} \quad \forall h \in L^2(0,1) \tag{4.15}$$

with a positive constant c. From this we get $\varphi_1(t) \leq c\varphi_2(t)$ out of a set of zero measure. Namely, if there would exist such a set Ω with $\mathrm{meas}(\Omega) > 0$ with $\varphi_1(t) > c\varphi_2(t)$ in Ω, then the function $\tilde{h}$ with

$$\tilde{h}(t) := \begin{cases} 1 \text{ in } \Omega \\ 0 \text{ elsewhere} \end{cases} \tag{4.16}$$

gives

$$\|\varphi_1 \tilde{h}\|_{L^2(0,1)} > c\|\varphi_2 \tilde{h}\|_{L^2(0,1)}, \tag{4.17}$$

which contradicts to the assumption. Now we have for the distribution function $p_1(\lambda)$

$$p_1(\lambda) = \text{meas}\{t : \varphi_1(t) \in [0, \lambda]\} \sim \lambda^{\frac{1}{\nu_1}} \tag{4.18}$$

according to the assumption. Then it follows for the distribution function $p_2(\lambda)$

$$\begin{aligned} p_2(\lambda) = \text{meas}\{t : \varphi_2(t) \in [0, \lambda]\} &\le \text{meas}\{t : c^{-1}\varphi_1(t) \in [0, \lambda]\} \\ = \text{meas}\{t : \varphi_1(t) \in [0, c\lambda]\} &\sim (c\lambda)^{\frac{1}{\nu_1}} \sim \lambda^{\frac{1}{\nu_1}}. \end{aligned} \tag{4.19}$$

Due to $p_2(\lambda) \sim \lambda^{\frac{1}{\nu_2}} \le \lambda^{\frac{1}{\nu_1}}$ it follows $\frac{1}{\nu_2} \ge \frac{1}{\nu_1}$ or $\nu_2 \le \nu_1$. This ends the proof. ∎

Now we want to give a proposition, that characterizes some equivalent statements concerned in the ill-posedness.

Proposition 15 *Let $\mathcal{M}$ be an injective multiplication operator, i.e. it holds* $\text{meas}\{\varphi^{-1}(0)\} = 0$. *Then the following conditions are equivalent:*

1. *φ has at least one zero,*
2. *$\tilde{\varphi}(0) = 0$,*
3. *$\dfrac{1}{\varphi} \in L^\infty(0, 1)$,*
4. *$R(\mathcal{M}) \neq \overline{R(\mathcal{M})}$.*

Proof: At first, we should mention, that the zeros are considered as essential zeros. That means, the existence of at least one zero is equivalent to a essential infimum $\inf \text{ess}\, \varphi(t) = 0$.
1.⇒2. With respect to the definition of the essential infimum it follows from

$$\inf_{t\in(0,1)} \text{ess}\, \varphi(t) = 0, \tag{4.20}$$

that

$$\sup\{\lambda : \text{meas}\{t \in (0, 1) : \varphi(t) \le \lambda\} = 0\} = 0. \tag{4.21}$$

Hence we have

$$\text{meas}\{t \in (0, 1) : \varphi(t) \le \lambda\} > 0 \quad \forall \lambda > 0. \tag{4.22}$$

From the definition of the distribution function p ($p(\lambda) := \text{meas}\{\varphi^{-1}[0, \lambda]\}$) it follows $p(\lambda) > 0$ for all $\lambda > 0$. Since p is monotonous increasing it follows for the increasing rearrangement $\tilde{\varphi}$

$$\tilde{\varphi}(\varepsilon) = \sup\{\lambda : p(\lambda) \le \varepsilon\} \tag{4.23}$$

$\tilde{\varphi}(\varepsilon) \to 0$ whenever $\varepsilon \to 0$. Due to the right continuity of $\tilde{\varphi}$ then we have $\tilde{\varphi}(0) = 0$.

2.$\Rightarrow$1. It can be shown that the conclusions above can be inverted. Let $\tilde{\varphi}(0) = 0$, then the right continuity of $\tilde{\varphi}$ gives the existence of a ε, such that

$$\tilde{\varphi}(t) \leq \delta \quad \forall t \in [0, \varepsilon] \tag{4.24}$$

for any $\delta > 0$. Consequently,

$$\text{meas}\{t \in (0,1) : \varphi(t) \leq \delta\} \geq \varepsilon > 0, \tag{4.25}$$

with $\varepsilon = \varepsilon(\delta)$ for sufficiently small $\delta > 0$. Then we have again

$$\sup\{\delta : \text{meas}\{t \in (0,1) : \varphi(t) \leq \delta\} = 0\} = 0. \tag{4.26}$$

Finally, it follows

$$\inf_{t \in (0,1)} \text{ess}\, \varphi(t) = 0 \tag{4.27}$$

and therefore the existence of zeros.
1.$\Longleftrightarrow$3. This equivalence we get from the identity of $\frac{1}{\inf \text{ess}\, \varphi(t)}$ and $\sup \text{ess} \frac{1}{\varphi(t)}$. Namely, it is

$$\inf_{t \in (0,1)} \text{ess}\, \varphi(t) = 0 \tag{4.28}$$

equivalent to

$$\text{meas}\{t \in (0,1) : \varphi(t) \leq \delta\} > 0 \quad \forall \delta > 0 \tag{4.29}$$

and further to

$$\text{meas}\{t \in (0,1) : \frac{1}{\varphi(t)} \geq M\} > 0 \quad \forall M < \infty. \tag{4.30}$$

This is the same as

$$\inf\{M : \text{meas}\{t \in (0,1) : \frac{1}{\varphi(t)} \geq M\} = 0\} = \infty \tag{4.31}$$

or

$$\sup_{t \in (0,1)} \text{ess} \frac{1}{\varphi(t)} = \infty, \tag{4.32}$$

hence

$$\frac{1}{\varphi} \notin L^\infty(0,1). \tag{4.33}$$

3.$\Longleftrightarrow$4. This statement is only a problem of linear operators in general. We used it already in previous considerations. We give the proof for completeness. Here $\frac{1}{\varphi} \notin L^\infty(0,1)$ means, that there exists no continuous inverse of the operator. By definition, we have

$$0 \in \sigma(\mathcal{M}). \tag{4.34}$$

Since we assumed injectivity, it follows $0 \in \sigma_c(\mathcal{M})$, hence

$$R(\mathcal{M}) \neq \overline{R(\mathcal{M})}. \tag{4.35}$$

This completes the proof. ■

Remark 16 The proposition is completely proved. Nevertheless, we will show, how lemma 5 is involved in this proposition. We will derive 4. from 1. using this lemma. Now 1. implies

$$\inf_{t\in(0,1)} \operatorname{ess} \varphi(t) = 0 \tag{4.36}$$

again. Now we suppose $0 \in G_\varphi$. Then it exists an open set $G \subset G_\varphi \subset \mathbb{R}$, such that meas$\{\varphi^{-1}(G)) = 0$. Now let g be the supremum of all real numbers in the connection component of G containing 0. That means,

$$g := \sup\{g' \in G : [0, g') \subset G\}. \tag{4.37}$$

The value g is positive, since G is open. Therefore it is

$$\inf_{t\in(0,1)} \operatorname{ess} \varphi(t) \geq g, \tag{4.38}$$

because we assumed $\varphi(t) \geq 0$. But this is a contradiction to the assumption. Consequently, $0 \in \mathcal{C}G_\varphi = R_e(\varphi)$. Applying lemma 5 gives $0 \in \sigma(\mathcal{M})$ and due to the injectivity $R(\mathcal{M}) \neq \overline{R(\mathcal{M})}$.

5 Regularization of multiplication equations

At first we consider any linear operator A in a Hilbert space. Then it is known, that the generalized solution can be expressed by the solution of the normal equation

$$A^*Ax = A^*y \tag{5.1}$$

in $\overline{R(A^*)} = N(A)^\perp$. Using Tikhonov regularization we consider instead of this equation the regularized equation

$$(A^*Ax + \alpha I) = A^*y \tag{5.2}$$

with the regularization parameter α.

Now we come back to the multiplication operators. Here we need not the assumption, that φ is real and non-negative. Then the equation (5.2) can be written as

$$(|\varphi(t)|^2 + \alpha)x(t) = \overline{\varphi(t)}y(t) \text{ a.e. } t \in (0,1). \tag{5.3}$$

The solution x_α of this can be simply derived by

$$x_\alpha(t) = \frac{\overline{\varphi(t)}}{|\varphi(t)|^2 + \alpha} y(t) \text{ a.e. in}(0,1). \tag{5.4}$$

The division is always possible and gives $x_\alpha \in L^2(0,1)$ whenever $y \in L^2(0,1)$. The regularized solutions vanishes outside of the support of φ. Therefore, it coincides there with the generalized solution $x^\dagger$. However, for any $t \in \operatorname{supp}\varphi$ we have

$$x_\alpha - x^\dagger = \left(\frac{\overline{\varphi}}{|\varphi|^2+\alpha} - \frac{1}{\varphi}\right) y = -\alpha\frac{1}{\varphi}\frac{1}{|\varphi|^2+\alpha}y = -\alpha\frac{1}{|\varphi|^2+\alpha}x^\dagger. \tag{5.5}$$

To obtain rates for the convergence we need a source condition for the generalized solution. Namely, we demand that

$$x^\dagger \in R((A^*A)^\gamma), \quad 0 \le \gamma \le 1 \tag{5.6}$$

is fulfilled. That means the existence of a constant $\gamma \in [0,1]$ and a function $w \in L^2(0,1)$, such that

$$x^\dagger = (A^*A)^\gamma w. \tag{5.7}$$

In our case we need the existence of γ and w with

$$x^\dagger = |\varphi|^{2\gamma} w. \tag{5.8}$$

Due to equation (5.5) we may estimate the regularization error by

$$\|x_\alpha - x^\dagger\|_{L^2(0,1)} = \left\|\alpha\frac{1}{|\varphi|^2+\alpha}x^\dagger\right\|_{L^2(0,1)} = \left\|\frac{\alpha|\varphi|^{2\gamma}}{|\varphi|^2+\alpha}w\right\|_{L^2(0,1)}. \tag{5.9}$$

Using a special case of Young's inequality

$$a^\varepsilon b^{1-\varepsilon} \le \varepsilon a + (1-\varepsilon)b \le a+b, \quad a,b \ge 0, \quad 0 \le \varepsilon \le 1 \tag{5.10}$$

by setting $\varepsilon := \gamma$, $a := |\varphi|^2$ and $b := \alpha$, we obtain

$$\|x_\alpha - x^\dagger\|_{L^2(0,1)} = \left\|\alpha^\gamma\frac{\alpha^{1-\gamma}|\varphi|^{2\gamma}}{|\varphi|^2+\alpha}w\right\|_{L^2(0,1)} \le \alpha^\gamma\|w\|_{L^2(0,1)}. \tag{5.11}$$

This gives the convergence rate

$$\|x_\alpha - x^\dagger\|_{L^2(0,1)} = \mathcal{O}(\alpha^\gamma). \tag{5.12}$$

This coincides with the theorem given by Neubauer for all linear bounded operators in Hilbert spaces in [6]. There we can find the statement, that for any linear bounded operator with generalized solution $x^\dagger \in R((A^*A)^\gamma)$, $0 < \gamma \le 1$, the error of the regularized solution can be expressed by

$$\|x_\alpha - x^\dagger\| = \mathcal{O}(\alpha^\gamma). \tag{5.13}$$

However, in [7] it is shown that the converse is only true for $\gamma = 1$. Nevertheless, there is given a condition, which is sufficient and necessary for the convergence rate α^γ. The condition is

$$\|x_\alpha - x^\dagger\| = \mathcal{O}(\alpha^\gamma) \iff \int_0^\mu d\|E_\lambda x\|^2 = \mathcal{O}(\mu^{2\nu}). \tag{5.14}$$

Here A is any linear bounded operator and E_λ the spectral family of A^*A.

Now we want to give a connection between the degree of ill-posedness and the convergence rate of the Tikhonov regularized solutions. For this we cite a proposition in [2].

Proposition 17 *If $\nu = \nu(\mathcal{M}) \geq \frac{1}{4}$ is the degree of ill-posedness of a multiplication operator $\mathcal{M}$, then the Tikhonov regularized solutions x_α converge to the exact solution $x^\dagger \in L^\infty(0,1)$, where $x^\dagger \geq 0$ a.e., at least with the order $\frac{1}{4\nu(\mathcal{M})}$.*

The proof is an immediate result of the convergence rate estimation (5.14), we refer to [2].

Remark 18 We only considered the injective case, in the non-injective case the regularized solution as well as the generalized solution vanish in the complement of $\operatorname{supp}\varphi$ and we may restrict the problem to $\operatorname{supp}\varphi$.
For $\nu < \frac{1}{4}$ we have saturation, that means we have only the convergence rate as for $\nu = \frac{1}{4}$, namely $\mathcal{O}(\alpha^1)$, not $\mathcal{O}(\alpha^{\frac{1}{4\nu}})$.

6 Compositions with multiplication operators

As we saw for example in [2], the multiplication operators often occur in composition with other operators. This may be other linear operators, e.g. compact operators as embedding operators or nonlinear operators, too. A typical case are the so-called decomposition cases of nonlinear operators given in [4]. There the composition $\mathcal{M}\mathcal{E}$ with a linear embedding operator $\mathcal{E}$ is considered. We want to investigate its ill-posedness properties.

We have the embedding operator $\mathcal{E} : H^1(0,1) \to L^2(0,1)$ and the multiplication operator $\mathcal{M} : L^2(0,1) \to L^2(0,1)$. Then the composition $\mathcal{M}\mathcal{E}$ is a compact operator, since $\mathcal{E}$ is compact and $\mathcal{M}$ is bounded. For such an operator the degree of ill-posedness is defined by the decay rate of the singular values of the operator. For this we need the adjoint operator $\mathcal{E}^*$ of the embedding operator $\mathcal{E}$. We define a operator B on a dense set in $H^1(0,1)$ and show the identity $\mathcal{E}^* = B^{-1}$. Let

$$B : D(B) \subset H^1(0,1) \to L^2(0,1) \tag{6.1}$$

with

$$D(B) = \{u \in H^2(0,1) : u'(0) = u'(1) = 0\}. \tag{6.2}$$

Indeed, it holds for every $x \in H^1(0,1)$ and $y \in L^2(0,1)$

$$\begin{aligned} (x, \mathcal{E}^* y)_{H^1(0,1)} &= (\mathcal{E}x, y)_{L^2(0,1)} = (x,y)_{L^2(0,1)} \\ &= (x, BB^{-1}y)_{L^2(0,1)} = (x, B^{-1}y - (B^{-1}y)'')_{L^2(0,1)} \\ &= (x, B^{-1}y)_{L^2(0,1)} + (x', (B^{-1}y)')_{L^2(0,1)} = (x, B^{-1}y)_{H^1(0,1)}. \end{aligned} \tag{6.3}$$

This gives the assertion above.

Remark 19 We can also consider the operator B as a mapping from $L^2(0,1)$ to $L^2(0,1)$ or from $H^1(0,1)$ to $H^1(0,1)$. We still have to take into account the appropriate embeddings. In the first case we obtain $B^{-1} = \mathcal{E}\mathcal{E}^*$, in the second case $B^{-1} = \mathcal{E}^*\mathcal{E}$.

To compute the singular values of $\mathcal{M}\mathcal{E} : H^1(0,1) \to L^2(0,1)$, we need the operator $(\mathcal{M}\mathcal{E})^*\mathcal{M}\mathcal{E}$. For it holds

$$(\mathcal{M}\mathcal{E})^*\mathcal{M}\mathcal{E} : H^1(0,1) \to H^1(0,1). \tag{6.4}$$

If we assume self-adjointness of $\mathcal{M}$, we obtain

$$(\mathcal{M}\mathcal{E})^*\mathcal{M}\mathcal{E} = \mathcal{E}^*\mathcal{M}^2\mathcal{E}. \tag{6.5}$$

The ill-posedness properties of the problem are not changed, if we change the spaces and additionally fits the embedding operators. So we can the operator (6.5) consider as a mapping from $L^2(0,1)$ into $H^1(0,1)$. Then of course the inner embedding from $H^1(0,1)$ to $L^2(0,1)$ has to be omitted. Therefore, we have

$$\mathcal{E}^*\mathcal{M}^2\mathcal{E} : H^1(0,1) \to H^1(0,1) \cong \mathcal{E}^*\mathcal{M}^2 : L^2(0,1) \to H^1(0,1). \tag{6.6}$$

Both operators are actually the same. Analogously the space $H^1(0,1)$ can be embedded into $L^2(0,1)$ again. Then

$$\mathcal{E}^*\mathcal{M}^2\mathcal{E} : H^1(0,1) \to H^1(0,1) \cong \mathcal{E}\mathcal{E}^*\mathcal{M}^2 : L^2(0,1) \to L^2(0,1). \tag{6.7}$$

With respect to $B^{-1} = \mathcal{E}\mathcal{E}^* : L^2(0,1) \to L^2(0,1)$ we may consider now the operator

$$B^{-1}\mathcal{M}_{L^2(0,1)} \to L^2(0,1). \tag{6.8}$$

Summarized, the ill-posedness properties of the operators

$$\mathcal{E}^*\mathcal{M}^2\mathcal{E} : H^1(0,1) \to H^1(0,1) \tag{6.9}$$

and

$$B^{-1}_{[L^2(0,1)\to L^2(0,1)]}\mathcal{M}^2 : L^2(0,1) \to L^2(0,1) \tag{6.10}$$

coincide, since both operators are identical.

In the following, we want to illustrate our consideration by treating some examples.

Example 20 *At first we consider the well-posed case $\varphi \equiv 1$, i.e. $\mathcal{M} = I$. Then the ill-posedness only depends on the embedding operator $\mathcal{E}$. During the computation of the eigenvalues λ we obtain for any $x \in D(B)$*

$$\begin{aligned} B^{-1}x &= \lambda x \\ x &= \lambda B x \\ x &= \lambda(-x'' + x),\ x'(0) = x'(1) = 0 \\ \lambda x'' &= (\lambda - 1)x \\ x'' &= (1-\mu)x, \end{aligned} \tag{6.11}$$

where we set $\mu := \frac{1}{\lambda}$*. The general solution of this differential equation is*

$$x(t) = c_1 \sin(\sqrt{\mu - 1}t) + c_2 \cos(\sqrt{\mu - 1}t) \tag{6.12}$$

with certain constants c_1 *and* c_2*, which has to be adapted to the boundary conditions. Obviously we have* $c_1 = 0$ *and from*

$$x(t) = c_2 \cos(\sqrt{\mu - 1}t),\ x'(1) = 0, \tag{6.13}$$

it follows

$$x'(1) = -c_2\sqrt{\mu - 1}\sin(\sqrt{\mu - 1}t)\Big|_{t=0} = 0 \tag{6.14}$$

or

$$\sqrt{\mu - 1}t = j\pi,\ j = 1, 2, \ldots \tag{6.15}$$

This includes

$$\mu_j = 1 + j^2\pi^2 \tag{6.16}$$

or

$$\lambda_j = \frac{1}{1 + j^2\pi^2} \sim \frac{1}{j^2}, \tag{6.17}$$

respectively. Therefore, for the singular values of $\mathcal{E}$ *we have a behaviour of*

$$\sigma_j(\mathcal{E}) = \sqrt{\lambda_j(\mathcal{E})} \sim \frac{1}{j}, \tag{6.18}$$

That means a degree of ill-posedness 1*. This result is well-known, we find it for example in [3].*

We want to consider another example, which is connected with increasing rearrangements. We saw, that in many cases the rearranged function can be estimated from below and from above by a potential function of the same power, namely the degree of ill-posedness. So it might be useful to consider multiplication operators with potential functions.

Example 21 *We have the function* $\varphi(t) = t^\alpha$ *with the real constant* $\alpha > 0$*. Let* $\mathcal{M}$ *the corresponding multiplication operator, i.e.*

$$\mathcal{M}x := t^\alpha x. \tag{6.19}$$

Analogously as in the previous example we have to solve

$$t^\alpha x = \lambda Bx, \tag{6.20}$$

that means

$$x'' + (\mu t^{2\alpha} - 1)x = 0 \quad \textit{w.r.t. boundary conditions.} \tag{6.21}$$

The -1 *can be omitted for sufficiently large* μ*. Then we have the equation*

$$x'' + \mu t^{2\alpha} x = 0 \quad \textit{w.r.t. boundary conditions.} \tag{6.22}$$

Its solution is given as

$$x(t) = c_1\sqrt{t}\,J_{\frac{1}{2(1+\alpha)}}\left(\tfrac{\sqrt{\mu}}{1+\alpha}t^{1+\alpha}\right) + c_2\sqrt{t}\,J_{-\frac{1}{2(1+\alpha)}}\left(\tfrac{\sqrt{\mu}}{1+\alpha}t^{1+\alpha}\right). \tag{6.23}$$

Here J_ν is Bessel's function of the order ν and c_1, c_2 are constants. According to the boundary condition $x'(0) = 0$ and the fact that the derivative of $\sqrt{t}\,J_{-\frac{1}{2(1+\alpha)}}\left(\tfrac{\sqrt{\mu}}{1+\alpha}t^{1+\alpha}\right)$ vanishes for $t = 0$ we find $c_1 = 0$. Due to the second boundary condition $x'(1) = 0$ we obtain

$$J_{-\frac{1}{2(1+\alpha)}}\left(\tfrac{\sqrt{\mu}}{1+\alpha}\right) = 0. \tag{6.24}$$

However, the function $J_{-\frac{1}{2(1+\alpha)}}(t)$ behaves for large t asymptotically as

$$J_{-\frac{1}{2(1+\alpha)}}(t) \sim \sqrt{\frac{2}{\pi t}}\cos\left(t - \frac{\pi\alpha}{4(1+\alpha)}\right), \tag{6.25}$$

therefore its zeros t_j are distributed by $t_j \sim -\frac{\pi(2+\alpha)}{4(1+\alpha)} + j\pi$, $j = 1, 2, \ldots$. Therefore, here is

$$\sqrt{\mu_j} \sim \sqrt{2}\left(-\frac{\pi(2+\alpha)}{4(1+\alpha)}\right) \sim j \tag{6.26}$$

and finally

$$\mu_j \sim j^2 \tag{6.27}$$

as well as

$$\lambda_j \sim \frac{1}{j^2}. \tag{6.28}$$

This means for the sequence of the singular values

$$\sigma(\mathcal{M}\mathcal{E}) \sim \frac{1}{j}, \tag{6.29}$$

which includes a degree of il-posedness of 1 again. The degree of ill-posedness did not change, although we had to solve an additional ill-posed problem (the multiplication equation). This shows up to a certain degree the fact, that the multiplication operators (as well as other non-compact operators) are less ill-posed than compact operators. At least the degree of ill-posedness, defined by the decay rate of the singular values is not fine enough to show an alteration.

Now we have to prove this statement for a general class of functions φ, not only potential functions. At first, we need

Assumption 22

$$\varphi \in C[0,1],\ \varphi \text{ has only a finite number of zeros.} \tag{6.30}$$

Then we have the following

Proposition 23 *Let φ a function fulfilling assumption 22 and $\mathcal{M}$ the appropriate multiplication operator as well as $\mathcal{E}$ the embedding operator from $H^1(0,1)$ into $L^2(0,1)$. Then the operator equation with the operator composition $\mathcal{ME}$ is ill-posed with a degree of ill-posedness 1.*

Proof: We have to determine the eigenvalues of a Sturm-Liouville boundary-eigenvalue problem. This we obtain from the following equations:

$$
\begin{aligned}
& B^{-1}\mathcal{M}^2 x = \lambda x \\
\Longleftrightarrow \quad & \varphi^2 x = \lambda B x \\
\Longleftrightarrow \quad & \varphi^2 x = \lambda(-x'' + x), \quad x'(0) = x'(1), \\
\Longleftrightarrow \quad & \lambda x'' = (\lambda - \varphi^2)x, \quad x'(0) = x'(1), \\
\Longleftrightarrow \quad & x'' = (1 - \mu\varphi^2)x, \quad x'(0) = x'(1),
\end{aligned}
\tag{6.31}
$$

if we set $\mu := \frac{1}{\lambda}$ again. In the next section we will show, that also in this general case $\sqrt{\mu_j} \sim j$ is true, that means for the singular values

$$\sigma_j(\mathcal{ME}) \sim \frac{1}{j}. \tag{6.32}$$

This includes a degree of ill-posedness of 1. Therefore we can generalize the assertion, that the degree of ill-posedness does not change by composition of the embedding operator with a multiplication operator. ∎

7 The eigenvalue problem

In this section we want to deal with the solution of the boundary-eigenvalue problem needed in the proof of proposition 23. In our case there is to solve the differential equation

$$u''(t) + (\lambda\varphi^2(t) - 1)u(t) = 0, \quad t \in [0,1] \tag{7.1}$$

with the boundary conditions

$$u'(0) = u'(1) = 0. \tag{7.2}$$

Here λ is a (real) parameter such that for certain values (the eigenvalues) the differential equation has a non-trivial solution. To generalize the problem we consider the following self-adjoint Sturm-Liouville problem

$$(p(t)u'(t))' + (q(t) + \lambda r(t))u(t) = 0, \quad t \in [0,1] \tag{7.3}$$

with the boundary conditions

$$R_1 u = R_2 u = 0, \tag{7.4}$$

where the boundary operators R_1 and R_2 are defined by

$$R_1 u := \alpha_1 u(0) + \alpha_2 p(0) u'(0) \tag{7.5}$$

and

$$R_2 u := \beta_1 u(1) + \beta_2 p(1) u'(1), \tag{7.6}$$

respectively. Then we are searching a solution $u \in C^2[0,1]$.

At first, we assume the following

Assumption 24 *We assume $p \in C^1[0,1]$ and $q, r \in C[0,1]$. Furthermore, we demand $p(t) > 0$ in $[0,1]$ and $r(t) \geq 0$ in $[0,1]$. Additionally we suppose that r vanishes at most on a set of a finite number of isolated points.*

Remark 25 In the case $r(t) > 0$ in $[0,1]$, the whole theory was already done in [10]. However, in [1] it is mentioned, that the proof is also true for functions r with isolated zeros, but the proof is not given. This we will do here. We follow the books mentioned above.

At first we want to scale the boundary conditions to obtain a canonical form. This is always possible, since a multiplication with a constant does not change the boundary conditions. Hence we have

$$u(0) \cos \alpha - p(0) u'(0) \sin \alpha = 0 \tag{7.7}$$

and

$$u(1) \cos \beta - p(1) u'(1) \sin \beta = 0. \tag{7.8}$$

We may assume without loss of generality: $0 \leq \alpha < \pi$ and $0 < \beta \leq \pi$. Now we want to apply the Prüfer transformation. Since the differential equation of second order is equivalent to the system

$$\begin{aligned} y_1 &= u, \\ y_2 &= p u', \end{aligned} \tag{7.9}$$

we introduce the following new coordinates:

$$\xi(t) := p(t) u'(t) \text{ and } \eta(t) := u(t). \tag{7.10}$$

These we can express in polar coordinates,

$$\xi(t) := \varrho(t) \cos \varphi(t) \text{ and } \eta(t) := \varrho(t) \sin \varphi(t). \tag{7.11}$$

Except the non-eigenfunction case $u \equiv 0$ the solution curve does not contain the origin, since due to $p(t) > 0$ the conditions $u(t_0) = u'(t_0) = 0$ for any $t_0 \in [0,1]$ imply with respect to the well-known uniqueness theorems for solutions of ordinary differential equations, that $u(t) = 0$ everywhere in $[0, 1]$. Now we find $\varrho(t) > 0$. Furthermore, $\varrho, \varphi \in C^1[0, 1]$ with

$$\varrho(t) = \sqrt{\xi^2(t) + \eta^2(t)} \tag{7.12}$$

and

$$\varphi(t) = \arctan \frac{\eta(t)}{\xi(t)}. \tag{7.13}$$

The phase of arctan has to be chosen in such a way, that φ is a continuous function of t. By differentiation we obtain

$$\xi' = \varrho' \cos \varphi - \varrho \varphi' \sin \varphi \tag{7.14}$$

and

$$\eta' = \varrho' \sin \varphi + \varrho \varphi' \cos \varphi \tag{7.15}$$

and from these equations

$$\eta' \cos \varphi - \xi' \sin \varphi = \varrho \varphi' \tag{7.16}$$

and

$$\eta' \sin \varphi + \xi' \cos \varphi = \varrho'. \tag{7.17}$$

Now we may insert the following expressions

$$\xi' = (pu')' = -(q + \lambda r) \varrho \sin \varphi \tag{7.18}$$

and

$$\eta' = \frac{\xi}{p} = \frac{\varrho}{p} \cos \varphi \tag{7.19}$$

to get the identities

$$\varphi' = \frac{1}{p} \cos^2 \varphi + (q + \lambda r) \sin^2 \varphi \tag{7.20}$$

and

$$\varrho' = \left(\frac{1}{p} - (q + \lambda r) \right) \varrho \cos \varphi \sin \varphi. \tag{7.21}$$

We found a ordinary differential equation of first order for φ and with the solution of this equation a ordinary differential equation of first order for ϱ.

Due to the boundary condition (7.7) it makes sense to search a solution $u(t, \lambda)$ of the differential equation (7.3) with the *initial conditions*

$$u(0) = \sin \alpha \text{ and } p(0)u'(0) = \cos \alpha. \tag{7.22}$$

Thus the first boundary condition is fulfilled automatically. Now we have to find all values λ fulfilling also the second boundary condition. Obviously, (7.22) is equivalent to

$$\varphi(0, \lambda) = \alpha, \tag{7.23}$$

if we assume $\alpha \in [0, \pi)$ again. Therefore we found an initial condition for the differential equation (7.20).

From (7.20) we get the following: If for any pair (t_0, λ_0) holds $\varphi(t_0, \lambda_0) = k\pi$, k a whole number, then due to $p > 0$ we have $\varphi'(t_0, \lambda_0) > 0$. That means, that the curve $y = \varphi(t, \lambda_0)$ intersects the straight line $y = k\pi$ at most once, and that from below to above. From this it follows due to $\alpha \geq 0$ also $\varphi(t, \lambda) > 0$ for $t > 0$ and all real λ.

Now we need a comparison theorem, which we will cite from [1]. For the proof we refer to the same book.

Theorem 26 *Let two differential equations given by*

$$(p_i u_i')' + g_i u_i = 0, \quad i = 1, 2. \tag{7.24}$$

Furthermore, let

$$0 < p_2(t) \leq p_1(t) \text{ and } g_2(t) \geq g_1(t). \tag{7.25}$$

Now let φ_1 and φ_2 the appropriate functions obtained by Prüfer transformation. Let

$$\varphi_2(0) \geq \varphi_1(0). \tag{7.26}$$

Then it holds

$$\varphi_2(t) \geq \varphi_1(t) \text{ for } t \in [0, 1]. \tag{7.27}$$

Moreover, if $g_2(t) > g_1(t)$, then follows

$$\varphi_2(t) > \varphi_1(t) \text{ for } t \in [0, 1]. \tag{7.28}$$

If we set in the theorem $p_1 = p_2 = p$ and $g_1 = q + \lambda_1 r$ as well as $g_2 = q + \lambda_2 r$ with $\lambda_1 < \lambda_2$, so it follows $\varphi(t, \lambda_1) < \varphi(t, \lambda_2)$. That means, $\varphi(t, \lambda)$ is strictly monotonously increasing for $t \in [0, 1]$.

Remark 27 In the zeros of r it holds $g_1 = g_2$. However, in the proof of the theorem we see, that only integrals of these functions are relevant. So a finite set of zeros plays no role.

Now we show the following lemma according to [1]:

Lemma 28 *Let c any value in $(0, 1]$. Then it holds*

$$\varphi(c, \lambda) \to \infty \text{ if } \lambda \to \infty \tag{7.29}$$

and

$$\varphi(c, \lambda) \to 0 \text{ if } \lambda \to -\infty. \tag{7.30}$$

Proof: At first we consider (7.29). We have to show for any $t_0 \in (0, c)$, that $\varphi(c, \lambda) - \varphi(t_0, \lambda)$ tends to ∞ whenever $\lambda \to \infty$. Now let c chosen such that $r(c) > 0$ is valid, we demand $r(c) \geq 2R$ with a constant $R > 0$. Due to continuity it exists a environment around c with $r(t) \geq R$. Let t_0 in this environment with $t_0 < c$. Further, we have $p(t) \leq P$ and $|q(t)| \leq Q$ in (t_0, c) with certain positive constants P and Q. We consider the following comparison differential equation

$$P\tilde{u}'' + (\lambda R + Q)\tilde{u} = 0 \tag{7.31}$$

with the initial conditions

$$\tilde{u}(t_0, \lambda) = u(t_0, \lambda) \text{ and } P\tilde{u}'(t_0, \lambda) = p(t_0)u'(t_0, \lambda). \tag{7.32}$$

Here u is again the solution of our differential equation (7.3) with the initial conditions (7.22). For the comparison differential equation we may apply the Prüfer transformation. There we obtain $\tilde{\varphi}$. Due to

$$\varphi = \arctan\left(\frac{u}{pu'}\right) \text{ and } \tilde{\varphi} = \arctan\left(\frac{\tilde{u}}{P\tilde{u}'}\right) \tag{7.33}$$

we see $\varphi(t_0, \lambda) = \tilde{\varphi}(t_0, \lambda)$. Because of theorem 26 it follows

$$\varphi(c, \lambda) - \varphi(t_0, \lambda) \geq \tilde{\varphi}(c, \lambda) - \tilde{\varphi}(t_0, \lambda). \tag{7.34}$$

Now it can simply be shown, that the zeros of $\tilde{u}$ have the distance $\pi\sqrt{\frac{P}{(\lambda R+Q)}}$. Therefore, the zeros go to the left for increasing λ and their distance tends to zero. Therefore we have $\tilde{\varphi} \equiv 0 \pmod{\pi}$ in infinite many points and due to $\tilde{\varphi} > 0$ we have $\tilde{\varphi} \to \infty$. Hence, the right-hand side of (7.34) tends to infinity and therefore the left-hand side, too. We will express the last sentences as some formulae. Since the zeros of $\tilde{u}$ are equidistant, we have $c = \frac{\gamma k}{\sqrt{\lambda}} + \delta$ with certain constants γ and δ dependent on P, Q and R, but independent of k and λ, if c is the k-th zero of $\tilde{\varphi} \pmod{\pi}$. That means, we have

$$\tilde{\varphi}\left(\frac{\gamma k}{\sqrt{\lambda}} + \delta, \lambda\right) = k\pi. \tag{7.35}$$

From this it follows immediately

$$\tilde{\varphi}(c, \lambda) = \sqrt{\lambda}\frac{c - \delta}{\gamma}\pi \tag{7.36}$$

and finally

$$\varphi(c, \lambda) > \bar{c}\sqrt{\lambda} \tag{7.37}$$

with a constant $\bar{c}$. This includes $\varphi(c, \lambda) \to \infty$ whenever $\lambda \to \infty$.
For the values of c with $r(c) = 0$ we obtain nevertheless the statement, since $\varphi(t, \lambda)$ is continuous both with respect to t and to λ and R can be chosen arbitrarily small. This proves the first part of the lemma.

Now we have to show (7.30). We again assume for c that $r(c) > 0$ and define the constants P, Q and R, if t is situated in a environment U of c. Additionally, we choose a $\delta > 0$ sufficiently small such that $\alpha < \pi - \delta$. Assumed, we had $\delta \leq \varphi(t, \lambda) \leq \pi - \delta$ for $t \in U$ with $t < c$ and $\lambda \to -\infty$. Then we have due to (7.20)

$$\varphi'(t, \lambda) < \frac{1}{P} + Q - |\lambda| R \sin^2 \delta. \tag{7.38}$$

Now it holds $\varphi'(t, \lambda) \to -\infty$ if $\lambda \to -\infty$. Therefore we have $\varphi(c, \lambda) \leq \delta$ for $-\lambda$ sufficiently large. Since δ can be chosen arbitrarily, the assertion follows. With respect to the continuity the statement is also true for c with $r(c) = 0$. ■

With this lemma we are able to characterize the properties of the Sturm-Liouville problem. If we set in the previous lemma $c = 1$, we have $\varphi(1, \lambda) \to 0$ for $\lambda \to -\infty$. Since $\beta > 0$ and $\varphi(1, \lambda)$ is monotonous increasing with respect to λ, it must exist a value λ_0 with $\varphi(1, \lambda_0) = \beta$. Due to $0 \leq \alpha < \pi$ and $\beta \leq \pi$ it follows $0 < \varphi(t, \lambda_0) < \pi$ in the open interval $(0, 1)$. Therefore, $u(t, \lambda_0)$ fulfills the second boundary condition and has no zeros in $(0, 1)$. Due to $\varphi(1, \lambda) \to \infty$ for $\lambda \to \infty$ it must exist an unique λ_1 with $\varphi(1, \lambda_1) = \beta + \pi$. In this case, all boundary conditions for u are also fulfilled. Now we are able to find an infinite number of eigenvalues λ_n with

$$\varphi(1, \lambda_n) = \beta + n\pi. \tag{7.39}$$

Note, that all these accomplishments are according to [1].

During the proof of the previous lemma we found, that there exists a positive constant $\bar{c}$ with

$$\varphi(1, \lambda) > \bar{c}\sqrt{\lambda}, \tag{7.40}$$

if λ is sufficiently large. Now we show the existance of a constant $\bar{C}$ such that

$$\varphi(1, \lambda) < \bar{C}\sqrt{\lambda}. \tag{7.41}$$

We know from (7.20) that

$$\varphi' = \frac{1}{p} \cos^2 \varphi + (q + \lambda r) \sin^2 \varphi \tag{7.42}$$

is valid. Due to the positivity of p and r and the boundedness of p, q and r there exist positive constants A and B such that

$$\varphi' \leq A + \lambda B \sin^2 \varphi \tag{7.43}$$

holds. Now we to proceed analogously to [10]. For this we integrate the estimation

$$\frac{\varphi'}{A + \lambda B \sin^2 \varphi} \leq 1 \tag{7.44}$$

from 0 to 1 and obtain (if we substitute $\varphi(t) =: s$)

$$\int_{\alpha}^{\varphi(1)} \frac{ds}{A + \lambda B \sin^2 s} \leq 1. \tag{7.45}$$

Now let $k\pi \leq \varphi(1) < (k+1)\pi$ for a fixed value $k \in \mathbb{N}$. If we take the integral only from π to $k\pi$, so it follows

$$\begin{aligned} 1 &\geq (k-1) \int_0^{\pi} \frac{ds}{A+\lambda B \sin^2 s} \geq (k-1) \int_0^{\pi} \frac{ds}{A+\lambda B s^2} \\ &= \frac{k-1}{\sqrt{\lambda}} \int_0^{\sqrt{\lambda}\pi} \frac{d\tau}{A+B\tau^2} \geq \frac{\tilde{C}(k-1)}{\sqrt{\lambda}}, \end{aligned} \tag{7.46}$$

where $\tau := \sqrt{\lambda} s$ and $\tilde{C} > 0$. This includes the assertion. For further details we refer to [10].

Now we have the following estimation for the function φ:

$$\bar{c}\sqrt{\lambda} < \varphi(1, \lambda) < \bar{C}\sqrt{\lambda}. \tag{7.47}$$

From

$$\varphi(1, \lambda_n) = \beta + n\pi \tag{7.48}$$

it follows

$$\bar{c}\sqrt{\lambda_n} < \beta + n\pi < \bar{C}\sqrt{\lambda_n}. \tag{7.49}$$

This immediately yields the existence of two constants d and D with

$$dn^2 \leq \lambda_n \leq Dn^2. \tag{7.50}$$

This is true for all Sturm-Liouville problems of the type (7.3). If we set $p \equiv 1$, $q \equiv -1$ and $r = \varphi^2$ as well as $\alpha_1 = \beta_1 = 0$ and $\alpha_2 = \beta_2 = 1$ in (7.5) and (7.6), then due to assumption 22 the assumption 24 is fulfilled. Therefore (7.50) can be found.

References

[1] Coddington, E.A., Levinson, N., Theory of Ordinary Differential Equations. McGraw-Hill Book Co., Inc., New York, Toronto, London, 1955.

[2] Fleischer, G., Hofmann, B., Ill-posedness aspects of some nonlinear inverse problems. In: Troch, I., Breitenecker, F. (Eds.), Proceedings, IMACS Symposium on mathematical modelling, Vienna, Austria. ARGESIM Report No. 11, Vienna, 1997.

[3] Hofmann, B., Regularization for applied inverse and ill-posed problems. B. G. Teubner, Leipzig, 1986.

[4] Hofmann, B., On the degree of ill-posedness for nonlinear problems. Journal of Inverse and Ill-Posed Problems, 2 (1994), 61 – 76.

[5] Hofmann, B., Tautenhahn, U., On ill-posedness measures and space change in Sobolev spaces. Preprint IP 7, TU Chemnitz-Zwickau, Faculty of Mathematics, Chemnitz, 1996.

[6] Neubauer, A., Tikhonov-Regularization of Ill-Posed Linear Operator Equations on Closed Convex Sets. PhD-Thesis, University of Linz, 1985.

[7] Neubauer, A., On Converse and Saturation Results for Tikhonov Regularization of Linear Ill-Posed Problems. SIAM J. Numer. Anal., 34 (1997), 517 – 727.

[8] Rudin, W., Functional Analysis – Second Edition. International series in pure and applied mathematics. McGraw-Hill Book Co., Singapore, 1991.

[9] Smirnow, W.I., Lehrgang der höheren Mathematik, Teil V. Verlag der Wissenschaften, Berlin, 10. Auflage, 1988.

[10] Walter, W., Gewöhnliche Differentialgleichungen. Springer, Berlin, Heidelberg, New York, 5. Auflage, 1993.