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Local theory of projection methods for Cauchy singular integral equations on an interval

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Abstract

We consider a finite section (Galerkin) and a collocation method for Cauchy singular integral equations on the interval based on weighted Chebyshev polynomials, where the coefficients of the operator are piecewise continuous. Stability conditions are derived using Banach algebra techniques, where also the system case is mentioned. With the help of appropriate Sobolev spaces a result on convergence rates is proved. Computational aspects are discussed in order to develop an effective algorithm. Numerical results, also for a class of nonlinear singular integral equations, are presented.

1 Introduction

The subject of the present paper is the investigation of a collocation method based on weighted polynomials for the approximate solution of singular integral equations on $(-1,1)$ of the type

$$a(x)u(x) + \frac{b(x)}{\pi i}\int_{-1}^{1}\frac{u(t)}{t-x}\,dt = f(x), \qquad x \in (-1,1), \tag{1.1}$$

where u is the unknown function and a, b, f are given. All functions involved are assumed to be complex-valued.

A lot of attention has been paid to investigating polynomial collocation and quadrature methods for this and similar types of equations (see, for example, Prössdorf & Silbermann [14, Chapter 9] and the literature cited there). These are essentially based on special mapping properties of the operator $A\omega I$, where A denotes the operator on the left-hand side of (1.1), and ω is a generalized Jacobi weight depending on the coefficients a and b, which are required to satisfy a Hölder condition.

We are going to give a somewhat different approach using weighted polynomials as ansatz functions. The original equation as well as stability and convergence of a finite section and a collocation method are considered in the weighted $\mathbf{L}^2$-space $\mathbf{L}^2_\sigma(-1,1)$ with respect to the Chebyshev weight of first kind $\sigma(x) = (1-x^2)^{-\frac{1}{2}}$. For the non-weighted space $\mathbf{L}^2(-1,1)$, a finite section (Galerkin) method with the same type of ansatz functions was investigated in Junghanns, Roch & Weber [10]. In Junghanns & Weber [12] a collocation method is considered in $\mathbf{L}^2_\sigma(-1,1)$, where $\sigma(x)$ belongs to a class of Jacobi weights, but the coefficient b is restricted by $b(\pm 1) = 0$.

There are two essential advantages of, in particular, our collocation method in comparison with the above mentioned polynomial approximation methods using generalized Jacobi weights.

1. Also in the case of variable coefficients a and b we will always use the same (Chebyshev) collocation points independently of the coefficients. This is very useful if a Newton method for a nonlinear singular integral equation results in a sequence of linear equations of type (1.1), the coefficients of which are different in each step. Furthermore, we can admit arbitrary

piecewise continuous coefficients. Such coefficients with jumps occur, for example, when considering seepage problems for channels or dams with corners (see Junghanns [9]).

2. The finite section and the collocation method can be easily generalized to the case of a system of Cauchy singular integral equations.

As a disadvantage of the methods introduced in this paper we will observe that, in general, the unique solvability of the original equation (1.1) is not sufficient for the stability.

We will give necessary and sufficient conditions for the stability of the proposed approximation methods using Banach algebra techniques, which have proved to be an efficient tool in stability analysis (see, for example, Hagen, Roch & Silbermann [6], Junghanns, Roch & Weber [10], Junghanns & Silbermann [11], Prössdorf & Silbermann [14], Silbermann [16]). We obtain necessary and sufficient stability conditions in the case of piecewise continuous coefficients.

In Section 2 we give some notations and define the numerical methods we are going to deal with. Section 3 provides some basic facts on Banach algebra techniques we will need for the stability analysis. In Section 4 we assign to the sequences of approximating operators on the interval a sequence of operators on the unit circle, which enables us to use a lot of known results on finite section and collocation methods for Cauchy singular integral operators on the unit circle. In Sections 5 and 6 we derive the main results concerning the stability of the finite section and the collocation method, respectively. In Sections 7 we make some remarks on the generalization to the system case. Finally, Section 8 is devoted to the proof of some convergence rates of the methods using appropriate Sobolev spaces, and in Section 9 we describe the computer implementation and present some numerical results for the collocation method, where we also consider nonlinear Cauchy singular integral equations.

2 Notations and preliminaries

Let $\sigma(x) = (1-x^2)^{-\frac{1}{2}}$ denote the Chebyshev weight of the first kind. We consider equation (1.1) in the weighted Lebesgue space $\mathbf{L}^2_\sigma := \mathbf{L}^2_\sigma(-1,1)$ of all (classes of) measurable functions $u : (-1,1) \to \mathbb{C}$ for which

$$\|u\|^2_\sigma := \int_{-1}^{1} |u(x)|^2 \sigma(x)\, dx$$

is finite, equipped with the inner product

$$\langle u, v\rangle_\sigma := \int_{-1}^{1} u(x)\overline{v(x)}\sigma(x)\, dx,$$

which turns $\mathbf{L}^2_\sigma$ into a Hilbert space. The Cauchy singular integral operator $S : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$ defined by

$$(Su)(x) = \frac{1}{\pi i}\int_{-1}^{1} \frac{u(t)}{t-x}\, dt \tag{2.1}$$

is bounded, i.e. $S \in \mathcal{L}(\mathbf{L}^2_\sigma)$ (Gohberg & Krupnik [5], Theorem I.4.1). The coefficients a, b are assumed to belong to the algebra $\mathbf{PC}[-1,1]$ of all piecewise continuous functions. The latter is defined as the closure (in the space of all bounded functions, equipped with the supremum norm) of the set of those functions being continuous on $[-1,1]$ with the possible exception of a finite number of jumps in $(-1,1)$, where the value of the function coincides with the left-sided limit. For definiteness we agree that $a(-1+0) = a(-1)$ for $a \in \mathbf{PC}[-1,1]$. Note that $\mathbf{PC}$-functions possess finite one-sided limits at all points. Under these assumptions, the operator on the left-hand side of (1.1), which in the following will be briefly referred to as $A := aI + bS$, is bounded on $\mathbf{L}^2_\sigma$.

Let $\varphi(x) = \sqrt{1-x^2}$ denote the Chebyshev weight of second kind and let U_n be the orthonormal polynomial of degree n (with positive leading coefficient) with respect to the inner product $\langle .,. \rangle_\varphi$, that is $\langle U_n, U_m \rangle_\varphi = \delta_{mn}$, where δ_{mn} is the Kronecker symbol. Remember the trigonometric representation

$$U_n(\cos s) = \sqrt{\frac{2}{\pi}} \frac{\sin(n+1)s}{\sin s}, \qquad n = 0, 1, 2, \ldots .$$

Obviously, the multiplication operator φI is an isometric isomorphism from $\mathbf{L}^2_\varphi$ onto $\mathbf{L}^2_\sigma$. Thus, the functions

$$\widetilde{u}_n := \varphi U_n, \qquad n = 0, 1, 2, \ldots, \tag{2.2}$$

form an orthonormal basis in $\mathbf{L}^2_\sigma$, because the same is true for U_n in the space $\mathbf{L}^2_\varphi$. For the approximation method we want to apply to (1.1), the functions (2.2) will be used as ansatz functions. That means we look for an approximate solution u_n of equation (1.1) of the form

$$u_n(x) = \sum_{k=0}^{n-1} \xi_{kn} \widetilde{u}_k(x) = \varphi(x) \sum_{k=0}^{n-1} \xi_{kn} U_k(x) .$$

As a first numerical method we consider a Galerkin method, which is also referred to as the finite section method and where u_n is the solution of

$$\langle f - Au_n, \widetilde{u}_k \rangle_\sigma = 0 , \qquad k = 0, 1, \ldots, n-1 . \tag{2.3}$$

If we define the sequence $\{P_n\}_{n=1}^\infty$ of Fourier projections $P_n : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$ by

$$P_n u = \sum_{k=0}^{n-1} \langle u, \widetilde{u}_k \rangle_\sigma \widetilde{u}_k ,$$

then (2.3) is equivalent to

$$A_{n,P} u_n = P_n f , \tag{2.4}$$

where $A_{n,P} := P_n A P_n$. As a further approximation method for equation (1.1) we will investigate the collocation method

$$(Au_n)(x^\varphi_{jn}) = f(x^\varphi_{jn}) , \quad j = 1, \ldots, n , \tag{2.5}$$

where x_{jn}^φ denote the Chebyshev nodes of second kind,

$$x_{jn}^\varphi = \cos \frac{j\pi}{n+1} .$$

Define the respective Lagrange interpolation operator

$$L_n^\varphi f = \sum_{j=1}^{n} f(x_{jn}^\varphi) l_{jn}^\varphi ,$$

where

$$l_{jn}^\varphi(x) = \frac{U_n(x)}{(x - x_{jn}^\varphi) U_n'(x_{jn}^\varphi)}$$

are the respective fundamental Lagrange polynomials. Then (2.5) is equivalent to

$$A_{n,M} u_n = M_n f , \tag{2.6}$$

where $A_{n,M} := M_n A P_n$ and $M_n := \varphi L_n^\varphi \varphi^{-1} I$. In (2.6) we use the modified interpolation operators M_n instead of L_n^φ, since the image space of M_n is the same as the image space of P_n, which is important for our further theoretical considerations concerning stability and convergence of the described collocation method.

Let $\{A_n\}$, $A_n \in \mathcal{L}(\mathrm{im}\ P_n)$, be one of the sequences $\{A_{n,P}\}$ or $\{A_{n,M}\}$. The sequence $\{A_n\}$ is said to be stable, if there is an n_0 such that $A_n : \mathrm{im}\, P_n \longrightarrow \mathrm{im}\, P_n$ is invertible for all $n \ge n_0$ and $\sup \left\{ \left\| A_n^{-1} P_n \right\|_{\mathcal{L}(\mathbf{L}^2_\sigma)} : n \ge n_0 \right\} < \infty$. Let R_n denote one of the projections P_n or M_n, and let $Au^* = f$, $A_n u_n^* = R_n f$. Assume that $\{A_n\}$ is stable. Because of the estimation

$$\|P_n u^* - u_n^*\|_{\mathbf{L}^2_\sigma} \le \left\| A_n^{-1} P_n \right\|_{\mathbf{L}^2_\sigma} \|A_n P_n u^* - R_n f\|_{\mathbf{L}^2_\sigma}$$

and the strong convergence $P_n \longrightarrow I$, we have that $u_n^* \longrightarrow u^*$ in $\mathbf{L}^2_\sigma$ if $A_n P_n \longrightarrow A$ strongly and $R_n f \longrightarrow f$ in $\mathbf{L}^2_\sigma$. Thus, our main concern is the proof of the stability of the sequence $\{A_n\}$. For this end we will use a Banach algebra technique described in the following section.

3 The Banach algebra technique

Define $W_n : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$ by

$$W_n u = \sum_{k=0}^{n-1} \langle u, \widetilde{u}_{n-1-k} \rangle_\sigma \widetilde{u}_k .$$

Then $W_n = W_n^*$ converges weakly to 0. Moreover, $W_n P_n = W_n$ and $W_n^2 = P_n$. By $\mathcal{A}$ we denote the unital Banach algebra of all sequences $\{A_n\}$, $A_n : \mathrm{im}\, P_n \longrightarrow \mathrm{im}\, P_n$, for which $A_n P_n$, $A_n^* P_n$, $\widetilde{A}_n P_n$ with $\widetilde{A}_n := W_n A_n W_n$, and $\widetilde{A}_n^* P_n$ converge strongly, equipped with componentwise algebraic operations and the norm

$$\|\{A_n\}\|_{\mathcal{A}} := \sup \left\{ \|A_n P_n\|_{\mathcal{L}(\mathbf{L}^2_\sigma)} : n = 1, 2, \ldots \right\}$$

(comp., for example, Prössdorf & Silbermann [14, p. 268]). The set

$$\mathcal{I} := \left\{ \{P_n K_1 P_n + W_n K_2 W_n + C_n\} : K_i \in \mathcal{K}(\mathbf{L}^2_\sigma),\ \|C_n\|_{\mathcal{L}(\mathbf{L}^2_\sigma)} \longrightarrow 0 \right\},$$

where $\mathcal{K}(\mathbf{L}^2_\sigma)$ denotes the ideal in $\mathcal{L}(\mathbf{L}^2_\sigma)$ of all compact operators, is a two-sided closed ideal in $\mathcal{A}$ (Silbermann [16, Prop. 2], Prössdorf & Silbermann [14, Prop. 7.6]). Thus, the quotient algebra $\mathcal{A}/\mathcal{I}$ is again a Banach algebra. If $\mathcal{B}$ is a Banach algebra then by $\mathcal{GB}$ we denote the subset of all invertible elements of $\mathcal{B}$. Now, the following theorem is fundamental for our investigations.

Theorem 3.1 ([16], Prop. 3, [14], Theorem 7.7) *If the sequence $\{A_n\}$ belongs to $\mathcal{A}$, where $A_n \longrightarrow A$ and $\widetilde{A}_n \longrightarrow \widetilde{A}$ strongly, then $\{A_n\}$ is stable if and only if $A, \widetilde{A} \in \mathcal{GL}(\mathbf{L}^2_\sigma)$ and $\{A_n\} + \mathcal{I} \in \mathcal{G}(\mathcal{A}/\mathcal{I})$.*

We will investigate the invertibility of the coset $\{A_n\} + \mathcal{I}$ with the help of the following local principle of Gohberg and Krupnik. Let $\mathcal{B}$ be a unital Banach algebra. A subset $\mathcal{M} \subset \mathcal{B}$ is called a localizing class if $0 \notin \mathcal{M}$ and if for all a_1, $a_2 \in \mathcal{M}$ there exists an element $a \in \mathcal{M}$ such that

$$aa_j = a_j a = a \quad \text{for} \quad j = 1, 2.$$

In the following let $\mathcal{M}$ be a localizing class. Two elements x, $y \in \mathcal{B}$ are called $\mathcal{M}$-equivalent (in symbols: $x \overset{\mathcal{M}}{\sim} y$), if

$$\inf_{a \in \mathcal{M}} \|a(x-y)\| = \inf_{a \in \mathcal{M}} \|(x-y)a\| = 0.$$

Further, $x \in \mathcal{B}$ is called $\mathcal{M}$-invertible if there exist a_1, $a_2 \in \mathcal{M}$, z_1, $z_2 \in \mathcal{B}$ such that

$$z_1 x a_1 = a_1, \qquad a_2 x z_2 = a_2.$$

(Note that an invertible element is also $\mathcal{M}$-invertible.) A system $\{\mathcal{M}_\tau\}_{\tau \in \Omega}$ of localizing classes (Ω is an arbitrary index set) is said to be covering if for each system $\{a_\tau\}_{\tau \in \Omega}$, $a_\tau \in \mathcal{M}_\tau$, there exists a finite subsystem $a_{\tau_1}, \ldots, a_{\tau_n}$ such that $a_{\tau_1} + \cdots + a_{\tau_n}$ is invertible in the algebra $\mathcal{B}$.

Theorem 3.2 ([5], Theorem XII.1.1) *Let $\mathcal{B}$ be a unital Banach algebra, $\{\mathcal{M}_\tau\}_{\tau \in \Omega}$ a covering system of localizing classes in $\mathcal{B}$, $x \in \mathcal{B}$ and $x \overset{\mathcal{M}_\tau}{\sim} x_\tau$ for all $\tau \in \Omega$. Further, assume that x commutes with all elements from $\bigcup_{\tau \in \Omega} \mathcal{M}_\tau$. Then x is invertible in $\mathcal{B}$ if and only if x_τ is $\mathcal{M}_\tau$-invertible for all $\tau \in \Omega$.*

4 Associated operator sequences on the unit circle

In this section we introduce two mappings $J : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2(\mathbb{T})$ and $F : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2(\mathbb{T})$, where $\mathbf{L}^2(\mathbb{T})$ is the Hilbert space of square integrable (complex-valued) functions on the unit circle $\mathbb{T} := \{t \in \mathbb{C} : |t| = 1\}$ with the inner product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{is}) \overline{g(e^{is})}\, ds\,.$$

These mappings will help us to associate an appropriate sequence $\{A_n^{\mathbb{T}}\}$ to our original sequence $\{A_n\}$ in order to study the properties of $\{A_n\}$ with the help of partial results concerning finite section and collocation methods for singular integral equations on the unit circle (see Hagen, Roch & Silbermann [6, Chapt.s 4 and 6], Prössdorf & Silbermann [14, Chapt. 7], and Roch [15]).

Let $e_n(t) = t^n$, $n = 0, \pm 1, \pm 2, \ldots$, $t \in \mathbb{T}$. Then, $\{e_n\}_{n=-\infty}^{\infty}$ forms an orthonormal basis in $\mathbf{L}^2(\mathbb{T})$. The operators $J : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2(\mathbb{T})$ and $F : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2(\mathbb{T})$ are defined by

$$Ju = \sum_{n=0}^{\infty} \langle u, \widetilde{u}_n \rangle_\sigma e_n \quad \text{and} \quad Fu = \frac{1}{\sqrt{2}i} \sum_{n=0}^{\infty} \langle u, \widetilde{u}_n \rangle_\sigma (e_{n+1} - e_{-n-1}) \,,$$

respectively. It is easily seen that $J : \mathbf{L}^2_\sigma \longrightarrow \mathbf{H}^2(\mathbb{T})$ and $F : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_{\mathrm{odd}}(\mathbb{T})$ are isometric isomorphisms, where $\mathbf{H}^2(\mathbb{T}) := \{f \in \mathbf{L}^2(\mathbb{T}) : \langle f, e_{-n} \rangle = 0\,,\ n = 1, 2, \ldots\}$ is the Hardy space and $\mathbf{L}^2_{\mathrm{odd}}(\mathbb{T}) := \operatorname{clos\,span}\{e_n - e_{-n} : n = 1, 2, \ldots\}$ the subspace of $\mathbf{L}^2(\mathbb{T})$ of "odd" functions. Moreover, because of

$$\begin{aligned} (Fu)(e^{\pm is}) &= \pm\sqrt{2} \sum_{n=0}^{\infty} \langle u, \widetilde{u}_n \rangle_\sigma \sin(n+1)s = \pm\sqrt{\pi} \sum_{n=0}^{\infty} \langle u, \widetilde{u}_n \rangle_\sigma \widetilde{u}_n(\cos s) \\ &= \pm\sqrt{\pi}\, u(\cos s) \,, \end{aligned}$$

$0 < s < \pi$, the relation

$$(Fu)(t) = \sqrt{\pi}\, \chi(t)\, u(\Re t) \,,$$

is true, where

$$\chi(t) = \begin{cases} 1 & , \quad \Im t > 0 \,, \\ -1 & , \quad \Im t < 0 \,, \\ 0 & , \quad t = \pm 1 \,. \end{cases} \tag{4.1}$$

Of course, at this place we could give $\chi(\pm 1)$ other values. But, in particular for the investigation of the collocation method, we need exactly the above definition of $\chi(t)$.
Define the Fourier projections $P_n^{\mathbb{T}} : \mathbf{L}^2(\mathbb{T}) \longrightarrow \mathbf{L}^2(\mathbb{T})$ by

$$P_n^{\mathbb{T}} f = \sum_{k=-n-1}^{n} \langle f, e_k \rangle e_k$$

and the operators $W_n^{\mathbb{T}} : \mathbf{L}^2(\mathbb{T}) \longrightarrow \mathbf{L}^2(\mathbb{T})$ by

$$W_n^{\mathbb{T}} f = \sum_{k=0}^{n} \langle f, e_{n-k} \rangle e_k + \sum_{k=-n-1}^{-1} \langle f, e_{-n-2-k} \rangle e_k \,.$$

The Multhopp interpolation operator $M_n^{\mathbb{T}}$ can be defined by

$$M_n^{\mathbb{T}} f = \sum_{k=-n-1}^{n} m_{kn}(f)\, e_k \,,$$

where

$$m_{kn}(f) = \frac{1}{2n+2} \sum_{j=-n-1}^{n} f(t_{jn}^M)\,(t_{jn}^M)^{-k}$$

and $t_{jn}^M = \exp\left(\frac{\pi i j}{n+1}\right)$, $j = -n-1, \ldots, n$. Note that

$$(M_n^{\mathbb{T}} f)(t_{jn}^M) = f(t_{jn}^M)\,, \quad j = -n-1, \ldots, n\,. \tag{4.2}$$

Let $\tau_n := \exp\left(\frac{\pi i}{n+1}\right)$. Then $t_{kn}^M = \tau_n^k$, $k = -n-1, \ldots, n$, and it follows, for $0 < |m| \le 2n+1$,

$$\begin{aligned}\sum_{k=-n-1}^{n} (t_{kn}^M)^m &= \sum_{k=-n-1}^{n} \tau_n^{km} = (\tau_n^m)^{-n-1} \sum_{k=0}^{2n+1} (\tau_n^m)^k \\ &= (\tau_n^m)^{-n-1} \frac{1-(\tau_n^m)^{2n+2}}{1-\tau_n^m} = 0\,.\end{aligned}$$

This proves

$$\sum_{k=-n-1}^{n} (t_{kn}^M)^m = 2(n+1)\delta_{0m} \quad \text{for} \quad m = -2n-1, \ldots, 2n+1\,. \tag{4.3}$$

An immediate consequence of (4.3) is the interpolation property (4.2). Moreover, with the help of (4.3) one can prove that, for each bounded and measurable function $a : \mathbb{T} \longrightarrow \mathbb{C}$ (comp. Junghanns & Silbermann [11, Lemma 2.2] or Prössdorf & Silbermann [14, 7.3(b)]),

$$\left\| M_n^{\mathbb{T}} a p_n \right\|_{\mathbf{L}^2(\mathbb{T})} \le \|a\|_\infty \, \|p_n\|_{\mathbf{L}^2(\mathbb{T})} \quad \text{for all} \quad p_n \in \operatorname{im} P_n^{\mathbb{T}}\,, \tag{4.4}$$

where $\|a\|_\infty := \sup\{|a(t)| : t \in \mathbb{T}\}$. Finally, let us remark that

$$\lim_{n\to\infty} \left\| M_n^{\mathbb{T}} f - f \right\|_{\mathbf{L}^2(\mathbb{T})} = 0$$

for all bounded Riemann integrable functions $f : \mathbb{T} \longrightarrow \mathbb{C}$.

As usually, together with the singular integral operator $S_{\mathbb{T}} : \mathbf{L}^2(\mathbb{T}) \longrightarrow \mathbf{L}^2(\mathbb{T})$ given by

$$(S_{\mathbb{T}} u)(t) = \frac{1}{\pi i} \int_{\mathbb{T}} \frac{u(\tau)}{\tau - t}\, d\tau\,,$$

we consider the projections $P_{\mathbb{T}} := \frac{1}{2}(I + S_{\mathbb{T}})$ and $Q_{\mathbb{T}} := \frac{1}{2}(I - S_{\mathbb{T}})$. Remember that

$$S_{\mathbb{T}} u = \sum_{n=0}^{\infty} \langle u, e_n \rangle e_n - \sum_{n=-\infty}^{-1} \langle u, e_n \rangle e_n\,,$$

which implies $S_{\mathbb{T}}^2 = I$ and

$$P_{\mathbb{T}} u = \sum_{n=0}^{\infty} \langle u, e_n \rangle e_n \quad \text{and} \quad Q_{\mathbb{T}} u = \sum_{n=-\infty}^{-1} \langle u, e_n \rangle e_n\,.$$

For a function $a:[-1,1]\longrightarrow\mathbb{C}$ we define $\widehat{a}:\mathbb{T}\longrightarrow\mathbb{C}$ by the formula $\widehat{a}(e^{is}):=a(\cos s)$. If $u\in\mathbf{L}^2_\sigma$ then

$$(Fau)(e^{is})=\sqrt{\pi}\chi(e^{is})a(\cos s)u(\cos s)=\widehat{a}(e^{is})(Fu)(e^{is}),$$

which implies that the multiplication operator $aI:\mathbf{L}^2_\sigma\longrightarrow\mathbf{L}^2_\sigma$ can be represented as

(A) $\quad aI=F^{-1}\widehat{a}F,$

where F^{-1} denotes the inverse mapping of $F:\mathbf{L}^2_\sigma\longrightarrow\mathbf{L}^2_{\mathrm{odd}}(\mathbb{T})$. Note $(F^{-1}f)(\cos s)=\dfrac{1}{\sqrt{\pi}}f(e^{is})$, $0<s<\pi$. It can be easily checked that

(B) $\quad P_n=F^{-1}P_n^{\mathbb{T}}FP_n.$

Since (for $u:[-1,1]\longrightarrow\mathbb{C}$) $M_n^{\mathbb{T}}Fu\in\mathbf{L}^2_{\mathrm{odd}}(\mathbb{T})\cap\operatorname{im}P_n^{\mathbb{T}}$ and, for $j=1,\dots,n$, $(M_n^{\mathbb{T}}Fu)(t_{jn}^M)=(Fu)(t_{jn}^M)=\sqrt{\pi}u(x_{jn}^\varphi)$, it follows $F^{-1}M_n^{\mathbb{T}}Fu\in\operatorname{im}P_n$ and $(F^{-1}M_n^{\mathbb{T}}Fu)(x_{jn}^\varphi)=u(x_{jn}^\varphi)$. This implies

(C) $\quad M_n=F^{-1}M_n^{\mathbb{T}}F.$

Now, let us look for the associated operator to the Cauchy singular integral operator $S\in\mathcal{L}(\mathbf{L}^2_\sigma)$. Remember the well-known relation

$$\frac{1}{\pi}\int_{-1}^1\frac{U_n(y)}{y-x}\sqrt{1-y^2}\,dy=-T_{n+1}(x),\qquad -1<x<1,\quad n=0,1,\dots,\tag{4.5}$$

where $T_n(x)$ denotes the normed Chebyshev polynomial of first kind with the trigonometric representation

$$T_0=\frac{1}{\sqrt{\pi}},\quad T_n(\cos s)=\sqrt{\frac{2}{\pi}}\cos ns,\quad n=1,2,\dots$$

Consequently,

$$S\widetilde{u}_n=iT_{n+1}\quad\text{and}\quad(FS\widetilde{u}_n)(e^{is})=\sqrt{2}i\,\chi(e^{is})\cos(n+1)s.$$

Otherwise, we have

$$(S_{\mathbb{T}}F\widetilde{u}_n)(e^{is})=\frac{1}{\sqrt{2}i}\left(S_{\mathbb{T}}(e_{n+1}-e_{-n-1})\right)(e^{is})=\frac{1}{\sqrt{2}i}\left(e^{i(n+1)s}+e^{-i(n+1)s}\right),$$

which equals $-\sqrt{2}i\cos(n+1)s$. It follows

(D) $\quad S=-F^{-1}\chi S_{\mathbb{T}}F.$

Taking into account

$$e_1W_n^{\mathbb{T}}FP_nu=\frac{1}{\sqrt{2}i}\,e_1\sum_{k=0}^{n-1}\langle u,\widetilde{u}_k\rangle_\sigma(e_{n-k-1}-e_{-n+k-1})=FW_nu$$

we obtain

(E) $\quad W_n = F^{-1} e_1 W_n^{\mathbb{T}} F P_n$

and, analogously,

(F) $\quad W_n = F^{-1} W_n^{\mathbb{T}} e_{-1} F P_n \,.$

For the proof of the following lemma we refer to Junghanns & Silbermann [11, Lemma 2.4, Lemma 2.7, and Theorem 2.3] and Prössdorf & Silbermann [14, Theorem 7.17, Corollary 7.18, and Theorem 7.19]. For $a : \mathbb{T} \to \mathbb{C}$ we define $\widetilde{a} := a(\overline{t})$, and by $\mathbf{PC}(\mathbb{T})$ we denote the set of all piecewise continuous functions $a : \mathbb{T} \to \mathbb{C}$ that are continuous from the left, that is, $a(t-0) = a(t)$ for all $t \in \mathbb{T}$.

Lemma 4.1 *Assume that a and b are bounded Riemann integrable functions.*

(a) Then the strong convergences

$$M_n^{\mathbb{T}}(aP_{\mathbb{T}} + bQ_{\mathbb{T}})P_n^{\mathbb{T}} \longrightarrow aP_{\mathbb{T}} + bQ_{\mathbb{T}} \,,$$

and

$$W_n^{\mathbb{T}} M_n^{\mathbb{T}}(aP_{\mathbb{T}} + bQ_{\mathbb{T}})W_n^{\mathbb{T}} \longrightarrow \widetilde{a}P_{\mathbb{T}} + \widetilde{b}Q_{\mathbb{T}}$$

hold true. Moreover, the strong limits of the respective sequences of the adjoint operators exist.

(b) For any continuous function $f : \mathbb{T} \longrightarrow \mathbb{C}$ there exist compact operators $K_1, K_2 \in \mathcal{K}(\mathbf{L}^2(\mathbb{T}))$ such that

$$M_n^{\mathbb{T}} f P_n^{\mathbb{T}} M_n^{\mathbb{T}}(aP_{\mathbb{T}} + bQ_{\mathbb{T}})P_n^{\mathbb{T}} - M_n^{\mathbb{T}}(aP_{\mathbb{T}} + bQ_{\mathbb{T}})P_n^{\mathbb{T}} M_n^{\mathbb{T}} f P_n^{\mathbb{T}}$$

$$= P_n^{\mathbb{T}} K_1 P_n^{\mathbb{T}} + P_n^{\mathbb{T}} K_2 P_n^{\mathbb{T}} + C_n$$

and $\lim_{n\to\infty} \|C_n\|_{\mathcal{L}(\mathbf{L}^2(\mathbb{T}))} = 0 \,.$

(c) For $a, b \in \mathbf{PC}(\mathbb{T})$ the sequence $\{M_n^{\mathbb{T}}(aP_{\mathbb{T}} + bQ_{\mathbb{T}})P_n^{\mathbb{T}}\}$ is stable if and only if $aP_{\mathbb{T}} + bQ_{\mathbb{T}} \in \mathcal{GL}(\mathbf{L}^2(\mathbb{T}))$.

Using the relations (A), (B), (C), and (D) we can write the operators $A_{n,M}$ in the form $A_{n,M} = F^{-1} A_{n,M}^{\mathbb{T}} F P_n$, where $A_{n,M}^{\mathbb{T}} = M_n^{\mathbb{T}} A_{\mathbb{T}} P_n^{\mathbb{T}}$ and $A_{\mathbb{T}} = \widehat{a} I - \widehat{b} \chi S_{\mathbb{T}}$.

Remark 4.2 *We point out that for the validity of this transformation it is essential to define χ just the way we did in (4.1). Hence, χ is not an element of $\mathbf{PC}(\mathbb{T})$ in the sense of the above definition, and Lemma 4.1(c) does not apply. Consider for instance $A = S \in \mathcal{L}(\mathbf{L}^2_\sigma)$, which is not invertible. Then $\{A_{n,M}\} = \{-F^{-1} M_n^{\mathbb{T}} \chi S_{\mathbb{T}} P_n^{\mathbb{T}} F P_n\}$ is not stable. If we, however, modified the function χ in ± 1 to obtain the function $\varrho \in \mathbf{PC}(\mathbb{T})$,*

$$\varrho(e^{is}) = \begin{cases} 1 \,, & 0 < s \le \pi \,, \\ -1 \,, & \pi < s \le 2\pi \,, \end{cases}$$

we would arrive at $\{M_n^{\mathbb{T}}\varrho S_{\mathbb{T}}P_n^{\mathbb{T}}\}$ which is stable due to Lemma 4.1(c). (The inverses are $\{S_{\mathbb{T}}M_n^{\mathbb{T}}\varrho P_n^{\mathbb{T}}\}$.) This shows that one cannot simply reduce the stability of $\{A_{n,M}\}$ to the conditions of this lemma. The transformation to the unit circle can merely be employed as an auxiliary tool in the local theory based on Theorem 3.1 and the local principle.

As a consequence of Lemma 4.1(a) the sequence $\{A_{n,M}\}$ belongs to the algebra $\mathcal{A}$, where $s-\lim A_{n,M}=A$. Indeed, the following proposition holds true.

Proposition 4.3 *Let the coefficients a,b be bounded and Riemann integrable. Then, the strong limit of the operator sequence $\{W_nA_{n,M}W_n\}$ exists and is equal to $\widetilde{A}_M$, where*

$$\widetilde{A}_M = aI - bS\,.$$

Proof. With the help of (E), (F), and Lemma 4.1(a) we get

$$W_nM_n(aI+bS)W_n = F^{-1}e_1W_n^{\mathbb{T}}M_n^{\mathbb{T}}(\widehat{a}I-\widehat{b}\chi S_{\mathbb{T}})W_n^{\mathbb{T}}e_{-1}FP_n$$

$$\longrightarrow F^{-1}e_1(\widetilde{\widehat{a}}I-\widetilde{\widehat{b}}\widetilde{\chi}S_{\mathbb{T}})e_{-1}F = aI-bS$$

taking into account $\widetilde{\widehat{a}}=\widehat{a}$, $\widetilde{\chi}=-\chi$, and $e_1S_{\mathbb{T}}e_{-1}F=S_{\mathbb{T}}F$. ■

For $a\in\mathbf{L}^\infty(\mathbb{T})$, the Toeplitz operator $T(a):\mathbf{H}^2(\mathbb{T})\longrightarrow\mathbf{H}^2(\mathbb{T})$ and the Hankel operator $H(a):\mathbf{H}^2(\mathbb{T})\longrightarrow\mathbf{H}^2(\mathbb{T})$ are defined by

$$T(a):=P_{\mathbb{T}}aP_{\mathbb{T}} \quad\text{and}\quad H(a):=P_{\mathbb{T}}ae_{-1}W_{\mathbb{T}}P_{\mathbb{T}}\,,$$

respectively, where the operator $W_{\mathbb{T}}:\mathbf{L}^2(\mathbb{T})\longrightarrow\mathbf{L}^2(\mathbb{T})$ is given by $(W_{\mathbb{T}}f)(t):=f(\overline{t})$. Note that $J=e_{-1}P_{\mathbb{T}}F$ and $J^{-1}=F^{-1}(I-W_{\mathbb{T}})P_{\mathbb{T}}e_1$. Since $\widehat{a}Fu\in\operatorname{im}F$ for all $u\in\mathbf{L}^2_\sigma$ we get

$$\begin{aligned}
F^{-1}\widehat{a}F &= F^{-1}(I-W_{\mathbb{T}})P_{\mathbb{T}}\widehat{a}(P_{\mathbb{T}}+Q_{\mathbb{T}})F\\
&= F^{-1}(I-W_{\mathbb{T}})P_{\mathbb{T}}e_1(P_{\mathbb{T}}\widehat{a}e_{-1}P_{\mathbb{T}}+P_{\mathbb{T}}\widehat{a}e_{-1}Q_{\mathbb{T}})F\\
&= F^{-1}(I-W_{\mathbb{T}})P_{\mathbb{T}}e_1(P_{\mathbb{T}}\widehat{a}P_{\mathbb{T}}e_{-1}P_{\mathbb{T}}-P_{\mathbb{T}}\widehat{a}e_{-1}W_{\mathbb{T}}P_{\mathbb{T}})F\\
&= F^{-1}(I-W_{\mathbb{T}})P_{\mathbb{T}}e_1(P_{\mathbb{T}}\widehat{a}P_{\mathbb{T}}-P_{\mathbb{T}}\widehat{a}e_{-1}e_{-1}W_{\mathbb{T}}P_{\mathbb{T}})e_{-1}P_{\mathbb{T}}F\,.
\end{aligned}$$

Thus, in view of (A)

$$\text{(G)}\qquad aI=J^{-1}\left[T(\widehat{a})-H(\widehat{a}e_{-1})\right]J\,.$$

Analogously, we obtain

$$\begin{aligned}
F^{-1}\chi S_{\mathbb{T}}F &= F^{-1}(I-W_{\mathbb{T}})P_{\mathbb{T}}\chi(P_{\mathbb{T}}-Q_{\mathbb{T}})F\\
&= F^{-1}(I-W_{\mathbb{T}})P_{\mathbb{T}}e_1(P_{\mathbb{T}}\chi e_{-1}P_{\mathbb{T}}-P_{\mathbb{T}}\chi e_{-1}Q_{\mathbb{T}})F\\
&= F^{-1}(I-W_{\mathbb{T}})P_{\mathbb{T}}e_1(P_{\mathbb{T}}\chi P_{\mathbb{T}}+P_{\mathbb{T}}\chi e_{-1}e_{-1}W_{\mathbb{T}})e_{-1}P_{\mathbb{T}}F
\end{aligned}$$

and, taking into account (D),

$$\text{(H)} \qquad S = -J^{-1}\,[T(\chi) + H(\chi e_{-1})]\,J\,.$$

If we assign to the sequence $\{A_{n,P}\}$ the sequence $\{A_{n,P}^{\mathbb{T}}\}$, where

$$A_{n,P}^{\mathbb{T}} := P_n^{\mathbb{T}}(JAJ^{-1}P_{\mathbb{T}} + Q_{\mathbb{T}})P_n^{\mathbb{T}}\,,$$

then we can make the following observation.

Lemma 4.4 *The sequence $\{A_{n,P}\}$ is stable if and only if the sequence $\{A_{n,P}^{\mathbb{T}}\}$ is stable.*

Proof. ¿From $BP_{\mathbb{T}} + Q_{\mathbb{T}} = (I + Q_{\mathbb{T}}BP_{\mathbb{T}})(P_{\mathbb{T}}BP_{\mathbb{T}} + Q_{\mathbb{T}})$, $B \in \mathcal{L}(\mathbf{L}^2(\mathbb{T}))$, it follows that $A_{n,P}^{\mathbb{T}}$ is invertible in im $P_n^{\mathbb{T}}$ if and only if $P_{\mathbb{T}}P_n^{\mathbb{T}}JAJ^{-1}P_n^{\mathbb{T}}P_{\mathbb{T}} = J^{-1}P_nAP_nJ$ is invertible in (im $P_n^{\mathbb{T}}$) $\cap\, \mathbf{H}^2(\mathbb{T})$ or, which is the same, if $A_{n,P}$ is invertible in im P_n. Moreover, it holds $(A_{n,P}^{\mathbb{T}})^{-1} = JA_{n,P}^{-1}P_nJ^{-1} + Q_{\mathbb{T}}P_n^{\mathbb{T}}$, which also yields the equivalence of the uniform boundedness of the inverses. ■

5 Stability of the finite section method

To get necessary and sufficient conditions for the stability of the finite section method we will apply a general result on a finite section method for operators $B \in \mathcal{L}(\mathbf{L}^2(\mathbb{T}))$ formulated in the following proposition.

By $\mathcal{C}$ we denote the smallest closed subalgebra of the algebra of all bounded sequences $\{B_n\}_{n=1}^{\infty}$, $B_n \in \mathcal{L}(\mathbf{L}^2(\mathbb{T}))$, (equipped with component-wise algebraic operations and the supremum norm) containing the constant sequences $\{P_{\mathbb{T}}\}$, $\{e_{-1}W_{\mathbb{T}}\}$, and $\{aI\}$ for $a \in \mathbf{PC}(\mathbb{T})$, as well as the sequences $\{P_{kn}^{\mathbb{T}}\}$ for every positive integer k. Furthermore, define the operator $W : \mathbf{L}^2(\mathbb{R}) \longrightarrow \mathbf{L}^2(\mathbb{R})$ by $(Wf)(t) = f(-t)$ and let $\chi_{\alpha,\beta}$ denote the characteristic function of the (bounded or unbounded) interval (α,β). For the generating elements of the algebra $\mathcal{C}$ we define the mappings W_t, $t \in \mathbb{T}$, $\Im t \geq 0$, and W^l, $l = 0,1,2,\ldots$, in the following way:

$$W_t\{P_{\mathbb{T}}\} = \begin{cases} \chi_{0,\infty}I & , \quad t = \pm 1\,, \\[2ex] \begin{bmatrix} \chi_{0,\infty}I & 0 \\ 0 & \chi_{-\infty,0}I \end{bmatrix} & , \quad \Im t > 0\,, \end{cases}$$

$$W^l\{P_{\mathbb{T}}\} = \begin{cases} P_{\mathbb{T}} & , \quad l = 0\,, \\[2ex] \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} & , \quad l > 0\,, \end{cases}$$

$$W_t\{P_{kn}^{\mathbb{T}}\} = \begin{cases} \chi_{-k,k} I & , \quad t = \pm 1\,, \\ \begin{bmatrix} \chi_{-k,k} I & 0 \\ 0 & \chi_{-k,k} I \end{bmatrix} & , \quad \Im t > 0\,, \end{cases}$$

$$W^l\{P_{kn}^{\mathbb{T}}\} = \begin{cases} I & , \quad l = 0\,, \\ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} & , \quad 0 < l < k\,, \\ \begin{bmatrix} Q_{\mathbb{T}} & 0 \\ 0 & Q_{\mathbb{T}} \end{bmatrix} & , \quad l = k\,, \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & , \quad l > k\,, \end{cases}$$

$$W_t\{aI\} = \begin{cases} a(t+0)Q_{\mathbb{R}} + a(t-0)P_{\mathbb{R}}\,, \qquad t = \pm 1\,, \\ \begin{bmatrix} a(t+0)Q_{\mathbb{R}} + a(t-0)P_{\mathbb{R}} & 0 \\ 0 & \widetilde{a}(t+0)Q_{\mathbb{R}} + \widetilde{a}(t-0)P_{\mathbb{R}} \end{bmatrix}, \\ \qquad \Im t > 0\,, \end{cases}$$

$$W^l\{aI\} = \begin{cases} aI & , \quad l = 0\,, \\ \begin{bmatrix} aI & 0 \\ 0 & \widetilde{a}I \end{bmatrix} & , \quad l > 0\,, \end{cases}$$

$$W_t\{e_{-1}W_{\mathbb{T}}\} = \begin{cases} \pm W & , \quad t = \pm 1\,, \\ \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} & , \quad \Im t > 0\,, \end{cases}$$

$$W^l\{e_{-1}W_{\mathbb{T}}\} = \begin{cases} e_{-1}W_{\mathbb{T}} & , \quad l = 0\,, \\ \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} & , \quad l > 0\,. \end{cases}$$

Here, $P_{\mathbb{R}}$ and $Q_{\mathbb{R}}$ are the projections $P_{\mathbb{R}} := \frac{1}{2}(I+S_{\mathbb{R}})$ and $Q_{\mathbb{R}} := \frac{1}{2}(I-S_{\mathbb{R}})$, where $S_{\mathbb{R}}$ denotes the Cauchy singular integral operator on the real line, $(S_{\mathbb{R}}f)(x) := \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{f(y)}{y-x}\,dy$.

Proposition 5.1 ([15] or [6], Chaps. 4,6) *The mappings W_t and W^l can be extended to continuous *-homomorphisms $W_t : \mathcal{C} \longrightarrow \mathcal{L}(\mathbf{L}^2(\mathbb{R}))$ for $t = \pm 1$, $W^0 : \mathcal{C} \longrightarrow \mathcal{L}(\mathbf{L}^2(\mathbb{T}))$ and $W_t : \mathcal{C} \longrightarrow \mathcal{L}(\mathbf{L}^2(\mathbb{R}) \times \mathbf{L}^2(\mathbb{R}))$ for $\Im t > 0$, $W^l : \mathcal{C} \longrightarrow \mathcal{L}(\mathbf{L}^2(\mathbb{T}) \times \mathbf{L}^2(\mathbb{T}))$ for $l = 1, 2, \ldots$ The sequence $\{B_n\}$ of finite sections $B_n = P_n^{\mathbb{T}} B P_n^{\mathbb{T}} + Q_n^{\mathbb{T}}$ with $B \in \mathcal{L}(\mathbf{L}^2(\mathbb{T}))$ and $\{B_n\} \in \mathcal{C}$, where $Q_n^{\mathbb{T}} := I - P_n^{\mathbb{T}}$, is stable if and only if all operators $W_t\{B_n\}$, $\Im t \geq 0$, and $W^l\{B_n\}$, $l = 0, 1, 2, \ldots$, are invertible.*

Taking into account Lemma 4.4 as well as (G) and (H) we can apply the last proposition to $B = JAJ^{-1}P_{\mathbb{T}} + Q_{\mathbb{T}}$ with

$$JAJ^{-1} = T(\widehat{a}) - H(\widehat{a}e_{-1}) - [T(\widehat{b}) - H(\widehat{b}e_{-1})]\,[T(\chi) + H(\chi e_{-1})]$$

and $a, b \in \mathbf{PC}[-1,1]$. For this end we determine the respective operators $W_t\{B_n\}$ and $W^l\{B_n\}$.

At first, for $f, g \in \mathbf{PC}(\mathbb{T})$, we directly compute

$$W^l\{P_n^{\mathbb{T}} T(f) P_n^{\mathbb{T}}\} = \begin{cases} T(f) & , \quad l = 0, \\[2ex] \begin{bmatrix} Q_{\mathbb{T}} f Q_{\mathbb{T}} & 0 \\ 0 & 0 \end{bmatrix} & , \quad l = 1, \\[2ex] \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & , \quad l > 1, \end{cases}$$

$$W^l\{P_n^{\mathbb{T}} H(f) P_n^{\mathbb{T}}\} = \begin{cases} H(f) & , \quad l = 0, \\[2ex] \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & , \quad l > 0, \end{cases}$$

and

$$W^l\{P_n^{\mathbb{T}}[T(f) - H(f)]\,[T(g) + H(g)]P_n^{\mathbb{T}}\}$$

$$= \begin{cases} [T(f) - H(f)]\,[T(g) + H(g)] & , \quad l = 0\,, \\ \begin{bmatrix} Q_{\mathbb{T}} f g Q_{\mathbb{T}} & 0 \\ 0 & 0 \end{bmatrix} & , \quad l = 1\,, \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & , \quad l > 1\,, \end{cases}$$

It follows

$$W^l\{B_n\} = \begin{cases} JAJ^{-1} P_{\mathbb{T}} + Q_{\mathbb{T}} & , \quad l = 0\,, \\ \begin{bmatrix} Q_{\mathbb{T}}(\widehat{a} - \widehat{b}\chi) Q_{\mathbb{T}} + P_{\mathbb{T}} & 0 \\ 0 & I \end{bmatrix} & , \quad l = 1\,, \\ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} & , \quad l > 1\,. \end{cases} \tag{5.1}$$

Since

$$W_{\pm 1}\{T(\widehat{a}) - H(\widehat{a} e_{-1})\} = a(\pm 1) \chi_{0,\infty} I$$

and

$$W_{\pm 1}\{T(\chi) + H(\chi e_{-1})\} = \mp \chi_{0,\infty} S_{\mathbb{R}}(I + W) \chi_{0,\infty}$$

we get

$$W_{\pm 1}\{B_n\} = \chi_{0,1}\left[a(\pm 1) \pm b(\pm 1) S_{\mathbb{R}}(I + W)\right] \chi_{0,1} I + \chi_{\mathbb{R}\setminus(0,1)} I\,. \tag{5.2}$$

For $\Im t > 0$, $x = \Re t$, and $T_a := a(x - 0) Q_{\mathbb{R}} + a(x + 0) P_{\mathbb{R}}$, we have

$$W_t\{T(\widehat{a})\} = \begin{bmatrix} \chi_{0,\infty} T_a \chi_{0,\infty} I & 0 \\ 0 & \chi_{-\infty,0} T_a \chi_{-\infty,0} I \end{bmatrix},$$

$$W_t\{H(\widehat{a} e_{-1})\} = \begin{bmatrix} 0 & t^{-1} \chi_{0,\infty} T_a \chi_{-\infty,0} I \\ t \chi_{-\infty,0} T_a \chi_{0,\infty} I & 0 \end{bmatrix},$$

$$W_t\{T(\chi)\} = \begin{bmatrix} \chi_{0,\infty} & 0 \\ 0 & -\chi_{-\infty,0} \end{bmatrix} \quad \text{and} \quad W_t\{T(\widehat{a})\} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Consequently, for $\Im t > 0$,

$$\begin{aligned} W_t\{B_n\} \quad = \quad & \begin{bmatrix} \chi_{0,1} T_{c_-} \chi_{0,1} I & -t^{-1} \chi_{0,1} T_{c_+} \chi_{-1,0} I \\ -t \chi_{-1,0} T_{c_-} \chi_{0,1} I & \chi_{-1,0} T_{c_+} \chi_{-1,0} I \end{bmatrix} \\ & + \begin{bmatrix} \chi_{\mathbb{R}\setminus(0,1)} & 0 \\ 0 & \chi_{\mathbb{R}\setminus(-1,0)} \end{bmatrix} \end{aligned} \tag{5.3}$$

with $c_\pm := a \pm b$. Now, we are able to prove the following theorem.

Theorem 5.2 *The finite section method is stable if and only if the following four conditions are satisfied:*

(a) The operator $A : \mathbf{L}^2_\sigma \longrightarrow \mathbf{L}^2_\sigma$ *is invertible.*

(b) The operator $(\widehat{a} - \widehat{b}\chi)P_{\mathbb{T}} + Q_{\mathbb{T}} : \mathbf{L}^2(\mathbb{T}) \longrightarrow \mathbf{L}^2(\mathbb{T})$ *is invertible.*

(c) The point 0 *lies outside the half-circle, which is formed by the segment* $[c_+(1), c_-(1)]$ *and the half-circle line from* $c_-(1)$ *to* $c_+(1)$ *that lies to the left of the line from* $c_-(1)$ *to* $c_+(1)$, *and outside the the half-circle, which is formed by the segment* $[c_-(-1), c_+(-1)]$ *and the half-circle line from* $c_+(-1)$ *to* $c_-(-1)$ *that lies to the left of the line from* $c_+(-1)$ *to* $c_-(-1)$.

(d) For every $x \in (-1,1)$, *the point* 0 *lies outside the triangle, which is formed by the points* $\dfrac{c_-(x+0)}{c_-(x-0)}$, $\dfrac{c_+(x+0)}{c_+(x-0)}$, *and* 1.

Proof. We apply Proposition 5.1 together with Lemma 4.4. Thus, we have to show that the invertibility of the operators $W^l\{B_n\}$ and $W_t\{B_n\}$ from (5.1), (5.2), and (5.3) is equivalent to the conditions (a)-(d) of the theorem.

Obviously, $W^0\{B_n\} = JAJ^{-1}P_{\mathbb{T}} + Q_{\mathbb{T}}$ is invertible if and only if A is invertible. Also the equivalence of the invertibility of $W^1\{B_n\}$ and condition (b) is easy to see. Now, let $c_0 := a(1)$ and $c_1 := b(1)$. Then, $W_1\{B_n\} = \chi_{0,1}\left[c_0 + c_1 S_{\mathbf{R}}(I+W)\right]\chi_{0,1}I + \chi_{\mathbf{R}\setminus(0,1)}I$ is invertible if and only if

$$\chi_{0,1}\left[c_0 + c_1 S_{\mathbf{R}}(I+W)\right]\chi_{0,1}I : \mathbf{L}^2(0,1) \longrightarrow \mathbf{L}^2(0,1)$$

is invertible. This operator can be written in the form

$$c_0 I + c_1\left(S_{(0,1)} - N\right), \tag{5.4}$$

where $(S_{(0,1)}u)(x) = \dfrac{1}{\pi i}\displaystyle\int_0^1 \dfrac{u(y)}{y-x}\,dy$ is a Cauchy singular integral operator and $(Nu)(x) = \dfrac{1}{\pi i}\displaystyle\int_0^1 \dfrac{u(y)}{y+x}\,dy$ is a singular integral operator with a fixed singularity. Following Duduchava [3, Sect. 8] we assign to the operator (5.4) the symbol

$$\mathbf{a}(t,\xi) = \begin{cases} c_0 + c_1\left(\tanh \pi t + \dfrac{i}{\cosh \pi t}\right) & , \quad -\infty \le t \le \infty,\ \xi = -\infty, \\ \\ c_0 - c_1 \tanh \pi\xi & , \quad t = \infty,\ -\infty \le \xi \le \infty, \end{cases}$$

which can also be written as

$$\mathbf{a}(\mu,\nu) = \begin{cases} c_0 + c_1(\mu + i\sqrt{1-\mu^2}) & , \quad -1 \le \mu \le 1,\ \nu = -1, \\ \\ c_0 - c_1\nu & , \quad \mu = 1,\ -1 \le \nu \le 1. \end{cases} \tag{5.5}$$

The image $\{\mathbf{a}(\mu,-1) : -1 \le \mu \le 1\} \cup \{\mathbf{a}(1,\nu) : -1 \le \nu \le 1\}$ of the symbol consists of a half-circle line from $c_0 - c_1$ to $c_0 + c_1$ and the diameter from $c_0 + c_1$ to $c_0 - c_1$. Thus, by Duduchava [3, Theor. 8.1] the operator (5.4) is invertible if and only if the first part of condition (c) is fulfilled. Analogously, the invertibility of $W_{-1}\{B_n\}$ is equivalent to the second part of condition (c).

The equivalence of the invertibility of $W_t\{B_n\}$ from (5.3) for all $t \in \mathbb{T}$ with $\Im t > 0$ and of condition (d) was proved in Junghanns, Roch & Weber [10, Lemma 3.6]. ■

6 The stability of the collocation method

To prove the stability of the collocation method (2.5) we apply Theorem 3.1. Having shown that the sequence $\{A_{n,M}\} = \{M_n(aI + bS)P_n\}$ belongs to the algebra $\mathcal{A}$ described in Section 3 and having computed $\widetilde{A} = aI - bS$ (cf. Prop. 4.3), we are left with investigating the invertibility of the coset $\{A_{n,M}\}+\mathcal{I} \in \mathcal{A}/\mathcal{I}$, which will be done by the local principle of Gohberg and Krupnik (cf. Theorem 3.2).

For $\tau \in [-1,1]$ let

$$m_\tau := \{f \in C[-1,1] : 0 \le f(x) \le 1\,,\, f(x) \equiv 1 \text{ in some neighbourhood of } \tau\}$$

and define

$$\mathcal{M}_\tau := \{\{M_n f P_n\} + \mathcal{I} : f \in m_\tau\}.$$

Lemma 6.1 *$\{\mathcal{M}_\tau\}_{\tau\in[-1,1]}$ is a covering system of localizing classes in $\mathcal{A}/\mathcal{I}$. If a and b are bounded and Riemann integrable functions then the coset $\{A_{n,M}\} + \mathcal{I}$ commutes with all elements of* $\displaystyle\bigcup_{\tau\in[-1,1]} M_\tau$.

Proof. For the proof of the first part of the lemma compare the proof of Junghanns & Silbermann [11, Lemma 2.6]. Let $f \in m_\tau$. With the help of (A)-(F) and Lemma 4.1(b) we obtain

$$M_n f P_n M_n(aI + bS)P_n - M_n(aI + bS)P_n M_n f P_n$$

$$= F^{-1}\left[M_n^{\mathbb{T}} \widehat{f} P_n^{\mathbb{T}} M_n^{\mathbb{T}}(\widehat{a}I - \widehat{b}\chi S_{\mathbb{T}})P_n^{\mathbb{T}} - M_n^{\mathbb{T}}(\widehat{a}I - \widehat{b}\chi S_{\mathbb{T}})P_n^{\mathbb{T}} M_n^{\mathbb{T}} \widehat{f} P_n^{\mathbb{T}}\right] F P_n$$

$$= P_n F^{-1} T(P_n^{\mathbb{T}} K_1 P_n^{\mathbb{T}} + W_n^{\mathbb{T}} K_2 W_n^{\mathbb{T}} + C_n) F P_n$$

with $K_1, K_2 \in \mathcal{K}(\mathbf{L}^2(\mathbb{T}))$, $\lim_{n\to\infty} \|C_n\|_{\mathcal{L}(\mathbf{L}^2(\mathbb{T}))} = 0$, and the projection $T : \mathbf{L}^2(\mathbb{T}) \longrightarrow \mathbf{L}^2(\mathbb{T})$, $(Tg)(t) = \frac{1}{2}[g(t) - g(\overline{t})]$. (We insert the projection T to be able to consider the three summands individually, since it is not guaranteed that each summand inside the parentheses maps $\operatorname{im} F$ into $\operatorname{im} F$.) If we use the relations $P_n F^{-1} T P_n^{\mathbb{T}} = P_n F^{-1} T$ and $P_n F^{-1} T W_n^{\mathbb{T}} = W_n F^{-1} T(e_1 P_{\mathbb{T}} + Q_{\mathbb{T}} e_1 I)$ as well as (B) and (E), we see that

$$\{P_n F^{-1} T(P_n^{\mathbb{T}} K_1 P_n^{\mathbb{T}} + W_n^{\mathbb{T}} K_2 W_n^{\mathbb{T}} + C_n) F P_n\}$$

$$= \{P_nF^{-1}TK_1FP_n + W_nF^{-1}T(e_1P_{\mathbb{T}} + Q_{\mathbb{T}}e_1I)K_2e_{-1}FW_n + P_nF^{-1}TC_nFP_n\} ,$$

which is obviously an element of $\mathcal{I}$. ■

Now we are able to give local representatives for $\{A_{n,M}\} + \mathcal{I}$, where we assume that the coefficients a and b of A are piecewise continuous.

Lemma 6.2 *Let $\tau \in [-1,1]$ and $a, a_\tau, b, b_\tau \in \mathbf{PC}[-1,1]$ such that*

$$a_\tau(\tau \pm 0) = a(\tau \pm 0) \quad \text{and} \quad b_\tau(\tau \pm 0) = b(\tau \pm 0) .$$

Then

$$\{M_n(a_\tau I + b_\tau S)P_n\} + \mathcal{I} \overset{\mathcal{M}_\tau}{\sim} \{M_n(aI + bS)P_n\} + \mathcal{I} .$$

Proof. Let $f \in m_\tau$. By $M_nfM_ngI = M_nfgI$, (A)-(D), and (4.4) we have

$$\|\{M_nfP_n\}\{M_n[(a-a_\tau)I + (b-b_\tau)S]P_n\} + \mathcal{I}\|_{\mathcal{A}/\mathcal{I}}$$

$$\leq \sup_n \left\| F^{-1}M_n^{\mathbb{T}} \left[\widehat{f}(\widehat{a} - \widehat{a_\tau})I + \widehat{f}(\widehat{b} - \widehat{b}_\tau)\chi S_{\mathbb{T}} \right] P_n^{\mathbb{T}} FP_n \right\|_{\mathcal{L}(\mathbf{L}^2_\sigma)}$$

$$\leq \|f(a-a_\tau)\|_\infty + \|f(b-b_\tau)\|_\infty$$

($\|g\|_\infty = \sup\{|g(x)| : x \in [-1,1]\}$ for a function $g : [-1,1] \longrightarrow \mathbb{C}$), which can be made arbitrarily small by a suitable choice of f. Thus, taking into account Lemma 6.1, the assertion is proved. ■

Lemma 6.3 *Assume that $a, b \in \mathbf{PC}[-1,1]$ and that $A = aI + bS$ and $\widetilde{A}_M = aI - bS$ are invertible in $\mathbf{L}^2_\sigma$. Then $|a(-1)| > |b(-1)|$ and $|a(1)| > |b(1)|$.*

Proof. The invertibility of A in $\mathbf{L}^2_\sigma$ implies $a(x \pm 0) + b(x \pm 0) \neq 0$ and $a(x \pm 0) - b(x \pm 0) \neq 0$ for all $x \in [-1,1]$ as well as $\mathbf{c}(x,\mu) \neq 0$, $x \in [-1,1]$, $\mu \in [0,1]$, where

$$\mathbf{c}(x,\mu) := \begin{cases} c(x-0)\mu + c(x+0)(1-\mu) & , \quad \mu \in [0,1],\ x \in (-1,1) , \\[2ex] c(1)[1-f(\mu)] + f(\mu) & , \quad \mu \in [0,1],\ x = 1 , \\[2ex] 1 - f(\mu) + c(-1)f(\mu) & , \quad \mu \in [0,1],\ x = -1 , \end{cases} \tag{6.1}$$

$c(x) = \dfrac{a(x)+b(x)}{a(x)-b(x)}$, and $f(\mu) = \sin\dfrac{\pi\mu}{2}\exp\left(\dfrac{i\pi(\mu-1)}{2}\right)$. Note that $z_1[1-f(\mu)] + z_2f(\mu)$, $\mu \in [0,1]$, describes the half-circle line from z_1 to z_2 that lies to the right of the straight line from z_1 to z_2. Thus, the image of $\mathbf{c}(x,\mu)$ is a closed curve in the complex plane, which possesses a natural orientation, and by wind $\mathbf{c}(x,\mu)$ we denote the winding number of this curve with respect to the origin 0. Then, the invertibility of A implies wind $\mathbf{c}(x,\mu) = 0$ (cf. Gohberg & Krupnik [5, Theorem

IX.4.1]). Since $\widetilde{A}_M$ is also assumed to be invertible in $\mathbf{L}^2_\sigma$, analogous relations hold for $d(x) := 1/c(x)$ instead of $c(x)$.

We will show now that under the assumptions of the lemma both $c(1)$ and $c(-1)$ are located in the right half plane, from which the assertion of the lemma follows immediately. Evidently, the real parts of $c(1)$ and $c(-1)$ cannot vanish because of $\mathbf{c}(\pm 1, \mu) \neq 0$, $\mu \in [0,1]$. Consider for instance the case $\Re c(1) < 0$, $\Re c(-1) > 0$. By arg we denote a continuous branch of the argument defined on $\{\mathbf{c}(x,\mu) : -1 < x < 1,\ \mu \in [0,1]\} \cup \{c(-1), c(1)\}$. Let $c(1) = |c(1)| \exp(i\gamma)$, $c(-1) = |c(-1)| \exp(i\delta)$, $0 \le \gamma < 2\pi$. Then the argument increase of the closed curve described by $\mathbf{c}(x,\mu)$ equals $2\pi - \gamma + \delta + \arg c(1) - \arg c(-1)$, which must be zero. On the other hand, the argument increase of $\mathbf{d}(x,\mu)$ is $\gamma - \delta - \arg c(1) + \arg c(-1) = 2\pi$ in contradiction to the invertibility of $\widetilde{A}_M$. All other cases can be treated analogously. ■

Now we are able to prove the following theorem on the stability of the collocation method.

Theorem 6.4 *For piecewise continuous coefficients a and b, the collocation method (2.5) is stable in $\mathbf{L}^2_\sigma$ if and only if the operators $aI \pm bS$ are invertible in $\mathbf{L}^2_\sigma$.*

Proof. Due to Theorem 3.1 and Proposition 4.3, we only have to consider the invertibility of the coset $\{A_{n,M}\} + \mathcal{I}$ in $\mathcal{A}/\mathcal{I}$. First, let $\tau \in (-1,1)$. We choose $a_\tau, b_\tau \in \mathbf{PC}[-1,1]$ such that $a_\tau(\tau \pm 0) = a(\tau \pm 0)$, $b_\tau(\tau \pm 0) = b(\tau \pm 0)$, $b_\tau(\pm 1) = 0$, and $a_\tau I \pm b_\tau S$ are invertible in $\mathbf{L}^2_\sigma$. If $\mathbf{c}_\tau(x,\mu)$ is defined via (6.1) with a_τ, b_τ instead of a, b, then (because of $b_\tau(\pm 1) = 0$) $\mathbf{c}_\tau(\pm 1, \mu) \equiv 1$, $\mu \in [0,1]$. This leads to the invertibility of $A^{\mathbb{T}}_\tau := \widehat{a_\tau} I - \widehat{b_\tau} \chi S_{\mathbb{T}}$ in $\mathbf{L}^2(\mathbb{T})$. Lemma 4.1(c) shows that the sequence $\{M^{\mathbb{T}}_n A^{\mathbb{T}}_\tau P^{\mathbb{T}}_n\}$ is stable in $\mathbf{L}^2(\mathbb{T})$, which means that there exists a sequence $B^{\mathbb{T}}_{n,\tau} \in \mathcal{L}(\operatorname{im} P^{\mathbb{T}}_n)$ such that $B^{\mathbb{T}}_{n,\tau} M^{\mathbb{T}}_n A^{\mathbb{T}}_\tau P^{\mathbb{T}}_n = P^{\mathbb{T}}_n$ for all sufficiently large n and $\left\| B^{\mathbb{T}}_{n,\tau} P^{\mathbb{T}}_n \right\|_{\mathcal{L}(\mathbf{L}^2(\mathbb{T}))} \le \text{const}$. Now we put $B^\tau_n := F^{-1} T B^{\mathbb{T}}_{n,\tau} F P_n$. Because of Lemma 6.2 $\{A^\tau_{n,M}\} + \mathcal{I}$ with $A^\tau_{n,M} := M_n(a_\tau I + b_\tau S) P_n$ is an $\mathcal{M}_\tau$-equivalent local representative of $\{A_{n,M}\} + \mathcal{I}$. Using (A)-(D) we obtain

$$B^\tau_n A^\tau_{n,M} = F^{-1} T B^{\mathbb{T}}_{n,\tau} F P_n F^{-1} M^{\mathbb{T}}_n A^{\mathbb{T}}_\tau P^{\mathbb{T}}_n F P_n = P_n$$

for all sufficiently large n. Moreover, $\|B^\tau_n P_n\|_{\mathcal{L}(\mathbf{L}^2_\sigma)} \le \left\| B^{\mathbb{T}}_{n,\tau} P^{\mathbb{T}}_n \right\|_{\mathcal{L}(\mathbf{L}^2(\mathbb{T}))} \le \text{const}$.

For $\tau = \pm 1$, we choose $A^\tau_{n,M} = M_n[a(\pm 1) I + b(\pm 1) S] P_n$. In this case, the stability of $\{M^{\mathbb{T}}_n [a(\pm 1) I - b(\pm 1) \chi S_{\mathbb{T}}] P^{\mathbb{T}}_n\}$ is easily seen by Lemma 6.3 and relation (4.4). Thus, the local principle of Gohberg and Krupnik (Theorem 3.2) yields the invertibility of the coset $\{A_{n,M}\} + \mathcal{I}$. ■

7 The system case

One of the advantages of our collocation method based on weighted polynomials is the fact that it can be easily generalized to the system case, that is, to problems of the form

$$\sum_{j=1}^{k}(a_{ij}I + b_{ij}S)u_j = f_i, \qquad i = 1, \dots, k, \tag{7.1}$$

where a_{ij}, $b_{ij} \in \mathbf{PC}[-1,1]$ and $f_i \in \mathbf{L}^2_\sigma$ are given functions and the functions $u_j \quad (j = 1, \dots, n)$ are unknown. The usual polynomial approximation methods based on certain mapping properties of weighted singular integral operators with respect to orthogonal polynomials are not suitable for this kind of problems if the matrices (a_{ij}), (b_{ij}) are not diagonal.

To avoid technical difficulties, we will restrict ourselves to deriving a stability result for the case of piecewise continuous coefficients under the additional assumption that $b_{ij}(\pm 1) = 0$ for all i, j.

Let $k > 1$ be an integer. By $(\mathbf{L}^2_\sigma)^k$ denote the cross product of k copies of the space $\mathbf{L}^2_\sigma$, equipped with the inner product

$$\langle \underline{u}, \underline{v} \rangle := \sum_{j=1}^{k} \langle u_j, v_j \rangle_\sigma ,$$

where $\underline{u} = (u_j)_{j=1}^k$, $\underline{v} = (v_j)_{j=1}^k \in (\mathbf{L}^2_\sigma)^k$. By $\mathbf{PC}^{k\times k}[-1,1]$ we denote the set of all $k \times k$-matrices with entries from $\mathbf{PC}[-1,1]$. By $\underline{S}$ we denote the diagonal operator $(\delta_{ij}S)_{i,j=1}^k \in \mathcal{L}((\mathbf{L}^2_\sigma)^k)$. Analogously, $\underline{P}_n$, $\underline{M}_n$, $\underline{W}_n$ etc. are defined. We have

$$\operatorname{im} \underline{P}_n = \operatorname{span}\{\underline{\tilde{u}}_{jm} : j = 0, \dots, n-1;\ m = 1, \dots, k\},$$

where $\underline{\tilde{u}}_{jm} = (\delta_{im}\widetilde{u}_j)_{i=1}^k$. Thus, we can write (7.1) in the form

$$(\underline{a}\,\underline{I} + \underline{b}\,\underline{S})\underline{u} = \underline{f}$$

($\underline{a}, \underline{b}, \in \mathbf{PC}^{k\times k}$, $\underline{f} \in (\mathbf{L}^2_\sigma)^k$ given, $\underline{u} \in (\mathbf{L}^2_\sigma)^k$ unknown), and we will consider the collocation method

$$\underline{M}_n(\underline{a}\,\underline{I} + \underline{b}\,\underline{S})\underline{P}_n\underline{u}_n = \underline{M}_n\underline{f}\,, \qquad \underline{u}_n \in \operatorname{im} \underline{P}_n\,,$$

for its approximate solution.

The algebra $\mathcal{A}$ and the ideal $\mathcal{I}$ as well as the associated operator sequences on the unit circle are defined analogously to the scalar case using the mapping $\underline{F} := (\delta_{ij}F)_{i,j=1}^k : (\mathbf{L}^2_\sigma)^k \to (L^2(\mathbb{T}))^k$. Beside the algebra $\mathcal{A}$ and the ideal $\mathcal{I}$ we also consider the respective algebra $\mathcal{A}^{\mathbb{T}}$ with the ideal $\mathcal{I}^{\mathbb{T}}$ constructed in the same way with $\underline{P}_n^{\mathbb{T}}$ and $\underline{W}_n^{\mathbb{T}}$ instead of $\underline{P}_n$ and $\underline{W}_n$, respectively. From the proof of Junghanns & Silbermann [11, Theorem 3.1] one can verify the following result.

Lemma 7.1 *Let* $\underline{c}, \underline{d} \in \mathbf{PC}^{k\times k}(\mathbb{T})$ *and* $A_{\mathbb{T}} := \underline{c}\,\underline{I} + \underline{d}\,\underline{S}_{\mathbb{T}} : (\mathbf{L}^2(\mathbb{T}))^k \longrightarrow (\mathbf{L}^2(\mathbb{T}))^k$ *be a* Φ*-operator. Then the coset* $\{\underline{M}_n^{\mathbb{T}} A_{\mathbb{T}} \underline{P}_n^{\mathbb{T}}\} + \mathcal{I}^{\mathbb{T}}$ *is invertible in* $\mathcal{A}^{\mathbb{T}}/\mathcal{I}^{\mathbb{T}}$.

Now we are in a position to prove the necessary and sufficient stability result.

Theorem 7.2 *Let* $\underline{a}, \underline{b} \in \mathbf{PC}^{k\times k}[-1,1]$, $\underline{b}(\pm 1) = 0$, $A = \underline{a}\,\underline{I} + \underline{b}\,\underline{S}$. *Then the sequence* $\{A_n\} = \{\underline{M}_n A \underline{P}_n\}$ *is stable if and only if the operators* A *and* $\widetilde{A} = \underline{a}\,\underline{I} - \underline{b}\,\underline{S}$ *are invertible in* $(\mathbf{L}^2_\sigma)^k$.

Proof. In the same way as in the scalar case one can show that $\underline{M}_n A \underline{P}_n$ and $\underline{W}_n \underline{M}_n A \underline{W}_n$ converge strongly to A and $\widetilde{A}$, respectively. Thus, as in the proof of Theorem 6.4, we only have to show that the two invertibility conditions imply the invertibility of the coset $\{A_n\} + \mathcal{I}$. At first we observe that $A = \underline{F}^{-1} A_{\mathbb{T}} \underline{F}$, where $A_{\mathbb{T}} = \underline{c}\,\underline{I} + \underline{d}\,\underline{S}_{\mathbb{T}}$ with $\underline{c} = \widehat{\underline{a}}$, $\underline{d} = \widehat{\underline{b}}\,\underline{\chi}$. Since $\underline{c}$ and $\underline{d}$ are continuous in ± 1 (because of $\underline{b}(\pm 1) = 0$), the invertibility of A and $\widetilde{A}$ implies that $A_{\mathbb{T}}$ is a Φ-operator (see Michlin & Prössdorf [13, Theorem 6.1]). By Lemma 7.1 we have the invertibility of the coset $\{\underline{M}^{\mathbb{T}}_n A_{\mathbb{T}} \underline{P}^{\mathbb{T}}_n\} + \mathcal{I}^{\mathbb{T}}$, from which we can conclude the invertibility of $\{A_n\} + \mathcal{I}$. ∎

8 Weighted Sobolev spaces and convergence rates

The aim of this section is to introduce an appropriate scale of Sobolev spaces and to study the mapping properties of the Cauchy singular integral operator involved in equation (1.1) in order to give a convergence rate for the error of the collocation method. (For the finite section method, the considerations are analogous.)

How to define these Sobolev spaces is suggested by the orthonormal system $\{\widetilde{u}_n\}$ in $\mathbf{L}^2_\sigma$, which we use as ansatz functions for the considered approximation methods. Thus, analogously to Berthold, Hoppe & Silbermann [1] we define

$$\widetilde{\mathbf{L}}^2_{\sigma,s} := \left\{ u \in \mathbf{L}^2_\sigma : \|u\|_{s,\sim} := \left(\sum_{n=0}^{\infty} (1+n)^{2s} |\langle u, \widetilde{u}_n \rangle_\sigma|^2 \right)^{\frac{1}{2}} < \infty \right\}$$

for all $s \geq 0$. Since

$$\langle u, \widetilde{u}_n \rangle_\sigma = \langle \sigma u, U_n \rangle_\varphi \tag{8.1}$$

we have $\|u\|_{s,\sim} = \|\sigma u\|_{\varphi,s}$, where

$$\mathbf{L}^2_{\varphi,s} := \left\{ v \in \mathbf{L}^2_\varphi : \|v\|_{\varphi,s} := \left(\sum_{n=0}^{\infty} (1+n)^{2s} |\langle v, U_n \rangle_\varphi|^2 \right)^{\frac{1}{2}} < \infty \right\}$$

is a special case of the Sobolev spaces studied in Berthold, Hoppe & Silbermann [1]. This means that the multiplication operator $\sigma I : \widetilde{\mathbf{L}}^2_{\sigma,s} \longrightarrow \mathbf{L}^2_{\varphi,s}$ is an isometric isomorphism. Relation (4.5) shows that

$$S : \widetilde{\mathbf{L}}^2_{\sigma,s} \longrightarrow \mathbf{L}^2_{\sigma,s} \quad \text{is continuous for all} \quad s \geq 0\,, \tag{8.2}$$

where $\mathbf{L}^2_{\sigma,s}$ is defined in the same way as $\mathbf{L}^2_{\varphi,s}$ with the weight σ instead of φ and the polynomials T_n instead of U_n. Thus, to find sufficient conditions on a and b

such that $A = aI + bS$ belongs to $\mathcal{L}(\widetilde{\mathbf{L}}^2_{\sigma,s})$, we have to study the multiplication operators $aI : \widetilde{\mathbf{L}}^2_{\sigma,s} \longrightarrow \widetilde{\mathbf{L}}^2_{\sigma,s}$ and $bI : \mathbf{L}^2_{\sigma,s} \to \widetilde{\mathbf{L}}^2_{\sigma,s}$. At first we remember that, for any integer $r \geq 0$, the norm $\|u\|_{\sigma,r}$ is equivalent to

$$\|u\|_{\sigma,(r)} := \sum_{k=0}^{r} \left\| \varphi^k u^{(k)} \right\|_\sigma$$

(Berthold, Hoppe & Silbermann [1, pp. 196,197]) and that $aI \in \mathcal{L}(\mathbf{L}^2_{\varphi,s})$, $0 \leq s \leq r$, if $\varphi^k a^{(k)} \in \mathbf{L}^\infty(-1,1)$, $k = 0, \ldots, r$ (Junghanns [8, Lemma 3.5]).

Lemma 8.1 *Let $r \geq 0$ be an integer. If $\varphi^k a^{(k)} \in \mathbf{L}^\infty(-1,1)$ and $\varphi^{k+1}(b\sigma)^{(k)} \in \mathbf{L}^\infty(-1,1)$, $k = 0, \ldots, r$, then $aI \in \mathcal{L}(\widetilde{\mathbf{L}}^2_{\sigma,s})$ and $bI \in \mathcal{L}(\mathbf{L}^2_{\sigma,s}, \widetilde{\mathbf{L}}^2_{\sigma,s})$, $0 \leq s \leq r$.*

Proof. From $\|au\|_{s,\sim} = \|a\sigma u\|_{\varphi,s}$ and $\|u\|_{s,\sim} = \|\sigma u\|_{\varphi,s}$ one can see that $aI \in \mathcal{L}(\widetilde{\mathbf{L}}^2_{\sigma,s})$ if and only if $aI \in \mathcal{L}(\mathbf{L}^2_{\varphi,s})$. The second assertion follows from

$$\begin{aligned}
\|bu\|_{r,\sim} &= \|b\sigma u\|_{\varphi,r} \\
&\sim \sum_{k=0}^{r} \left\| \varphi^k (b\sigma u)^{(k)} \right\|_\varphi = \sum_{k=0}^{r} \left\| \varphi^{k+1} (b\sigma u)^{(k)} \right\|_\sigma \\
&\leq \sum_{k=0}^{r} \sum_{j=0}^{k} \binom{k}{j} \left\| \varphi^{j+1} (b\sigma)^{(j)} \right\|_\infty \left\| \varphi^{k-j} u^{(k-j)} \right\|_\sigma \\
&\leq \text{const} \sum_{k=0}^{r} \sum_{j=0}^{k} \binom{k}{j} \left\| \varphi^{j+1} (b\sigma)^{(j)} \right\|_\infty \|u\|_{\sigma,r}
\end{aligned}$$

and the interpolation property of the Sobolev spaces (comp. Junghanns [8, Remark 1.5]). ■

Lemma 8.2 *For $s \geq 0$ and $f \in \widetilde{\mathbf{L}}^2_{\sigma,s}$ the following assertions hold:*

(a) $\lim_{n\to\infty} \|f - P_n f\|_{s,\sim} = 0$,

(b) $\|f - P_n f\|_{t,\sim} \leq (1+n)^{t-s} \|f\|_{s,\sim}$, $0 \leq t \leq s$,

(c) $\|P_n f\|_{t,\sim} \leq n^{t-s} \|f\|_{s,\sim}$, $t \geq s$,

(d) $\lim_{n\to\infty} \|f - M_n f\|_{s,\sim} = 0$ *if* $s > \frac{1}{2}$,

(e) $\|f - M_n f\|_{t,\sim} \leq \text{const}\, n^{t-s} \|f\|_{s,\sim}$, $0 \leq t \leq s$, $s > \frac{1}{2}$.

Proof. Define, for $u \in \mathbf{L}^2_\varphi$, $P_n^\varphi u = \sum_{k=0}^{n-1} \langle u, U_k \rangle_\varphi U_k$. Then, in view of (8.1), $P_n = \varphi P_n^\varphi \sigma$. Since also M_n is defined as $M_n = \varphi L_n^\varphi \sigma$, the assertions (a)-(e) are immediate consequences of the respective properties of P_n^φ and L_n^φ in the scale $\mathbf{L}^2_{\varphi,s}$, $s \geq 0$ (Berthold, Hoppe & Silbermann [1, Lemma 2.2 and Theorem 3.4]). ■

Theorem 8.3 *Assume that* $\frac{1}{2} < s < r$, r *an integer, and that* $\varphi^k a^{(k)}$, $\varphi^{k+1}(b\sigma)^{(k)} \in \mathbf{L}^\infty(-1,1)$, $k = 0, \ldots, r$. *If the collocation method (2.5) is stable and if the solution* u^* *of equation (1.1) belongs to* $\widetilde{\mathbf{L}}^2_{\sigma,s}$, *then*

$$\|u_n^* - u^*\|_{t,\sim} \leq \text{const}\, n^{t-s} \|u^*\|_{s,\sim}\,, \quad 0 \leq t \leq s\,,$$

where $u_n^* \in \operatorname{im} P_n$ *is the solution of (2.5).*

Proof. Since $\{A_{n,M}\}$ is assumed to be stable we have, in view of Lemma 8.2(c),(e),

$$\begin{aligned}
\|P_n u^* - u_n^*\|_{t,\sim} &\leq n^t \|P_n u^* - u_n^*\|_\sigma \\
&\leq \text{const}\, n^t \|A_{n,M} P_n u^* - M_n f\|_\sigma \\
&= \text{const}\, n^t \|M_n A(P_n u^* - u^*)\|_\sigma \\
&\leq \text{const}\, n^t \left(\|(M_n - I)A(P_n u^* - u^*)\|_\sigma + \|A(P_n u^* - u^*)\|_\sigma\right) \\
&\leq \text{const}\, n^t \left(n^{-s} \|A(P_n u^* - u^*)\|_{s,\sim} + \|P_n u^* - u^*\|_\sigma\right).
\end{aligned}$$

By Lemma 8.1 and (8.2) $A \in \mathcal{L}(\widetilde{\mathbf{L}}^2_{\sigma,s})$. Thus, taking into account Lemma 8.2(b), the assertion follows. ∎

9 Implementation of the collocation method and numerical results

A suitable implementation of the collocation method enables us to solve the resulting system of linear equations with a fast algorithm that requires only $O(n^2)$ operations and $O(n)$ storage due to the special structure of the system matrix. For this end, we have to choose an even number n of collocation points.

We search for the values of the approximate solution u_n of (2.5) in the Chebyshev nodes of first kind $x_{kn} = \cos \dfrac{2k-1}{2n}\pi$, $k = 1, \ldots, n$, that is, the zeros of $T_n(x)$. Therefore, since n is even, none of these nodes coincides with one of the collocation points x_{jn}^φ. We now write the weighted polynomial u_n in the form

$$u_n(x) = \varphi(x) w_n(x) = \varphi(x) \sum_{k=1}^n \xi_{kn} l_{kn}(x), \tag{9.1}$$

where

$$l_{kn}(x) = \frac{T_n(x)}{(x - x_{kn}) T_n'(x_{kn})}$$

is the k-th fundamental polynomial of Lagrange interpolation with respect to the nodes x_{kn}. Let

$$\lambda_{kn} = \frac{1 - x_{kn}^2}{i\,n}.$$

Then we have (using the Gaussian rule with respect to $\sigma(x)$)

$$\frac{1}{\pi i}\int_{-1}^{1} \varphi(t) w_n(t)\,dt = \frac{1}{\pi i}\int_{-1}^{1} (1-t^2) w_n(t)\sigma(t)\,dt = \sum_{k=1}^{n} \lambda_{kn} w_n(x_{kn})$$

if w_n is a polynomial of degree less than $2n-2$.

Let us consider now the action of the operator S on a weighted polynomial of the form (9.1). Let $x \neq x_{kn}$ for all $k = 1, \ldots, n$. Then

$$\begin{aligned}(S\varphi w_n)(x) &= w_n(x)(S\varphi)(x) + \frac{1}{\pi i}\int_{-1}^{1} \frac{w_n(t) - w_n(x)}{t-x}\varphi(t)\,dt \\ &= w_n(x)(S\varphi)(x) + \sum_{k=1}^{n} \lambda_{kn} \frac{\xi_{kn} - w_n(x)}{x_{kn} - x},\end{aligned} \tag{9.2}$$

where $\xi_{kn} = w_n(x_{kn})$. Obviously, this formula still holds if w_n is a polynomial of degree $2n-2$. Thus, if we put $w_n = T_n$ and note that $T_n(x_{kn}) = 0$, (9.2) yields

$$\varrho_n(x) := (S\varphi T_n)(x) = T_n(x)\left[(S\varphi)(x) - \sum_{k=1}^{n} \frac{\lambda_{kn}}{x_{kn} - x}\right].$$

Note that $\lambda_{kn} = \dfrac{\varrho(x_{kn})}{T_n'(x_{kn})}$. Using the latter relation, we obtain that (9.2) equals

$$(S\varphi w_n)(x) = \sum_{k=1}^{n} \lambda_{kn} \frac{\xi_k}{x_{kn} - x} + w_n(x)\frac{\varrho_n(x)}{T_n(x)}.$$

Now we can write down equation (2.5) in the form

$$\begin{aligned}&a(x_{jn}^{\varphi})\varphi(x_{jn}^{\varphi})w_n(x_{jn}^{\varphi}) + b(x_{jn}^{\varphi})(S\varphi w_n)(x_{jn}^{\varphi}) \\ &\qquad = \sum_{k=1}^{n} \frac{a_k e_j - b_j}{c_k - d_j}\frac{1}{T_n'(x_{kn})}\,\xi_{kn} = f(x_{jn}^{\varphi}),\end{aligned}$$

where

$$\begin{aligned}a_k &= \lambda_{kn} T_n'(x_{kn}) = \varrho_n(x_{kn}), \\ b_j &= b(x_{jn}^{\varphi})\varrho_n(x_{jn}^{\varphi}) + a(x_{jn}^{\varphi})\varphi(x_{jn}^{\varphi})T_n(x_{jn}^{\varphi}), \\ c_k &= x_{kn}, \quad d_j = x_{jn}^{\varphi}, \quad e_j = b(x_{jn}^{\varphi}).\end{aligned}$$

Note that

$$\varrho_n = S\varphi T_n = \frac{1}{2} S\varphi(U_n - U_{n-2}) = \frac{i}{2}(T_{n+1} - T_{n-1}).$$

In the following we will show how to solve efficiently a system of linear equations with a Löwner-like matrix $\mathbf{A}_n = (a_{jk})_{j,k=1}^n$, where

$$a_{jk} = \frac{a_k e_j - b_j}{c_k - d_j}. \tag{9.3}$$

Having solved this system, we only have to multiply its solution by the diagonal matrix $\operatorname{diag}(T_n'(x_{kn}))_{k=1}^n$ to obtain the ξ_{kn}. We assume that $\mathbf{A}_n$ is strongly regular, that means all sections $\mathbf{A}_m = (a_{jk})_{j,k=1}^m$ are invertible for $m = 1, \ldots, n$. The fast algorithm is essentially based on the following two lemmata (comp. Heinig & Rost [7]).

Lemma 9.1 *Let $\mathbf{A}_n = (a_{jk})_{j,k=1}^n$ be regular, and let $x^{n-1} = (x_k^{n-1})_{k=1}^{n-1}$ and $x_0^{n-1} = (x_{0k}^{n-1})_{k=1}^{n-1}$ be solutions of*

$$\mathbf{A}_{n-1} x^{n-1} = (f_j)_{j=1}^{n-1} \text{ and } \mathbf{A}_{n-1} x_0^{n-1} = (a_{jn})_{j=1}^{n-1}.$$

Then the solution x_n of $\mathbf{A}_n x_n = (f_j)_{j=1}^n$ is given by

$$x_n = \begin{pmatrix} x^{n-1} \\ 0 \end{pmatrix} + \frac{\gamma_n}{\eta_n} \begin{pmatrix} x_0^{n-1} \\ -1 \end{pmatrix},$$

where

$$\gamma_n = f_n - \sum_{k=1}^{n-1} a_{nk} x_k^{n-1}, \qquad \eta_n = \sum_{k=1}^{n-1} a_{nk} x_{0k}^{n-1} - a_{nn}.$$

(Note that the regularity of $\mathbf{A}_n$ implies $\eta_n \neq 0$.) The special structure of the matrix (9.3) allows to determine the vectors x_0^m recursively from the solutions x_1^m, x_2^m of the fundamental equations

$$\mathbf{A}_m x_1^m = (e_j)_{j=1}^m \text{ and } \mathbf{A}_m x_2^m = (-b_j)_{j=1}^m.$$

Lemma 9.2 (cf. [2], Equ. (2.31)) *For $m = 1, \ldots, n-1$ let*

$$\beta_{m1} = 1 - \sum_{k=1}^m \frac{a_{m+1} - a_k}{c_{m+1} - c_k} x_{1k}^m, \qquad \beta_{m2} = a_{m+1} + \sum_{k=1}^m \frac{a_{m+1} - a_k}{c_{m+1} - c_k} x_{2k}^m,$$

and

$$\alpha_m = \beta_{m1} \left[1 + \sum_{k=1}^m \frac{x_{2k}^m}{c_{m+1} - c_k} \right] + \beta_{m2} \sum_{k=1}^m \frac{x_{1k}^m}{c_{m+1} - c_k}.$$

Then

$$x_{0k}^m = \frac{\beta_{m2} x_{1k}^m + \beta_{m1} x_{2k}^m}{\alpha_m (c_{m+1} - c_k)}, \qquad k = 1, \ldots, m.$$

Now we can alternately apply Lemma 9.1 and Lemma 9.2 to obtain the following algorithm:

- Put $x_1^1 := f_1/a_{11}$, $x_{01}^1 := a_{12}/a_{11}$, $x_{11}^1 := e_1/a_{11}$, $x_{21}^1 := -b_1/a_{11}$
- FOR $m := 2$ TO $n-1$ DO
  - Compute x^m, x_1^m, x_2^m from x^{m-1}, x_1^{m-1} and x_2^{m-1} by Lemma 9.1
  - Compute x_0^m from x_1^m and x_2^m by Lemma 9.2
- Compute x_n by Lemma 9.1

In the following we present some numerical results for the collocation method. At first, we consider three examples for the coefficients $a(x)$ and $b(x)$ with various solutions u^* of the original equation (1.1) and of different smoothness in order to verify the assertion of Theorem 8.3.

(A) $a(x) = \operatorname{sgn}(x)$, $b(x) = ix$.

(A1) $f(x) = x\left(1 + \frac{x}{\pi}\ln\frac{1-x^2}{x^2}\right)$, $u^*(x) = |x|$.

(A2) $f(x) = (1-x)\left[\operatorname{sgn}(x) + \frac{x}{\pi}\ln\frac{1-x}{1+x}\right] - \frac{2x}{\pi}$, $u^*(x) = 1-x$.

(B) $a(x) \equiv 2$, $b(x) = i(1-x^2)$. The assumptions of Theorem 8.3 are fulfilled for $r = 3$.

(B1) $f(x) = 2|x| + \frac{x(1-x^2)}{\pi}\ln\frac{1-x^2}{x^2}$, $u^*(x) = |x|$.

(B2) $f(x) = 2(1-x) + \frac{1-x^2}{\pi}\left[(1-x)\ln\frac{1-x}{1+x} - 2\right]$, $u^*(x) = 1-x$.

(B3) $f(x) = (1-x^2)\left(2 + \frac{1-x^2}{\pi}\ln\frac{1-x}{1+x} - \frac{2x}{\pi}\right)$, $u^*(x) = 1-x^2$.

Remark that $u^* \in \widetilde{\mathbf{L}}^2_{\sigma,s}$ for $s < \frac{1}{2}$ in examples (B1), (B2) and for $s < \frac{5}{2}$ in example (B3).

(C) $a(x) \equiv 2$, $b(x) = i(1-x^2)^{\frac{3}{2}}$. Here, in Theorem 8.3, the integer $r \geq 0$ can be chosen arbitrarily. If

$$f(x) = \sqrt{1-x^2}\left[2|x| + (1-x^2)\left(\frac{x\sqrt{1-x^2}}{\pi}\ln\frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}} - \frac{2x}{\pi}\right)\right]$$

then $u^*(x) = |x|\sqrt{1-x^2}$ and $u^* \in \widetilde{\mathbf{L}}^2_{\sigma,s}$ for $s < \frac{3}{2}$.

Taking into account Lemma 8.2(b) it is sufficient to compute

$$\varepsilon_{n,t}(u_n^*) := \|P_n u^* - u_n^*\|_{t,\sim} = \sqrt{\sum_{k=0}^{n-1}(1+k)^{2t}|\alpha_{kn} - \alpha_k|^2}$$

in order to check the convergence rate, where

$$\alpha_k := \langle u^*, \widetilde{u}_k\rangle_\sigma \quad \text{and} \quad \alpha_{kn} := \langle u_n^*, \widetilde{u}_k\rangle_\sigma$$

are the Fourier coefficients of the exact and the approximate solution, respectively.

Using the Gaussian rule with respect to the Chebyshev nodes of first kind we find

$$\alpha_{jn} = \frac{\pi}{n}\sum_{k=1}^{n}(1-x_{kn}^2)\xi_{kn}U_j(x_{kn})\,, \qquad j=0,1,\ldots,n-2\,,$$

with ξ_{kn} from (9.1). Furthermore, with the help of the three-term recurrence relation

$$U_{j+1}(x) = 2xU_j(x) - U_{j-1}(x)\,, \qquad j=1,2,\ldots\,, \tag{9.4}$$

and $T_{n+1}(x) = \frac{1}{2}[U_{n+1}(x) - U_{n-1}(x)]$ we obtain

$$(1-x^2)U_{n-1}(x) = \frac{1}{2}[xU_{n-2}(x) - U_{n-3}(x)] - \frac{1}{2}T_{n+1}(x)$$

and, consequently,

$$\alpha_{n-1,n} = \frac{\pi}{2n}\sum_{k=1}^{n}[x_{kn}U_{n-2}(x_{kn}) - U_{n-3}(x_{kn})]\xi_{kn}\,.$$

(Note that T_{n+1} is orthogonal with respect to $\langle\cdot,\cdot\rangle_\sigma$ to all polynomials of lower degree.) Thus, again by (9.4) and by $U_0(x) = \sqrt{\frac{2}{\pi}}$, $U_1(x) = 2xU_0(x)$, we find the following algorithm to compute the Fourier coefficients of the approximate solution effectively:

- Put $b_{k0} = \frac{\sqrt{2\pi}}{n}(1-x_{kn}^2)\xi_{kn}$, $b_{k1} = 2x_{kn}b_{k0}$, $k=1,\ldots,n$.
- Compute $\alpha_{0n} = \sum_{k=1}^{n} b_{k0}$, $\alpha_{1n} = \sum_{k=1}^{n} b_{k1}$.
- FOR j=2 TO n-2 DO
  - Put $b_{kj} = 2x_{kn}b_{k,j-1} - b_{k,j-2}$, $k=1,\ldots,n$.
  - Compute $\alpha_{jn} = \sum_{k=1}^{n} b_{kj}$.
- Compute $\alpha_{n-1,n} = \frac{1}{2}\sum_{k=1}^{n}\frac{x_{kn}b_{k,n-2} - b_{k,n-3}}{1-x_{kn}^2}$.

As the following tables show, in all examples we can observe a convergence rate depending from the smoothness of the solution of the original equation (1.1) although not all examples are covered by Theorem 8.3. We use the notation $\varepsilon_n(u_n^*) := \varepsilon_{n,0}(u_n^*)$.

Example (A1)		
n	$\varepsilon_n(u_n^*)$	$n^{0.5}\varepsilon_n(u_n^*)$
20	0.14450	0.646
40	0.10500	0.664
60	0.08654	0.670
80	0.07530	0.673
100	0.06754	0.675
500	0.03049	0.682
1000	0.02158	0.682
3000	0.01247	0.683
5000	0.00966	0.683

Example (A2)		
n	$\varepsilon_n(u_n^*)$	$n^{0.5}\varepsilon_n(u_n^*)$
20	0.27786	1.243
40	0.20108	1.272
60	0.16543	1.281
80	0.14381	1.286
100	0.12892	1.289
500	0.05807	1.298
1000	0.04110	1.300
3000	0.02374	1.300
5000	0.01839	1.300

Example (B1)		
n	$\varepsilon_n(u_n^*)$	$n^{0.5}\varepsilon_n(u_n^*)$
20	0.12130	0.542
40	0.09000	0.569
60	0.07476	0.579
80	0.06532	0.584
100	0.05873	0.587
500	0.02674	0.598
1000	0.01895	0.599
3000	0.01096	0.600
5000	0.00849	0.600

Example (B2)		
n	$\varepsilon_n(u_n^*)$	$n^{0.5}\varepsilon_n(u_n^*)$
20	0.18066	0.808
40	0.13050	0.825
60	0.10746	0.832
80	0.09349	0.836
100	0.08386	0.839
500	0.03788	0.847
1000	0.02682	0.848
3000	0.01550	0.849
5000	0.01201	0.849

Example (B3)		
n	$\varepsilon_n(u_n^*)$	$n^{2.5}\varepsilon_n(u_n^*)$
20	0.0004078075	0.730
40	0.0000813336	0.823
60	0.0000307953	0.859
80	0.0000153296	0.878
100	0.0000088909	0.889
500	0.0000001660	0.928
1000	0.0000000295	0.933
3000	0.0000000019	0.936
5000	0.0000000005	0.941

Example (C)		
n	$\varepsilon_n(u_n^*)$	$n^{1.5}\varepsilon_n(u_n^*)$
20	0.00658347	0.589
40	0.00240712	0.609
60	0.00132479	0.616
80	0.00086516	0.619
100	0.00062106	0.621
500	0.00005611	0.627
1000	0.00001986	0.628
3000	0.00000383	0.629
5000	0.00000178	0.629

With the following two examples we show that the condition of the invertibility of the operator $\widetilde{A} = aI - bS$ is essential for the stability of the collocation method.

(D1) $a(x) = 2 + \sqrt{1-x^2}\,,\ b(x) = -ix\,,$

$$f(x) = |x|\sqrt{1-x^2}(2+\sqrt{1-x^2}) - \frac{x^2}{\pi}\left(\sqrt{1-x^2}\ln\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}} - 2\right),$$

$$u^*(x) = |x|\sqrt{1-x^2}\,.$$

(D2) $a(x) = \sqrt{1-x^2}\,,\ b(x) = -ix\,,$
$$f(x) = |x|(1-x^2) - \frac{x^2}{\pi}\left(\sqrt{1-x^2}\ln\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}} - 2\right)\,,$$
$$u^*(x) = |x|\sqrt{1-x^2}\,.$$

In both examples (D1) and (D2) the operator $A = aI + bS$ is invertible in $\mathbf{L}^2_\sigma$. The operator $\widetilde{A} = aI - bS$ is invertible in $\mathbf{L}^2_\sigma$ in example (D1), but not in example (D2). The following tables compare the results for both examples and show the instabilities in example (D2).

Example (D1)		
n	$\varepsilon_n(u_n^*)$	$n^{1.5}\varepsilon_n(u_n^*)$
5000	0.00000189	0.669111
6000	0.00000144	0.669199
7000	0.00000114	0.669262
8000	0.00000094	0.669310
9000	0.00000078	0.669346
10000	0.00000067	0.669376
11000	0.00000058	0.669400
12000	0.00000051	0.669420
13000	0.00000045	0.669437
14000	0.00000040	0.669452
15000	0.00000036	0.669464

Example (D2)		
n	$\varepsilon_n(u_n^*)$	$n^{1.5}\varepsilon_n(u_n^*)$
5000	0.00000189	0.668841
6000	0.00000144	0.668974
7000	0.00000114	0.669077
8000	0.00000094	0.669147
9000	0.00000079	0.673460
10000	0.00000098	0.980731
11000	0.00000069	0.791523
12000	0.00000055	0.726934
13000	0.00000427	6.325547
14000	0.00000058	0.953810
15000	0.00000036	0.669447

The following table shows the behaviour of the algorithm for example (D2) between $n = 12000$ and $n = 14000$ in more detail.

Example (D2)		
n	$\varepsilon_n(u_n^*)$	$n^{1.5}\varepsilon_n(u_n^*)$
12100	0.00000050	0.671
12200	0.00000200	2.694
12400	0.00000183	2.525
12600	0.00003123	44.175
12800	0.00040888	592.127
13000	0.00000427	6.326
13200	0.00000413	6.270
13400	0.00000103	1.596
13600	0.00000080	1.268
13800	0.00000043	0.702
13900	0.00000041	0.670

In the following example (D3) both operators $A = iS$ and $\widetilde{A} = -iS$ are not invertible, and one can observe instabilities.

(D3) $a(x) \equiv 0\,,\ b(x) \equiv i\,,$

$$f(x) = \frac{x}{\pi}\left(\sqrt{1-x^2}\ln\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}} - 2\right)\,,$$

$$u^*(x) = |x|\sqrt{1-x^2}\,.$$

Example (D3)		
n	$\varepsilon_n(u_n^*)$	$n^{1.5}\varepsilon_n(u_n^*)$
200	0.00074038	2.094
300	0.00040362	2.097
400	0.00026282	2.103
1000	0.00006671	2.110
1500	0.00003630	2.109
2000	0.00002356	2.107
2500	0.00001689	2.111
3000	0.00001283	2.108
3500	0.00001025	2.123
4000	0.00000720	1.821
4500	0.00000698	2.106

Example (D3)		
n	$\varepsilon_n(u_n^*)$	$n^{1.5}\varepsilon_n(u_n^*)$
10300	0.00000493	5.151
10310	0.00000199	2.084
10320	0.00000502	5.267
10330	0.00000218	2.292
10340	0.00000204	2.147
10350	0.00002361	24.864
10360	0.00000190	2.001
10370	0.00000128	1.350
10380	0.00000230	2.427
10390	0.00000179	1.894
10400	0.00008048	85.360

With our last two examples we want to demonstrate that the approximation methods investigated in this paper can also be applied in Newton-like methods for the numerical solution of nonlinear Cauchy singular integral equations. For this aim we consider an equation of the type

$$F(x,u(x)) + \frac{1}{\pi}\int_{-1}^{1}\frac{u(y)}{y-x}\,dy = 0 \tag{9.5}$$

and look for an approximate solution $u_n(x)$ of the form (9.1), which satisfies

$$F(x_{jn}^{\varphi}, u_n(x_{jn}^{\varphi})) + \frac{1}{\pi}\int_{-1}^{1}\frac{u_n(y)}{y-x_{jn}^{\varphi}}\,dy = 0\,, \qquad j=1,\ldots,n\,. \tag{9.6}$$

The solution of (9.6) is approximated by $\{u_n^{(m)}\}_{m=0}^{\infty} \subset \mathrm{im}\,P_n$, where $u_n^{(m+1)} = u_n^{(m)} + v_n^{(m)}$ and

$$F_u(x_{jn}^{\varphi}, u_n^{(m)}(x_{jn}^{\varphi}))v_n^{(m)}(x_{jn}^{\varphi}) + \frac{1}{\pi}\int_{-1}^{1}\frac{v_n^{(m)}(y)}{y-x_{jn}^{\varphi}}\,dy$$

$$= -F(x_{jn}^{\varphi}, u_n^{(m)}(x_{jn}^{\varphi})) - \frac{1}{\pi}\int_{-1}^{1}\frac{u_n^{(m)}(y)}{y-x_{jn}^{\varphi}}\,dy\,, \quad j=1,\ldots,n\,.$$

Remark that the last system of equations is of type (2.5). Let us consider the examples

(E1) $F(x,u) = (3+x)u^2 - (1-x)\left[(3+x)(1-x) + \frac{1}{\pi}\ln\frac{1-x}{1+x}\right] + \frac{2}{\pi}$

and

(E2) $F(x,u) = (3+x)u^2 - (1-x^2)\left[(3+x)(1-x^2) + \frac{1}{\pi}\ln\frac{1-x}{1+x}\right] + \frac{2x}{\pi}$

with $u^*(x) = 1-x$ and $u^*(x) = |x|\sqrt{1-x^2}$ as the solution of (9.5), respectively. The following tables show the values $u_n^{(8)}(x_{jn}^{\varphi})$ respective $u_n^{(7)}(x_{jn}^{\varphi})$ in comparison with $u^*(x_{jn}^{\varphi})$ for $n = 20$, where the iteration with respect to m was started with $u^{(0)}(x) \equiv 0.5$ and was stopped, when $\|v_n^{(m)}\|_\sigma < 10^{-5}$.

Nonlinear SIE (9.5), Example (E1)

j	$u_n^{(8)}(x_{jn}^{\varphi})$	$u^*(x_{jn}^{\varphi})$
1	0.004738	0.011169
2	0.044585	0.044427
3	0.098166	0.099031
4	0.172867	0.173761
5	0.267069	0.266948
6	0.375692	0.376510
7	0.500191	0.500000
8	0.633953	0.634659
9	0.777670	0.777479
10	0.924564	0.925270

j	$u_n^{(8)}(x_{jn}^{\varphi})$	$u^*(x_{jn}^{\varphi})$
11	1.074944	1.074730
12	1.221685	1.222521
13	1.365625	1.365341
14	1.498809	1.500000
15	1.623959	1.623490
16	1.730910	1.733052
17	1.827287	1.826239
18	1.895318	1.900969
19	1.959763	1.955573
20	1.944654	1.988831

Nonlinear SIE (9.5), Example (E2)

j	$u_n^{(7)}(x_{jn}^{\varphi})$	$u^*(x_{jn}^{\varphi})$
1	0.021202	0.022214
2	0.087643	0.086881
3	0.187964	0.188255
4	0.317469	0.317329
5	0.462583	0.462635
6	0.611303	0.611260
7	0.749985	0.750000
8	0.866549	0.866526
9	0.950480	0.950484
10	0.994435	0.994415

j	$u_n^{(7)}(x_{jn}^{\varphi})$	$u^*(x_{jn}^{\varphi})$
11	0.994418	0.994415
12	0.950509	0.950484
13	0.866538	0.866526
14	0.750046	0.750000
15	0.611306	0.611260
16	0.462769	0.462635
17	0.317531	0.317329
18	0.188801	0.188255
19	0.087618	0.086881
20	0.024697	0.022214

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