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Exceptional vector bundles, tilting sheaves and tilting complexes on weighted projective lines

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Chapter 1

Introduction

This work deals with relationships between representation theory of finite dimensional algebras and some aspects of algebraic geometry. Sometimes classification problems of modules over certain algebras are largely equivalent to those for coherent sheaves on appropriate projective algebraic varieties. Such a correspondence is often given by what is called a tilting sheaf which can be understood in some sense as a basis of the derived category of coherent sheaves. The study of tilting sheaves and tilting modules allows therefore to transfer information from one category to another. Further the knowledge of the tilting objects in a fixed category also gives valuable information about its global structure. It is important to realize that the indecomposable direct summands of tilting objects are exceptional objects, which are rigid. Further, a natural generalization of the notion of tilting objects leads to the concept of tilting complexes in the derived category.

In this paper we investigate exceptional vector bundles, tilting sheaves and tilting complexes for weighted projective lines which were introduced by Geigle and Lenzing in order to give a geometrical approach to Ringel's canonical algebras.

We start with a brief historical sketch on the interplay between coherent sheaves and modules over finite dimensional algebras and give a survey on the contents of the paper afterwards.

Coherent sheaves for algebraic varieties were studied first by Serre in 1955 [111]. Later the concept of sheaves was used by Grothendieck to develop a general theory of schemes which belongs to the foundations of modern algebraic geometry [35]. In this context the following connections between modules over commutative noetherian rings and coherent sheaves play an important role. Let A be a commutative noetherian ring and $\mathbb{X} = \mathrm{Spec}(A)$ the affine scheme associated to A. Then the category of coherent sheaves $\mathrm{coh}(\mathbb{X})$ on $\mathbb{X}$ is equivalent to the category of finitely generated A-modules $\mathrm{mod}(A)$, mutually inverse functors are given by sheafification and the functor of global sections. On the other hand, let S be a $\mathbb{Z}$-graded noetherian ring and $\mathbb{X} = \mathrm{Proj}(S)$ the associated projective scheme. Then the sheafification functor

$$^{\sim} : \mathrm{mod}^{\mathbb{Z}}(S) \to \mathrm{coh}(\mathbb{X}),\ M \mapsto M^{\sim}$$

annihilates exactly the finitely generated $\mathbf{Z}$-graded S-modules $M \in \mathrm{mod}^{\mathbf{Z}}(S)$, which are of finite length and induces an equivalence $\mathrm{mod}^{\mathbf{Z}}(S)/\mathrm{mod}_0^{\mathbf{Z}}(S) \xrightarrow{\cong} \mathrm{coh}(\mathbf{X})$. Here $\mathrm{mod}^{\mathbf{Z}}(S)/\mathrm{mod}_0^{\mathbf{Z}}(S)$ is the quotient category of $\mathrm{mod}^{\mathbf{Z}}(S)$ with respect to the Serre subcategory $\mathrm{mod}_0^{\mathbf{Z}}(S)$ of finite length modules. This result is usually referred as Serre's theorem.

In their fundamental papers [10] and [9] (see also [31]) Bernstein-Gelfand-Gelfand and Beilinson gave descriptions of the derived category of coherent sheaves on a projective space $\mathbf{P}^n = \mathbf{P}(V)$ in terms of modules over finite dimensional algebras. These results translate problems of vector bundles on projective spaces to linear algebra and generalize the methods of monads introduced by Horrocks [54], which belong to the central techniques in the study of vector bundles.

Beilinson's result can be described in terms of tilting theory as follows. Let T_1 (resp. T_2) be the direct sum of sheaves of twisted differential forms $T_1 = \bigoplus_{0\le j\le n} \Omega^j(j)$ (resp. of twisted structure sheaves $T_2 = \bigoplus_{0\le j\le n} \mathcal{O}(j)$). Denote by $A_i = \mathrm{End}(T_i)$, $i = 1,2$, the endomorphism rings. Observe that A_1 and A_2 are isomorphic to triangular matrix rings

$$A_1 \cong \begin{pmatrix} \Lambda^0(V^*) & & & \\ \Lambda^1(V^*) & \Lambda^0(V^*) & & 0 \\ \vdots & & \ddots & \\ \Lambda^n(V^*) & \Lambda^{n-1}(V^*) & \dots & \Lambda^0(V^*) \end{pmatrix} \quad A_2 \cong \begin{pmatrix} S^0(V) & & & \\ S^1(V) & S^0(V) & & 0 \\ \vdots & & \ddots & \\ S^n(V) & S^{n-1}(V) & \dots & S^0(V) \end{pmatrix}$$

where $\Lambda(V^*) = \bigoplus_{0\le j\le n} \Lambda^j(V^*)$ and $S(V) = \bigoplus_{0\le j\le n} S^j(V)$ are the exterior algebra of the dual space V^* and the symmetric algebras of V, endowed with the natural grading, respectively. Then, for $i = 1,2$, the indecomposable direct summands of T_i generate the derived category $\mathcal{D}^b(\mathrm{coh}(\mathbf{P}^n))$ and, moreover, we have $\mathrm{Ext}^s(T_i, T_i) = 0$ for $s > 0$. A coherent sheaf satisfying these conditions is called nowadays a tilting sheaf. It follows that the derived functor

$$\mathbf{L}G_i : \mathcal{D}^b(\mathrm{mod}(A_i)) \to \mathcal{D}^b(\mathrm{coh}(\mathbf{P}^n))$$

of the functor $G_i = - \otimes_{A_i} T : \mathrm{mod}(A_i) \to \mathrm{coh}(\mathbf{P}^n)$ is an equivalence of triangulated categories.

In the simplest case $n = 1$ the algebras A_1 and A_2 coincide and are equal to the path algebra A of the quiver

$$\circ \rightrightarrows \circ$$

The problem of classifying of all modules over this algebra of a fixed class in the Grothendieck group $\mathrm{K}_0(\mathrm{mod}(A))$ represented by a dimension vector (n, m) is equivalent to the classification of all orbits of pairs of $m \times n$ matrices under the natural action of the group $Gl(n,k) \times Gl(m,k)$. This problem which has an application in the theory of differential equations was formulated and partially solved by Weierstraß in 1867 [118]. The final classification of all indecomposable modules in this case was given by Kronecker in 1890 [66], and the algebra is called nowadays the Kronecker algebra.

On the other hand, for the corresponding variety $\mathbf{P}^1$ Grothendieck described in 1957 the structure of the vector bundles by showing that that the indecomposable ones are the line bundles $\mathcal{O}(i)$, $i \in \mathbf{Z}$ [36]. Since each coherent sheaf on $\mathbf{P}^1$ is a direct sum of a vector bundle and a finite length sheaf and the structure of the latter ones is rather easy to determine, we can say that Kronecker's classification is equivalent to Grothendieck's result via the correspondence of Beilinson. Also, it was noticed by Seshadri [113] that the splitting theorem of Grothendieck is equivalent to a theorem on holomorphic invertible matrices on $\mathbf{C}^*$, which was proved by Birkhoff in 1913 [8] and which was already known to Plemelj in 1908 [94], to Hilbert in 1905 [50] and to Dedekind and Weber in 1892 [21] (see [91, Chapter I, §2, 2.4.]).

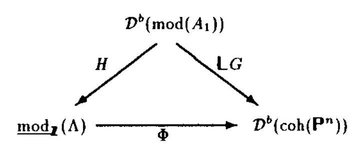
An alternative description of the derived category for coherent sheaves on projective spaces was given by Bernstein-Gelfand-Gelfand. They proved that the functor $\Phi : \mathrm{mod}^{\mathbf{Z}}(\Lambda) \to \mathcal{D}^b(\mathrm{coh}(\mathbf{P}^n))$, associating the complex

$$\Phi(M): \quad \dots \to M_j \otimes \mathcal{O}(j) \to M_{j+1} \otimes \mathcal{O}(j) \to \dots$$

to each $\mathbf{Z}$-graded module $M = \bigoplus_{j\in\mathbf{Z}} M_j$ over the exterior algebra $\Lambda = \bigoplus_{0\le j\le n} \Lambda^j(V)$ induces an equivalence $\underline{\mathrm{mod}}^{\mathbf{Z}}(\Lambda) \cong \mathcal{D}^b(\mathrm{coh}(\mathbf{P}^n))$, where $\underline{\mathrm{mod}}_{\mathbf{Z}}(\Lambda)$ denotes the category of $\mathbf{Z}$-graded Λ-modules modulo the projectives.

This construction has been generalized in [82] to tilting sheaves T on a nonsingular projective variety $\mathbf{X}$ of dimension d such that T has $d+1$ indecomposable direct summands. Later Polishchuk [95] gave an analogue result in the abstract setting of triangulated categories. Bernstein-Gelfand-Gelfand equivalences in the case that there exists a tilting sheaf such that the quiver of A has $d+1$ levels were given in [81] and by Hille in [51], [52].

The two rather different descriptions of Bernstein-Gelfand-Gelfand and Beilinson are related by a result of Happel which states that for any finite dimensional k-algebra A of finite global dimension there is an equivalence $H : \mathcal{D}^b(\mathrm{mod}(A)) \to \underline{\mathrm{mod}}_{\mathbf{Z}}(T(A))$ where $T(A) = A \ltimes \mathrm{D}(A)$ is the trivial extension of A by its minimal cogenerator $\mathrm{D}(A) = \mathrm{Hom}_k(A,k)$ [37]. Now, since $\mathrm{mod}_{\mathbf{Z}}(T(A_1)) \cong \mathrm{mod}_{\mathbf{Z}}(\Lambda)$ [38] we obtain the following triangle of equivalences



The functors $\Phi \circ H$ and $\mathbf{L}G$ are not isomorphic, however in [27] a correction automorphism in each vertex is given that makes the diagram commutative.

Beilinson's concept has several generalizations. In algebraic geometry, using the technique of a resolution of the diagonal as in Beilinson's proof, tilting sheaves were constructed for other varieties like Grassmanians, flag varieties, and certain intersections of quadrics by Kapranov [59], [60], [61], other examples related to the concept of exceptional sequences will be explained below.

On the other hand one should remark that tilting sheaves are quite rare, in particular a nonsingular projective curve $\mathbf{X}$ admits a tilting sheaf if and only if $\mathbf{X}$ is the projective line. An obvious condition for the existence of a tilting sheaf on a variety is that the Grothendieck group of the sheaf category is free of finite rank; recently Bondal has given a necessary condition in terms of the Hodge numbers (oral communication).

There is an analogous concept of tilting modules for finite dimensional algebras which was introduced by Brenner-Butler [17] and Happel-Ringel [45]. Since that time tilting theory has been developed rapidly by several mathematicians and has become a major tool in the representation theory. The basic idea of this concept is that if A is a finite dimensional algebra and T a tilting module in mod(A) then the categories of modules over A and $B = \mathrm{End}(T)$ are closely related in the sense that some nice subcategories are equivalent. It was also shown by Happel [37] that in this case mod(A) and mod(B) have equivalent derived categories which led to a systematic study of derived categories for finite dimensional algebras. The situation is particularly well understood if the algebra A one starts with is hereditary, because in this situation detailed information for the modules over A are available. In this case B is called a tilted algebra. In several papers (see for example [45], [100], [101], [63], [64]) specific results concerning the module category of a tilted algebra, including the shape of the Auslander-Reiten components, were obtained. Further, tilted algebras provide interesting classes of critical algebras, as the so called tame concealed algebras which coincide with the minimal representation-infinite algebras having a preprojective component [47], [16] and give a useful criterion for finite representation type [15] .

Recently Happel Reiten and Smalø [42] have developed an extensive theory of quasitilted algebras by investigating tilting theory in arbitrary hereditary abelian k categories and raised the question about those categories admitting a tilting object. The only known examples, up to derived equivalence, are module categories over hereditary algebras and categories of sheaves on weighted projective lines and it was shown that under certain additional assumptions there are no other possibilities [72], [40], [41].

Another important development for the connections between sheaves on projective varieties and modules over finite dimensional algebras has started with the systematic study of exceptional vector bundles on the projective plane $\mathbf{P}^2$ and other varieties. The notion of an exceptional object was introduced by Drezet and Le Portier [26] in order to classify vector bundles on the projective plane. They described the possible Chern classes of stable vector bundles on $\mathbf{P}^2$ and gave a construction of the moduli spaces of semi-stable sheaves with given rank and given Chern classes. An important step in the proofs is the investigation of vector bundles E with $\mathrm{End}(E) = k$ and $\mathrm{Ext}^1(E, E) = 0$ (it follows that $\mathrm{Ext}^2(E, E) = 0$), these bundles were called exceptional. Later Drezet used exceptional bundles to obtain more concrete information about moduli spaces [22] [23], [24].

Since the middle of the eighties exceptional vector bundles on $\mathbf{P}^2$ and other varieties have been studied by a group of algebraic geometers in Moscow and deep results have been obtained by Rudakov, Gorodentsev, Bondal, Kuleshov, Orlov and others. One of their basic ideas is to put exceptional sheaves into sequences, whose lengths equal the rank of the Grothendieck group, and then to extend these sequences to infinite ones, called helices, with some form of periodicity [34]. To be more precise, a sequence $\epsilon = (E_1, E_2, \ldots, E_n)$ of sheaves on a variety $\mathbf{X}$ is called an exceptional sequence if $\mathrm{End}(E_i) = k$, $\mathrm{Ext}^s(E_i, E_i) = 0$, for $s > 0$ and $0 \leq i \leq n$, and moreover $\mathrm{Ext}^s(E_i, E_j) = 0$, for $i > j$ and $s \geq 0$. Further,

a helix is an infinite sequence $(E_i)_{i\in\mathbf{Z}}$ obtained from ϵ in a specific way where in addition the E_i's are related by the Serre functor [12].

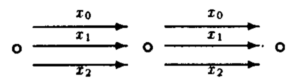
Exceptional sequences were constructed explicitly for the projective spaces $\mathbf{P}^n$, Grassmanians, quadrics, del Pezzo surfaces, i.e. surfaces with ample anticanonical sheaf, some Fano threefolds and some projective space bundles [34], [33], [53], [69], [62], [89], [92], [93] [90], [103], [104] yielding therefore interesting descriptions of the derived category of sheaves for those varieties.

In general it seems to be difficult to determine all exceptional objects in a given category. The central technique developed by Gorodentsev and Rudakov in this context is the method of mutations, a way of constructing exceptional sheaves from given ones. Also problems of interesting diophantine equations appear in these considerations. In particular, it was observed by Gorodentsev and Rudakov [34] that the ranks of a triple of exceptional vector bundles on $\mathbf{P}^2$ satisfy the Markov equation [80]

$$X^2 + Y^2 + Z^2 = 3XYZ$$

for which an algorithm yielding all solutions is known [18]. To be more precise, all solutions can be obtained from the trivial solution $(1, 1, 1)$ applying two standard transformations, which allow to associate with the set of solutions a binary tree called the Markov tree. Based on the fact that the mutation procedure corresponds exactly to these transformations Rudakov was able to construct all exceptional vector bundles on $\mathbf{P}^2$ starting with the triple of line bundles $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$ [103].

As an example for the transfer of information we used Rudakov's classification in a joint paper with Unger [87] to deduce specific information on tilting modules over the truncated symmetric algebra $A = kQ/I$ where Q is the quiver



and I is the ideal generated by all $x_ix_j - x_jx_i$. The result is a countable number of infinite trees similar to the Markov tree.

The techniques of mutations and helices were generalized to sheaf categories for other varieties by Gorodentsev [32] and to triangulated k-categories by Bondal [11].

Bondal studied also properties of some algebras A, as to be Koszul, in terms of mutations of exceptional sequences in $\mathcal{D}^b(\mathrm{mod}(A))$, this in particular applies if A is the endomorphism algebra on a tilting sheaf on a Fano variety. Later Bondal and Polishchuk [14] introduced the concept of geometricity of a helix and studied the related homological properties.

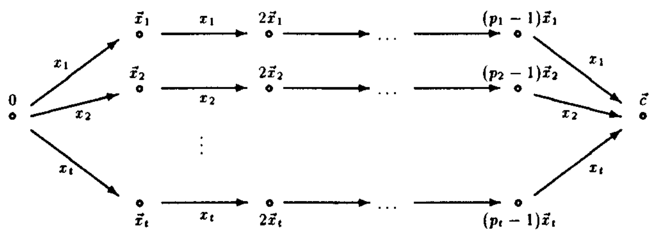
Mutations define an interesting action of the braid group B_r on r strings on the set of exceptional sequences of length r. This makes sense for an arbitrary triangulated k-category $\mathcal{C}$, moreover, invoking the translation functor of $\mathcal{C}$ one also has an action of the semidirect product $\mathbf{Z}^r \ltimes B_r$. Bondal and Polishchuk conjecture in [14] that the group $\mathbf{Z}^n \ltimes B_n$ acts transitively on the set of exceptional sequences of length n for any triangulated category $\mathcal{C}$ which is generated by an exceptional sequence of length n. In this situation an exceptional sequence of length n is said to be complete.

The conjecture is proved now in the case that $\mathcal{C}$ is the derived category of coherent sheaves on a del Pezzo surface $\mathbb{X}$. After Rudakov's result dealing with the case $\mathbb{X} = \mathbb{P}^2$ mentioned above, the transitivity of the braid group action was shown, by Rudakov [104], for $\mathbb{X} = \mathbb{P}^1 \times \mathbb{P}^1$ and, by Nogin [89], for $\mathbb{X}$ being the blowing up of $\mathbb{P}^2$ in a point, both yielding again interesting diophantine equations. The case of an arbitrary del Pezzo surface was solved recently by Kuleshov and Orlov [69]. In the same paper it is also shown that for a del Pezzo surface each exceptional sequence can be enlarged to a complete one. Recent results of Kuleshov [68] allow to hope that the transitity of the braid group action is also valid in the case of a projective space $\mathbb{P}^n$. We mention also that some results and several questions in the higher dimensional case were stated by Drezet [25].

The results in algebraic geometry have led Crawley-Boevey to study the braid group operation in representation theory of algebras [20]. He developed the corresponding theory and showed that for the path algebra over an algebraically closed field k of a quiver Q with n vertices the braid group B_n acts transitively on the set of complete exceptional sequences of kQ-modules. The result was generalized by Ringel to hereditary artin algebras [102].

Relationships between representation theory of algebras and algebraic geometry were also investigated by Lenzing and collaborators. Baer worked out the general concept of a tilting sheaf and stressed the analogy between tilting procedures from sheaves to modules and from modules to modules [5]. In particular she proved that a tilting sheaf T on a nonsingular weighted (or classical) projective variety $\mathbb{X}$ always gives rise to an derived equivalence $\mathcal{D}^b(\mathrm{coh}(\mathbb{X})) \xrightarrow{\cong} \mathcal{D}^b(\mathrm{mod}((\mathrm{End}\, T)))$.

Using a graded theory of sheaves, Geigle and Lenzing [29] created a new class of curves, called weighted projective lines, which are related to Ringel's canonical algebras [100]. A canonical algebra Λ is defined as the quotient of the path algebra of the quiver



modulo the ideal generated by the relations $x_i^{p_i} - x_2^{p_2} + \lambda_i x_1^{p_1}, i = 3, ..., t$, where λ_i are pairwise distinct elements from $k\backslash\{0\}$. Note that Λ depends on the integers $p_1, p_2, \ldots, p_t$, which will be called weights furtheron, and on the parameters λ_i.

The key observation of Geigle and Lenzing was that a weighted projective line $\mathbb{X}$ admits a tilting sheaf T such that the endomorphism ring $\mathrm{End}(T)$ is the corresponding canonical algebra. Hence the category of finite dimensional modules $\mathrm{mod}(\Lambda)$ and the category of coherent sheaves $\mathrm{coh}(\mathbb{X})$ share the same derived category $\mathcal{D}$ and, because $\mathrm{coh}(\mathbb{X})$ is a hereditary abelian category, the structure of $\mathcal{D}$ is known as soon as we know the structure

of $\mathrm{coh}(\mathbb{X})$. Further, the complexity of the classification for $\mathrm{coh}(\mathbb{X})$ depends mainly on an invariant $g_{\mathbb{X}}$, called the virtual genus of $\mathbb{X}$, which is given by the formula

$$g_{\mathbb{X}} = 1 + \frac{1}{2}\left((t-2)p - \sum_{i=1}^{t} p/p_i\right).$$

For $g_{\mathbb{X}} < 1$, the algebra Λ is concealed of extended Dynkin type. In this case the problem to classify the indecomposable coherent sheaves on $\mathbb{X}$ is equivalent to the classification of indecomposable modules over a tame hereditary algebra. Moreover, this problem is also related to the classification of indecomposable Cohen-Macaulay modules over a simple surface singularity [30].

For $g_{\mathbb{X}} = 1$, the algebra Λ the algebra Λ is of tubular representation type. Invoking an appropriate group action on weighted projective lines of type $(2,2,2,2)$, Geigle and Lenzing described in [29, 5.8] a nice link between Atiyah's classification of vector bundles on smooth elliptic curves [3] and Ringel's classification of modules over canonical algebras of tubular type [100, Chapter 5]. The latter was a significant step after the classification for the tame hereditary algebras.

For $g_{\mathbb{X}} > 1$, the algebra Λ is wild. In this case the vector bundles on $\mathbb{X}$ are, if k is the field of complex numbers, related to $\mathbb{Z}$-graded Cohen-Macaulay modules over an algebra of entire automorphic forms attached to a certain Fuchsian group [70].

The investigation of coherent sheaves on weighted projective lines was continued by several authors in [30], [57], [58], [70], [74], [75], [77], [78], [79].

We now discuss our main results and give a survey of the contents of this paper. Chapter 2 contains some basic concepts and a short outline on weighted projective lines. In view of later applications we treat the special case of weighted projective lines of type $(2, ..., 2)$, t-entries, $t \geq 5$, which will be called hyperelliptic.

In particular, we provide the following useful version of the Riemann-Roch theorem: For coherent sheaves A, B on a hyperelliptic weighted projective line $\mathbb{X}$ the following equality holds

$$\begin{aligned}\dim_k\mathrm{Hom}_{\mathbb{X}}(A,B) - \dim_k\mathrm{Ext}^1_{\mathbb{X}}(A,B) &+ \dim_k\mathrm{Hom}_{\mathbb{X}}(A,\tau B) - \dim_k\mathrm{Ext}^1_{\mathbb{X}}(A,\tau B) \\ &= \begin{vmatrix} \mathrm{rk}(A) & \mathrm{rk}(B) \\ \deg(A) & \deg(B) \end{vmatrix}\end{aligned}$$

Here τ is the Auslander-Reiten translation in $\mathrm{coh}(\mathbb{X})$ Mas rk and deg denote the rank and degree of a coherent sheaf, respectively. The formula is of particular interest if (A, B) is an exceptional pair, because in this case the last two terms vanish by Serre duality and moreover, only one of the spaces $\mathrm{Hom}_{\mathbb{X}}(A,B)$, $\mathrm{Ext}^1_{\mathbb{X}}(A,B)$ is nonzero.

We further discuss some important properties of exceptional sheaves and show that each exceptional vector bundle on a hyperelliptic weighted projective line is stable. Finally we summarize some results concerning perpendicular categories which will be essential in our investigation.

Chapter 3 deals with exceptional sequences and their mutations. The main result is the following

Theorem. *Let* $\mathbf{X}$ *be a weighted projective line of arbitrary weight type and* n *the rank of the Grothendieck group of the category of coherent sheaves* $\mathrm{coh}(\mathbf{X})$. *Then the braid group* B_n *acts transitively on the set of complete exceptional sequences in* $\mathrm{coh}(\mathbf{X})$.

The proof of the theorem is in the spirit of the results of Crawley-Boevey [20] and Ringel [102] and uses also the techniques of weight reduction for weighted projective lines by means of forming perpendicular categories with respect to simple exceptional sheaves of finite length [30].

The transitivity of the braid group operation has some important consequences which will be applied later to the study of tilting complexes. It shows, first of all, that each exceptional sheaf on $\mathbf{X}$ can be constructed from line bundles by applying mutations. It follows that the theory of exceptional sheaves on a weighted projective line depends only on the weights but not on the parameters. More precisely, for weighted projective lines $\mathbf{X} = \mathbf{X}(\mathbf{p}, \lambda)$ and $\mathbf{X}' = \mathbf{X}(\mathbf{p}, \lambda')$ there is a bijection between the exceptional sheaves on $\mathbf{X}$ and $\mathbf{X}'$, respectively, which preserves the class in the Grothendieck group under the natural identification of $K_0(\mathbf{X})$ and $K_0(\mathbf{X}')$. Similarly, the exceptional objects are independent of the ground field.

The transitivity of the braid group operation gives also rise to inductive proof methods for exceptional sheaves. We show here the following corollary which will be applied in further investigations, too.

Corollary. *Let* E *be an exceptional sheaf on a hyperelliptic weighted projective line. Then* $\mathrm{rk}(E)$ *and* $\deg(E)$ *are coprime.*

We mention that the coprimeness of rank and degree is known in other situations, for example for exceptional vector bundles on the projective plane [103] and for vector bundles with trivial endomorphism ring on smooth elliptic curves [3], however it is proved in these classical situations by completely different methods.

In Chapter 4 we study mutations with respect to an Auslander-Reiten orbit of a quasi-simple sheaf for a weighted projective line of tubular type. We show that these mutations can be considered as equivalences of the derived category and we characterize how they work on indecomposable objects. We further illustrate their use for the classification of indecomposable sheaves for weighted projective lines of genus one [74]. The basic tool for this purpose is the concept of telescopic functors which are compositions of two chosen tubular mutations playing an analogous role as Ringel's shrinking functors [100].

We remark that tubular mutations, in the functorial sense, exist also for the category of coherent sheaves on a smooth elliptic curve [85]. In some sense Atiyah's classification of vector bundles on those curves contains implicitly the idea of tubular mutations and telescopic functors.

We finally give an overview of the description of the automorphism group of the derived category of coherent sheaves on a weighted projective line. Details will appear in a joint paper with Lenzing [76].

A third kind of mutations is considered in Chapter 5. We introduce the concept of admissible pairs of exceptional objects for tubular weighted projective lines and define for them, invoking the Auslander-Reiten translation, a variant of twisted mutations. These problems are related to the question how the homomorphisms from one quasi-simple object to the objects on the mouth of a tube $\mathcal{T}$ are distributed to the various quasi-simples of $\mathcal{T}$. In the case that the dimension of this homomorphism-space is one, the twisted mutations generate a braid group B_3. The main result states that each stable exceptional sheaf can be constructed by starting with the structure sheaf and a simple exceptional sheaf of finite length using twisted mutations, line bundle shifts and automorphisms of the curve $\mathbf{X}$.

Chapter 6 is devoted to the question on the number of exceptional vector bundles with some fixed invariants. In contrast to the tame situation little is known in the case of categories of coherent sheaves for wild weighted projective lines as well as for module categories for wild hereditary algebras. Here we show that there are, up to isomorphism, only finitely many exceptional vector bundles of fixed rank r and degree d on a weighted projective line of arbitrary type and we give give a bound, which is polynomial in terms of the weights, for this number.

As a consequence, for a hyperelliptic weighted projective line the number of exceptional vector bundles of fixed slope $q = \frac{d}{r}$ is bounded polynomially in r. Moreover, using an embedding of a perpendicular category $\mathcal{C}$ with respect to a system of simple exceptional finite length sheaves such that $\mathcal{C}$ is equivalent to a sheaf category for a weighted projective line of tubular type $(2,2,2,2)$, we see that for each $q \in \mathbf{Q} \cup \{\infty\}$ there are exceptional vector bundles of slope q.

The more interesting exceptional bundles are of course those, which do not come from such embeddings. Thus we introduce the notion of an omnipresent exceptional vector bundle E by requiring that there is a nonzero map $E \to S$ to each finite length sheaf S.

Theorem. *For a wild weighted projective line there exists an omnipresent exceptional vector bundle.*

It turns out that the rank of an omnipresent exceptional bundle is greater than or equal to the number of weights less one. In the hyperelliptic case such bundles with "minimal" rank always exist. In fact we show more

Theorem. *On a hyperelliptic weighted projective line with* t *weights there is, up to line bundle shift, a unique omnipresent exceptional vector bundle of minimal rank* $t-1$.

The problems appearing here are related to the question how many roots of the quadratic form associated to $\mathbf{X}$ can be realized by exceptional vector bundles. We also obtain information concerning the number of exceptional components of the Auslander-Reiten quiver which contain an exceptional vector bundle of a fixed rank.

Chapter 7 deals with endomorphism rings of tilting bundles and tilting sheaves on weighted projective lines. Such algebras were studied first in [75] and were called concealed-canonical and almost concealed canonical algebras, respectively. Concealed-canonical and almost concealed canonical algebras are important classes of quasitilted algebras.

We summarize some basic properties of concealed-canonical and almost concealed canonical algebras. We also prove that an almost concealed canonical algebra realized on a wild weighted projective line is wild, again.

Next we describe the general structure of the module category of a concealed-canonical and an almost concealed-canonical algebra and determine the shape of its Auslander-

Reiten-components in detail. It is shown, in particular, that for a wild almost concealed-canonical algebra Σ, the stable part of a component in the category of Σ-modules of negative rank or zero rank and negative degree, which is different from the preinjective component, is of type $\mathbb{Z}A_\infty$. Moreover we construct bijections between components of the following three sets:

- $\Omega_-(\Sigma)$ of components of Σ-modules of negative rank,
- $\Omega(\mathbb{X})$ of components of vector bundles on $\mathbb{X}$,
- $\Omega(\Sigma_I)$ of regular components of modules over a concealed wild algebra Σ_I defining the unique preinjective component of $\mathrm{mod}(\Sigma)$.

A similar result for the modules of positive rank is true if Σ is a wild concealed-canonical algebra. For an almost concealed-canonical algebra Σ this part can be "smaller", depending on the decomposition $T = T' \oplus T''$ of the tilting sheaf in a vector bundle T' and a sheaf of finite length T''.

The bijections above are established by showing that corresponding components agree on a cone in τ or τ^--direction, respectively. Our results are similar to those of Kerner [63], [64], [65] concerning the situation of tilted algebras and of Lenzing and de la Peña [78] dealing with the case of wild canonical algebras. We follow here the general philosophy that the vector bundles in $\mathrm{coh}(\mathbb{X})$ have the same behaviour as regular modules over wild hereditary algebras. Some proofs however become easier in the geometrical situation. Moreover, in contrast to the situation of tilted algebras we can characterize special summands in the sense of Strauß [116] using the rank and degree of vector bundles appearing in the wing decomposition of the tilting sheaf.

Chapter 8 is concerned with several aspects of tilting complexes. An object T in the derived category $\mathcal{D} = \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ is called a tilting complex if $\mathrm{Hom}_{\mathcal{D}}(T, T[i]) = 0$ for $i \neq 0$ and the indecomposable direct summands of T generate $\mathcal{D}$ as a triangulated category. The notion of a tilting complex generalizes that of a tilting module and that of a tilting sheaf in a natural way because, as was shown by Rickard [97], the tilting complexes are exactly those complexes which induce derived equivalences. Therefore, in our situation, the categories $\mathrm{coh}(\mathbb{X})$ and $\mathrm{mod}((\mathrm{End}(T)))$ are derived equivalent.

Since the category $\mathrm{coh}(\mathbb{X})$ has global dimension one, each multiplicity-free tilting complex can be considered as a complete exceptional sequence. Unfortunately the converse is not true, in general there are complete exceptional sequences of coherent sheaves which by no way can be distributed to suitable copies of $\mathrm{coh}(\mathbb{X})$ in the derived category such that the direct sum of them forms a tilting complex.

Nevertheless we can apply our previous results on exceptional sequences. In particular, it follows from the transitivity of the braid group action that for two weighted projective lines $\mathbb{X}$ and $\mathbb{X}'$ having the same weights but different parameters there is a bijection between the tilting complexes on $\mathbb{X}$ and $\mathbb{X}'$, respectively. This implies that the quiver and the global dimension of the endomorphism ring of a tilting complex are independent of the parameters. Moreover, for an exceptional vector bundle E on a weighted projective line $\mathbb{X}$ the right perpendicular category $E^\perp$, formed in $\mathrm{coh}(\mathbb{X})$, is known to be equivalent to a category of modules over a hereditary finite dimensional algebra H [56]; we show that H is independent of the parameters, too. As a consequence we obtain the following result

Theorem. *The endomorphism ring of a tilting complex on a weighted projective line with at least four weights is representation-infinite.*

Next we investigate branch enlargements of concealed canonical algebras. We characterize these algebras as endomorphism algebras of tilting complexes of the form

$$T = T_{-n}[-n] \oplus \ldots \oplus T_{-1}[-1] \oplus T_+ \oplus T_0 \oplus T_1[1] \oplus \ldots \oplus T_m[m]$$

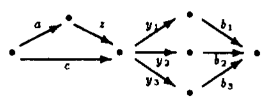
where T_+ is a vector bundle and the T_i are finite length sheaves for $-n \leq i \leq m$. Furthermore, we show that these algberas admit separating families which generalizes results of Lenzing and the author [75], Lenzing and de la Peña [78], and Lenzing and Skowroński [79].

Branch enlargements have been studied intensively by Assem and Skowroński. In particular, they proved that an algebra is representation-infinite and derived equivalent to a tame hereditary (resp. tubular) algebra if and only if it is a domestic (resp. tubular) branch enlargement of a tame concealed algebras [2]. We give here for tilting complexes on tame domestic and tubular weighted projective lines criteria whether or not their endomorphism rings are representation-infinite. This provides an easy and more conceptional proof of Assem's and Skowroński's result mentioned above.

We finally turn to our results in Chapter 9. Here we study tilting complexes on hyperelliptic weighted projective lines in detail and give a description of the general structure of their endomorphism rings. We first prove that the indecomposable direct summands of such tilting complexes are contained in two consecutive copies of the category $\mathrm{coh}(\mathbb{X})$ and that for two such summands T_i, T_j the k-dimension of $\mathrm{Hom}_{\mathbb{X}}(T_i, T_j)$ or $\mathrm{Ext}^1_{\mathbb{X}}(T_i, T_j)$ (in fact only one of them can be nonzero) depends only on the slopes $\mu(T_i)$ and $\mu(T_j)$. Moreover, we associate to such a tilting complex T diophantine equations in terms of the ranks and the degrees of the indecomposable direct summands of T and in terms of the entries of the Cartan matrix of the endomorphism algebra.

This leads to the concept of a layered algebra, for which the indecomposable projective modules are distributed to certain layers in such a way that there are nonzero morphisms only from objects of the former layers to the later ones, and in this case the dimension of the Hom-space depends only on the layers but not on the individual modules. We attach to a layered algebra Σ the layer triangle consisting of the k-dimensions $\mathrm{Hom}_\Sigma(P_i, P_j)$ between the indecomposable projective Σ-modules, one for each layer, and the Cartan triangle which, in addition, contains the information how many projectives there are in each layer.

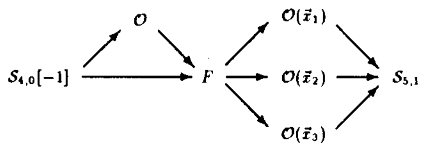
We provide a complete classification of all finite dimensional k-algebras which are tame and derived equivalent to a canonical algebra of hyperelliptic type. Roughly speaking, an algebra A is tame if and only if all but finitely many modules of a given dimension can be parametrized by a finite number of one-parameter families. The class of tame algebras studied here is described by a list of families of algebras given by quivers and relations, depending on parameters. We give an example of such a family in order to illustrate the structure of these algebras.



$$b_3y_3 - b_2y_2 + b_1y_1 = 0$$
$$(b_2y_2 - \lambda_5 b_1 y_1)z = 0$$
$$y_1c = 0$$
$$y_2(za - c) = 0$$
$$y_3(za - \tfrac{\lambda_4 - 1}{\lambda_4}c) = 0$$

The proof is divided into two steps. In a first K-theoretical part we classify all layer and Cartan triangles of layered algebras which are tame. Then a more refined analysis, based on the interplay between vector bundles and finite length sheaves, shows how the possible candidates can be realized by tilting complexes.

For an algebra of the family above a realization is given by a tilting complex T on a weighted projective line $\mathbf{X} = \mathbf{X}((2,2,2,2,2,),(\infty,0,1,\lambda_4,\lambda_5))$ (observe that the parameters have changed) of the form



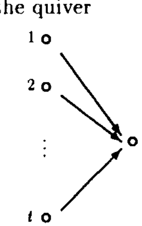
where the middle part can be considered as a tilting bundle on a weighted projective line of type $(2,2,2)$ and $\mathcal{S}_{4,0}$, $\mathcal{S}_{5,1}$ are finite length sheaves corresponding to the parameters λ_4 and λ_5. This structure is typical for the tilting complexes in question. They all consist of a tilting bundle on a domestic weighted projective line of type $(2,2,2)$, $(2,2)$, (2) or $\mathbf{P}^1$ and of simple finite length sheaves from the tubes for the vertices in the first and the last layer.

As a consequence we obtain

Theorem. *Let Σ be a tame algebra derived equivalent to a canonical algebra of type $(2,...,2)$, t entries. Then $t \leq 8$. Moreover, the algebra Σ is quasitilted.*

We hope that using additional methods our results can be generalized to other weight types.

Recall that the path algebra of the quiver



is called the t-subspace problem algebra, because for all but finitely many indecomposable representations the vector spaces corresponding to the vertices on the left hand side can be considered as subspaces of the vector space corresponding to the vertex on the right hand side. The description of the indecomposable objects in the case $t = 4$ by Nazarova [88] was one of the first complete classifications of indecomposable modules for a tame hereditary algebra and played an important role in the development of representation theory.

Applying the fact that the perpendicular category to a line bundle on a weighted projective line of type $(2,...,2)$, t entries, is equivalent to the category of modules over the t-subspace problem algebra, we give a complete classification of all finite dimensional k-algebras which are tame and derived equivalent to a t-subspace problem algebra. We see again that tame algebras can occur only if $t \leq 8$.

Parts of the results of Chapter 3, Chapter 4 and Chapter 7 were published in [83], [85] and [84].

I would like to express my thanks to Dieter Happel and Helmut Lenzing for their interest, support and many fruitful discussions on the material presented here.

Chapter 2

Exceptional vector bundles

2.1 Basic notations

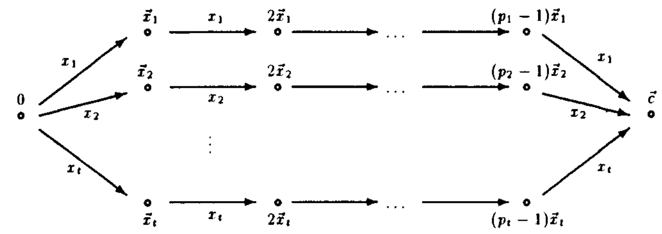
2.1.1 Throughout this work k will denote an algebraically closed field. By an algebra A we mean an associative, finite dimensional k-algebra with identity. We shall usually assume A to be basic and connected. It is well known that for a basic algebra A there exists a bound quiver (Q, I) and an isomorphism $A \xrightarrow{\cong} kQ/I$, called a presentation of A, where I is an admissible ideal in the path algebra kQ of Q. Equivalently, $A = kQ/I$ may be considered as a k-linear category, whose object class is the set of points of Q, and where the morphism set $A(x, y)$ from x to y is the quotient of the k-vector space $kQ(x, y)$ of all linear combinations of paths from x to y by the subspace $I(x, y) = I \cap kQ(x, y)$.

By an A-module we usually mean a finite dimensional right A-module. We shall denote by $\mathrm{mod}(A)$ the category of finite dimensional A-modules and by $\mathrm{ind}(A)$ a full subcategory consisting of a complete set of non-isomorphic indecomposable A-modules. Finally, $\mathrm{D} = \mathrm{Hom}_k(-, k)$ will denote the standard duality of $\mathrm{mod}(A)$.

2.1.2 For an abelian category $\mathcal{A}$ we denote by $C^b(\mathcal{A})$ the category of bounded complexes over $\mathcal{A}$, and by $K^b(\mathcal{A})$ (resp. $D^b(\mathcal{A})$) the corresponding homotopy (resp. derived) category. Moreover, if $\mathcal{A}'$ is a full abelian subcategory of $\mathcal{A}$ then $C^b_{\mathcal{A}'}(\mathcal{A})$ (resp. $K^b_{\mathcal{A}'}(\mathcal{A})$, $D^b_{\mathcal{A}'}(\mathcal{A})$) denotes the full subcategory of $C^b(\mathcal{A})$ (resp. $K^b(\mathcal{A})$, $D^b(\mathcal{A})$) formed by all complexes with cohomology in $\mathcal{A}'$. For details concerning derived and triangulated categories we refer to [49] and [117]. We denote the translation functor of a triangulated category by $X \mapsto X[1]$.

Furthermore, for an abelian category $\mathcal{A}$ we consider the Grothendieck group $\mathrm{K}_0(\mathcal{A})$ of $\mathcal{A}$. To simplify notation we write $\mathrm{K}_0(A)$ instead of $\mathrm{K}_0(\mathrm{mod}(A))$ in the case of a module category $\mathrm{mod}(A)$ and $\mathrm{K}_0(\mathbb{X})$ instead of $\mathrm{K}_0(\mathrm{coh}(\mathbb{X}))$, in the case of a category of coherent sheaves $\mathrm{coh}(\mathbb{X})$.

2.1.3 We are interested in the class of *canonical algebras* which was introduced by Ringel [100]. Let $\mathbf{p} = (p_1, \ldots, p_t)$ be a sequence of positive integers and $\boldsymbol{\lambda} = (\lambda_3, \ldots, \lambda_t)$ a sequence of elements of $k \setminus \{0\}$, called parameters, which are assumed to be pairwise distinct. Then the canonical algebra $\Lambda = \Lambda(\mathbf{p}, \boldsymbol{\lambda})$ is given by the quiver



bound by the relations $x_i^{p_i} - x_2^{p_2} + \lambda_i x_1^{p_1}, i = 3, ..., t$. (The notations for the vertices will be explained in the next section.) Of course, we can assume $p_1 \geq p_2 \geq \ldots \geq p_t$. For $t = 2$ the canonical algebra of type (p_1, p_2) is hereditary, in particular the algebra of type $(1,1)$ is the Kronecker algebra.

2.2 Weighted projective lines

Geigle and Lenzing have related a weighted projective line to a canonical algebra. In this section we recall the definition and summarize some basic properties. For details we refer to [29]. We further put emphasis to the special case of weighted projective lines of type $(2, ..., 2)$, t-entries.

2.2.1 Let $\mathbf{p} = (p_1, p_2, \ldots, p_t)$ be a t-tuple of of positive integers, called *weight sequence*. Denote by $\mathbf{L}(\mathbf{p})$ the rank one abelian group on generators $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_t$ with relations $p_1\vec{x}_1 = p_2\vec{x}_2 = \ldots = p_t\vec{x}_t$. Consider the polynomial algebra $k[X_1, X_2, \ldots, X_t]$ as an $\mathbf{L}(\mathbf{p})$-graded algebra, where the graduation is given by defining X_i to be homogeneous of degree $\vec{x}_i$.

Furthermore, let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_t)$ be a sequence of pairwise distinct elements of the projective line $\mathbf{P}^1(k)$, called *parameter sequence*. We usually assume that $\boldsymbol{\lambda}$ is normalized in the sense that $\lambda_1 = \infty$, $\lambda_2 = 0$, $\lambda_3 = 1$. To $\mathbf{p}$ and $\boldsymbol{\lambda}$ we attach the algebra

$$S = S(\mathbf{p}, \boldsymbol{\lambda}) = k[X_1, X_2, \ldots, X_t]/I(\mathbf{p}, \boldsymbol{\lambda})$$

where $I(\mathbf{p}, \boldsymbol{\lambda})$ is the ideal generated by the elements $X_i^{p_i} - X_2^{p_2} + \lambda_i X_1^{p_1}, i = 3, ..., t$. Because $I(\mathbf{p}, \boldsymbol{\lambda})$ is a homogeneous ideal, the algebra $S = S(\mathbf{p}, \boldsymbol{\lambda})$ is $\mathbf{L}(\mathbf{p})$-graded.

2.2.2 We call $\vec{c} = p_1\vec{x}_1 = p_2\vec{x}_2 = \ldots = p_t\vec{x}_t$ the *canonical element* of $\mathbf{L}(\mathbf{p})$ whereas $\vec{\omega} = (t-2)\vec{c} - \sum_{i=1}^{t} \vec{x}_i$ is called the dualizing element. $\mathbf{L}(\mathbf{p})$ is an ordered group with $\mathbf{L}^+ = \sum_{i=1}^{t} \mathbf{N}\vec{x}_i$ as its set of positive elements. Since $\mathbf{L}(\mathbf{p})/\mathbf{Z}\vec{c} \cong \prod_{i=1}^{t} \mathbf{Z}/\mathbf{Z}p_i$, each $\vec{l} \in \mathbf{L}(\mathbf{p})$ can be uniquely written in normal form

$$\vec{l} = \sum_{i=1}^{t} l_i\vec{x}_i + l\vec{c} \qquad \text{with} \qquad 0 \leq l_i < p_i \qquad \textit{and} \qquad l \in \mathbf{Z}.$$

We further put $p = \mathrm{l.c.m.}(p_1, p_2, \ldots, p_t)$. The *degree homomorphism* $\delta : \mathbf{L}(\mathbf{p}) \to \mathbf{Z}$ is given on generators by $\delta(\vec{x}_i) = \frac{p}{p_i}$, its kernel is the finite torsion group $\mathbf{tL}(\mathbf{p})$ of $\mathbf{L}(\mathbf{p})$.

2.2.3 The *weighted projective line* $\mathbf{X} = \mathbf{X}(\mathbf{p}, \boldsymbol{\lambda})$ is by definition the projective spectrum of the $\mathbf{L}(\mathbf{p})$-graded algebra $S(\mathbf{p}, \boldsymbol{\lambda})$. As a set $\mathbf{X}$ consists of all $\mathbf{L}(\mathbf{p})$-graded prime ideals of $S(\mathbf{p}, \boldsymbol{\lambda})$ and this set is equipped with the Zariski topology and an $\mathbf{L}(\mathbf{p})$-graded *structure sheaf* $\mathcal{O} = \mathcal{O}_{\mathbf{X}}$. This is the sheaf arising from the presheaf which associates to a standard open set $D(f)$, corresponding to a homogeneous element $f \in S$, the $\mathbf{L}(\mathbf{p})$-graded quotient of S with respect to the multiplicative system $\{f^n, n \in \mathbf{N}\}$. Note that the map

$$\mathbf{X}(\mathbf{p}, \boldsymbol{\lambda}) \to \mathbf{P}^1(k), \qquad [x_1, x_2, \ldots, x_t] \mapsto [x_1^{p_1}, x_2^{p_2}]$$

is a bijection of sets. By means of this correspondence $\mathbf{X}(\mathbf{p}, \boldsymbol{\lambda})$ can be understood as the usual projective line, where weights $p_1, p_2, \ldots, p_t$ are attached to the t points $\lambda_1, \lambda_2, \ldots, \lambda_t$. We say that the λ_i are *exceptional points* whereas the remaining points are called *ordinary*.

2.2.4 By a sheaf on $\mathbf{X}$ we always mean an $\mathbf{L}(\mathbf{p})$-graded sheaf of $\mathcal{O}_{\mathbf{X}}$-modules. Denote by $\mathrm{Mod}^{\mathbf{L}(\mathbf{p})}(\mathcal{O}_{\mathbf{X}})$ the category of $\mathbf{L}(\mathbf{p})$-graded $\mathcal{O}_{\mathbf{X}}$-modules. The group $\mathbf{L}(\mathbf{p})$ acts on $\mathrm{Mod}^{\mathbf{L}(\mathbf{p})}(\mathcal{O}_{\mathbf{X}})$ by grading shift $(\vec{l}, M) \mapsto M(\vec{l})$, where $M(\vec{l})_{\vec{x}} = M(\vec{l} + \vec{x})$. Since $S(\mathbf{p}, \boldsymbol{\lambda})$ is $\mathbf{L}(\mathbf{p})$-graded factorial, hence each line bundle L on $\mathbf{X}$ has the form $L \cong \mathcal{O}_{\mathbf{X}}(\vec{x})$ for some uniquely determined $\vec{x} \in \mathbf{L}(\mathbf{p})$, the grading group $\mathbf{L}(\mathbf{p})$ can be identified with the *Picard group* $\mathrm{Pic}(\mathbf{X})$ on $\mathbf{X}$. For two line bundles $\mathcal{O}_{\mathbf{X}}(\vec{x})$ and $\mathcal{O}_{\mathbf{X}}(\vec{y})$ the space of homomorphisms is given by $\mathrm{Hom}(\mathcal{O}_{\mathbf{X}}(\vec{x}), \mathcal{O}_{\mathbf{X}}(\vec{y})) = S_{\vec{y}-\vec{x}}$.

A sheaf M on $\mathbf{X}$ is called *quasi-coherent* if for each point in $\mathbf{X}$ there is neighbourhood U and an exact sequence

$$\bigoplus_{j \in J} \mathcal{O}_{\mathbf{X}}(\vec{l}_j)|_U \to \bigoplus_{i \in I} \mathcal{O}_{\mathbf{X}}(\vec{l}_i)|_U \to M|_U \to 0.$$

If both sets I and J are finite, then we say that M is a *coherent* sheaf on $\mathbf{X}$.

We denote the categories of quasi-coherent and coherent $\mathbf{L}(\mathbf{p})$-graded $\mathcal{O}_{\mathbf{X}}$-modules by $\mathrm{Qcoh}(\mathbf{X})$ and $\mathrm{coh}(\mathbf{X})$, respectively. By graded sheafification $\mathrm{coh}(\mathbf{X})$ is equivalent to the localization of the category of finitely presented $\mathbf{L}(\mathbf{p})$-graded S-modules with respect to the Serre subcategory of $\mathbf{L}(\mathbf{p})$-graded S-modules of finite length.

2.2.5 Each coherent sheaf $F \in \mathrm{coh}(\mathbf{X})$ splits into a direct sum $F = F_+ \oplus F_0$ of a coherent sheaf F_0 of finite length and a locally free coherent sheaf, i.e. a *vector bundle*, F_+. We will denote by $\mathrm{coh}_0(\mathbf{X})$ the category of finite length sheaves and by $\mathrm{vect}(\mathbf{X})$ the category of vector bundles. The category $\mathrm{coh}_0(\mathbf{X})$ decomposes into a coproduct $\coprod_{\lambda \in \mathbf{X}} \mathcal{U}_\lambda$, where $\mathcal{U}_\lambda$ denotes the uniserial category of finite length sheaves concentrated at the point λ.

If λ is an ordinary point, then there is exactly one simple object in $\mathcal{U}_\lambda$ while for an exceptional point λ_i this category has exactly p_i simple objects (up to isomorphism). The simple finite length sheaf at an ordinary point λ is given as the cokernel term of the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{X}} \overset{X_2^{p_2} - \lambda X_1^{p_1}}{\longrightarrow} \mathcal{O}_{\mathbf{X}}(\vec{c}) \longrightarrow \mathcal{S}_\lambda \longrightarrow 0$$

while the p_i exceptional simple sheaves concentrated at λ_i arise as the cokernel terms of exact sequences

$$0 \longrightarrow \mathcal{O}_{\mathbf{X}}(j\vec{x}_i) \xrightarrow{X_i} \mathcal{O}_{\mathbf{X}}((j+1)\vec{x}_i) \longrightarrow \mathcal{S}_{i,j} \longrightarrow 0 \qquad j \in \mathbf{Z}/p_i\mathbf{Z}.$$

2.2.6 The Hom-and Ext-spaces for $\mathrm{coh}(\mathbf{X})$ are finite dimensional, moreover $\mathrm{coh}(\mathbf{X})$ is a hereditary category, i.e. $\mathrm{Ext}^2_{\mathbf{X}}(-,-)=0$. The Hom-and Ext^1-spaces are related by *Serre duality* [109], [110] : $\mathrm{Ext}^1_{\mathbf{X}}(E,F) \cong \mathrm{DHom}_{\mathbf{X}}(F,E(\vec{\omega}))$. As a consequence the category $\mathrm{coh}(\mathbf{X})$ admits almost split sequences with the automorphism $F \mapsto F(\vec{\omega})$ serving as the Auslander-Reiten translation $\tau_{\mathbf{X}}$. We refer to [4] for the notion of Auslander-Reiten sequences and the Auslander-Reiten quiver and to [39] for the concept of Auslander-Reiten triangles in a triangulated category.

It follows from the heredity of $\mathrm{coh}(\mathbf{X})$ that each indecomposable object in the derived category $\mathcal{D} = \mathcal{D}^b(\mathrm{coh}(\mathbf{X}))$ is, up to isomorphism, of the form $F[i]$ for an indecomposable sheaf $F \in \mathrm{coh}(\mathbf{X})$ and some $i \in \mathbf{Z}$. Moreover, for each Auslander-Reiten component $\mathcal{C}$ in $\mathrm{coh}(\mathbf{X})$, $\mathcal{C}[n]$ is an Auslander-Reiten component in $\mathcal{D}^b(\mathrm{coh}(\mathbf{X}))$ and each Auslander-Reiten component in $\mathcal{D}^b(\mathrm{coh}(\mathbf{X}))$ has this form [77].

2.2.7 A coherent sheaf T on $\mathbf{X}$ is called a *tilting sheaf* [5], [29] if the following properties hold:

(1) $\mathrm{Ext}^i_{\mathbf{X}}(T,T)=0$ for all $i>0$

(2) T generates $\mathcal{D}^b(\mathrm{coh}(\mathbf{X}))$ as a triangulated category, i.e. $\mathcal{D}^b(\mathrm{coh}(\mathbf{X}))$ is the smallest triangulated category of $\mathcal{D}^b(\mathrm{coh}(\mathbf{X}))$ containing T.

It can be shown that the endomorphism algebra $\mathrm{End}(T)$ has finite global dimension. By a *tilting bundle* we mean a tilting sheaf which belongs to the subcategory $\mathrm{vect}(\mathbf{X})$. The main result of [29] states that

$$T = \bigoplus_{0 \le \vec{x} \le \vec{c}} \mathcal{O}_{\mathbf{X}}(\vec{x})$$

is a tilting bundle for a weighted projective line $\mathbf{X} = \mathbf{X}(\mathbf{p},\boldsymbol{\lambda})$ whose endomorphism ring is the canonical algebra $\Lambda = \Lambda(\mathbf{p},\boldsymbol{\lambda})$. T will be called the *canonical tilting sheaf* on $\mathbf{X}$ furtheron. As a consequence the right derived functor $\mathbf{R}\mathrm{Hom}(T,-)$ induces an equivalence of triangulated categories

$$\mathcal{D}^b(\mathrm{coh}(\mathbf{X})) \xrightarrow{\cong} \mathcal{D}^b(\mathrm{mod}(\Lambda)).$$

The mapping $[F] \mapsto [\mathrm{Hom}_{\mathbf{X}}(T,F)]-[\mathrm{Ext}^1_{\mathbf{X}}(T,F)]$, $F \in \mathrm{coh}(\mathbf{X})$, induces an isomorphism of the Grothendieck groups $\mathrm{K}_0(\mathbf{X}) \to \mathrm{K}_0(\Lambda)$. For coherent sheaves there are well known notions of *rank* and *degree* which by means of the isomorphism above correspond to linear forms $\mathrm{rk}, \deg : \mathrm{K}_0(\Lambda) \to \mathbf{Z}$, again called rank and degree. On a Λ-module M the rank and the degree are given by the expressions

$$\mathrm{rk}(M) = \dim_k M_0 - \dim_k M_{\vec{c}}$$

$$\deg(M) = \sum_{i=1}^{t} \frac{p}{p_i}\left(\sum_{j=1}^{p_i-1} \dim_k M_{j\vec{x}_i}\right) - \delta(\vec{\omega}+\vec{c})\dim_k M_{\vec{c}}.$$

We further define for an object M in $\mathcal{D}^b(\mathrm{coh}(\mathbf{X})) = \mathcal{D}^b(\mathrm{mod}(\Lambda))$ the *slope* $\mu(M) = \frac{\deg(M)}{\mathrm{rk}(M)}$ as an element of $\mathbf{Q} \cup \{\infty\}$.

2.2.8 There are three kinds of indecomposable Λ-modules M: those having $\mathrm{rk}(M)>0$, $\mathrm{rk}(M)=0$, resp. $\mathrm{rk}(M)<0$. We denote by $\mathrm{mod}_+(\Lambda)$ ($\mathrm{mod}_0(\Lambda)$, resp. $\mathrm{mod}_-(\Lambda)$) the full subcategories of $\mathrm{mod}(\Lambda)$ formed by all Λ-modules whose indecomposable summands have positive rank (zero rank, resp. negative rank). Further, $\mathrm{mod}_{\ge}(\Lambda)$ denotes the additive closure of $\mathrm{mod}_+(\Lambda) \cup \mathrm{mod}_0(\Lambda)$.

Let $\mathrm{coh}_+(\mathbf{X})$ (resp. $\mathrm{coh}_-(\mathbf{X})$) be the full subcategory of $\mathrm{vect}(\mathbf{X})$ formed by all vector bundles whose indecomposable summands F satisfy the condition $\mathrm{Ext}^1_{\mathbf{X}}(T,F)=0$ (resp. $\mathrm{Hom}_{\mathbf{X}}(T,F)=0$) and denote by $\mathrm{coh}_{\ge}(\mathbf{X})$ the additive closure of $\mathrm{coh}_+(\mathbf{X}) \cup \mathrm{coh}_0(\mathbf{X})$. Under the equivalence $\mathcal{D}^b(\mathrm{coh}(\mathbf{X})) \xrightarrow{\cong} \mathcal{D}^b(\mathrm{mod}(\Lambda))$

- $\mathrm{coh}_+(\mathbf{X})$ corresponds to $\mathrm{mod}_+(\Lambda)$ by means of $F \mapsto \mathrm{Hom}_{\mathbf{X}}(T,F)$,
- $\mathrm{coh}_0(\mathbf{X})$ corresponds to $\mathrm{mod}_0(\Lambda)$ by means of $F \mapsto \mathrm{Hom}_{\mathbf{X}}(T,F)$,
- $\mathrm{coh}_-(\mathbf{X})[1]$ corresponds to $\mathrm{mod}_-(\Lambda)$ by means of $F[1] \mapsto \mathrm{Ext}^1_{\mathbf{X}}(T,F)$.

2.2.9 The virtual *genus* $g_{\mathbf{X}}$ of $\mathbf{X}$ is defined by

$$g_{\mathbf{X}} = 1 + \frac{1}{2}\delta(\vec{\omega}) = 1 + \frac{1}{2}\left((t-2)p - \sum_{i=1}^{t} p/p_i\right).$$

For $g_{\mathbf{X}} < 1$, the algebra Λ is *concealed* of extended Dynkin type Δ. Therefore the classification problems for $\mathrm{coh}(\mathbf{X})$ and $\mathrm{mod}(\Lambda)$ are largely equivalent to the classification of indecomposable modules over a tame hereditary algebra. In particular, there is exactly one component of vector bundles on $\mathbf{X}$ and this component is of type $\mathbf{Z}\Delta$.

If $g_{\mathbf{X}} = 1$, then Λ is a *tubular algebra*. It is easy to see that $\mathbf{X}$ has genus one if and only if the weight sequence is up to permutation one of the following $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$ and $(2,3,6)$. The representation theory of a tubular algebra was investigated in [100] and the indecomposable sheaves over a weighted projective line of genus one were classified in [74]. In this case all components in the Auslander-Reiten quiver of $\mathrm{coh}(\mathbf{X})$ are stable tubes [29, Theorem 5.6]. We will summarize this classification in 4.2. Note further that each tubular algebra is derived equivalent to a tubular canonical algebra [46].

If $g_{\mathbf{X}} > 1$, then the algebra Λ is of *wild* representation type. This situation was studied in detail in [77]. In particular it was shown:

Proposition 2.2.9 [77, 4.8] *Let $\mathbf{X}$ be a weighted projective line of genus $g_{\mathbf{X}} > 1$. Then each Auslander-Reiten component in* $\mathrm{vect}(\mathbf{X})$ *has shape* $\mathbf{Z}A_\infty$. □

Note that, independently of the genus, the category $\mathrm{coh}_0(\mathbf{X})$ is closed under Auslander-Reiten sequences and that each Auslander-Reiten component in $\mathrm{coh}_0(\mathbf{X})$ is a tube.

A weighted projective line of genus $g_{\mathbf{X}} < 1$ (resp. $g_{\mathbf{X}} = 1$, $g_{\mathbf{X}} > 1$) will be called of domestic (resp. tubular, wild) type.

Definition 2.2.10 *A weighted projective line of type* $(2,\ldots,2)$, *t entries, $t \ge 5$, is called a hyperelliptic weighted projective line. The corresponding canonical algebra is said to be a hyperelliptic algebra.*

Recall that a curve $\mathbf{Y}$ is called hyperelliptic if it is of genus $g \ge 2$ and if there exists a finite morphism $f : \mathbf{Y} \to \mathbf{P}^1$ of degree 2 [48].

2.2.11 For a weighted projective line $\mathbb{X}$ of arbitrary weight type the *Euler form* on $K_0(\mathbb{X})$ is given on (classes of) sheaves by $\chi(A,B) = \dim_k \mathrm{Hom}_{\mathbb{X}}(A,B) - \dim_k \mathrm{Ext}^1_{\mathbb{X}}(A,B)$. The weighted form of *Riemann-Roch's theorem* [29, 2.9] states that

$$\begin{aligned}\overline{\chi}(A,B) &= p(1-g_{\mathbb{X}})\,\mathrm{rk}(A)\,\mathrm{rk}(B) + \begin{vmatrix} \mathrm{rk}(A) & \mathrm{rk}(B) \\ \deg(A) & \deg(B) \end{vmatrix} \\ &= \mathrm{rk}(A)\,\mathrm{rk}(B)\,(p(1-g_{\mathbb{X}})) + (\mu(B) - \mu(A))),\end{aligned}$$

where $\overline{\chi}(A,B) = \sum_{j=0}^{p-1} \chi(\tau^j_{\mathbb{X}} A, B)$ is the averaged Euler form.

In the case of a hyperelliptic weighted projective line the formula simplifies substantially which will play a central role in our investigations.

Proposition 2.2.11 *Let $\mathbb{X}$ be a weighted projective line $\mathbb{X}$ of type $(2,\dots,2)$. Then*

$$\overline{\chi}(A,\tau_{\mathbb{X}} B) = \begin{vmatrix} \mathrm{rk}(A) & \mathrm{rk}(B) \\ \deg(A) & \deg(B) \end{vmatrix} = \mathrm{rk}(A)\,\mathrm{rk}(B)\,(\mu(B) - \mu(A))$$

holds for all $A, B \in \mathrm{coh}(\mathbb{X})$.

Proof. We have

$$\overline{\chi}(A,\tau_{\mathbb{X}} B) = 2(1-g_{\mathbb{X}})\,\mathrm{rk}(A)\,\mathrm{rk}(\tau_{\mathbb{X}} B) + \begin{vmatrix} \mathrm{rk}(A) & \mathrm{rk}(\tau_{\mathbb{X}} B) \\ \deg(A) & \deg(\tau_{\mathbb{X}} B) \end{vmatrix}.$$

Applying $1 - g_{\mathbb{X}} = -\frac{1}{2}\deg(\omega)$, $\mathrm{rk}(\tau_{\mathbb{X}} B) = \mathrm{rk}(B)$ and $\deg(\tau_{\mathbb{X}} B) = \deg(\omega)\,\mathrm{rk}(B) + \deg(B)$ we get the result. □

2.3 Exceptional sheaves and exceptional pairs for weighted projective lines

2.3.1 The concept of an exceptional object makes sense in any abelian and any triangulated k-category. The term "exceptional" was introduced in sheaf theory and is now also widely accepted in representation theory of algebras. Synonymous terms are "Schur module", "stone", "brick without selfextensions" and " indecomposable partial tilting module".

We recall the terminology. An object E in an abelian k-category $\mathcal{A}$ is called *exceptional* if $\mathrm{End}_{\mathcal{A}}(E) = k$ and $\mathrm{Ext}^i_{\mathcal{A}}(E,E) = 0$ for $i > 0$. An object E in a triangulated k-category $\mathcal{C}$ is called exceptional if $\mathrm{End}_{\mathcal{C}}(E) = k$ and $\mathrm{Hom}_{\mathcal{C}}(E, E[i]) = 0$ for $i \neq 0$. An object in an abelian category $\mathcal{A}$ is therefore exceptional if and only if it is exceptional in the derived category $\mathcal{D}^b(\mathcal{A})$.

2.3.2 Let us summarize some basic properties of exceptional objects in the classical case $\mathcal{C} = \mathcal{D}^b(\mathrm{coh}(\mathbb{P}^2))$.

(1) Each exceptional object in $\mathcal{D}^b(\mathrm{coh}(\mathbb{P}^2))$ is of the form $E[i]$ for some exceptional coherent sheaf $\mathbb{P}^2$ and some $i \in \mathbb{Z}$ [32].

(2) Exceptional sheaves on $\mathbb{P}^2$ are locally free hence can be identified with vector bundles [34].

(3) Exceptional vector bundles on $\mathbb{P}^2$ are stable [34].

(4) An exceptional vector bundle on $\mathbb{P}^2$ is uniquely determined by its slope [22].

We will apply the notion of exceptional objects to $\mathcal{A} = \mathrm{coh}(\mathbb{X})$ and $\mathcal{C} = \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$. We know from heredity that each indecomposable object $E \in \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ can be viewed, up to translation, as a coherent sheaf. Therefore each exceptional object in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ is of the form $E[i]$ for some exceptional coherent sheaf on $\mathbb{X}$ and some $i \in \mathbb{Z}$.

It is easily checked that in the case of a weighted projective line $\mathbb{X}$ the analogous statements to (2), (3) and (4) are not true. However there is only a finite number of exceptional sheaves which do not belong to $\mathrm{vect}(\mathbb{X})$ and we will see later that there is a bound for the number of exceptional vector bundles of a given slope (see Chapter 6). Moreover, in the case of a hyperelliptic weighted projective line each exceptional sheaf is stable (see Proposition 2.3.7).

2.3.3 We will use frequently the following lemma and its corollary which was proved in the situation of modules over a hereditary algebra [45, 4.1, 4.2]. It is easily checked that the proof works also for sheaves on weighted projective lines.

Lemma 2.3.3 *Let E, F be indecomposable sheaves in $\mathrm{coh}(\mathbb{X})$. If $\mathrm{Ext}^1_{\mathbb{X}}(F,E) = 0$ then any nonzero morphism $E \to F$ is an epimorphism or a monomorphism. In particular, for an indecomposable sheaf E in $\mathrm{coh}(\mathbb{X})$, the condition $\mathrm{Ext}^1_{\mathbb{X}}(E,E) = 0$ implies $\mathrm{End}(E) = k$.* □

Thus an indecomposable sheaf E in $\mathrm{coh}(\mathbb{X})$ is exceptional if and only if $\mathrm{Ext}^1_{\mathbb{X}}(E,E) = 0$ and an indecomposable object E in $\mathcal{D} = \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ is exceptional if and only if $\mathrm{Hom}_{\mathcal{D}}(E, E[1]) = 0$.

Corollary 2.3.3 *Let T be a sheaf in $\mathrm{coh}(\mathbb{X})$ satisfying $\mathrm{Ext}^1_{\mathbb{X}}(T,T) = 0$. Then the quiver of $\Sigma = \mathrm{End}(T)$ has no oriented cycles.* □

2.3.4 A pair (E,F) of exceptional objects in $\mathcal{A}$ (resp. in $\mathcal{C}$) is called an *exceptional pair* provided we have in addition $\mathrm{Hom}_{\mathcal{A}}(F,E) = 0$ and $\mathrm{Ext}^i_{\mathcal{A}}(F,E) = 0$ for all $i > 0$ (resp. $\mathrm{Hom}_{\mathcal{C}}(F, E[i]) = 0$ for all $i \in \mathbb{Z}$). The following lemma shows that for an exceptional pair (E,F) in $\mathrm{coh}(\mathbb{X})$ at most one of the spaces $\mathrm{Hom}_{\mathbb{X}}(E,F)$, $\mathrm{Ext}^1_{\mathbb{X}}(E,F)$ is nonzero.

Lemma 2.3.4 *Let $\mathcal{H}$ be a hereditary abelian category and (E,F) be an exceptional pair in $\mathcal{H}$. Then $\mathrm{Hom}_{\mathcal{H}}(E,F) = 0$ or $\mathrm{Ext}^1_{\mathcal{H}}(E,F) = 0$.*

Proof. Suppose that there is a nonzero map $f : E \to F$. We have $\mathrm{Ext}^1_{\mathcal{H}}(F,E) = 0$ by assumption, consequently f is a monomorphism or an epimorphism. Assume first that f is a monomorphism. Then the induced map $\mathrm{Ext}^1_{\mathcal{H}}(F,F) \twoheadrightarrow \mathrm{Ext}^1_{\mathcal{H}}(E,F)$ is an epimorphism, and it follows that $\mathrm{Ext}^1_{\mathcal{H}}(E,F) = 0$, because F is exceptional.

If f is an epimorphism the argument is similar: The epimorphism f induces an epimorphism $\mathrm{Ext}^1_{\mathcal{H}}(E,E) \twoheadrightarrow \mathrm{Ext}^1_{\mathcal{H}}(E,F)$. Because E is exceptional, we obtain again $\mathrm{Ext}^1_{\mathcal{H}}(E,F) = 0$. □

Proposition 2.3.5 *Let $\mathbf{X}$ be a weighted projective line of type $(2,\dots,2)$ and let (E,F) be an exceptional pair in* $\mathrm{coh}(\mathbf{X})$. *Then*

(i) $\chi(E,F) = \begin{vmatrix} \mathrm{rk}(E) & \mathrm{rk}(F) \\ \deg(E) & \deg(F) \end{vmatrix} = \mathrm{rk}(E)\,\mathrm{rk}(F)\,(\mu(F) - \mu(E)))$

(ii) If $\mu(E) < \mu(F)$, *then* $\mathrm{Hom}_{\mathbf{X}}(E,F) \neq 0$ *and* $\dim_k \mathrm{Hom}_{\mathbf{X}}(E,F) = \begin{vmatrix} \mathrm{rk}(E) & \mathrm{rk}(F) \\ \deg(E) & \deg(F) \end{vmatrix}$

(iii) If $\mu(E) > \mu(F)$, *then* $\mathrm{Ext}^1_{\mathbf{X}}(E,F) \neq 0$ *and* $\dim_k \mathrm{Ext}^1_{\mathbf{X}}(E,F) = -\begin{vmatrix} \mathrm{rk}(E) & \mathrm{rk}(F) \\ \deg(E) & \deg(F) \end{vmatrix}$

(iv) If $\mu(E) = \mu(F)$, *then* $\mathrm{Hom}_{\mathbf{X}}(E,F) = 0$ *and* $\mathrm{Ext}^1_{\mathbf{X}}(E,F) = 0$.

Proof. According to 2.2.11,

$$\begin{vmatrix} \mathrm{rk}(E) & \mathrm{rk}(F) \\ \deg(E) & \deg(F) \end{vmatrix} = \overline{\chi}(E, \tau_{\mathbf{X}} F) =$$

$$= \dim_k \mathrm{Hom}_{\mathbf{X}}(E,F) - \dim_k \mathrm{Ext}^1_{\mathbf{X}}(E,F) + \dim_k \mathrm{Hom}_{\mathbf{X}}(E,\tau_{\mathbf{X}} F) - \dim_k \mathrm{Ext}^1_{\mathbf{X}}(E,\tau_{\mathbf{X}} F).$$

Since (E,F) is an exceptional pair, in view of Serre duality the last two terms vanish. This proves (i). The other statements follow directly from the previous lemma. □

2.3.6 Let E be an indecomposable object in $\mathrm{coh}(\mathbf{X})$ lying in a component which is either a tube or of type $\mathbb{Z}A_\infty$. Then the *quasi-length* l of E is the largest integer such that there exists a sequence $E = E_l \twoheadrightarrow E_{l-1} \twoheadrightarrow \dots \twoheadrightarrow E_2 \twoheadrightarrow E_1 = F$ of irreducible epimorphisms. In this case we call F the *quasi-top* of E and we write $E = F^{[l]}$ Further, if E has quasi-length l, then there is also a sequence $G = G_1 \hookrightarrow G_2 \hookrightarrow \dots \hookrightarrow G_{l-1} \hookrightarrow G_l = E$ of irreducible monomorphisms. We call G the *quasi-socle* of E and we write $E =^{[l]} G$

Note that for any finite length sheaf its quasi-length agrees with the usual length.

It is easily seen that a finite length sheaf $E \in \mathrm{coh}_0(\mathbf{X})$ is exceptional if and only if its quasi-length is smaller than its τ-period, in particular the exceptional finite length sheaves belong to non-homogeneous tubes. In case $\mathbf{X}$ is of domestic type each indecomposable vector bundle is exceptional. If $\mathbf{X}$ is of tubular type, then an indecomposable vector bundle is exceptional if and only if its quasi-length is smaller than its τ-period. In the wild case the following result goes back to Strauß [116] (see also [77, 7.4]).

Proposition 2.3.6 *Let $\mathbf{X}$ be a weighted projective line of genus $g_{\mathbf{X}} > 1$. Then for an indecomposable vector bundle $E \in \mathrm{vect}(\mathbf{X})$ the following conditions are equivalent:*

(i) E is exceptional of quasi-length l.

(ii) The quotients $F, \tau_{\mathbf{X}} F, \dots \tau_{\mathbf{X}}^{l-1} F$ of the Auslander-Reiten filtration for E are exceptional and satisfy $\mathrm{Hom}_{\mathbf{X}}(F, \tau_{\mathbf{X}}^h F) = 0$ for all $1 \leq h \leq l$.

(iii) Each object E' of quasi-length $\leq l$ from the Auslander-Reiten component of E is exceptional. □

Moreover, it was shown in [77] that the maximal quasi-length for an exceptional vector bundle on a wild weighted projective line is universally bounded. This bound equals 1, 2, 3, or 5 depending on the weight type of $\mathbf{X}$, see [77, 10.5] for details.

2.3.7 A nonzero vector bundle $E \in \mathrm{vect}(\mathbf{X})$ is called *semi-stable* [resp. *stable*] if for each nonzero proper subbundle F of E we have $\mu(F) \leq \mu(E)$ [resp. $\mu(F) < \mu(E)$]. If $\mathbf{X}$ is of domestic type, then each indecomposable vector bundle $E \in \mathrm{vect}(\mathbf{X})$ is stable. In the tubular case each indecomposable vector bundle $E \in \mathrm{vect}(\mathbf{X})$ is semi-stable and the stable bundles coincide with the quasi-simple ones [29]. If $\mathbf{X}$ is wild, then an exceptional vector bundle E on $\mathbf{X}$ need not to be semi-stable. In particular, if E is not quasi-simple then it is not semi-stable. However in the hyperelliptic situation we have the following result.

Proposition 2.3.7 *Let $\mathbf{X}$ be a hyperelliptic weighted projective line. Then each exceptional vector bundle E on $\mathbf{X}$ is stable.*

Proof. Suppose that E is an exceptional vector bundle and F is a nonzero subbundle of smaller rank. Applying the functor $\mathrm{Hom}_{\mathbf{X}}(-,E)$ to the embedding $F \hookrightarrow E$ and using the fact that E is exceptional we obtain that $\mathrm{Ext}^1_{\mathbf{X}}(F,E) = 0$. Furthermore, $\mathrm{End}_{\mathbf{X}}(E) = k$ implies $\mathrm{Hom}_{\mathbf{X}}(E,F) = 0$, and consequently $\mathrm{Ext}^1_{\mathbf{X}}(F, \tau_{\mathbf{X}} E) = 0$ by Serre duality. Therefore $\overline{\chi}(F, \tau_{\mathbf{X}} E) = \dim_k \mathrm{Hom}_{\mathbf{X}}(F, \tau_{\mathbf{X}} E) + \dim_k \mathrm{Hom}_{\mathbf{X}}(F,E) > 0$. On the other hand, by 2.2.11, $\overline{\chi}(F, \tau_{\mathbf{X}} E) = \mathrm{rk}(F)\,\mathrm{rk}(E)\,(\mu(E) - \mu(F)))$, which proves that $\mu(E) > \mu(F)$. □

It is easy to see that for a wild weighted projective line each indecomposable semi-stable vector bundle is quasi-simple [77, 8.1]. Combining this with the proposition above we obtain a new proof for the maximal quasi-length of an exceptional bundle on a hyperelliptic weighted projective line (compare [77, 10.5]):

Corollary 2.3.7 *Each exceptional vector bundle on a hyperelliptic weighted projective line is quasi-simple.* □

Observe that the same holds true for a tubular weighted projective line of type $(2,2,2,2)$.

2.4 Perpendicular categories

2.4.1 If $\mathcal{B}$ is a system of objects in an abelian category $\mathcal{A}$, the category $\mathcal{B}^\perp$ *right perpendicular* to $\mathcal{B}$ is defined as the full subcategory of $\mathcal{A}$ consisting of all objects $A \in \mathcal{A}$ satisfying the following conditions

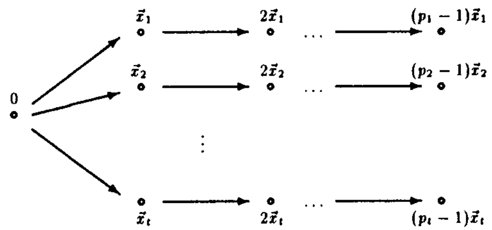
$$\mathrm{Hom}(B,A) = 0, \qquad \mathrm{Ext}^1(B,A) = 0 \qquad \text{for all } B \in \mathcal{B}$$

Dually the *left perpendicular category* is defined. Perpendicular categories were introduced in [30], see also [108]. If A is a hereditary algebra with n simple modules and X is an exceptional A-module, then the right perpendicular category $X^\perp$, formed in $\mathrm{mod}(A)$, is equivalent to a module category $\mathrm{mod}(A_0)$ for some hereditary algebra A_0 having $n-1$ simple modules [30]. For exceptional objects in $\mathrm{coh}(\mathbf{X})$ we have the following results which are basic for the rest of our investigations.

Theorem 2.4.2 [30] *Let $\mathbb{X} = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ be a weighted projective line and $\mathcal{S}$ a simple exceptional finite length sheaf concentrated at λ_i. Then the right perpendicular category $\mathcal{S}^{\perp}$, formed in* coh($\mathbb{X}$), *is equivalent to a category of coherent sheaves* coh($\mathbb{X}'$) *on a weighted projective line $\mathbb{X}' = \mathbb{X}(\mathbf{p}', \boldsymbol{\lambda})$ with weight sequence $\mathbf{p}' = (p_1, ...p_{i-1}, p_i - 1, p_{i+1}, ..., p_n)$ and where the parameter sequence $\boldsymbol{\lambda}$ remains unchanged. In particular,* $\mathrm{rk}(\mathrm{K}_0(\mathbb{X}')) = \mathrm{rk}(\mathrm{K}_0(\mathbb{X})) - 1$. □

Theorem 2.4.3 [56] *Let E be an exceptional vector bundle on a weighted projective line $\mathbb{X}$ with $\mathrm{rk}(\mathrm{K}_0(\mathbb{X})) = n$. Then the right perpendicular category $E^{\perp}$, formed in* coh($\mathbb{X}$), *is equivalent to a module category* mod(H) *where H is a (not necessarily connected) hereditary k-algebra with $n - 1$ simple modules.* □

If L is a line bundle on $\mathbb{X}$, then the algebra H is explicitly known. Up to a shift we can assume that $L = \mathcal{O}_{\mathbb{X}}(\vec{c})$. Then the right perpendicular category $\mathcal{O}_{\mathbb{X}}(\vec{c})^{\perp}$ is equivalent to a module category over the path algebra Λ_0 which is given by the quiver



and, under the identification $\mathcal{O}_{\mathbb{X}}(\vec{c})^{\perp} = \mathrm{mod}(\Lambda_0)$, the line bundles $\mathcal{O}(\vec{x})$, $0 \leq \vec{x} < \vec{c}$, form a complete system of indecomposable projective modules in $\mathrm{mod}(\Lambda_0)$ [77, Proposition 3.6]. In the special case of a weighted projective line of type $(2, ..., 2)$, t entries, the dual of Λ_0 is the t-subspace problem algebra.

Chapter 3

Mutations of exceptional sequences

Throughout this chapter, unless stated otherwise, $\mathbb{X}$ denotes a weighted projective line of arbitrary weight type.

3.1 Exceptional sequences for weighted projective lines

3.1.1 The concept of an exceptional sequence was developed in [34] and generalized in [11]. A sequence of exceptional objects $\epsilon = (E_1, ..., E_r)$ in an abelian category $\mathcal{A}$ (resp. in a triangulated k-category $\mathcal{C}$) is called an *exceptional sequence of length* r provided $\mathrm{Ext}^s_{\mathcal{A}}(E_i, E_j) = 0$ (resp. $\mathrm{Hom}_{\mathcal{C}}(E_i, E_j[s]) = 0$) for all $i > j$ and all $s \in \mathbb{Z}$. Actually we are interested only in isomorphism classes of objects, not in the objects themselves, thus an exceptional sequence will be considered as a sequence of isomorphism classes.

3.1.2 Let ϵ be an exceptional sequence of length r in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$. If r equals the rank of $\mathrm{K}_0(\mathbb{X})$, we call ϵ a *complete exceptional sequence.* Observe that the classes of the objects of a complete exceptional sequence form a basis of the Grothendieck group $\mathrm{K}_0(\mathbb{X})$.

Lemma 3.1.2 *Let $(E_1, ..., E_n)$ be a complete exceptional sequence in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$. Then the smallest full triangulated subcategory $\langle E_1, ..., E_n \rangle$ containing all E_i coincides with* $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$.

Proof. Consider the full subcategory $\mathcal{C} =: \langle E_1, ..., E_n \rangle$ of $\mathcal{D} = \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$. $\mathcal{C}$ is an admissible subcategory in $\mathcal{D}$ in the sense of [12, 2.10]. This means that for any object $Y \in \mathcal{D}$ there is a triangle $C \to Y \to B$ with $C \in \mathcal{C}$ and $B \in \mathcal{C}^{\perp}$. Here $\mathcal{C}^{\perp}$ denotes the full subcategory of $\mathcal{D}$ formed by all objects $Z \in \mathcal{D}$ such that $\mathrm{Hom}_{\mathcal{D}}(X, Z[n]) = 0$ for every $X \in \mathcal{C}$ and every $n \in \mathbb{Z}$.

We claim that $\mathcal{C}^{\perp} = 0$. To prove this, we can assume that all E_i are sheaves. Then it is sufficient to show that the right perpendicular category, formed in coh($\mathbb{X}$), to $\{E_1, \ldots, E_n\}$ vanishes. Assume to the contrary that there is an $0 \neq F \in (E_1, ..., E_n)^{\perp}$. By a line bundle shift we can assume that $F, E_1, ..., E_n$ belong to $\mathrm{coh}_{\geq}(\mathbb{X}) = \mathrm{mod}_{\geq}(\Lambda)$. But then $(E_1, ..., E_n)$ is an exceptional sequence of Λ-modules of projective dimension ≤ 1. Applying [30] we conclude that the perpendicular category $(E_1, ..., E_n)^{\perp}$, formed in mod(Λ), is zero, a contradiction. Therefore $B = 0$, and Y belongs to $\mathcal{C}$. □

3.1.3 We have the following enlargement property, which in the case of a category of modules over a hereditary algebra was proved by Crawley-Boevey [20, Lemma 1].

Lemma 3.1.3 *Any exceptional sequence* $(E_1, ..., E_a, F_1, ..., F_c)$ *in* $\mathrm{coh}(\mathbb{X})$ *can be enlarged to a complete exceptional sequence* $(E_1, ..., E_a, H_1, ..., H_b, F_1, ..., F_c)$.

Proof. It suffices to find a complete exceptional sequence $(H_1, ..., H_b, F_1, ..., F_c)$ in the category ${}^{\perp}(E_1, ... E_a)$ which is equivalent either to a module category $\mathrm{mod}(H)$ for some hereditary algebra H or to a sheaf category $\mathrm{coh}(\mathbb{X}')$ for a weighted projective line $\mathbb{X}'$ of smaller weight type. In the first case we can apply Crawley-Boevey's lemma and in the second one we can reduce, by 2.4.2, to the case $a = 0$. Then it suffices to find a complete exceptional sequence $(H_1, ..., H_b)$ in $(F_1, ... F_c)^{\perp}$. Repeating the previous argument we only need to consider the case $a = 0$ and $c = 0$. In this case the indecomposables from the canonical tilting sheaf, suitably ordered, give a complete exceptional sequence. □

Lemma 3.1.4 *If* $(E_1, ... E_n)$ *and* $(F_1, ..., F_n)$ *are complete exceptional sequences in* $\mathrm{coh}(\mathbb{X})$ *which differ in at most one place, say* $E_j \cong F_j$ *for* $j \neq i$, *then also* $E_i \cong F_i$.

Proof. By 2.4.2 and 2.4.3 ${}^{\perp}(E_1, ... E_{i-1}) \cap (E_{i+1}, ..., E_n)^{\perp}$ is equivalent to a category $\mathrm{mod}(A)$ with one simple module, hence $E_i \cong F_i$. □

3.1.5 An exceptional sequence $(E_1, ..., E_n)$ in an abelian category $\mathcal{A}$ is called *orthogonal* if $\mathrm{Hom}_{\mathcal{A}}(E_i, E_j) = 0$ for all $i \neq j$. In the situation of a module category Ringel has proved the following result

Theorem 3.1.5 [102] *The orthogonal complete exceptional sequences in a category of modules over a hereditary algebra are just those exceptional sequences which consist of the simple modules.* □

In contrast to this case for the category of sheaves on a weighted projective line we have the following

Proposition 3.1.5 *There are no orthogonal complete exceptional sequences in* $\mathrm{coh}(\mathbb{X})$.

Proof. Suppose that $\epsilon = (E_1, E_2, ..., E_n)$ is an orthogonal complete exceptional sequence in $\mathrm{coh}(\mathbb{X})$. Shifting with a line bundle we can assume that all E_i belong to $\mathrm{coh}_{\geq}(\mathbb{X}) = \mathrm{mod}_{\geq}(\Lambda)$.

Denote by $C(\epsilon)$ the smallest subcategory of $\mathrm{mod}(\Lambda)$ containing all E_i which is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms. Then $C(\epsilon)$ equals $U(\epsilon)$, the subcategory consisting of all Λ-modules which have a filtration by modules of the form E_i (see [98, 1.2]). Moreover, the same proof as in [20, Lemma 3] shows that $C(\epsilon) = \mathrm{mod}(\Lambda)$. Observe that we we can apply the induction arguments since the projective dimension of E_n is ≤ 1.

Now, $E_1, ..., E_n$ are the simple Λ-modules which is impossible, because there is one simple Λ-module belonging to $\mathrm{mod}_{-}(\Lambda)$. □

3.2 The braid group action on the set of exceptional sequences

3.2.1 We recall the concept of mutations and the braid group action. For further details we refer to [34], [11] and [14].

Let $\mathcal{C}$ be a triangulated k-category. For an exceptional pair (E, F) in $\mathcal{C}$ we consider in the derived category of vector spaces the complex $\mathrm{Hom}^{\bullet}(E, F) = \bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(E, F[j])[-j]$ with trivial differential. We assume that all spaces $\mathrm{Hom}^{\bullet}(E, F)$ are finite dimensional.

Then the *left mutation* $\mathcal{L}_E F$ of F by E is defined by the distinguished triangle

$$\mathcal{L}_E F[-1] \longrightarrow \mathrm{Hom}^{\bullet}(E, F) \otimes_k E \xrightarrow{can} F \longrightarrow \mathcal{L}_E F$$

where *can* is the canonical morphism which corresponds to the sequence of identities via the identification

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(\mathrm{Hom}^{\bullet}(E, F) \otimes E, F) &\cong \bigoplus_{j \in \mathbb{Z}} D\mathrm{Hom}_{\mathcal{C}}(E, F[j]) \otimes \mathrm{Hom}_{\mathcal{C}}(E, F[j]) \\ &\cong \bigoplus_{j \in \mathbb{Z}} \mathrm{End}_{\mathcal{C}}(\mathrm{Hom}_{\mathcal{C}}(E, F[j])) \end{aligned}$$

Dually, the *right mutation* $\mathcal{R}_F E$ of E by F is defined by a triangle

$$\mathcal{R}_F E \longrightarrow E \xrightarrow{can} D\mathrm{Hom}^{\bullet}(E, F) \otimes_k F \longrightarrow \mathcal{R}_F E[1].$$

Here duals of vector spaces have the reversed grading. The exceptional pairs $(\mathcal{L}_E F, E)$ and $(F, \mathcal{R}_F E)$ are called the left and right mutation of (E, F), respectively.

In other words, let (E, F) be an exceptional pair. Then $\mathrm{Hom}(E, F[j]) \neq 0$ for only finitely many $j \in \mathbb{Z}$. For each j choose a basis $f_1^{(j)}, \ldots, f_{i_j}^{(j)}$ of $\mathrm{Hom}_{\mathcal{C}}(E, F[j])$ and consider the cone C_f of the mapping $\bigoplus_{j \in \mathbb{Z}} (E[-j])^{i_j} \xrightarrow{f} F$, where f is the map with coordinates $f_1^{(j)}[-j], \ldots, f_{i_j}^{(j)}[-j]$. This construction is independent of the choice of the basis of $\mathrm{Hom}(E, F[j])$ and C_f is isomorphic to $\mathcal{L}_E F$.

Dually, let $C_{\tilde{f}}^{-}$ be the inverse cone of the map $E \xrightarrow{\tilde{f}} \bigoplus_{j \in \mathbb{Z}} (F[j])^{i_j}$, where $\tilde{f}$ is again the map with coordinates $f_1^{(j)}[-j], \ldots, f_{i_j}^{(j)}[-j]$. Then $C_{\tilde{f}}^{-}$ is isomorphic to $\mathcal{R}_F E$.

3.2.2 Let B_r be the *braid group* on r strings, so with generators $\sigma_1, ..., \sigma_{r-1}$ and with relations $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $j \geq i + 2$ and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $i = 1, ..., r - 2$. The group B_r acts on the set of exceptional sequences of length r in $\mathcal{C}$ by

$$\sigma_i(E_1, ..., E_r) = (E_1, ..., E_{i-1}, E_{i+1}, \mathcal{R}_{E_{i+1}} E_i, E_{i+2}, ..., E_r)$$

$$\sigma_i^{-1}(E_1, ..., E_r) = (E_1, ..., E_{i-1}, \mathcal{L}_{E_i} E_{i+1}, E_i, E_{i+2}, ..., E_r)$$

Moreover, denote by G_r the semidirect product $\mathbb{Z}^r \ltimes B_r$ defined by the group homomorphism $B_r \rightarrow S_r \rightarrow \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^r)$ being the composition of the map, given by $\sigma_i \mapsto (i, i+1)$ (the transposition in the symmetric group S_r), with the natural action of S_r on $\mathbb{Z}^r$. Then also the group G_r acts on the set of exceptional sequences of length r in $\mathcal{C}$; for the elements e_i of the natural basis of $\mathbb{Z}^r$ we define

$$e_i(E_1, ..., E_r) = (E_1, ..., E_{i-1}, E_i[1], E_{i+1}, ..., E_r).$$

We are interested in the situation where $\mathcal{C}$ is $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$. For any exceptional pair (E, F) in $\mathrm{coh}(\mathbb{X})$ there are uniquely determined exceptional sheaves $LF = L_F F$ and $RE = R_F E$ which, up to translation in the derived category, coincide with $\mathcal{L}_E F$ and $\mathcal{R}_F E$, respectively. Indeed, this is a consequence of the fact that $L_E F$ and $R_F E$ are exceptional in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$. Observe that as in the situation of [32], LF is defined by one of the three following exact sequences

$$0 \longrightarrow LF \xrightarrow{can} \mathrm{Hom}_{\mathbb{X}}(E, F) \otimes E \longrightarrow F \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{X}}(E, F) \otimes E \xrightarrow{can} F \longrightarrow LF \longrightarrow 0$$

$$0 \longrightarrow F \longrightarrow LF \longrightarrow \mathrm{Ext}^1_{\mathbb{X}}(E, F) \otimes E \longrightarrow 0$$

and a similar fact is true for RE.

There is also an action of the braid group on the set of exceptional sequences of length r in the abelian category $\mathrm{coh}(\mathbb{X})$. In fact, it is easy to see that

$$\sigma_i(E_1, ..., E_r) = (E_1, ..., E_{i-1}, E_{i+1}, R_{E_{i+1}} E_i, E_{i+2}, ..., E_r)$$

$$\sigma_i^{-1}(E_1, ..., E_r) = (E_1, ..., E_{i-1}, L_{E_i} E_{i+1}, E_i, E_{i+2}, ..., E_r)$$

define an action of B_r.

3.2.3 In [34] Gorodentsev and Rudakov introduced the concept of a *helix* by extending a complete exceptional sequence on $\mathbb{P}^n$ in both directions. This concept was generalized by Bondal [11] to arbitrary varieties and applies to weighted projective lines, as well.

Definition 3.2.3 *A sequence $(E_i)_{i \in \mathbb{Z}}$ of objects in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ is called a helix of period n if $E_i = E_{i+n}(\vec{\omega})[2-n]$ for all $i \in \mathbb{Z}$.*

Let $\epsilon = (E_1, ..., E_n)$ be an exceptional sequence in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$. Defining inductively new objects

$$E_{n+i} = \mathcal{R}^{n-1} E_i, \qquad i > 0$$

$$E_{-i} = \mathcal{L}^{n-1} E_{n-i}, \qquad i \geq 0$$

we associate to ϵ an infinite sequence $S = (E_i)_{i \in \mathbb{Z}}$. The exceptional sequence ϵ is called a *foundation of a helix* if the sequence S constructed in this way is a helix of period n. In the case of algebraic varieties Bondal proved the following

Theorem 3.2.3 [11] *Let $\mathbb{Y}$ be a variety with very ample anticanonical sheaf and $\epsilon = (E_1, ..., E_n)$ an exceptional sequence in $\mathcal{D}^b(\mathrm{coh}(\mathbb{Y}))$. Then the following are equivalent:*

(i) $E_1, ..., E_n$ generate $\mathcal{D}^b(\mathrm{coh}(\mathbb{Y}))$ as a triangulated category.

(ii) ϵ is a foundation of a helix. □

The proof of the implication (i) ⇒ (ii) uses only Serre duality and arguments in perpendicular categories and can be applied to the situation of weighted projective lines. Using Lemma 3.1.2 we obtain

Proposition 3.2.3 *Let $\mathbb{X}$ be a weighted projective line and $\epsilon = (E_1, ..., E_n)$ a complete exceptional sequence in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$. Then ϵ is a foundation of a helix.* □

Corollary 3.2.3 *Let $\epsilon = (E_1, ..., E_n)$ be a complete exceptional sequence in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$. Then*

$$\sigma_1^{-1}\sigma_2^{-1} \ldots \sigma_{n-1}^{-1}\epsilon = (E_n(\vec{\omega})[2-n], E_1, ..., E_{n-1})$$

$$\sigma_{n-1}\sigma_{n-2} \ldots \sigma_1 \epsilon = (E_2, ..., E_n, E_1(-\vec{\omega})[n-2]).$$

□

Using the proposition above we can, up to translation in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$, always assume that, without changing the ranks of the sheaves involved, a fixed sheaf of an exceptional sequence stands at the beginning or the end of an exceptional sequence.

3.3 Transitivity of the braid group action

Theorem 3.3.1 *The braid group B_n acts transitively on the set of complete exceptional sequences in $\mathrm{coh}(\mathbb{X})$.*

Corollary 3.3.2 *The group $G_n = \mathbb{Z}^n \ltimes B_n$ acts transitively on the set of complete exceptional sequences in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$.*

3.3.3 The proof of the theorem is by induction and rank-reduction. We need some preparation. We will also use the following theorem of Crawley-Boevey dealing with the braid group operation on the set of exceptional sequences of modules for a hereditary algebra.

Theorem 3.3.4 [20] *Let H be a finite dimensional hereditary algebras with n simple modules. Then the braid group B_n acts transitively on the set of complete exceptional sequences in $\mathrm{mod}(H)$.* □

Lemma 3.3.5 *Let $(E_1, E_2, ..., E_n)$ be an exceptional sequence in $\mathrm{coh}(\mathbb{X})$ such that* $\dim_k \mathrm{Hom}_{\mathbb{X}}(E_1, E_2) \geq 2$.

(i) Suppose that $LE_2 = L_{E_1} E_2$ is defined by an exact sequence

$$0 \to LE_2 \to \mathrm{Hom}_{\mathbb{X}}(E_1, E_2) \otimes E_1 \to E_2 \to 0.$$

Then morphisms $0 \neq h \in \mathrm{Hom}_{\mathbb{X}}(LE_2, E_1)$ and $0 \neq f \in \mathrm{Hom}_{\mathbb{X}}(E_1, E_2)$ are either both monomorphisms or both epimorphisms.

(ii) Suppose that $RE_1 = R_{E_2} E_1$ is defined by an exact sequence

$$0 \to E_1 \to \mathrm{DHom}_{\mathbb{X}}(E_1, E_2) \otimes E_2 \to RE_1 \to 0.$$

Then morphisms $0 \neq h \in \mathrm{Hom}_{\mathbb{X}}(E_2, RE_1)$ and $0 \neq f \in \mathrm{Hom}_{\mathbb{X}}(E_1, E_2)$ are either both monomorphisms or both epimorphisms.

epimorphism. Since f is an epimorphism we conclude that $\dim_k \mathrm{Hom}_{\mathbb{X}}(E_1, E_2) \geq 2$, and then by Lemma 3.3.5, h is an epimorphism.

Now, in this case,

$$\mathrm{rk}(E_1) > \mathrm{rk}(E_2) > \mathrm{rk}(RE_1)$$

and therefore again $\|\sigma_1 \epsilon\| < \|\epsilon\|$, which completes the proof. □

Lemma 3.3.7 *Assume that a complete exceptional sequence ϵ in $\mathrm{coh}(\mathbb{X})$ contains a finite length sheaf. Then in the orbit of ϵ under the braid group action there is an exceptional sequence containing a simple finite length sheaf.*

Proof. Let s be minimal with the property that the orbit of ϵ contains an exceptional sequence with a sheaf F of length t. By 3.2.3 we can assume that this exceptional sequence is of the form $(E_1, ..., E_{n-1}, F)$.

We have to show that $s = 1$. Assume that F is not simple and denote by $\mathcal{S}$ the socle of F. We claim that $(E_1, ..., E_{n-1}, \mathcal{S})$ is an exceptional sequence, too. Indeed, we have $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{S}, E_i) = 0$ for $1 \leq i \leq n-1$, because the embedding $\mathcal{S} \hookrightarrow F$ induces epimorphisms $\mathrm{Ext}^1_{\mathbb{X}}(F, E_i) \twoheadrightarrow \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{S}, E_i)$ and the first Ext-group vanishes by assumption. On the other hand, $\mathrm{Hom}_{\mathbb{X}}(\mathcal{S}, E_i) = 0$ for $1 \leq i \leq n-1$, because the existence of a nonzero morphism from $\mathcal{S}$ to some E_i implies that E_i also has finite length, and equals therefore ${}^{[r]}\mathcal{S}$, for some r, the unique indecomposable finite length sheaf with socle $\mathcal{S}$ and length r. Then $r \geq s$ by minimality of s. But this implies $\mathrm{Hom}_{\mathbb{X}}(F, E_i) \neq 0$, contrary to the fact that $(E_1, ..., E_{n-1}, F)$ is an exceptional sequence. Thus we have two exceptional sequences which coincide in the first $n-1$ terms but are different in the last one. By Lemma 3.1.4 this is impossible. □

Proposition 3.3.8 *Let $\mathbb{X}$ be a weighted projective line such that at least one weight is greater than one, i.e. $\mathbb{X} \not\cong \mathbf{P}^1$. Then the orbit of the braid group action of any complete exceptional sequence contains an exceptional sequence with a simple sheaf.*

Proof. By Propositions 3.3.6 and 3.2.3 we can assume that the exceptional sequence is of the form $(E_1, ..., E_{n-1}, L)$ where L is a sheaf of rank ≤ 1. By Lemma 3.3.7 it remains to consider the case that L is a line bundle. In this situation the Riemann-Roch theorem implies the existence of a simple exceptional sheaf $\mathcal{S}$ such that $\mathrm{Hom}_{\mathbb{X}}(L, \mathcal{S}) = 0$. Now, the perpendicular category $L^\perp$ is equivalent to the module category for the path algebra Λ_0 given by the star in 2.4.3, and $\mathcal{S}$ belongs to $L^\perp$.

Applying the results of Crawley-Boevey mentioned in 3.1.3 and 3.3.3 we see that the exceptional sequence $(\mathcal{S})$ can be completed to an exceptional sequence $(Y_1, ..., Y_{n-2}, \mathcal{S})$ in $\mathrm{mod}(\Lambda_0)$, and moreover, $(E_1, ..., E_{n-1})$ and $(Y_1, ..., Y_{n-2}, \mathcal{S})$ are in the same orbit under the braid group action on the set of complete exceptional sequences in $\mathrm{mod}(\Lambda_0)$. It follows that $(E_1, ..., E_{n-1}, L)$ and $(Y_1, ..., Y_{n-2}, \mathcal{S}, L)$ are in the same orbit of the acton of the braid group B_{n-1}. □

3.3.9 Proof of Theorem 3.3.1 We show by induction on the rank of $\mathrm{K}_0(\mathbb{X})$ that the group B_n acts transitively on the set of complete exceptional sequences in $\mathrm{coh}(\mathbb{X})$.

If $n = 2$ then $\mathbb{X} = \mathbf{P}^1$. In this case an exceptional sequence is of the form $(\mathcal{O}(i), \mathcal{O}(i+1))$ and the braid group $B_2 \cong \mathbb{Z}$ obviously acts transitively on the set of these exceptional sequences.

Now, suppose $n > 2$ and assume that $\epsilon = (E_1, E_2, ..., E_n)$ is an exceptional sequence in $\mathrm{coh}(\mathbb{X})$. By Lemma 3.3.8 we have $g.\epsilon = (E'_1, ... E'_{n-1}, \mathcal{S})$ for some $g \in B_n$ and some simple sheaf $\mathcal{S}$. Denote by $\mathbf{K} = (\mathcal{O}, \mathcal{O}(\vec{x}_1), ..., \mathcal{O}((p_t - 1)\vec{x}_t), \mathcal{O}(\vec{c}))$ the exceptional sequence corresponding to the canonical tilting sheaf. Since $\mathcal{S}$ is exceptional simple, $\mathcal{S} = \mathcal{S}_{i,j}$ for some i, j (see 2.2.5). From the exact sequence

$$0 \to \mathcal{O}(j\vec{x}_i) \to \mathcal{O}((j+1)\vec{x}_i) \to \mathcal{S}_{i,j} \to 0$$

we see that the right mutation of the pair $(\mathcal{O}(j\vec{x}_i), \mathcal{O}((j+1)\vec{x}_i))$ equals $(\mathcal{O}((j+1)\vec{x}_i), \mathcal{S}_{i,j})$. Thus, for some $g_1 \in B_n$ we get $g_1.\mathbf{K} = (\mathcal{O}, ..., \mathcal{O}((j+1)\vec{x}_i), \mathcal{S}_{i,j}, ...)$. Observe that in case $j = p_i - 1$, $i \neq t$, we first can apply transpositions in order to arrange that $\mathcal{O}(j\vec{x}_i)$ and $\mathcal{O}((j+1)\vec{x}_i)$ are neighbours. Applying if necessary 3.2.3, we obtain $g_2.\mathbf{K} = (F_1, ..., F_{n-1}, \mathcal{S})$ for some $g_2 \in B_n$ and line bundles $F_1, ..., F_{n-1}$. Now, the right perpendicular category $\mathcal{S}^\perp$ is equivalent to a sheaf category $\mathrm{coh}(\mathbb{X}')$ for a weighted projective line $\mathbb{X}' = \mathbb{X}(\mathbf{p}', \boldsymbol{\lambda})$ with weight sequence $\mathbf{p}' = (p_1, ... p_{i-1}, p_i - 1, p_{i+1}, ..., p_n)$. By induction $(E'_1, ..., E'_{n-1})$ and $(F_1, ..., F_{n-1})$, considered as complete exceptional sequences in $\mathcal{S}^\perp$, are in the same orbit under the action of the braid group B_{n-1} on the set of complete exceptional sequences in $\mathrm{coh}(\mathbb{X}')$). We conclude that ϵ and $\mathbf{K}$ are in the same orbit, which finishes the proof. □

3.4 Bijections between exceptional sheaves

The transitivity of the braid group action has important consequences. In particular we are going to show that the exceptional sheaves on a weighted projective line do not depend on the parameters. An application to tilting complexes will be given in 8.2.

Lemma 3.4.1 *An exceptional sheaf E on a weighted projective line $\mathbb{X}$ is uniquely determined by its class $[E]$ in $\mathrm{K}_0(\mathbb{X})$.*

Proof. Suppose that E and E' are exceptional sheaves in $\mathrm{coh}(\mathbb{X})$ such that $[E] = [E']$. Since $1 = \chi(E, E) = \chi(E, E')$ there is a nonzero map $f : E \to E'$. Write $K = \mathrm{ker} f$, $B = \mathrm{im} f$, $Q = \mathrm{coker} f$ and consider the exact sequences

$$(1) \qquad 0 \to K \longrightarrow E \xrightarrow{\alpha} B \longrightarrow 0,$$

$$(2) \qquad 0 \to B \longrightarrow E' \longrightarrow Q \longrightarrow 0.$$

Applying the functor $\mathrm{Hom}_{\mathbb{X}}(E, -)$ to the exact sequence (1), we obtain $\mathrm{Hom}_{\mathbb{X}}(E, K) = 0$, because $\mathrm{End}(E) = k$ and $\mathrm{Hom}_{\mathbb{X}}(E, \alpha)$ is nonzero. Moreover, application of $\mathrm{Hom}_{\mathbb{X}}(E', -)$ to (2) gives $\mathrm{Ext}^1_{\mathbb{X}}(E', Q) = 0$, since $\mathrm{Ext}^1_{\mathbb{X}}(E', E') = 0$. Note further that the assumption $[E] = [E']$ implies that $[K] = [Q]$. Now,

$$
\begin{aligned}
-\dim_k \mathrm{Ext}^1_{\mathbb{X}}(E,K) &= \dim_k \mathrm{Hom}_{\mathbb{X}}(E,K) - \dim_k \mathrm{Ext}^1_{\mathbb{X}}(E,K)\\
&= \chi(E,K)\\
&= \chi(E',Q)\\
&= \dim_k \mathrm{Hom}_{\mathbb{X}}(E',Q) - \dim_k \mathrm{Ext}^1_{\mathbb{X}}(E',Q)\\
&= \dim_k \mathrm{Hom}_{\mathbb{X}}(E',Q).
\end{aligned}
$$

This is possible only in case $\mathrm{Hom}_{\mathbb{X}}(E',Q) = 0$. Therefore $Q = 0$, and consequently f is an epimorphism. It follows that $[K] = 0$. Hence $K = 0$ and f is an isomorphism. □

3.4.2 Let $\mathbb{X} = \mathbb{X}(\mathbf{p},\lambda)$ and $\mathbb{X}' = \mathbb{X}(\mathbf{p},\lambda')$ be two weighted projective lines of the same weight type. The assignement $\Phi_0[\mathcal{O}_{\mathbb{X}}(\vec{x})] = [\mathcal{O}_{\mathbb{X}'}(\vec{x})]$, $0 \le \vec{x} \le \vec{c}$, defines an isomorphism $\mathrm{K}_0(\mathbb{X}) \to \mathrm{K}_0(\mathbb{X}')$ which preserves the Euler form. We will identify the two Grothendieck groups by means of this isomorphism. Further, we denote by $\mathrm{Ex}(\mathbb{X})$ and $\mathrm{Ex}(\mathbb{X}')$ the set of exceptional sheaves in $\mathrm{coh}(\mathbb{X})$ and $\mathrm{coh}(\mathbb{X}')$, respectively.

Theorem 3.4.2 *There is a bijection* $\Phi : \mathrm{Ex}(\mathbb{X}) \to \mathrm{Ex}(\mathbb{X}')$ *such that* $[\Phi E] = [E]$ *for all* $E \in \mathrm{Ex}(\mathbb{X})$.

Proof. Let $\mathbf{K}$ and $\mathbf{K}'$ be exceptional sequences corresponding to the canonical tilting sheaves on $\mathbb{X}$ and $\mathbb{X}'$, respectively.

For a braid group element $g \in B_n$ we define the length as the smallest natural number l such that g can be written as a product of l elements in the generators σ_i and their inverses. We can order the elements $(g_m)_{m\in\mathbb{N}}$ of B_n in such a way that for $m_1 < m_2$ the length of g_{m_1} is smaller than or equal to the length of g_{m_2}. Further, we denote by $\mathcal{M}_m$ (resp. $\mathcal{M}'_m$) the set of exceptional sheaves E on $\mathbb{X}$ (resp. E' on $\mathbb{X}'$) with the property that there exist an index $i \le m$ such that E (resp. E') appears in an exceptional sequence of the form $g_i.\mathbf{K}$ (resp. $g_i.\mathbf{K}'$).

We define inductively maps $\Phi_m : \mathcal{M}_m \to \mathcal{M}'_m$ such that $\Phi_m|_{\mathcal{M}_{m-1}} = \Phi_{m-1}$ and $[\Phi E] = [E]$ for all $E \in \mathcal{M}_m$. Φ_0 is already given. Assume now that Φ_{m-1} is defined and E is in $\mathcal{M}_m$ but not in $\mathcal{M}_{m-1}$. Then E can be constructed by a left or right mutation from an exceptional pair (A,B) such that $A, B \in \mathcal{M}_{m-1}$. In the first case E is defined by one of the three exact sequences

$$0 \to E \to \mathrm{Hom}_{\mathbb{X}}(A,B) \otimes A \to B \to 0$$

$$0 \to \mathrm{Hom}_{\mathbb{X}}(A,B) \otimes A \to B \to E \to 0$$

$$0 \to B \to E \to \mathrm{Ext}^1_{\mathbb{X}}(A,B) \otimes A \to 0.$$

The left mutation of the pair $(\Phi_{m-1}A, \Phi_{m-1}B)$ is of the same type and defines an exceptional sheaf E' on $\mathbb{X}'$ such that $[E] = [E']$. We define $\Phi_m(E) = E'$. If E is given by a right mutation, we proceed dually. The Φ_m's define a map Φ with the required property. Indeed, Φ is surjective, according to the transitivity of the braid group action on the set of exceptional sequences in $\mathrm{coh}(\mathbb{X}')$, and it is injective by the preceding lemma. □

Remark 3.4.3 *Similar considerations show that the exceptional objects for a weighted projective line are independent of the ground field* k. □

3.5 Coprimeness of rank and degree

3.5.1 For an exceptional vector bundle E on $\mathbf{P}^2$ the greatest common divisor of $\mathrm{rk}(E)$ and $\deg(E)$ is one [103, Corollary 2.1]. This is not true in the situation of a weighted projective line of arbitrary weight type. However, we will show that for an exceptional bundle on a hyperelliptic weighted projective line rank and degree are again coprime. This result will be applied in Chapter 9 to the study of tilting complexes for those curves.

Theorem 3.5.1 *Let* E *be an exceptional sheaf on a hyperelliptic weighted projective line. Then* $\mathrm{rk}(E)$ *and* $\deg(E)$ *are coprime.*

Proof. We prove the theorem by induction on the minimal length l of a braid group element $g \in B_n$ such that E appears in an exceptional sequence of the form $g.\mathbf{K}$, where $\mathbf{K}$ is, as before, an exceptional sequence corresponding to the canonical tilting sheaf.

Since all sheaves of $\mathbf{K}$ have rank one, the statement holds true for $l = 0$. Assume that $l > 1$ and that E is constructed by a left mutation

$$E[-1] \to \mathrm{Hom}^\bullet(A,B) \otimes A \to B \to E$$

from an exceptional pair (A,B) which is contained in $g.\mathbf{K}$ for some $g \in B_n$, where the length of g is smaller than l. By induction hypothesis we have $\gcd(\mathrm{rk}(A),\deg(A)) = 1$ and $\gcd(\mathrm{rk}(B),\deg(B)) = 1$. Since (A,B) is an exceptional pair, we deduce from Corollary 2.3.5 that $[E] = [B] - h[A]$ with

$$h = \chi(A,B) = \begin{vmatrix} \mathrm{rk}(A) & \mathrm{rk}(B) \\ \deg(A) & \deg(B) \end{vmatrix}.$$

Therefore $\mathrm{rk}(E) = \mathrm{rk}(B) - h\,\mathrm{rk}(A)$ and $\deg(E) = \deg(B) - h\deg(A)$. Suppose now that $c \in \mathbb{Z}$ divides both $\mathrm{rk}(E)$ and $\deg(E)$. Then c divides also

$$\begin{vmatrix} \mathrm{rk}(E) & \mathrm{rk}(A) \\ \deg(E) & \deg(A) \end{vmatrix} = -h.$$

As a consequence, c divides both $\mathrm{rk}(B)$ and $\deg(B)$, and therefore $c = 1$ or $c = -1$. The case that E is constructed by a right mutation is proved by dual arguments. □

3.5.2 The proof works also for a weighted projective line $\mathbb{X}$ of type $(2,\dots,2)$, t entries, with $t \le 4$. In this case the result is known and can be shown not involving the transitivity of the braid group action. In fact, for $t < 4$ the assertion is an easy consequence of the shape of the vector bundle component $\mathrm{vect}(\mathbb{X})$ and for $t = 4$ one can apply the telescopic functors to be defined in chapter 4.

3.5.3 Example. Let $\mathbb{X}$ be a weighted projective line of weight type $(2,2,2,3)$. Then $\deg(\tau_\Lambda^- \mathcal{O}(\vec{x}_1)) = 8$ and $\mathrm{rk}(\tau_\Lambda^- \mathcal{O}(\vec{x}_1)) = 2$. Moreover, $\deg(\tau_\Lambda^{-8}\mathcal{O}) = 780$ and $\mathrm{rk}(\tau_\Lambda^{-8}\mathcal{O}) = 169$, hence $\gcd(\deg(\tau_\Lambda^{-8}\mathcal{O}), \mathrm{rk}(\tau_\Lambda^{-8}\mathcal{O})) = 13$. Here τ_Λ^- denotes the inverse of Auslander-Reiten-translation in $\mathrm{mod}(\Lambda)$. Note that rank and degree of the two exceptional vector bundles above can be calculated in the preprojective component for the algebra Λ_0 (see Chapter 7).

Chapter 4

Tubular mutations

4.1 Mutations as derived equivalences for sheaves on weighted projective lines of genus one

4.1.1 The mutations of pairs considered in Chapter 3 can be considered as functors if we fix one exceptional object. More precisely let $\mathcal{C}$ be a triangulated category and let A be an exceptional object in $\mathcal{C}$. For an arbitrary object $X \in \mathcal{C}$ we form the triangle

$$\mathrm{Hom}^\bullet(A,X)\otimes A \xrightarrow{can} X \to \mathcal{L}_A(X).$$

It is proved in [32, 3.4.3] that these triangles are functorial in X, thus yielding a functor $\mathcal{L}_A : \mathcal{C} \to \mathcal{C}$. Observe that $\mathcal{L}_A$ is not an equivalence, because it vanishes on A.

If the object A is not exceptional, then we can also consider the triangles above, however in general it is not clear how to extend the assignment $X \mapsto \mathcal{L}_A(X)$ to a functor.

In this chapter we will investigate mutations with respect to an Auslander-Reiten orbit of a quasi-simple sheaf for a weighted projective line of tubular type, a particular case where a functorial extension is possible. These mutations were introduced in [85] and will be called *tubular mutations* furtheron.

4.1.2 For a weighted projective line $\mathbb{X}$ of arbitrary weight type and for each $q \in \mathbb{Q}\cup\{\infty\}$ we denote by $\mathcal{C}_q$ the subcategory of $\mathrm{coh}(\mathbb{X})$ consisting of the zero bundle and all semi-stable sheaves of slope q. As in the case of nonsingular projective curves we have the following result due to Narashiman and Seshadri [114] (see also [29, Proposition 5.2]).

Proposition 4.1.2 *(i) Each $\mathcal{C}_q$ is an exact abelian subcategory of* $\mathrm{coh}(\mathbb{X})$*, closed under extensions.*

(ii) Each $F \in \mathcal{C}_q$ has finite length in $\mathcal{C}_q$. The simple objets in $\mathcal{C}_q$ are just the stable sheaves; in particular $\mathrm{End}(F) = k$ *if F is stable.*

(iii) If $F \in \mathcal{C}_q$ and $F' \in \mathcal{C}_{q'}$ and $\mathrm{Hom}_{\mathbb{X}}(F,F') \neq 0$*, then $q \leq q'$.* □

4.1.3 Assume now that $\mathbb{X}$ is of tubular weight type. The following proposition was proved by Geigle and Lenzing [29, Theorem 5.6].

Proposition 4.1.3 (*i*) *Each indecomposable sheaf on* $\mathbb{X}$ *is semistable.*

(*ii*) *Each* $\mathcal{C}_q$ *is closed under the formation of Auslander-Reiten sequences, in particular* $\mathcal{C}_q(\vec{\omega}) = \mathcal{C}_q$.

(*iii*) *Each* $\mathcal{C}_q$ *is a uniserial category. Accordingly* $\mathrm{ind}(\mathcal{C}_q)$ *decomposes into Auslander-Reiten components, which all are tubes of finite rank.* □

Theorem 4.1.4 *Let* $\mathbb{X}$ *be a tubular weighted projective line and let* $\mathcal{U}$ *be a* τ*-orbit of a quasi-simple sheaf in* $\mathrm{coh}(\mathbb{X})$. *Then there exists an equivalence* $L : \mathcal{D}^b(\mathrm{coh}(\mathbb{X})) \to \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ *and a natural transformation* $\eta : \mathrm{id} \to L$ *such that*

$$\bigoplus_{U \in \mathcal{U}} \mathrm{Hom}^\bullet(U, X) \otimes U \xrightarrow{\mathrm{can}_X} X \xrightarrow{\eta_X} L(X)$$

is a distinguished triangle for each object $X \in \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$.

Proof. Step 1: Let $\mathcal{U}$ be an arbitrary τ-orbit of a quasi-simple sheaf in $\mathrm{coh}(\mathbb{X})$. We first show the existence of a functor L such that $L(X)$ appears in a distinguished triangle as above.

We consider the functor $F : \mathrm{Qcoh}(\mathbb{X}) \to \mathrm{Qcoh}(\mathbb{X})$ defined on objects by

$$F(X) = \bigoplus_{U \in \mathcal{U}} \mathrm{Hom}(U, X) \otimes U$$

and the morphism of functors $\alpha : F \to \mathrm{id}_{\mathrm{Qcoh}(\mathbb{X})}$ which is given by the canonical maps. Obviously, F extends to a functor $\overline{F} : \mathcal{K}^b(\mathcal{Q}\mathrm{coh}(\mathbb{X})) \to \mathcal{K}^b(\mathcal{Q}\mathrm{coh}(\mathbb{X}))$ and α extends to a morphism of functors $\overline{\alpha} : \overline{F} \to \mathrm{id}_{\mathcal{K}^b(\mathcal{Q}\mathrm{coh}(\mathbb{X}))}$. Define $\overline{L} : \mathcal{K}^b(\mathcal{Q}\mathrm{coh}(\mathbb{X})) \to \mathcal{K}^b(\mathcal{Q}\mathrm{coh}(\mathbb{X}))$ as the mapping cone of $\overline{\alpha}$, thus for $X^\bullet = (X^n, d^n) \in \mathcal{K}^b(\mathcal{Q}\mathrm{coh}(\mathbb{X}))$ we get $\overline{L}(X^\bullet)^n = F(X^{n+1}) \oplus X^n$ and the differential in $\overline{L}(X^\bullet)$ is given by

$$\begin{pmatrix} -F(d^{n+1}) & 0 \\ \alpha(X^{n+1}) & d^n \end{pmatrix} : F(X^{n+1}) \oplus X^n \to F(X^{n+2}) \oplus X^{n+1}.$$

Clearly $\overline{L}$ is a functor.

Let $\mathcal{I}$ be the full subcategory of $\mathrm{Qcoh}(\mathbb{X})$ consisting of all injective quasi-coherent sheaves and let $\mathcal{C}^b(\mathcal{I})$ (resp. $\mathcal{K}^b(\mathcal{I})$) be the full subcategory of $\mathcal{K}^b(\mathcal{Q}\mathrm{coh}(\mathbb{X}))$ (resp. $\mathcal{K}^b(\mathcal{Q}\mathrm{coh}(\mathbb{X}))$) formed by all complexes of objects from $\mathcal{I}$. Moreover, let $\mathcal{K}^b_{\mathrm{coh}(\mathbb{X})}(\mathcal{I})$ be the full subcategory of $\mathcal{K}^b(\mathcal{I})$ having all cohomology sheaves in $\mathrm{coh}(\mathbb{X})$.

We show that if $I^\bullet \in \mathcal{K}^b_{\mathrm{coh}(\mathbb{X})}(\mathcal{I})$, then $\overline{L}(I^\bullet) \in \mathcal{K}^b_{\mathrm{coh}(\mathbb{X})}(\mathrm{Qcoh}(\mathbb{X}))$. Let $I^\bullet = (I^n, d^n) \in \mathcal{K}^b_{\mathrm{coh}(\mathbb{X})}(\mathcal{I})$. It is sufficient to show that $\overline{F}(I^\bullet)$ has coherent cohomology. We write $B^n = \mathrm{im}(d^{n-1})$, $Z^n = \ker(d^n)$, $H^n = Z^n/B^n$ and put $F'(X) = \bigoplus_{j=1}^p \mathrm{Ext}^1_{\mathbb{X}}(\tau^j\mathcal{O}, X) \otimes \tau^j\mathcal{O}$ for $X \in \mathrm{coh}(\mathbb{X})$.

Observe that $F(Z^n) \cong \ker(F(d^n))$, $F'(B^n) = 0$ and $F(H^n) \cong F(Z^n)/F(B^n)$ for all $n \in \mathbb{Z}$. Now, for the exact sequence

$$0 \to F(B^n)/\mathrm{im}(F(d^{n-1})) \to F(Z^n)/\mathrm{im}(F(d^{n-1})) \to F(Z^n)/F(B^n) \to 0$$

we get that the end terms $F(B^n)/\mathrm{im}(F(d^{n-1})) \cong F'(Z^{n-1}) \cong F'(H^{n-1})$ and $F(Z^n)/F(B^n)$ are coherent, therefore the middle term is coherent, too. Hence the complex $\overline{F}(I^\bullet)$ has coherent cohomology.

Thus, by restriction we obtain a functor $L' : \mathcal{K}^b_{\mathrm{coh}(\mathbb{X})}(\mathcal{I}) \to \mathcal{K}^b_{\mathrm{coh}(\mathbb{X})}(\mathrm{Qcoh}(\mathbb{X}))$. Now, the composition $\mathcal{K}^b_{\mathrm{coh}(\mathbb{X})}(\mathcal{I}) \xrightarrow{i} \mathcal{K}^b_{\mathrm{coh}(\mathbb{X})}(\mathrm{Qcoh}(\mathbb{X})) \xrightarrow{\kappa} \mathcal{D}^b_{\mathrm{coh}(\mathbb{X})}(\mathcal{Q}\mathrm{coh}(\mathbb{X}))$ is an equivalence, let φ be a quasi-inverse of $\kappa \circ i$. Let us consider the composition

$$\mathcal{D}^b_{\mathrm{coh}(\mathbb{X})}(\mathcal{Q}\mathrm{coh}(\mathbb{X})) \xrightarrow{\varphi} \mathcal{K}^b_{\mathrm{coh}(\mathbb{X})}(\mathcal{I}) \xrightarrow{L'} \mathcal{K}^b_{\mathrm{coh}(\mathbb{X})}(\mathrm{Qcoh}(\mathbb{X})) \xrightarrow{\kappa} \mathcal{D}^b_{\mathrm{coh}(\mathbb{X})}(\mathcal{Q}\mathrm{coh}(\mathbb{X})).$$

Identifying $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ with $\mathcal{D}^b_{\mathrm{coh}(\mathbb{X})}(\mathcal{Q}\mathrm{coh}(\mathbb{X}))$, we get a functor $L : \mathcal{D}^b(\mathrm{coh}(\mathbb{X})) \to \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$. Moreover, the obvious natural transformation $\mathrm{id}_{\mathcal{K}^b(\mathcal{Q}\mathrm{coh}(\mathbb{X}))} \to \overline{F}$ induces a natural transformation $\eta : \mathrm{id}_{\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))} \to L$. The existence of triangles as stated in the theorem is a consequence of the definition of L.

Step 2: Next we show that $L = L_{\mathcal{U}}$ is an equivalence in the special case that $\mathcal{U}$ is the τ-orbit of the structure sheaf. For this we apply Beilinson's lemma [9], stating that if $G : \mathcal{C} \to \mathcal{D}$ is an exact functor between triangulated categories and if $\mathcal{X} = \{X_i\}_{i \in I}$ is a generating system (in the sense of triangulated categories) of $\mathcal{C}$ such that $\{G(X_i)\}_{i \in I}$ is a generating system of $\mathcal{D}$ and G induces equivalences $\mathrm{Hom}^\bullet(X_i, X_j) \xrightarrow{\cong} \mathrm{Hom}^\bullet(G(X_i), G(X_j))$ for all $X_i, X_j \in \mathcal{X}$, then G is an equivalence.

Now, if $0 \longrightarrow I_1^\bullet \xrightarrow{\alpha} I_2^\bullet \xrightarrow{\beta} I_3^\bullet \longrightarrow 0$ is an exact sequence in $\mathcal{C}^b(\mathrm{Qcoh}(\mathbb{X}))$ with terms in $\mathcal{C}^b_{\mathrm{coh}(\mathbb{X})}(\mathcal{I})$, then the complex $0 \longrightarrow L'(I_1^\bullet) \xrightarrow{L'(\alpha)} L'(I_2^\bullet) \xrightarrow{L'(\beta)} L'(I_3^\bullet) \longrightarrow 0$ is a pointwise split exact sequence. Interpreting the homotopy categories as Frobenius categories, it follows from [39, Chapter I, Lemma 2.8] that L' and hence L is an exact functor of triangulated categories.

Consider the generating system $\{\mathcal{O}, \{\mathcal{S}_{i,j}\}_{i=1,\dots,t,\ j=0,\dots,p_i-1}\}$ where the $\mathcal{S}_{i,j}$ are the simple sheaves concentrated at the exceptional points. From the exact sequences

$$0 \to \mathcal{O}(j\vec{x}_i) \to \mathcal{O}((j+1)\vec{x}_i) \to \mathcal{S}_{i,j} \to 0$$

we conclude that the only non-vanishing Ext-spaces between the sheaves of the generating system are $\mathrm{Hom}_{\mathbb{X}}(\mathcal{O}, \mathcal{S}_{i,p_i-1}) = k$, $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{S}_{i,0}, \mathcal{O}) = k$, $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{S}_{i,j}, \mathcal{S}_{i,j-1}) = k$ and $\mathrm{Hom}_{\mathbb{X}}(X, X) = k$ for $X = \mathcal{O}$ or $\mathcal{S}_{i,j}$. Choosing injective resolutions for $\mathcal{O}$ and $\mathcal{S}_{i,j}$ one easily calculates that $L(\mathcal{O}) \cong \omega^{-1}$ and $L(\mathcal{S}_{i,j}) \cong \mathcal{O}(-\vec{x}_i + (p_i - 1 - j)\vec{\omega})[1]$. It follows that

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{X}}(L(\mathcal{O}), L(\mathcal{S}_{i,p_i-1})) &= k,\\
\mathrm{Hom}_{\mathbb{X}}(L(\mathcal{S}_{i,0}), L(\mathcal{O})[1]) &= k,\\
\mathrm{Hom}_{\mathbb{X}}(L(\mathcal{S}_{i,j}), L(\mathcal{S}_{i,j-1})[1]) &= k,\\
\mathrm{Hom}_{\mathbb{X}}(L(\mathcal{O}), L(\mathcal{O})) &= k,\\
\mathrm{Hom}_{\mathbb{X}}(L(\mathcal{S}_{i,j}), L(\mathcal{S}_{i,j})) &= k,
\end{aligned}$$

and that the other Hom-spaces vanish. Furthermore, it is easy to check that L induces non-zero maps and therefore isomorphisms between the corresponding one-dimensional Hom-spaces. Finally, $L(\mathcal{O})$ and the $L(\mathcal{S}_{i,j})$ form again a generating system. Thus by Beilinson's lemma, L is an equivalence.

Step 3: Now we show that $L = L_{\mathcal{U}}$ is an equivalence in case that $\mathcal{U}$ is the τ-orbit of a simple sheaf of finite length. We proceed similarly as in the previous step.

Case (a) $\mathcal{U}$ is the τ orbit of a simple sheaf $\mathcal{S}_{i,0}$ concentrated at an exceptional point λ_i.

Choosing again injective resolutions one easily checks that $L(\mathcal{O}) \cong \mathcal{O}(\vec{x}_i)$, $L(\mathcal{S}_{i,j}) \cong \mathcal{S}_{i,j+1}$ and $L(\mathcal{S}_{i',j}) \cong \mathcal{S}_{i',j}$ for $i \neq i'$ and that L induces isomorphisms between those one-dimensional Hom-spaces, which do not vanish.

Case (b) $\mathcal{U}$ is the τ orbit of a simple sheaf $\mathcal{S}$ concentrated at an ordinary point.

In this case it follows easily that $L(\mathcal{O}) \cong \mathcal{O}(\vec{c})$ and $L(\mathcal{S}_{i,j}) \cong \mathcal{S}_{i,j}$ for all i, j and proceeds as before.

Step 4: In order to prove the theorem in the general case we apply the method of telescopic functors developed in [74, Section 4]. The proof will be finished in the next section.

4.2 Telescopic functors

We will give a short outline of the application of tubular mutations to the classification of indecomposable sheaves on a weighted projective line of genus one [74]. For corresponding K-theoretical results we refer to [74] and [70].

4.2.1 Let $\mathbb{X}$ be a weighted projective line of genus one. Denote by L a tubular mutation with respect to the τ-orbit of the structure sheaf and by R a quasi-inverse functor of L. Then we have triangles

$$R(X) \longrightarrow X \xrightarrow{\text{can}} \bigoplus_{j=1}^{p} \mathrm{DHom}^\bullet(X, \tau^j\mathcal{O}) \otimes \tau^j\mathcal{O}$$

for all $X \in \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$. Further, let S be a tubular mutation with respect to the τ-orbit of $\mathcal{S}_{t,0}$ and S^{-1} a quasi-inverse functor of S.

4.2.2 The following corollary follows easily from Theorem 4.1.4 invoking the semi-stability of indecomposable sheaves. It indicates how to calculate a left mutation L with respect to the τ-orbit of the structure sheaf in the abelian category, and shows in particular that in this case L coincides on indecomposable sheaves X such that $0 < \mu(X) \leq 1$ with the functor considered in [74].

Corollary 4.2.2 *Assume that $\mathbb{X}$ is a weighted projective line of genus one. Let X be an indecomposable sheaf on $\mathbb{X}$ and*

$$\bigoplus_{j=1}^{p} \mathrm{Hom}_{\mathbb{X}}(\tau^j\mathcal{O}, X) \otimes \tau^j\mathcal{O} \xrightarrow{can} X$$

the canonical map.

(a) If $\mu(X) > 1$, then $L(X) \cong \ker(\mathrm{can})[1]$.

(b) If $0 < \mu(X) \leq 1$, then $L(X) \cong \mathrm{coker}(\mathrm{can})$.

(c) If $\mu(X) = 0$, then $L(X) \cong \tau^-(X)$ provided X is in the Auslander-Reiten component of $\mathcal{O}$, and $L(X) \cong X$ otherwise.

(d) If $\mu(X) < 0$, then $L(X)$ coincides with the middle term of the universal extension of X with respect to the Auslander-Reiten orbit of $\mathcal{O}$.

4.2.3 Using the fact that R is a quasi-inverse functor of L, we deduce the dual result.

Corollary 4.2.3 *Assume that $\mathbb{X}$ is a weighted projective line of genus one. Let X be an indecomposable sheaf on $\mathbb{X}$ and*

$$X \xrightarrow{can} \bigoplus_{j=1}^{p} \mathrm{DHom}_{\mathbb{X}}(X, \tau^j\mathcal{O}) \otimes \tau^j\mathcal{O}$$

the co-canonical map.

(a) If $\mu(X) \leq -1$, then $R(X) \cong \mathrm{coker}(\mathrm{can})[-1]$.

(b) If $-1 < \mu(X) < 0$, then $R(X) \cong \ker(\mathrm{can})$.

(c) If $\mu(X) = 0$, then $R(X) \cong \tau(X)$ provided X is in the Auslander-Reiten component of $\mathcal{O}$, and $R(X) \cong X$ otherwise.

(d) If $\mu(X) > 0$, then $R(X)$ coincides with the middle term of the co-universal extension of X with respect to the Auslander-Reiten orbit of $\mathcal{O}$. □

4.2.4 According to Theorem 4.1.4 and Corollaries 4.2.2 and 4.2.3 we obtain

Corollary 4.2.4 *The tubular mutations*

$$L : \mathcal{D}^b(\mathrm{coh}(\mathbb{X})) \to \mathcal{D}^b(\mathrm{coh}(\mathbb{X})) \quad \text{and} \quad R : \mathcal{D}^b(\mathrm{coh}(\mathbb{X})) \to \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$$

induce equivalences

$$L_q : \mathcal{C}_q \xrightarrow{\cong} \mathcal{C}_{\frac{q}{1-q}} \quad \text{and} \quad R_q : \mathcal{C}_q \xrightarrow{\cong} \mathcal{C}_{\frac{q}{1+q}}.$$

□

4.2.5 For explicit calculations the following combinatorial description of the positive rationals [74] is useful.

Proposition 4.2.5 *There are natural bijections between the following four sets:*

(i) The set $\mathbb{Q}_+$ of all rationals $q > 0$.

(ii) The semigroup $SL(2, \mathbb{N})$ of all 2×2-matrices of determinant one with entries in $\mathbb{N}$.

(iii) The free semigroup $\mathrm{F}\{X_1, X_2\}$ in two letters X_1, X_2.

(iv) The binary tree T. □

Proof. (*a*) The morphism of semigroups

$$\varphi : \mathrm{F}\{X_1, X_2\} \to SL(2, \mathbb{N}), \quad X_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad X_2 \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

is an isomorphism.

(*b*) The semigroup $SL(2, \mathbb{N})$ acts on $\mathbb{Q}_+ = \{q \in \mathbb{Q}, q > 0\}$ by linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .q = \frac{aq + b}{cq + d}.$$

The mapping

$$\psi : SL(2,\mathbf{N}) \to \mathbf{Q}_+, \quad u \mapsto u.1$$

is a bijection.

(c) The graph Γ of the semigroup $F = \mathrm{F}\{X_1, X_2\}$ (with respect to the generators X_1, X_2) is the oriented graph whose vertices are the elements of F, i.e.the words in X_1, X_2. Further, as depicted below

$$\begin{array}{ccccc} & & w & & \\ & X_1 \swarrow & & \searrow X_2 & \\ X_1 w & & & & X_2 w \end{array}$$

for each vertex w there are exactly two arrows starting at w, one labeled X_1 from w to X_1w, the other labeled X_2 from w to X_2w. Obviously, Γ is the binary tree with root 1. □

By means of the bijection $\psi \circ \varphi : \mathrm{F}\{X_1, X_2\} \to \mathbf{Q}_+$, $w_q \mapsto q$, we can define on the positive rationals the structure of a free semigroup $(\mathbf{Q}_+, *)$ in the generators $X_1 = 2$ and $X_2 = 1/2$ such that

— 1 is the neutral element,

— $1/2 * q = q/(1+q)$,

— $2 * q = 1 + q$.

In this setting each $q \in \mathbf{Q}_+$ has a representation (unique up to factors equal to one)

$$q = (a_1 * 1/b_1) * (a_2 * 1/b_2) * \cdots * (a_n * 1/b_n),$$

where all a_i, b_i are integers ≥ 1.

Theorem 4.2.6 *Assume that* $\mathbf{X}$ *is a weighted projective line of genus one. Then, for each* $q, q' \in \mathbf{Q} \cup \{\infty\}$, *there is an equivalence* $\Phi_{q',q} : \mathcal{D}^b(\mathrm{coh}(\mathbf{X})) \to \mathcal{D}^b(\mathrm{coh}(\mathbf{X}))$ *such that* $\mathcal{C}_q$ *is mapped to* $\mathcal{C}_{q'}$. *Moreover, these functors satisfy the conditions* $\Phi_{q'',q} = \Phi_{q'',q'} \circ \Phi_{q',q}$ *and* $\Phi_{q,q} = Id$.

Proof. Let $q \in \mathbf{Q}_+$ and let w_q be the corresponding word in $\mathrm{F}\{X_1, X_2\}$ under the isomorphism established 4.2.5. Define $\Phi_{q,\infty} = w_q(S, R) \circ R : \mathcal{C}_\infty \to \mathcal{C}_1 \to \mathcal{C}_q$.

For $q \in \mathbf{Q}, q \leq 0$, choose $n \in \mathbf{Z}$ such that $q + n \in \mathbf{Q}_+$ and define $\Phi_{q,\infty} = S^{-n} \circ \Phi_{q+n,\infty}$, this is independent of the choice of n.

Finally, for $q, q' \in \mathbf{Q} \cup \{\infty\}$ define $\Phi_{q',q} = \Phi_{q',\infty} \circ \Phi^{-1}_{q,\infty}$. Then, by construction, all properties are easily verified. □

We refer to the functors of the theorem as *telescopic functors*. Observe that they are globally defined, in contrast to the telescopic functors considered in [74].

4.2.7 Since $\mathcal{C}_\infty$ coincides with the category of finite length sheaves $\mathrm{coh}_0(\mathbf{X})$ and is therefore explicitly known, we obtain a constructive description of all indecomposable sheaves on a weighted projective line $\mathbf{X}$ of genus one. In particular each indecomposable sheaf on $\mathbf{X}$ is uniquely determined by the following data: the slope q, a point λ on $\mathbf{X}$, a number $i \in \mathbf{Z}/p(\lambda)\mathbf{Z}$, where $p(\lambda)$ denotes the rank of the tube of finite length sheaves concentrated at λ, and the quasi-length $l \in \mathbf{N}$.

4.2.8 We continue the proof of Theorem 4.1.4. It remains to show that for each τ-orbit $\mathcal{U}_q$ of a quasi-simple sheaf of slope q there is an equivalence $L_{\mathcal{U}_q}$ such that

$$\bigoplus_{U \in \mathcal{U}_q} \mathrm{Hom}^\bullet(U, X) \otimes U \xrightarrow{\mathrm{can}_X} X \xrightarrow{\eta_X} L(X)$$

is a distinguished triangle for each object $X \in \mathcal{D}^b(\mathrm{coh}(\mathbf{X}))$.

Denote by $\mathcal{U}_\infty$ the image of $\mathcal{U}_q$ under $\Phi_{\infty,q}$ and by $L_{\mathcal{U}_\infty}$ the corresponding tubular mutation. Then the equivalence $\Phi_{q,\infty} \circ L_{\mathcal{U}_\infty} \circ \Phi_{\infty,q}$ satisfies the assertion which finishes the proof of the theorem. □

4.2.9 The idea of tubular mutations goes back to Atiyah [3] who classified the indecomposable vector bundles on a nonsingular elliptic curve $\mathbf{Y}$ (compare [67]). Using modern terminology, Atiyah's classification can be described as follows. The Auslander-Reiten quiver $\mathrm{coh}(\mathbf{Y})$ consists of homogeneous tubes, one for each point of $\mathbf{Y}$. Similarly as in 4.1.4, for each quasi-simple sheaf $\mathcal{S}$ in $\mathrm{coh}(\mathbf{Y})$ there is an equivalence $L : \mathcal{D}^b(\mathrm{coh}(\mathbf{Y})) \xrightarrow{\cong} \mathcal{D}^b(\mathrm{coh}(\mathbf{Y}))$ such that

$$\mathrm{Hom}^\bullet(\mathcal{S}, X) \otimes \mathcal{S} \xrightarrow{\mathrm{can}} \mathcal{S} \xrightarrow{\eta} L(X)$$

is a distinguished triangle for each object $X \in \mathcal{D}^b(\mathrm{coh}(\mathbf{Y}))$ [85]. Then telescopic functors relating the categories $\mathcal{C}_q$ of semi-stable sheaves on $\mathbf{Y}$, for each $q \in \mathbf{Q}$, to the category of finite length sheaves $\mathrm{coh}_0(\mathbf{Y})$, can be defined as in the case of weighted projective lines of genus one.

On the other hand, in Ringel's classification of modules over a tubular algebra the role of the equivalences R and S is played by certain functors defined individually for each weight type by suitable tilting modules, the so-called *shrinking functors.*

4.3 Automorphisms of the derived category

Tubular mutations allow to investigate the automorphism group of the derived category of coherent sheaves for a weighted projective line of genus one. We present here the main results and refer to the forthcoming joint paper with Lenzing [76] for details. Automorphism groups of derived categories of coherent sheaves for some classical varieties were described by Bondal and Orlov [13].

4.3.1 We first assume that $\mathbf{X}$ is a weighted projective line of arbitrary weight type.

Definition 4.3.1 *An automorphisms of* $\mathcal{D}^b(\mathrm{coh}(\mathbf{X}))$ *is an exact functor* $\mathcal{D}^b(\mathrm{coh}(\mathbf{X})) \to \mathcal{D}^b(\mathrm{coh}(\mathbf{X}))$ *which is an equivalence. The automorphism group* $\mathrm{Aut}(\mathcal{D}^b(\mathrm{coh}(\mathbf{X})))$ *consists of classes, with respect to equivalence of functors, of automorphisms of* $\mathcal{D}^b(\mathrm{coh}(\mathbf{X}))$.

An automorphisms of $\mathcal{D}^b(\mathrm{coh}(\mathbf{X}))$ *sending* $\mathrm{coh}(\mathbf{X})$ *to* $\mathrm{coh}(\mathbf{X})$ *and fixing* $\mathcal{O}$ *is called an automorphism of* $\mathbf{X}$. *The subgroup of* $\mathrm{Aut}(\mathcal{D}^b(\mathrm{coh}(\mathbf{X})))$ *consisting of all automorphisms of* $\mathbf{X}$ *is denoted by* $\mathrm{Aut}(\mathbf{X})$.

Recall that the Picard group $\mathrm{Pic}(\mathbf{X})$ of $\mathbf{X}$ can be identified with the grading group $\mathbf{L}(\mathbf{p})$ and that the torsion group $\mathbf{tL}(\mathbf{p})$ consists of all elements of $\mathbf{L}(\mathbf{p})$ of degree zero (see 2.2.2). Therefore, this group can be interpreted as the group of line bundles on $\mathbf{X}$ of degree zero and it is denoted by $\mathrm{Pic}_0(\mathbf{X})$.

Proposition 4.3.2 *The automorphism group* $\mathrm{Aut}(\mathbb{X})$ *of* $\mathbb{X}$ *is isomorphic to the subgroup of automorphisms of* $\mathbb{P}^1(k)$*, preserving weights. In particular,* $\mathrm{Aut}(\mathbb{X})$ *is finite if* $\mathbb{X}$ *has at least three exceptional points.* □

Theorem 4.3.3 *The group of rank preserving automorphisms (up to sign) has the form*

$$(\mathbb{Z} \times \mathrm{Pic}(\mathbb{X})) \ltimes \mathrm{Aut}(\mathbb{X}).$$

If $\mathbb{X}$ *has at least three exceptional points, then this group is finitely presented, and generators and relations can be explicitly given.*

Moreover, if the genus of $\mathbb{X}$ *is different from one, then each automorphism is rank preserving (up to sign).* □

4.3.4 We now assume that $\mathbb{X}$ is of tubular type. In this case the automorphisms R and S defined in 4.2.1 satisfy the relation

$$RS^{-1}R = S^{-1}RS^{-1}.$$

Furthermore, it can be shown that the subgroup of $\mathrm{Aut}(\mathcal{D}^b(\mathrm{coh}(\mathbb{X})))$ generated by R and S is isomorphic to the braid group B_3.

Theorem 4.3.4 *Let* $\mathbb{X}$ *be a weighted projective line of genus one. Then the automorphism group* $\mathrm{Aut}(\mathcal{D}^b(\mathrm{coh}(\mathbb{X})))$ *is isomorhic to*

$$(\mathrm{Pic}_0(\mathbb{X}) \ltimes \mathrm{Aut}(\mathbb{X})) \ltimes B_3.$$

In particular this group is finitely presented.

Moreover, the telescopic functors of the form $\Phi_{q,\infty}$ *provide a natural bijection between the set of left cosets* gU *in* $\mathrm{Aut}(\mathcal{D}^b(\mathrm{coh}(\mathbb{X})))$ *modulo the subgroup* U *of rank preserving automorphisms (up to sign) and the set* $\mathbb{Q} \cup \{\infty\}$. □

Also for these groups explicit presentations by generators and relations can be given. We emphasize that the automorphism group $\mathrm{Aut}(\mathbb{X})$ depends on the weight sequence and in the case of weight type $(2,2,2,2)$ also on the parameter sequence. In particular, it follows from 4.3.2 that $\mathrm{Aut}(\mathbb{X})$ is isomorphic to the symmetric group S_3 (resp. S_2) if $\mathbb{X}$ is of type $(3,3,3)$ (resp. $(2,4,4)$), and is trivial if $\mathbb{X}$ is of type $(2,3,6)$. If $\mathbb{X}$ is of weight type $(2,2,2,2)$ we can assume that the parameter sequence $\boldsymbol{\lambda}$ is of the form $(\infty, 0, 1, \lambda)$ for some $\lambda \in k$, $\lambda \notin \{0,1\}$. Then the group $\mathrm{Aut}(\mathbb{X})$ depends only on the j-invariant, $j = 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$, of $\mathbb{X}$. Explicit calculations show that $\mathrm{Aut}(\mathbb{X})$ is isomorphic to the alternating group A_4 in case $j = 0$, to the dihedral group D_4 in case $j = 1728$ and to the Klein fours group otherwise.

Chapter 5

Twisted mutations

Throughout this chapter $\mathbb{X}$ denotes a weighted projective line of tubular type. We will study exceptional pairs (A,B) of stable exceptional objects in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ of the same τ-order such that there are morphisms from A to B, but not from A to the Auslander-Reiten translates of B. In this case the dimension of the vector space $\mathrm{Hom}_{\mathcal{D}}(A,B)$ can be calculated using the Riemann-Roch formula. Combining the mutations of exceptional pairs with the Auslander-Reiten translation we define a new variant of mutations, which can be used for constructing the stable exceptional sheaves by starting with pairs consisting of a line bundle and a stable exceptional finite length sheaf.

We will interpret this kind of mutations as action of the braid group B_3 on the set of exceptional pairs having some special properties. This approach does not work in the domestic and in the wild case, where for the construction of the exceptional objects an action of the bigger braid group B_n, $n = \mathrm{rk}(\mathrm{K}_0(\mathbb{X})$, is needed (see Chapter 4). In the tubular case, however, we get an alternative method at least for the stable exceptional sheaves. In particular, in case the weight type is $(2,2,2,2)$ this produces all exceptional objects. A similar concept of mutations of exceptional pairs for coherent sheaves on nonsingular elliptic curves was studied by Kuleshov [67].

5.1 Admissible exceptional pairs for tubular weighted projective lines

5.1.1 An exceptional object A in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ for a tubular weighted projective line $\mathbb{X}$ is of finite τ-order which equals the rank of the tube (in the sense of [100]) of A. We will denote this number by $\rho = \rho(A)$. Note that ρ divides p. We know from 4.1.2 that the stable sheaves on $\mathbb{X}$ coincide with the quasi-simple ones. An indecomposable object $A \in \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ of the form $A = E[i]$, where E is a stable in $\mathrm{coh}(\mathbb{X})$ and $i \in \mathbb{Z}$, is said to be stable, too.

Definition 5.1.1 *A pair* (A,B) *of objects in* $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ *is called an admissible exceptional pair if it satisfies the following conditions:*

(i) A *and* B *are exceptional and stable and of the same* τ*-order* ρ*.*

(ii) $\mathrm{Hom}_{\mathcal{D}}(A,B) \neq 0$.

(iii) $\mathrm{Hom}_{\mathcal{D}}(A, \tau^j B) = 0$ *if* ρ *does not divide* j.

An admissible exceptional pair (A, B) *with* $\dim_k \mathrm{Hom}_{\mathcal{D}}(A, B) = m$ *is said to be an* m*-exceptional pair.*

Here τ denotes the Auslander-Reiten translation in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$.

Lemma 5.1.2 *Let* (A, B) *be an admissible exceptional pair in* $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$. *Then*

(i) $\mathrm{Hom}_{\mathcal{D}}(B, A[s]) = 0$ *for* $s \in \mathbb{Z}$, *thus* (A, B) *is an exceptional pair in the sense of* 3.1.1.

(ii) $\mathrm{Hom}_{\mathcal{D}}(A, B[s]) = 0$ *for* $s \neq 0$.

(iii) $\mathrm{Hom}_{\mathcal{D}}(A, \tau^j B[s]) = 0$ *for* $s \in \mathbb{Z}$, *provided* ρ *does not divide* j.

Proof. Up to shift in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ we have one of the following two cases.

(a) $A, B \in \mathrm{coh}(\mathbb{X})$ and $\mu(A) < \mu(B)$.

(b) $A \in \mathrm{coh}(\mathbb{X})$, $B \in \mathrm{coh}(\mathbb{X})[1]$ and $\mu(A) > \mu(B)$.

In the case (a) we infer $\mathrm{Hom}_{\mathbb{X}}(B, A) = 0$, by stability, and $\mathrm{Ext}^1_{\mathbb{X}}(B, A) \cong \mathrm{Hom}_{\mathbb{X}}(A, \tau_{\mathbb{X}} B) = 0$, by Serre duality and condition (iii) of the definition. This proves (i). Assertion (ii) follows from $\mathrm{Ext}^1_{\mathbb{X}}(A, B) \cong \mathrm{Hom}_{\mathbb{X}}(B, \tau A) = 0$ using again the fact that A and B are stable. Furthermore, $\mathrm{Ext}^1_{\mathbb{X}}(\tau^j A, B) = 0$, for all $j \in \mathbb{Z}$, and therefore (iii) is a consequence of condition (iii) of the definition.

In the case (b) the lemma is proved by similar arguments, the detailed verification is left to the reader. □

5.1.3 For an admissible exceptional pair (A, B) we write $M_{A,B} = \begin{pmatrix} \mathrm{rk}(A) & rk(B) \\ \deg(A) & \deg(B) \end{pmatrix}$. Furthermore, we denote by $\mathcal{P}_m$ the set of m-exceptional pairs in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ and set

$$M(\mathcal{P}_m) = \{M_{A,B} \mid (A, B) \in \mathcal{P}_m\}.$$

Lemma 5.1.3 *Let* (A, B) *be an* m*-exceptional pair. If* A *and* B *have* τ*-order* ρ, *then*

$$\frac{\rho}{p} \cdot |M_{A,B}| = m.$$

Proof. Riemann-Roch's theorem and Lemma 5.1.2 imply

$$|M_{A,B}| = \overline{\chi}(A, B) = \sum_{j=0}^{p-1} \chi(\tau^j A, B) = \frac{p}{\rho} \cdot \dim_k \mathrm{Hom}_{\mathcal{D}}(A, B).$$

□

Proposition 5.1.4 *Let* (A, B) *be an* m*-exceptional pair of objects of* τ*-order* ρ. *Then*

(i) $\rho = p$.

(ii) $m \leq 2$. *Moreover, if* $m = 2$ *then* $\mathbb{X}$ *is of weight type* $(2, 2, 2, 2)$.

Proof. (i) Assume to the contrary that (A, B) is an m-exceptional pair such that $\rho < p$. Applying an automorphism $\Phi \in \mathrm{Aut}(\mathcal{D}^b(\mathrm{coh}(\mathbb{X})))$ we obtain an m-exceptional pair $(C, \mathcal{S})$, where $\mathcal{S}$ is a simple exceptional sheaf of finite length and C is a vector bundle on $\mathbb{X}$. Now, $\sum_{j=1}^{\rho} [\tau^j C] = \sum_{i=1}^{p} [\tau^j D]$ for some indecomposable bundle D having maximal τ-order and the same slope as C. It follows that $\rho \cdot \mathrm{rk}(C) = p \cdot \mathrm{rk}(D)$, and consequently $\frac{p}{\rho}$ divides $\mathrm{rk}(C)$.

On the other hand C can be considered as an exceptional vector bundle in the perpendicular category ${}^{\perp}\{\tau\mathcal{S}, \ldots, \tau^{\rho-1}\mathcal{S}\}$, which is equivalent to a category of sheaves $\mathrm{coh}(\mathbb{X}')$ for some weighted projective line $\mathbb{X}'$. In any case, under the assumption $\rho < p$, the curve $\mathbb{X}'$ is of weight type (p_1, p_2). By a variant of Grothendieck's theorem (see [57, Chapter VI, §21]) we conclude $\mathrm{rk}(C) = 1$. This gives a contradiction and proves (i).

(ii) Keeping the notations of (i), we have $M_{C,\mathcal{S}} = \begin{pmatrix} \mathrm{rk}(C) & 0 \\ \deg(C) & 1 \end{pmatrix}$. From Lemma 5.1.3 we deduce that $\mathrm{rk}(C) = m$. Using the same argument as above, we see that the curve $\mathbb{X}'$ cannot be of weight type (p_1, p_2) provided $m > 1$. Hence the only possibility for $m > 1$ is $m = 2$, and in this case $\mathbb{X}$ is of type $(2, 2, 2, 2)$, accordingly $\mathbb{X}'$ is of type $(2, 2, 2)$. This proves (ii). □

5.1.5 Examples (i) For an arbitrary tubular weighted projective line the pair $(\mathcal{O}, \mathcal{S}_{t, p_t - 1})$ is an 1-exceptional pair.

(ii) Let $\mathbb{X}$ be of weighted type $(2, 2, 2, 2)$. Then $(\mathcal{O}, \mathcal{O}(\vec{c}))$ is a 2-exceptional pair. The corresponding matrix is $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$.

(iii) Let $\mathbb{X}$ be of weighted type $(2, 2, 2, 2)$. Denote by E_i the middle term of a non-split exact sequence of the form

$$0 \to \omega \to E_i \to \mathcal{O}(\vec{x}_i) \to 0.$$

Then $(E_i, \mathcal{S}_{i,0})$ is a 2-exceptional pair, for $i = 1, \ldots, 4$, with matrix $\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$. □

5.2 Twisted mutations of admissible exceptional pairs

5.2.1 Recall from 3.2.1 that for an exceptional pair (A, B) in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ the left and right mutations are given by triangles

$$(\Delta_l) \qquad \mathcal{L}_A B[-1] \longrightarrow \mathrm{Hom}^\bullet(A, B) \otimes A \xrightarrow{can} B \longrightarrow \mathcal{L}_A B,$$

$$(\Delta_r) \qquad \mathcal{R}_B A \longrightarrow A \xrightarrow{can} \mathrm{DHom}^\bullet(A, B) \otimes B \longrightarrow \mathcal{R}_B A[1]$$

If (A, B) is even m-exceptional, then $(B, \tau A)$ and $(\tau^- B, A)$ are exceptional pairs, by Lemma 5.1.2 (iii) and Serre duality, and we can consider the exceptional pairs $(\mathcal{L}_B(\tau A), B)$ and $(A, \mathcal{R}_A(\tau^- B))$. We define

$$f_1(A, B) = (A, \mathcal{R}_A(\tau^- B)), \quad f_2(A, B) = (\mathcal{L}_B(\tau A), B),$$

$$f_1^{-1}(A, B) = (A, \tau(\mathcal{L}_A B)), \quad f_2^{-1}(A, B) = (\tau^-(\mathcal{R}_B A), B)$$

Lemma 5.2.1 *(i) If (A,B) is an m-exceptional pair then so are $f_1(A,B)$, $f_2(A,B)$, $f_1^{-1}(A,B)$ and $f_2^{-1}(A,B)$.*
(ii) f_1 and f_1^{-1} [resp. f_2 and f_2^{-1}] are mutually inverse each other.

Proof. (i) We give the proof only for $f_1^{-1}(A,B) = (A, \tau(\mathcal{L}_A B))$, the other cases are left to the reader.

The left mutation of the exceptional pair (A,B) is given by the distinguished triangle (Δ_l) and, using the Auslander-Reiten-translation, we get a triangle

$$(\Delta') \qquad \tau(\mathcal{L}_A B)[-1] \longrightarrow \mathrm{Hom}^\bullet(A,B) \otimes \tau A \longrightarrow \tau B \longrightarrow \tau(\mathcal{L}_A B).$$

Let ρ be the τ-order of A and B. Applying the functor $\mathrm{Hom}(A,-)$ to the triangle (Δ') and using that $\mathrm{Hom}(A,\tau B) = 0 = \mathrm{Hom}(A,\tau B[1])$ by 5.1.2, we see that $\mathrm{Hom}(A,\tau(\mathcal{L}_A B))$ is m-dimensional.

Further, application of $\mathrm{Hom}(\tau^n A,-)$ to (Δ') gives, for $n = 2,\ldots,\rho-1$, $\mathrm{Hom}(\tau^{n-1}A, \mathcal{L}_A B) \cong \mathrm{Hom}(\tau^n A, \tau\mathcal{L}_A B) = 0$, because $\mathrm{Hom}(\tau^{n-1}A, B) = 0$ and $\mathrm{Hom}(\tau^{n-1}A, A[1]) = 0$ by assumption. Moreover, in the special case $n = 1$ we obtain an exact sequence $\mathrm{Hom}(\tau A, \mathrm{Hom}(A,B)\otimes\tau A) \xrightarrow{f} \mathrm{Hom}(\tau A, \tau B) \to \mathrm{Hom}(\tau A, \tau(\mathcal{L}_A B) \to \mathrm{Hom}(\tau A, \mathrm{Hom}(A,B)\otimes \tau A[1]) = 0$. The map f is an isomorphism, because it is induced from the canonical map. Hence we get also $\mathrm{Hom}(\tau A, \tau\mathcal{L}_A B) = 0$.

The object $\mathcal{L}_A B$ is defined by a mutation of an exceptional pair, hence $\mathcal{L}_A B$ is exceptional and $\tau(\mathcal{L}_A B)$ has the same property. It remains to show that $\tau(\mathcal{L}_A B)$ is quasi-simple and of τ-order ρ. For this it is sufficient to prove that $\mathrm{Hom}(\tau^n\mathcal{L}_A B, \mathcal{L}_A B) = 0$ for $n = 1,\ldots,\rho-1$.

We first apply the functor $\mathrm{Hom}(\tau^n B,-)$ to the triangle (Δ_l). Invoking the duality $\mathrm{Hom}(\tau^n B, \mathrm{Hom}(A,B)\otimes A[1]) \cong \mathrm{Hom}(A,B)\otimes \mathrm{DHom}(A,\tau^{n+1}B)$ we see that $\mathrm{Hom}(\tau^n B, \mathcal{L}_A B) = 0$, for $n = 1,\ldots,\rho-2$, and $\mathrm{Hom}(\tau^{\rho-1}B, \mathcal{L}_A B) \cong \mathrm{DHom}(A,B)\otimes \mathrm{Hom}(A,B)$.

Similarly, one easily proves that $\mathrm{Hom}(\tau^n A, \mathcal{L}_A B) = 0$, for $n = 1,\ldots,\rho-2$, and $\mathrm{Hom}(\tau^{\rho-1}A, \mathcal{L}_A B) \cong \mathrm{Hom}(A,B)$.

Now we apply the functor $\mathrm{Hom}(-,\mathcal{L}_A B)$ to the distinguished triangle

$$(\Delta'') \qquad \tau^n(\mathcal{L}_A B)[-1] \longrightarrow \mathrm{Hom}(A,B)\otimes\tau^n A \longrightarrow \tau^n B \longrightarrow \tau^n(\mathcal{L}_A B).$$

For $n = 1,\ldots,\rho-2$ we obtain
$0 = \mathrm{Hom}(\mathrm{Hom}(A,B)\otimes\tau^n A[1], \mathcal{L}_A B) \to \mathrm{Hom}(\tau^n\mathcal{L}_A B, \mathcal{L}_A B) \to \mathrm{Hom}(\tau^n B, \mathcal{L}_A B) = 0$, and consequently $\mathrm{Hom}(\tau^n\mathcal{L}_A B, \mathcal{L}_A B) = 0$. Finally, for $n = \rho-1$ we get the exact sequence

$$0 \to \mathrm{Hom}(\tau^{\rho-1}\mathcal{L}_A B, \mathcal{L}_A B) \to \mathrm{Hom}(\tau^{\rho-1}B, \mathcal{L}_A B) \xrightarrow{g} \mathrm{Hom}(\mathrm{Hom}(A,B)\otimes\tau^{\rho-1}A, \mathcal{L}_A B) \to 0.$$

Both terms $\mathrm{Hom}(\tau^{\rho-1}B, \mathcal{L}_A B)$ and $\mathrm{Hom}(\mathrm{Hom}(A,B)\otimes\tau^{\rho-1}A, \mathcal{L}_A B)$ can be identified with $\mathrm{DHom}(A,B)\otimes\mathrm{Hom}(A,B)$, hence g is an isomorphism. It follows that $\mathrm{Hom}(\tau^{\rho-1}\mathcal{L}_A B, \mathcal{L}_A B) = 0$, and consequently $\mathcal{L}_A B$ is quasi-simple and of τ-order ρ. Thus $(A, \tau(\mathcal{L}_A B))$ is an m-exceptional pair.

Note that we did not have to use Proposition 5.1.4.

(ii) We check at once that $f_1(f_1^{-1}(A,B)) = f_1(A, \tau(\mathcal{L}_A B)) = (A, \mathcal{R}_A(\tau^-(\tau(\mathcal{L}_A B)))) = (A, \mathcal{R}_A\mathcal{L}_A B) = (A,B)$ for each m-exceptional pair (A,B), thus $f_1 \circ f_1^{-1} = id$. The other identities are proved similarly. □

5.2.2 According to Lemma 5.2.1 we can consider f_1 and f_2 as bijections of $\mathcal{P}_m$. Let us denote by F_m the subgroup of the symmetric group of $\mathcal{P}_m$ generated by f_1 and f_2. Observe that f_1 and f_2 induce isomorphisms of $\mathrm{K}_0(\mathbf{X}) \times \mathrm{K}_0(\mathbf{X})$ given by multiplications from the right with

$$\begin{pmatrix} 1 & m \\ 0 & \tau^- \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tau & 0 \\ m & 1 \end{pmatrix},$$

respectively.

Lemma 5.2.2 *For $m = 1$ the identity $f_1 f_2^{-1} f_1 = f_2^{-1} f_1 f_2^{-1}$ holds.*

Proof. Since

$$\begin{pmatrix} 1 & 1 \\ 0 & \tau^- \end{pmatrix}\begin{pmatrix} \tau^- & 0 \\ -\tau^- & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & \tau^- \end{pmatrix} = \begin{pmatrix} 0 & \tau^- \\ -\tau^{-2} & 0 \end{pmatrix} = \begin{pmatrix} \tau^- & 0 \\ -\tau^- & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & \tau^- \end{pmatrix}\begin{pmatrix} \tau^- & 0 \\ -\tau^- & 1 \end{pmatrix},$$

$[f_1 f_2^{-1} f_1(A,B)]$ and $[f_2^{-1} f_1 f_2^{-1}(A,B)]$ coincide up to translations in the derived category. By 3.4.1, an exceptional object is uniquely determined, up to translation, by its class in the Grothendieck group. Through case by case inspection it is also easy to verify that $f_1 f_2^{-1} f_1$ and $f_2^{-1} f_1 f_2^{-1}$ send each pair to pairs such that the corresponding objects are in the same copy of $\mathrm{coh}(\mathbf{X})$. □

5.2.3 There is a group homomorphism

$$\epsilon_m : F_m \to SL(2,\mathbf{Z})$$

defined on the generators of F_m by $\epsilon_m(f_1) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ and $\epsilon_m(f_2) = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$. Note that $\epsilon_m(g) = |M_{A,B}^{-1}| \cdot |(M_{A,B}).g|$ for each m-exceptional pair (A,B).

Denote by G_m the image of ϵ_m. By classical results in group theory [96] we have

Proposition 5.2.3 *(i) $G_1 = SL(2,\mathbf{Z})$.*
(ii) G_2 is freely generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Furthermore,

$$G_2 = \{\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \in SL(2,\mathbf{Z}) |\, a_{1,1} \equiv a_{2,2} \equiv 1 \,(\mathrm{mod}\ 4),\ a_{1,2} \equiv a_{2,1} \equiv 0 \,(\mathrm{mod}\ 2)\}.$$

5.2.4 As a consequence we obtain

Proposition 5.2.4 *(i) F_1 is isomorphic to the braid group B_3.*
(ii) F_2 is a free group generated by f_1 and f_2. □

Proof. (i) Denote by T the automorphism $\mathcal{D}^b(\mathrm{coh}(\mathbf{X})) \to \mathcal{D}^b(\mathrm{coh}(\mathbf{X}))$, given by the translation $X \mapsto X[1]$. It is easily seen that $(f_2 f_1^{-1})^3 = T\tau^3$, thus $(f_2 f_1^{-1})^6 = T^2\tau^6$. By Lemma 5.2.2, there is a surjective group homomorphism $\psi : B_3 \to \langle f_1, f_2 \rangle$ mapping the generators σ_1 and σ_2 of B_3 to f_1 and f_2^{-1}, respectively. Further, we have a surjective group homomorphism $h : B_3 \to SL(2,\mathbf{Z})$ mapping the generators σ_1 and σ_2 to $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The kernel of h is isomorphic to $\mathbb{Z}$ and generated by $(\sigma_2\sigma_1)^6$, which follows from [112] and [19].

We thus obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} \cong \langle(\sigma_2\sigma_1)^6\rangle & \longrightarrow & B_3 & \xrightarrow{h} & SL(2,\mathbb{Z}) & \longrightarrow & 1 \\ & & \downarrow \psi' & & \downarrow \psi & & \| & & \\ 1 & \longrightarrow & N & \longrightarrow & F_1 & \xrightarrow{\epsilon} & SL(2,\mathbb{Z}) & \longrightarrow & 1. \end{array}$$

Since ψ is an epimorphism, this holds also for ψ'. Therefore N is infinite cyclic which implies that ψ', hence ψ is an isomorphism.

(ii) follows from Proposition 5.2.3. □

5.2.5 We will study the actions

$$\mathcal{P}_1 \times F_1 \to \mathcal{P}_1$$

for an arbitrary tubular weighted projective line and

$$\mathcal{P}_2 \times F_2 \to \mathcal{P}_2$$

for a weighted projective line of type $(2,2,2,2)$. Observe that the induced actions

$$M(\mathcal{P}_1) \times G_1 \to M(\mathcal{P}_1) \quad \text{and} \quad M(\mathcal{P}_2) \times G_2 \to M(\mathcal{P}_2)$$

are given by multiplication of matrices, and $\epsilon(f_1)$ and $\epsilon(f_2)$ act as multiplications with

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix},$$

respectively, from the right.

Note further that

$$M(\mathcal{P}_1) = SL(2,\mathbb{Z})$$

and

$$M(\mathcal{P}_2) = \{M = \begin{pmatrix} r & r' \\ d & d' \end{pmatrix} \in GL(2,\mathbb{Z}) \mid |M| = 2,\ \gcd(r,d) = 1,\ \gcd(r',d') = 1\},$$

as is easy to check by applying the telescopic functors of Chapter 4 and Theorem 3.5.

Obviously, we also have actions, for $m = 1,2$,

$$\mathrm{Aut}(\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))) \times \mathcal{P}_m \to \mathcal{P}_m, \quad (\Phi,(A,B)) \mapsto (\Phi A, \Phi B).$$

It is clear that the automorphisms R and S (see Chapter 4) act on the level of $M(\mathcal{P}_m)$ as multiplications from the left with

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

respectively. Moreover, it is easy to see that, considered as bijections of $\mathcal{P}_m$, the automorphisms commute with the elements of F_m.

5.2.6 We recall from [74, 2.7] the notion of the determinant homomorphism for a weighted projective line.

Denote by $\mathcal{U}_i$ the uniserial category of coherent sheaves concentrated at the exceptional point λ_i, $i = 1,\dots,t$. Then the inclusion $\mathcal{U}_i \hookrightarrow \mathrm{coh}(\mathbb{X})$ induces an embedding $\mathrm{K}_0(\mathcal{U}_i) \hookrightarrow \mathrm{K}_0(\mathbb{X})$. If further $h : \mathrm{K}_0(\mathbb{X}) \to G$ is a homomorphism into a group G, with the property $h(\mathcal{O}) = 0$ then h is uniquely determined by a family of homomorphisms $h_i : \mathrm{K}_0(\mathcal{U}_i) \to G$ such that $h_1(\mathbf{w}) = \dots = h_t(\mathbf{w})$, where $\mathbf{w}$ denotes the class of a simple sheaf concentrated at an ordinary point. The *determinant homomorphism* $\det : \mathrm{K}_0(\mathbb{X}) \to \mathbf{L}(\mathbf{p})$ is defined by requiring that each simple object from $\mathcal{U}_i$ is mapped to $\vec{x}_i \in \mathbf{L}(\mathbf{p})$ and $\det(\mathcal{O}) = 0$. Note that the degree homomorphism $\deg : \mathrm{K}_0(\mathbb{X}) \to \mathbb{Z}$ arises as the composition of the determinant homomorphism with the degree map $\delta : \mathbf{L}(\mathbf{p}) \to \mathbb{Z}$ that kills the torsion $\mathbf{tL}(\mathbf{p})$ of $\mathbf{L}(\mathbf{p})$.

5.2.7 For a 1-exceptional pair (A,B) in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ with $M_{A,B} = \begin{pmatrix} r_A & r_B \\ d_A & d_B \end{pmatrix}$ we denote

$$(A,B) = r_A \cdot \det B - r_B \cdot \det A.$$

According to 5.1.4 we have $\delta(\mathrm{Det}(A,B)) = 1$. We will identify the groups $\mathbf{L}(\mathbf{p})$ and $\mathrm{Pic}(\mathbb{X})$, thus $\mathrm{Det}(A,B)$ can be considered as a line bundle of degree one. The cyclic group generated by τ acts on the set of line bundles of degree one by shift because $\mathbb{X}$ is tubular. We denote by $[E]_\tau$ the class of a bundle E under this action.

Lemma 5.2.7 *(i)* $[\mathrm{Det}(A,B).g]_\tau = [\mathrm{Det}(A,B)]_\tau$ *for all* $g \in F_1$.
(ii) $[\mathrm{Det}(A(\vec{x}),B(\vec{x}))]_\tau = [\mathrm{Det}(A,B)]_\tau$ *for each* $\vec{x} \in \mathbf{L}(\mathbf{p})$.

Proof. (i) It is sufficient to show the equality for the generators of F_1. We check this only for $g = f_1^{-1}$, in the same manner the assertion is proved for f_1, f_2, f_2^{-1}.

Since $\dim_k \mathrm{Hom}(A,B) = 1$ we have a triangle $\mathcal{L}_A B[-1] \longrightarrow A \xrightarrow{can} B \longrightarrow \mathcal{L}_A B$, which gives rise to the formulas $\mathrm{rk}(\mathcal{L}_A B) = r_B - r_A$, and $\det(\mathcal{L}_A B) = \det B - \det A$. Then

$$\begin{aligned} \mathrm{Det}((A,B).f_1^{-1}) &= \mathrm{Det}(A, \tau\mathcal{L}_A B) \\ &= r_A \cdot \det(\mathcal{L}_A B(\vec{\omega})) + (r_A - r_B)\cdot \det A \\ &= r_A \cdot (\det(\mathcal{L}_A B) + (r_B - r_A)\cdot\vec{\omega}) + (r_A - r_B)\cdot \det A \\ &= r_A \cdot (\det B - \det A) + r_A(r_B - r_A)\cdot\vec{\omega} + (r_A - r_B)\cdot\det A \\ &= \mathrm{Det}(A,B) + r_A(r_B - r_A)\cdot\vec{\omega} \end{aligned}$$

The proof for (ii) is similar. □

Proposition 5.2.8 *Let* (A,B) *be a 1-exceptional pair with* $[\mathrm{Det}(A,B)]_\tau = [\mathcal{O}(\vec{x}_i)]_\tau$. *Then there are elements* $\vec{x} \in \mathbf{tL}(\mathbf{p})$ *and* $g \in F_1$ *such that* $(A,B) = (\mathcal{O}(\vec{x}_i), \mathcal{S}_{i,p_i-1}(\vec{x})).g$.

Proof. Let $(A,B) \in \mathcal{P}_1$. Applying Proposition 5.2.3 (i), the matrix $M_{A,B}$ can be written as a product u of matrices $M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $M_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and their inverses. Replacing in u the factors M_1, M_2, M_1^{-1} and M_2^{-1} by f_1, f_2, f_1^{-1} and f_2^{-1}, respectively, we obtain an element $g \in F_1$ such that $(A,B).g^{-1}$ is a 1-exceptional pair $(L,\mathcal{S})$ with matrix $M_{L,\mathcal{S}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus $L = \mathcal{O}(\vec{x})$ is a line bundle and $\mathcal{S}$ is a simple exceptional finite length sheaf. Then $(A,B) = (\mathcal{O}(\vec{x}), \mathcal{S}(-\vec{x})(\vec{x})).g$. Further by Lemma 5.2.7, $[\mathrm{Det}(A,B)]_\tau = [\mathcal{O}(\vec{x}_i)]_\tau$ implies $[\mathrm{Det}(\mathcal{O}, \mathcal{S}(-\vec{x}))]_\tau = [\mathcal{O}(\vec{x}_i)]_\tau$, therefore $\mathcal{S}(-\vec{x})) = \mathcal{S}_{i,p_i-1}$. □

5.2.9 We consider $\mathrm{Pic}(\mathbb{X})$, $\mathrm{Pic}_0(\mathbb{X})$, $\mathrm{Aut}(\mathbb{X})$ and F_1 as subgroups of the symmetric group of $\mathcal{P}_1$. It follows from 4.3.3 that the semidirect product $\mathrm{Pic}_0(\mathbb{X}) \ltimes \mathrm{Aut}(\mathbb{X})$ can be identified with the group of rank and degree preserving automorphisms of $\mathrm{coh}(\mathbb{X})$.

Lemma 5.2.9 *The subgroup generated by* $\mathrm{Pic}_0(\mathbb{X}) \ltimes \mathrm{Aut}(\mathbb{X})$ *and* F_1 *is a direct product of these two subgroups.*

Proof. Since f_1 and f_2 commute with automorphisms of $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$, it is sufficient to show that $(\mathrm{Pic}_0(\mathbb{X}) \ltimes \mathrm{Aut}(\mathbb{X})) \cap F_1 = \{1\}$. Assume that $g \in (\mathrm{Pic}_0(\mathbb{X}) \ltimes \mathrm{Aut}(\mathbb{X})) \cap F_1$. Then $\epsilon(g) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, hence $g = (f_2 f_1^{-1})^{6n}$ for some $n \in \mathbb{Z}$ (see 5.2.4). But $(A,B).f_2 f_1^{-1} = T^2\tau^6(A,B)$, which gives that $n = 0$. Consequently $g = 1$. □

Now, Proposition 5.2.8 implies that the orbits in $\mathcal{P}_1$ under the action of $\mathrm{Pic}_0(\mathbb{X}) \times F_1$ are in 1-1-correspondence with the the set $\{\mathcal{O}(\vec{x}_i),\ i = 1, \ldots t\}$. Allowing, in addition, geometrical automorphisms we obtain

Theorem 5.2.9 *The group* $(\mathrm{Pic}_0(\mathbb{X}) \ltimes \mathrm{Aut}(\mathbb{X})) \times F_1$ *acts transitively on the set of 1-exceptional pairs on a tubular weighted projective line* $\mathbb{X}$. □

5.2.10 In case $\mathbb{X}$ is of weight type $(2,2,2,2)$ we do not get a transitive group action on $\mathcal{P}_2$. However, it is easily seen that in this situation for each 2-exceptional pair (A,B) there exists an element $g \in F_2$ such that $M_{(A,B).g} = \begin{pmatrix} r_A & r_B \\ d_A & d_B \end{pmatrix}$ with $r_A + r_B = 2$. From this it follows that under the action of the group $(\mathrm{Pic}(\mathbb{X}) \ltimes \mathrm{Aut}(\mathbb{X})) \times F_1$ on $\mathcal{P}_2$ there are exactly 3 orbits, containing pairs with matrices

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ -1 & 1 \end{pmatrix},$$

respectively.

The 1-exceptional and 2-exceptional pairs on weighted projective lines of weight type $(2,2,2,2)$ will occur as partial tilting complexes for hyperelliptic weighted projective lines in Chapter 9.

Chapter 6

On the number of exceptional vector bundles

For a wild weighted projective line $\mathbb{X}$ the exceptional vector bundles are the objects of certain components in the Auslander-Reiten quiver of $\mathrm{coh}(\mathbb{X})$ for which the quasi-length is smaller than some integer. It follows from 3.4.1 that for a weighted projective line there are countably many exceptional sheaves. In this chapter we will show that there are only finitely many exceptional vector bundles of given rank and degree, moreover we give a bound for this number which is polynomial in the weights.

We are further interested in those exceptional vector bundles on $\mathbb{X}$ which cannot be realized as exceptional bundles on weighted projective lines of smaller weight type by forming perpendicular categories to simple exceptional finite length sheaves. This leads to the concept of an omnipresent exceptional vector bundle. We will prove that omnipresent exceptional vector bundles in the wild and in the tubular case always exist. Moreover, for a hyperelliptic weighted projective line we will study omnipresent exceptional vector bundles of minimal rank.

6.1 Equations and inequations for exceptional bundles

6.1.1 Let $\mathbb{X} = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ be a weighted projective line of weight type $\mathbf{p} = (p_1, p_2, \ldots, p_t)$. Throughout this chapter we assume that all $p_i \geq 2$. For an exceptional vector bundle E on $\mathbb{X}$ we define natural numbers

$$r_{i,j} = r_{i,j}(E) = \dim_k \mathrm{Hom}_{\mathbb{X}}(E, \mathcal{S}_{i,j}), \qquad i = 1, \ldots, t, \qquad j = 0, 1, \ldots, p_i - 1,$$

where the $\mathcal{S}_{i,j}$ are the exceptional simple sheaves defined in 2.2.5. Moreover, we denote by $\mathcal{S}$ a simple finite length sheaf concentrated at a fixed ordinary point. Observe that the rank r of E is equals $\dim_k \mathrm{Hom}_{\mathbb{X}}(E, \mathcal{S})$.

The classes $[\mathcal{O}(\vec{x})]$, $0 \leq \vec{x} \leq \vec{c}$, of the indecomposable summands of the canonical tilting sheaf form a basis of the Grothendieck group $\mathrm{K}_0(\mathbb{X})$. It is easy to see that $[\mathcal{O}]$, $[\mathcal{S}_{i,j}]$, $i = 1, \ldots, t$, $j = 0, 1, \ldots, p_i - 2$, $[\mathcal{S}]$, form a basis of $\mathrm{K}_0(\mathbb{X})$, as well.

Thus, the class $[E]$ of E can be written in the form

$$[E] = k_0[\mathcal{O}] + \sum_{\substack{i=1,\dots,t\\ j=1,\dots,p_i-1}} k_{i,j}[\mathcal{O}(j\vec{x}_i)] + k_c[\mathcal{O}(\vec{c})]$$
$$= n_0[\mathcal{O}] + \sum_{\substack{i=1,\dots,t\\ j=0,\dots,p_i-2}} n_{i,j}[\mathcal{S}_{i,j}] + n_S[\mathcal{S}]$$

for uniquely determined integers $k_0, k_{i,j}, k_c, n_0, n_{i,j}, n_S$.

The following equations hold:

(a) $n_0 = r$.

(b) $k_{i,j} = r_{i,j-1}, \quad i = 1,\dots,t,\ j = 1,\dots,p_i - 1$.

(c) $r_{i,j} = n_{i,j} - n_{i,j+1}, \quad i = 1,\dots,t,\ j = 0,\dots,p_i - 2$. (Set $n_{i,p_i-1} = 0$.)

(d) $n_{i,j} = r_{i,j} + r_{i,j+1} + \cdots + r_{i,p_i-2}, \quad i = 1,\dots,t,\ j = 0,\dots,p_i - 2$.

Indeed, assertion (a) is an immediate consequence of the additivity of the rank function. Furthermore, we have $r_{i,j} = \dim_k \mathrm{Hom}_{\mathbb{X}}(E, \mathcal{S}_{i,j}) = \chi([E],[\mathcal{S}_{i,j}]) = \chi(k_0[\mathcal{O}] + \sum k_{i,j}[\mathcal{O}(j\vec{x}_i)] + k_c[\mathcal{O}(\vec{c})], [\mathcal{S}_{i,j}])$, therefore (b) follows from the exact sequences of 2.2.5. Assertion (c) is proved similarly by writing $[E]$ as a linear combination in the second basis and (d) follows straightforward from (c).

Proposition 6.1.2 *Let E be an exceptional vector bundle on a weighted projective line $\mathbb{X}$. Then*

$$r^2 + \sum_{\substack{i=1,\dots,t\\ j=0,\dots,p_i-2}} r_{i,j}^2 + \sum_{\substack{i=1,\dots,t\\ 0\le j<j'\le p_i-2}} r_{i,j}r'_{i,j} - r\cdot \sum_{\substack{1,\dots,t\\ j=0,\dots,p_i-2}} r_{i,j} = 1.$$

In particular, if $\mathbb{X}$ is of weight type $(2,\dots,2)$, then

$$r^2 + \sum_{i=1,\dots,t} r_{i,0}^2 - r\cdot\sum_{i=1,\dots,t} r_{i,0} = 1,$$

Proof. Since E is exceptional, we have $\chi(E,E) = 1$. Writing $[E]$ as a linear combination in the basis containing the simple exceptional sheaves we obtain

$$\begin{aligned}
\chi([E],[E]) &= \chi(n_0[\mathcal{O}] + \sum_{\substack{1,\dots,t\\ j=0,\dots,p_i-2}} n_{i,j}[\mathcal{S}_{i,j}] + n_S[\mathcal{S}],\ n_0[\mathcal{O}] + \sum_{\substack{1,\dots,t\\ j=0,\dots,p_i-2}} n_{i,j}[\mathcal{S}_{i,j}] + n_S[\mathcal{S}])\\
&= r^2 - \sum_{i=1,\dots,t} rn_{i,0} + \sum_{\substack{i=1,\dots,t\\ j=0,\dots,p_i-2}} n_{i,j}^2 - \sum_{\substack{i=1,\dots,t\\ j=0,\dots,p_i-2}} n_{i,j}n_{i,j+1}\\
&= r^2 - \sum_{i=1,\dots,t} rn_{i,0} + \sum_{\substack{i=1,\dots,t\\ j=0,\dots,p_i-2}} n_{i,j}(n_{i,j} - n_{i,j+1})\\
&= r^2 - \sum_{i=1,\dots,t} r(r_{i,0} + \cdots + r_{i,p_i-2}) + \sum_{\substack{i=1,\dots,t\\ j=0,\dots,p_i-2}} (r_{i,j} + \cdots + r_{i,p_i-2})r_{i,j}\\
&= r^2 + \sum_{\substack{i=1,\dots,t\\ j=0,\dots,p_i-2}} r_{i,j}^2 + \sum_{\substack{i=1,\dots,t\\ 0\le j<j'\le p_i-2}} r_{i,j}r'_{i,j} - r\cdot\sum_{\substack{1,\dots,t\\ j=0,\dots,p_i-2}} r_{i,j}.
\end{aligned}$$

□

Remark. Let Λ_0 be the hereditary algebra such that $\mathrm{mod}(\Lambda_0)$ is equivalent to the right perpendicular category $\mathcal{O}_{\mathbb{X}}(\vec{c})^{\perp}$ (see 2.4.3). Denote by $\mathrm{K}_0(\Lambda_0)$ the Grothendieck group for Λ_0 and by U the subgroup of $\mathrm{K}_0(\mathbb{X})$ spanned by $[\mathcal{O}]$ and the $[\mathcal{S}_{i,j}]$, $i = 1,\dots,t, j = 0,\dots,p_i - 2$. Moreover, let $\varphi : \mathrm{K}_0(\Lambda_0) \to U$ be the $\mathbb{Z}$-linear map defined by $\varphi[S_0] = [\mathcal{O}]$, $\varphi[S_{j\vec{x}_i}] = [S_{i,j-1}]$, where S_0, $S_{j\vec{x}_i}$ are the simple Λ_0-modules. Then the proposition expresses the fact that φ is an isomorphism which preserves the Euler form.

6.1.3 let $\mathbb{X}$ be a weighted projective line of type $(2,\dots,2)$, t entries. In this case we simply write r_i instead of $r_{i,0}$ and k_i instead of $k_{i,1}$. Thus, for an exceptional vector bundle E we have

$$E = k_0[\mathcal{O}] + \sum_{i=1,\dots,t} k_i[\mathcal{O}(\vec{x}_i)] + k_c[\mathcal{O}(\vec{c})].$$

Observe that $k_i = r_i \ge 0$ for $i = 1\dots,t$.

In 6.3 we will need the following result

Lemma 6.1.3 *Keeping the notations above the following inequalities hold:*

(i) if $\mu(E) > 0$, then $k_0 \le 0$,

(ii) if $\mu(E) < 2$, then $k_c \le 0$,

(iii) if $\mu(E) > 1$, then $-k_0 \ge r_i$ for $i = 1,\dots,t$.

Proof. (i) We have $\chi([E],[\mathcal{O}]) = \chi(k_0[\mathcal{O}] + \sum_{i=1,\dots,t} k_i[\mathcal{O}(\vec{x}_i)] + k_c[\mathcal{O}(\vec{c})],[\mathcal{O}]) = k_0$. By assumption $\mu(E) > 0$, hence $\mathrm{Hom}_{\mathbb{X}}(E,\mathcal{O}) = 0$ using the stability of exceptional vector bundles, see Proposition 2.3.7. Therefore $k_0 = \chi([E],[\mathcal{O}]) = -\dim_k \mathrm{Ext}^1_{\mathbb{X}}(E,\mathcal{O}) \le 0$.

(ii) One proves by the same method as in (i) that $k_c = \chi([\mathcal{O}(\vec{c})],[E]) = -\dim_k \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{c}), E) \le 0$.

(iii) Application of the functor $\mathrm{Hom}_{\mathbb{X}}(E,-)$ to the exact sequence $0 \to \mathcal{O} \to \mathcal{O}(\vec{x}_i) \to \mathcal{S}_{i,0} \to 0$ yields a monomorphism $\mathrm{Hom}_{\mathbb{X}}(E,\mathcal{S}_{i,0}) \hookrightarrow \mathrm{Ext}^1_{\mathbb{X}}(E,\mathcal{O})$. In fact, $\mathrm{Hom}_{\mathbb{X}}(E,\mathcal{O}(\vec{x}_i))$ vanishes, because $\mu(E) > 1 = \mu(\mathcal{O}(\vec{x}_i))$ and both vector bundles are exceptional, hence stable. Now, $r_i = \dim_k \mathrm{Hom}_{\mathbb{X}}(E,\mathcal{S}_{i,0})$ by definition, and (iii) follows from the fact that $-k_0 = \dim_k \mathrm{Ext}^1_{\mathbb{X}}(E,\mathcal{O})$. □

6.2 A bound for the number of exceptional bundles of fixed rank and degree

Proposition 6.2.1 *The number of non-isomorphic exceptional vector bundles of fixed rank r and degree d on a weighted projective line $\mathbb{X} = \mathbb{X}(\mathbf{p},\boldsymbol{\lambda})$ of weight type $\mathbf{p} = (p_1,p_2,\dots,p_t)$ is bounded by $\binom{r+p_1-1}{p_1-1}\cdot\binom{r+p_2-1}{p_2-1}\cdot\ldots\cdot\binom{r+p_t-1}{p_t-1}$.*

Proof. Since $[E] = r[\mathcal{O}] + \sum k_{i,j}[\mathcal{O}(j\vec{x}_i)] + k_c[\mathcal{O}(\vec{c})]$, it follows from 3.4.1 that there is at most one exceptional vector bundle having the data r, $r_{i,j}$, d (observe that d can be calculated from the coefficients). Hence for fixed r and d, E is uniquely determined by the $r_{i,j}$. Now the natural numbers $r_{i,j}$ satisfy the equations $r_{i,0}+r_{i,1}+\ldots+r_{i,p_i-1} = r$. Further, for each $i = 1,\dots t$, the number of sequences of non-negative integers $(r_{i,0}, r_{i,1},\dots,r_{i,p_i-1})$ such that $r_{i,0} + r_{i,1} + \ldots + r_{i,p_i-1} = r$ equals $\binom{r+p_i-1}{p_i-1}$, which completes the proof □

Corollary 6.2.2 *The number of non-isomorphic exceptional vector bundles of fixed rank r and degree d on a weighted projective line $\mathbb{X}$ is bounded by $2(r+1)^{\mathrm{rk}(\mathrm{K}_0(\mathbb{X}))-3}$.*

Proof. All numbers $r_{i,j}$ satisfy $0 \leq r_{i,j} \leq r$. Moreover, according to 6.1.2, for one of the $r_{i,j}$ there are at most two choices if the others are fixed. □

Proposition 6.2.3 *Let $\mathbf{X}$ be a hyperelliptic weighted projective line with t weights.*
(i) For each $q \in \mathbf{Q}$ there is an exceptional vector bundle of slope q.
(ii) The number of non-isomorphic exceptional vector bundles of fixed slope $q \in \mathbf{Q}$ is bounded by $2(r+1)^{t-1}$.

Proof. (i) If $\mathbf{X}'$ is a weighted projective line of type $(2,...,2)$, t' entries, and $\mathcal{S}$ a simple exceptional sheaf, then the perpendicular category ${}^{\perp}\mathcal{S}$ is equivalent to a category of coherent sheaves $\operatorname{coh}(\mathbf{X}'')$ of weight type $(2,...,2)$, t'' entries. Moreover the exact embedding $\operatorname{coh}(\mathbf{X}'') \hookrightarrow \operatorname{coh}(\mathbf{X}')$ preserves the rank, and by [30, Chapter 9] in this situation also, the degree. Thus assertion (i) is a consequence of the fact that for each $q \in \mathbf{Q}$ there exists an exceptional vector bundle of slope q for a corresponding tubular weighted projective line of type $(2,2,2,2)$.
(ii) follows straightforward from Corollary 6.2.2 and Theorem 3.5.1. □

Remark. We note that for a hyperelliptic weighted projective line of type with t weights, the number of non-isomorphic exceptional vector bundles of rank 1 and fixed degree d is exactly 2^{t-1}. In fact, this number coincides with the number of elements in $\operatorname{Pic}_0(\mathbf{X}) \cong \mathbf{tL}(\mathbf{p})$.

Proposition 6.2.4 *Let $\mathbf{X}$ be a weighted projective line of arbitrary type. The number of components in $\operatorname{vect}(\mathbf{X})$ containing an exceptional vector bundle of rank r is bounded by $\binom{r+p_1-1}{p_1-1} \cdot \binom{r+p_2-1}{p_2-1} \cdot \ldots \cdot \binom{r+p_t-1}{p_t-1} \cdot \deg(\vec{\omega}) \cdot r$.*

Proof. According to the formula $\deg(\tau_{\mathbf{X}} E) = \deg(E(\vec{\omega})) = \deg E + \deg(\vec{\omega}) \cdot \operatorname{rk}(E)$ there are, up to τ-translation, $\deg(\vec{\omega}) \cdot r$ values for the degree of an exceptional bundle on $\mathbf{X}$. Then the assertion follows from Proposition 6.2.1. □

6.3 Omnipresent exceptional vector bundles

Definition 6.3.1 *We call an exceptional vector bundle E on a weighted projective line $\mathbf{X}$ omnipresent if for each finite length sheaf $\mathcal{S}$ there is a nonzero map $E \to \mathcal{S}$.*

Obviously it is sufficient to require that there is a nonzero map to each simple exceptional finite length sheaf. Thus an exceptional vector bundle is omnipresent if and only if the numbers $r_{i,j}$ defined in 6.1.1 are nonzero.

Each exceptional vector bundle can be considered as omnipresent on some weighted projective line. Indeed, assume that E is an exceptional vector bundle on $\mathbf{X}$ which is not omnipresent. Then there is a simple exceptional sheaf $\mathcal{S}$ such that $\operatorname{Hom}_{\mathbf{X}}(E,\mathcal{S}) = 0$, thus E belongs to the perpendicular category ${}^{\perp}\mathcal{S}$. Forming the perpendicular category with respect to all simple finite length sheaves with this property we see that E is an omnipresent exceptional vector bundle on a weighted projective line of smaller weight type.

6.3.2 For a weighted projective line of domestic type an omnipresent exceptional vector bundle does not necessarily exist, however in the other cases we have

Theorem 6.3.2 *Let $\mathbf{X}$ be a weighted projective line of wild or tubular weight type. Then there is an omnipresent exceptional vector bundle.*

Proof. Let E be an arbitrary exceptional vector bundle on $\mathbf{X}$. Assume that E is not omnipresent, thus $\operatorname{Hom}_{\mathbf{X}}(E,\mathcal{S}) = 0$ for some simple exceptional finite length sheaf $\mathcal{S}$. We will replace E by an exceptional bundle E' having the following two properties:
(i) $\operatorname{Hom}_{\mathbf{X}}(E',\mathcal{S}) \neq 0$,
(ii) $\operatorname{Hom}_{\mathbf{X}}(E',\mathcal{S}') \neq 0$ for all simple exceptional finite length sheaves $\mathcal{S}'$ such that $\operatorname{Hom}_{\mathbf{X}}(E,\mathcal{S}') \neq 0$.

The right perpendicular category $E^{\perp}$, formed in $\operatorname{coh}(\mathbf{X})$, is equivalent to a module category over a hereditary algebra. More precisely, by analogue results to those of Strauß [116, Theorem 3.5 and Theorem B], (compare [58, Proposition 7.5]) $E^{\perp}$ is equivalent to the coproduct of the category of a finite wing $\mathcal{W}$, consisting of bundles of the component of E in the Auslander-Reiten quiver, and a category of modules over a hereditary algebra H, which is connected and wild (resp. tame) provided $\mathbf{X}$ is wild (resp. tubular).

By assumption $\mathcal{S}$ is in $E^{\perp}$. Since the indecomposables of $\mathcal{W}$ are vector bundles, $\mathcal{S}$ belongs to $\operatorname{mod}(H)$. Further, $\mathcal{S}$ has only finitely many successors in $E^{\perp}$, hence it is a preinjective H-module.

Now, the preinjective component in $\operatorname{mod}(H)$ contains only finitely many nonsincere indecomposable modules [101]. It follows that we can choose an exceptional sequence $(F_1,\ldots,F_m)$ in $\operatorname{mod}(H)$, given by a complete slice in the preinjective component, such that $\operatorname{Hom}_{\mathbf{X}}(F_i,\mathcal{S}) \neq 0$ for each $i = 1,\ldots m$. Extending the exceptional sequence $(F_1,\ldots,F_m,E)$, considered in $\operatorname{coh}(\mathbf{X})$, to a complete exceptional sequence, we conclude by connectedness of the category $\operatorname{coh}(\mathbf{X})$ that there is an index i such that one of the spaces $\operatorname{Hom}_{\mathbf{X}}(F_i,E)$, $\operatorname{Ext}^1_{\mathbf{X}}(F_i,E)$ is nonzero. We set $F = F_i$.

According to 3.2.2, there are three possibilities for the left mutation of the exceptional pair (F,E) in $\operatorname{coh}(\mathbf{X})$.

Case (a) $\operatorname{Hom}_{\mathbf{X}}(F,E) \neq 0$ and the left mutation of ϵ is given by an exact sequence

$$0 \longrightarrow L \longrightarrow \operatorname{Hom}_{\mathbf{X}}(F,E) \otimes F \xrightarrow{can} E \longrightarrow 0.$$

In this case we can set $E' = F$. In fact, we have $\operatorname{Hom}_{\mathbf{X}}(F,\mathcal{S})$ by construction, and applying the functor $\operatorname{Hom}_{\mathbf{X}}(-,\mathcal{S})$ to the exact sequence above, we see that $\operatorname{Hom}_{\mathbf{X}}(E,\mathcal{S}') \neq 0$ implies $\operatorname{Hom}_{\mathbf{X}}(F,\mathcal{S}') \neq 0$.

Case (b) $\operatorname{Hom}_{\mathbf{X}}(F,E) \neq 0$ and the left mutation of ϵ is given by an exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathbf{X}}(F,E) \otimes F \xrightarrow{can} E \longrightarrow L \longrightarrow 0.$$

We claim that F is projective in $\operatorname{mod}(H)$. Indeed, since the canonical map is injective, we obtain, for any $X \in E^{\perp}$, $0 = \operatorname{Ext}^1_{\mathbf{X}}(E,X) \longrightarrow \operatorname{Ext}^1_{\mathbf{X}}(\operatorname{Hom}_{\mathbf{X}}(F,E) \otimes F, X) \longrightarrow 0$, consequently $\operatorname{Ext}^1_{E^{\perp}}(F,-) = 0$. But F was chosen as a preinjective H-module, therefore this case is impossible.

Case (c) $\operatorname{Ext}^1_{\mathbb{X}}(F,E)\neq 0$ and the left mutation of ϵ is given by an exact sequence

$$0\longrightarrow E\longrightarrow L\longrightarrow \operatorname{Ext}^1_{\mathbb{X}}(F,E)\otimes F\longrightarrow 0.$$

From the exactedness of the sequence

$$0\longrightarrow \operatorname{Hom}_{\mathbb{X}}(\operatorname{Ext}^1_{\mathbb{X}}(F,E)\otimes F,\mathcal{S})\longrightarrow \operatorname{Hom}_{\mathbb{X}}(L,\mathcal{S})\longrightarrow \operatorname{Hom}_{\mathbb{X}}(E,\mathcal{S})\longrightarrow 0$$

and the fact that $\operatorname{Hom}_{\mathbb{X}}(F,\mathcal{S})\neq 0$ we get $\operatorname{Hom}_{\mathbb{X}}(L,\mathcal{S})\neq 0$. Similarly, application of $\operatorname{Hom}_{\mathbb{X}}(-,\mathcal{S}')$ to the exact sequence above, for those $\mathcal{S}'$ which satisfy $\operatorname{Hom}_{\mathbb{X}}(E,\mathcal{S}')\neq 0$, yields $\operatorname{Hom}_{\mathbb{X}}(L,\mathcal{S}')\neq 0$. Therefore, in this case we can take L for the new exceptional bundle E'.

Iterating the procedure we obtain an exceptional vector bundle, which is omnipresent. □

6.3.3 For an omnipresent exceptional vector bundle on a weighted projective line $\mathbb{X}=\mathbb{X}(\mathbf{p},\boldsymbol{\lambda})$ of weight type $\mathbf{p}=(p_1,p_2,\ldots,p_t)$ the rank r is greater than or equal to p_i for each $i=1,\ldots t$. Furthermore we have

Proposition 6.3.3 *The rank r of an omnipresent exceptional vector bundle satisfies the inequation $r\geq t-1$. Moreover, if at least one weight p_i is greater than two, then $r>t-1$.*

Proof. We set $r_i=r_{i,0}+r_{i,1}+\ldots+r_{i,p_i-2}$. Applying 6.1.2 we obtain

$$\begin{aligned}0 &= r^2-1+\sum_{i=1}^{t}\Big(\sum_{j=0}^{p_i-2}r_{i,j}^2+\sum_{0\leq j<j'\leq p_i-2}r_{i,j}r_{i,j'}\Big)-r\sum_{i=1}^{t}\sum_{j=0}^{p_i-2}r_{i,j}\\ &= r^2-1+\sum_{i=1}^{t}(r_{i,0}+\ldots+r_{i,p_i-2})^2-r\sum_{i=1}^{t}(r_{i,0}+\ldots+r_{i,p_i-2})\\ &= r^2-1+\sum_{i=1}^{t}r_i^2-r\sum_{i=1}^{t}r_i\\ &= (*).\end{aligned}$$

Note that $0\leq(*)$ and $0<(*)$ in case at least one weight p_i is greater than two.

Since E is omnipresent, the numbers r_i satisfy $1\leq r_i\leq r-1$. Suppose that among the numbers $r_1,\ldots,r_t$ the number i appears exactly a_i-times, $i=1,\ldots,r-1$. Thus $a_{r-1}=t-a_1-\ldots-a_{r-2}$. We conclude that

$$\begin{aligned}(*) &= r^2-1+(r-1)^2(t-a_1-\ldots-a_{r-2})+\sum_{i=1}^{r-2}i^2a_i\\ &\quad -r((r-1)(t-a_1-\ldots-a_{r-2})+\sum_{i=1}^{r-2}ia_i)\\ &= r^2-1-t(r-1)+\sum_{s=1}^{r-2}c_sa_s,\end{aligned}$$

where $c_s=-sr+r+s^2-1=(s-1)(-r+1+s)$ for $s=1,\ldots,r-2$. The coefficients c_s satisfy $c_s\leq 0$, therefore $0\leq(r-1)(r+1-t)$, consequently $r\geq t-1$ and $r>t-1$ if one of the weights is greater than two. □

6.3.4 We keep the notations above. Furthermore, we consider the quadratic forms $q_{\mathbb{X}}$ and q_{Λ_0} associated to the Euler forms on $\mathrm{K}_0(\mathbb{X})$ and $\mathrm{K}_0(\Lambda_0)$, where Λ_0 is the hereditary algebra described in 2.4.3.

To each exceptional vector bundle E on $\mathbb{X}$ we associate its class

$$[E]=r[\mathcal{O}]+\sum_{\substack{i=1,\ldots,t\\ j=0,\ldots,p_i-2}}n_{i,j}[\mathcal{S}_{i,j}]+n_{\mathcal{S}}[\mathcal{S}]$$

in $\mathrm{K}_0(\mathbb{X})$, which is a root of $q_{\mathbb{X}}$. Moreover, by the remark in 6.1.2, a root $(r,n_{i,j},n_{\mathcal{S}})$ of $q_{\mathbb{X}}$ determines a root $(r,n_{i,j})$ of q_{Λ_0}.

Let r be a nonzero natural number. Denote by $Ex^{omn}_{\mathbb{X}}(d,r)$ the set of isoclasses of omnipresent exceptional vector bundles on $\mathbb{X}$ of rank r and degree d and by $\mathcal{R}^{omn}_{\mathbb{X}}(d,r)$ (resp. $\mathcal{R}^{omn}_{\Lambda_0}(r)$) the set of all roots $(r,n_{i,j},n_{\mathcal{S}})$ of $q_{\mathbb{X}}$ of rank r and degree d satisfying the condition $0<n_{i,j}<r$ for all i,j (resp. of all roots $(r,n_{i,j})$ of q_{Λ_0} of rank r satisfying the condition $0<n_{i,j}<r$ for all i,j). Thus we have maps

$$Ex^{omn}_{\mathbb{X}}(d,r)\xrightarrow{\Phi_1}\mathcal{R}^{omn}_{\mathbb{X}}(d,r)\xrightarrow{\Phi_2}\mathcal{R}^{omn}_{\Lambda_0}(r).$$

Proposition 6.3.4 *(i) Let $(r,n_{i,j})$ be a root of q_{Λ_0} of rank r such that $0<n_{i,j}<r$ for all i,j. Then $r\geq t-1$. Moreover, if Λ_0 is not a subspace problem algebra, then $r>t-1$.*

(ii) If Λ_0 is the t-subspace problem algebra and $t\geq 4$, then there are exactly 2^t roots of q_{Λ_0}, $\begin{pmatrix} & r_1\\ t-1 & \vdots\\ & r_t\end{pmatrix}$, such that $0<r_i<t-1$ for all i. In this case $r_i=1$ or $r_i=t-2$, for all i.

Proof. (i) follows immediately from the proof of Proposition 6.3.3 an the remark in 6.1.2.

(ii) Keeping the notations of this proof we have $0=(r-1)(r+1-t)+\sum_{s=1}^{r-2}c_sa_s$ with $c_s=(s-1)(-r+1-s)$ for $s=1,\ldots,r-2$. Since $r=t-1$, we deduce that $c_2=\ldots=c_{r-2}=0$, whereas c_1 and c_{r-1} can be chosen arbitrarily such that $c_1+c_{r-1}=t$. Further, all vectors $\begin{pmatrix} & r_1\\ t-1 & \vdots\\ & r_t\end{pmatrix}$ with $r_i\in\{1,t-2\}$ are roots of q_{Λ_0}, which proves (ii). □

6.3.5 Up to the end of the chapter $\mathbb{X}$ denotes a weighted projective line of type $(2,\ldots,2)$, t entries, and we assume that $t\geq 4$. We will study the map $\Phi=\Phi_2\circ\Phi_1$ for the minimal $r=t-1$. Clearly Φ_1 and Φ_2 are injective. By the proposition above, $\mathcal{R}^{omn}_{\Lambda_0}(r)$ has 2^t elements, but not all of these roots can be realized by omnipresent exceptional vector bundles on $\mathbb{X}$. In particular, if r and d are not coprime, then $Ex^{omn}_{\mathbb{X}}(d,r)=\emptyset$. We will see that Φ is never bijective, however for certain choices of d exactly the half of the 2^t roots will be realized by omnipresent exceptional bundles.

Recall that $\mathrm{Pic}_0(\mathbb{X})$ denotes the group of line bundles of degree zero. In our situation $\mathrm{Pic}_0(\mathbb{X})$ has exactly 2^{t-1} elements: $\mathcal{O}(l_1\vec{x}_1+\ldots+l_t\vec{x}_t)$, $0\leq l_i\leq 1$, $\sum_{i=1}^{t}l_i=0$. For an arbitrary r, the group $\mathrm{Pic}_0(\mathbb{X})$ acts on the set $Ex^{omn}_{\mathbb{X}}(d,r)$ of isoclasses of omnipresent exceptional vector bundles on $\mathbb{X}$ of rank r and degree d by shift, provided $Ex^{omn}_{\mathbb{X}}(d,r)\neq\emptyset$.

Lemma 6.3.5 *Let $r = t-1$. If $Ex^{omn}_{\mathbf{X}}(d,r)$ is not empty, then the group $\mathrm{Pic}_0(\mathbf{X})$ acts freely on this set.*

Proof. Since each $\vec{x} \in t\mathbf{L}(\mathbf{p})$ can be written uniquely in the form $\vec{x} = c_2(\vec{x}_1 - \vec{x}_2) + c_3(\vec{x}_1 - \vec{x}_3) + \ldots + c_t(\vec{x}_1 - \vec{x}_t)$, $c_i \in \{0,1\}$, it suffices to show the following statement. Let $E \in Ex^{omn}_{\mathbf{X}}(d,r)$ with $\Phi(E) = \begin{pmatrix} & r_1 \\ r & \vdots \\ & r_t \end{pmatrix}$. Then $\Phi(E(\vec{x}_1 - \vec{x}_m)) = \begin{pmatrix} & r'_1 \\ r & \vdots \\ & r'_t \end{pmatrix}$, where $r'_1 = r - r_1$, $r'_m = r - r_m$ and $r'_i = r_i$ for $i \neq 1, m$. (Recall that the entries r_i are 1 or $r-1$ by the preceding proposition)

To prove the statement we can assume $m = 2$. By [29], the vector bundle $E \in Ex^{omn}_{\mathbf{X}}(d,r)$ has a line bundle filtration

$$E = F_r \supset F_{r-1} \supset \ldots \supset F_2 \supset F_1, \qquad F_i/F_{i-1} \cong L_i.$$

Then $[E] = \sum_{j=1}^r [L_j]$, and therefore $r_i(E) = \dim_k \mathrm{Hom}_{\mathbf{X}}(E, \mathcal{S}_{i,0}) = \sum_{j=1}^r \dim_k \mathrm{Hom}_{\mathbf{X}}(L_j, \mathcal{S}_{i,0}) = \sum_{j=1}^r r_i(L_j)$. Writing $L_j = \mathcal{O}(a_1^{(j)}\vec{x}_1 + a_2^{(j)}\vec{x}_2 + \ldots + a_t^{(j)}\vec{x}_t)$ we conclude from the exact sequences in 2.2.5 that $r_i(L_j) = 0$ if $a_i^{(j)}$ is even and $r_i(L_j) = 1$ otherwise. It follows that $r_i(E) = \sum_{j=1}^r \alpha_i^{(j)}$ with $\alpha_i^{(j)} = 0$ if $a_i^{(j)}$ is even and $\alpha_i^{(j)} = 1$ otherwise.

On the other hand, because shifting by a line bundle is an exact functor, we obtain $[E(\vec{x}_1 - \vec{x}_2)] = \sum_{j=1}^r [L_j(\vec{x}_1 - \vec{x}_2)]$ with $L_j(\vec{x}_1 - \vec{x}_2) = \mathcal{O}((a_1^{(j)}+1)\vec{x}_1 + (a_2^{(j)}-1)\vec{x}_2 + a_3^{(j)}\vec{x}_3 + \ldots + a_t^{(j)}\vec{x}_t)$. We see at once that $r_i(E(\vec{x}_1 - \vec{x}_2)) = r_i(E)$ for $i = 3, \ldots, t$. Further, if $r_1(E) = 1$, then all but one of the numbers $\alpha_1^{(1)}, \alpha_1^{(2)}, \ldots, \alpha_1^{(r)}$ are even. Therefore all but one of the numbers $\alpha_1^{(1)}+1, \alpha_1^{(2)}+1, \ldots, \alpha_1^{(r)}+1$ are odd, and consequently $r_1(E(\vec{x}_1 - \vec{x}_2)) = r-1$. Similarly, $r_1(E) = r-1$ implies $r_1(E(\vec{x}_1 - \vec{x}_2)) = 1$. The same argument can be applied for r_2, which completes the proof. □

Theorem 6.3.6 *Let $\mathbf{X}$ be a hyperelliptic weighted projective line with t weights. Then there is, up to line bundle shift, a unique omnipresent exceptional vector bundle of minmal rank $r = t-1$ on $\mathbf{X}$.*

Remark. Obviously the assertion of the theorem holds also for $t = 3$. The statement of the uniqueness fails in the tubular case $t = 4$.

Proof. We first show the existence of a vector bundle with the desired properties. Consider the exceptional pair $\epsilon = (\mathcal{O}(\vec{c}), \mathcal{O}(-\vec{\omega}))$, which is obtained if in the exceptional sequence of the canonical tilting sheaf $\mathcal{O}$ is mutated to the right end. The left mutation of ϵ yields a new exceptional bundle M as the middle term of the exact sequence

$$0 \to \mathcal{O}(-\vec{\omega}) \to M \to \mathrm{Ext}^1_{\mathbf{X}}(\mathcal{O}(\vec{c}), \mathcal{O}(-\vec{\omega})) \otimes \mathcal{O}(\vec{c}) \to 0.$$

Since $\mathrm{Ext}^1_{\mathbf{X}}(\mathcal{O}(\vec{c}), \mathcal{O}(-\vec{\omega})) \cong \mathrm{Hom}_{\mathbf{X}}(\mathcal{O}, \mathcal{O}(\vec{c}+2\vec{\omega}))$ and $\vec{c}+2\vec{\omega} = (t-3)\vec{c}$, the vector space $\mathrm{Ext}^1_{\mathbf{X}}(\mathcal{O}(\vec{c}), \mathcal{O}(-\vec{\omega}))$ has dimension $t-2$. Thus $\mathrm{rk}(M) = t-1$. Applying the functors $\mathrm{Hom}_{\mathbf{X}}(-, \mathcal{S}_{i,0})$ to the exact sequence above we see conclude $r_i(M) = \dim_k \mathrm{Hom}_{\mathbf{X}}(M, \mathcal{S}_{i,0}) = 1$, for $i = 1, \ldots, t$. Hence the vector bundle M is omnipresent.

Furthermore, we deduce that $\deg(M) = (t-2)\deg(\mathcal{O}(\vec{c})) - \deg(\mathcal{O}(\vec{\omega})) = t$, and consequently $\mu(M) = \frac{t}{t-1}$. According to Lemma 6.3.5 the action of the group $\mathrm{Pic}_0(\mathbf{X})$ gives 2^{t-1} omnipresent exceptional vector bundles of slope $\frac{t}{t-1}$.

We have to show that any omnipresent exceptional vector bundle $E \in \mathrm{vect}(\mathbf{X})$ of rank $t-1$ is obtained from M by a line bundle shift.

Let $d = \deg(E)$. By Theorem 3.5.1 rank and degree of E are coprime, therefore we can assume that $1 < \mu(E) < 2$.

Now, under this assumption, it suffices to prove the following two assertions.

(a) There is an omnipresent exceptional vector bundle E with $r_i(E) = 1$, for $i = 1, \ldots, t$, if and only if $d = t$.

(b) There is no omnipresent exceptional vector bundle E with $r_1(E) = t-2$ and $r_i(E) = 1$ for $i = 2, \ldots, t$.

Indeed, then by Lemma 6.3.5 for each rational number of the form $\frac{t}{t-1} + n$, $n \in \mathbb{Z}$, there are exactly 2^{t-1} omnipresent exceptional vector bundles having this slope and all of them are obtained from M by line bundle shift.

In order to prove (a) we suppose that E is an omnipresent exceptional vector bundle of rank $r = t-1$ and degree d, such that $1 < \frac{d}{r} < 2$ and all $r_i = 1$. Since $E = k_0[\mathcal{O}] + \sum_{i=1,\ldots,t} r_i[\mathcal{O}(\vec{x}_i)] + k_c[\mathcal{O}(\vec{c})]$, we see by calculating the degree that $2k_c = d - t$. By assumption $d - t \geq 0$. On the other hand $k_c \leq 0$, by Lemma 6.1.3. Thus we get $d = t$, which proves (a).

Note that assertion (a) holds also in the tubular case $t = 4$.

Now suppose that E is an omnipresent exceptional vector bundle on a hyperelliptic weighted projective line of rank $r = t-1$ and degree d, such that $1 < \frac{d}{r} < 2$, $r_1 = t-2$ and $r_i = 1$ for $i = 2, \ldots, t$.

We claim that $d = 2t-3$. To show this write again $E = k_0[\mathcal{O}] + \sum_{i=1,\ldots,t} r_i[\mathcal{O}(\vec{x}_i)] + k_c[\mathcal{O}(\vec{c})]$. Calculating the degree and applying that $\sum_{i=1,\ldots,t} r_i = 2t-3$ we obtain $k_c = \frac{1}{2}(d+3) - t$. Next, calculating the rank and using the formula above, we conclude that $k_0 = -\frac{1}{2}(d-1)$. By assumption $d < 2t-3$. In Lemma 6.1.3 (iii) we have shown that $-k_0 \geq r_i$ for all i, in particular we get $-\frac{1}{2}(d-1) \geq t-2$. This gives $d \geq 2t-3$, and consequently $d = 2t-3$ as claimed.

It follows that $k_c = 0$, hence $\mathrm{Ext}^1_{\mathbf{X}}(\mathcal{O}(\vec{c}), E) = 0$ by Lemma 6.1.3. Furthermore $\mathrm{Hom}_{\mathbf{X}}(\mathcal{O}(\vec{c}), E) = 0$, because $\mu(E) < 2 = \mu(\mathcal{O}(\vec{c}))$ and exceptional vector bundles are stable in the hyperelliptic case, by Proposition 2.3.7. This means that $(E, \mathcal{O}(\vec{c}))$ is an exceptional pair. Moreover, from

$$\begin{vmatrix} \mathrm{rk}(E) & \mathrm{rk}(\mathcal{O}(\vec{c})) \\ \deg(E) & \deg(\mathcal{O}(\vec{c})) \end{vmatrix} = \begin{vmatrix} t-1 & 1 \\ 2t-3 & 2 \end{vmatrix} = 1.$$

we conclude by 2.3.5 that $\mathrm{Ext}^1_{\mathbf{X}}(E, \mathcal{O}(\vec{c})) = 0$ and $\mathrm{Hom}_{\mathbf{X}}(E, \mathcal{O}(\vec{c})) \cong k$. The left mutation of the exceptional pair $(E, \mathcal{O}(\vec{c}))$ defines a new exceptional bundle L as the kernel of the canonical morphism

$$0 \to L \to \mathrm{Hom}_{\mathbf{X}}(E, \mathcal{O}(\vec{c})) \otimes E \to \mathcal{O}(\vec{c}) \to 0.$$

Since $[L] = [E] - [\mathcal{O}(\vec{c})]$, we obtain $\mathrm{rk}(L) = t-2$ and $\deg(L) = 2t-5$. Moreover, applying the functors $\mathrm{Hom}_{\mathbf{X}}(-, \mathcal{S}_{i,0})$ to the exact sequence above and using the assumptions $r_1 =$

$t-2$ and $r_i = 1$ for $i = 2, \dots, t$, it is easily seen that $\dim_k \mathrm{Hom}_{\mathbb{X}}(L, \mathcal{S}_{i,0}) = 1$ for $i = 2, \dots, t$, $\dim_k \mathrm{Hom}_{\mathbb{X}}(L, \mathcal{S}_{1,0}) = t-2$, and consequently $\dim_k \mathrm{Hom}_{\mathbb{X}}(L, \mathcal{S}_{i,1}) = t-3$ for $i = 2, \dots, t$, $\dim_k \mathrm{Hom}_{\mathbb{X}}(L, \mathcal{S}_{1,1}) = 0$. Therefore L is in the perpendicular category ${}^{\perp}\mathcal{S}_{1,1}$, which is equivalent to a sheaf category $\mathrm{coh}(\mathbb{X}')$ for a weighted projective line $(2, ..., 2)$, $t-1$ entries. Considered as an exceptional bundle on $\mathbb{X}'$, L is omnipresent and satisfies $r_i(L) = 1$ for all i. But $\mu(L) = \frac{2t-5}{t-2}$ and $1 < \frac{2t-5}{t-2} < 2$, therefore $2t - 5 = \deg(L) = t - 1$ by (a). It follows that $t = 4$, a contradiction. This proves the theorem. □

Corollary 6.3.7 *For a hyperelliptic weighted projective line with t weights there are exactly $(t-4)(t-1)2^{t-1}$ components in* $\mathrm{vect}(\mathbb{X})$ *containing an omnipresent exceptional vector bundle of minimal rank $r = t-1$. Each of these components contains an omnipresent exceptional bundle of slope $q = \frac{t}{t-1} + n$ with $n \in \{0, 1, \cdots, (t-4)(t-1) - 1\}$.*

Proof. The corollary follows straightforward from the proof of the theorem and the formula $\deg(\tau_{\mathbb{X}} E) = \deg(E(\vec{\omega})) = \deg E + \deg(\vec{\omega})\,\mathrm{rk}(E) = \deg(E) + (t-4)(t-1)$. □

6.3.8 Example. Let $\mathbb{X}$ be a weighted projective line of weight type $(2,2,2,2,2)$. Then for each $q = \frac{d}{r} \in \mathbb{Q}$, with $r \geq 3$, there are exactly 80 exceptional vector bundles which are not omnipresent. In fact, we have 10 possibilities to realize a tubular weighted projective line of type $(2,2,2,2)$ as a perpendicular category to a simple exceptional finite length sheaf and for each of these embeddings there are 8 exceptional vector bundles of slope $q = \frac{d}{r}$. Moreover, a vector bundle E cannot be contained in the intersection of two perpendicular categories ${}^{\perp}\mathcal{S} \cap {}^{\perp}\mathcal{S}'$ for different simple exceptional finite length sheaves $\mathcal{S}$ and $\mathcal{S}'$, because this would give an exceptional vector bundle of rank $r \geq 3$ on a weighted projective line of type $(2,2,2)$, a contradiction.

There is no omnipresent exceptional bundle on $\mathbb{X}$ of rank 2 or 3. Also there is no omnipresent exceptional bundle of slope $q = \frac{7}{4}$. For $q = \frac{5}{4}$ there are 2^4 omnipresent exceptional vector bundles. Expressed in the basis $[\mathcal{O}], [\mathcal{S}_{i,0}]$, $i = 1, \dots, t$, $[\mathcal{S}]$, their classes in $\mathrm{K}_0(\mathbb{X})$, up to the choice of the 0, 2 or 4 places for which $r_i = 3$, are

$$\begin{pmatrix} & 1 & \\ & 1 & \\ -1 & 1 & 0 \\ & 1 & \\ & 1 & \end{pmatrix}, \begin{pmatrix} & 3 & \\ & 3 & \\ -3 & 1 & -2 \\ & 1 & \\ & 1 & \end{pmatrix}, \begin{pmatrix} & 3 & \\ & 3 & \\ -5 & 3 & -4 \\ & 3 & \\ & 1 & \end{pmatrix}.$$

□

Chapter 7

Tilting sheaves

7.1 Concealed-canonical and almost concealed-canonical algebras

In this section we recall the concept of concealed-canonical and almost concealed-canonical algebras and give a global view on their module categories. Proofs of this results can be found in [75]. We further show that for an almost concealed-canonical algebra $\Sigma = \mathrm{End}(T)$, where the tilting sheaf T is realized on a wild weighted projective line, the algebra Σ is strictly wild, which generalizes the case of a concealed-canonical algebra.

7.1.1 Remember that a coherent sheaf on a weighted projective line $\mathbb{X}$ is called a tilting sheaf if $\mathrm{Ext}^1_{\mathbb{X}}(T,T) = 0$ and T generates $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ as a triangulated category (see 2.2.7).

A finite dimensional k-algebra is called *concealed-canonical* (resp.*almost concealed-canonical*) if it is isomorphic to the endomorphism ring of a tilting bundle (resp. tilting sheaf) on a weighted projective line $\mathbb{X} = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$. In this case $(\mathbf{p}, \boldsymbol{\lambda})$ is called the *weight type of* Σ.

Let us summarize some basic properties of concealed-canonical and almost concealed-canonical algebras proved in [75].

(1) The weight type $(\mathbf{p}, \boldsymbol{\lambda})$ of an almost concealed-canonical algebra is uniquely determined up to equivalence.

(2) If Σ is a concealed-canonical algebra then the algebra Σ^{op} is also concealed-canonical of the same weight type.

(3) An algebra Σ is concealed-canonical if and only if Σ and Σ^{op} are almost concealed-canonical.

(4) The notions concealed-canonical and almost concealed-canonical coincide for tubular weight type and agree in this case with the notion of a tubular algebra.

Furthermore, almost concealed-canonical algebras were characterized in [75] as certain branch coenlargements of concealed-canonical algebras. In Chapter 8 we will present a more general concept for endomorphism algebras of tilting complexes of a special form, containing the case of almost concealed-canonical algebras as a special case.

For a tilting sheaf T we denote by $\mathrm{coh}_+(T)$ (resp. $\mathrm{coh}_0^+(T)$) the full subcategory of $\mathrm{vect}(\mathbb{X})$ (resp. $\mathrm{coh}_0(\mathbb{X})$) consisting of all F satisfying the condition $\mathrm{Ext}^1_{\mathbb{X}}(T, F) = 0$.

Similarly, let $\mathrm{coh}_-(T)$ (resp. $\mathrm{coh}_0^-(T)$) be the full subcategory of $\mathrm{vect}(\mathbb{X})$ (resp. $\mathrm{coh}_0(\mathbb{X})$) consisting of all F satisfying the condition $\mathrm{Hom}_{\mathbb{X}}(T,F) = 0$. Furthermore, let $\mathrm{coh}_{\geq}(T)$ (resp. $\mathrm{coh}_{\leq}(T)$) be the additive closure of $\mathrm{coh}_+(T) \cup \mathrm{coh}_0^+(T)$ (resp. $\mathrm{coh}_-(T) \cup \mathrm{coh}_0^-(T)$).

7.1.2 We will describe the global structure of the module category for a concealed-canonical and an almost concealed-canonical algebra $\Sigma = \mathrm{End}(T)$. Observe first that the indecomposable direct summands from T (resp. from $T(\vec{\omega})[1]$) are the indecomposable projectives (resp. injectives). As a consequence of the identification $\mathcal{D}^b(\mathrm{mod}(\Sigma)) \cong \mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ we have

Proposition 7.1.2 [75, 5.1] *Let $\Sigma = \mathrm{End}(T)$ be an almost concealed-canonical algebra. Then each indecomposable Σ-module M belongs to one of the following subcategories*

- $\mathrm{mod}_+(\Sigma)$, *consisting of all vector bundles X on $\mathbb{X}$ satisfying* $\mathrm{Ext}^1_{\mathbb{X}}(T,X) = 0$,
- $\mathrm{mod}_0(\Sigma)$, *consisting of all finite length sheaves X on $\mathbb{X}$ satisfying* $\mathrm{Ext}^1_{\mathbb{X}}(T,X) = 0$,
- $\mathrm{mod}_-(\Sigma)$, *consisting of all $Z[1]$, with Z a bundle on $\mathbb{X}$ satisfying* $\mathrm{Hom}_{\mathbb{X}}(T,Z) = 0$,
- $\mathrm{mod}_0^{[1]}(\Sigma)$, *consisting of all $Z[1]$, with Z a finite length sheaf on $\mathbb{X}$ satisfying* $\mathrm{Hom}_{\mathbb{X}}(T,Z) = 0$.

Moreover,

- $M \in \mathrm{mod}_+(\Sigma) \iff \mathrm{rk}(M) > 0$,
- $M \in \mathrm{mod}_0(\Sigma) \iff \mathrm{rk}(M) = 0$ *and* $\deg(M) > 0$,
- $M \in \mathrm{mod}_-(\Sigma) \iff \mathrm{rk}(M) < 0$,
- $M \in \mathrm{mod}_0^{[1]}(\Sigma) \iff \mathrm{rk}(M) = 0$ *and* $\deg(M) < 0$.

Further, in the ordering $\mathrm{mod}_+(\Sigma)$, $\mathrm{mod}_0(\Sigma)$, $\mathrm{mod}_-(\Sigma)$, $\mathrm{mod}_0^{[1]}(\Sigma)$ *there are no nonzero morphisms from the right to the left.* □

We further define $\mathrm{mod}_{\geq}(\Sigma)$ (resp. $\mathrm{mod}_{\leq}(\Sigma)$) as the additive closure of $\mathrm{mod}_+(\Sigma) \cup \mathrm{mod}_0(\Sigma)$ (resp. $\mathrm{mod}_-(\Sigma) \cup \mathrm{mod}_0^{[1]}(\Sigma)$). Obviously, for a concealed-canonical algebra we have $\mathrm{mod}_0^{[1]}(\Sigma) = 0$. We further denote the projective (resp. injective) dimension of a Σ-module X by $\mathrm{pd}_\Sigma X$ (resp. $\mathrm{id}_\Sigma X$).

Theorem 7.1.2 [75, 5.4] *A concealed-canonical algebra Σ is connected, and the trisection of $\mathrm{mod}(\Sigma)$ satisfies the following conditions:*

(*i*) *Each projective Σ-module belongs to $\mathrm{mod}_+(\Sigma)$. All modules X in $\mathrm{mod}_{\geq}(\Sigma)$ have* $\mathrm{pd}_\Sigma X \leq 1$.

(*ii*) *Each injective Σ-module belongs to $\mathrm{mod}_-(\Sigma)$. All modules Z in $\mathrm{mod}_-(\Sigma)$ have* $\mathrm{id}_\Sigma Y \leq 1$.

(*iii*) *Each module Y in $\mathrm{mod}_0(\Sigma)$ has $\mathrm{pd}_\Sigma Y \leq 1$ and $\mathrm{id}_\Sigma Y \leq 1$.*

(*iv*) $\mathrm{mod}_0(\Sigma) = \mathrm{coh}_0^+(T)$ *is a uniserial exact subcategory of $\mathrm{mod}(\Sigma)$, decomposing in a coproduct $\coprod_{x\in\mathbb{X}} \mathcal{U}_x$ of connected uniserial sincere subcategories (stable tubes of finite rank), separating $\mathrm{mod}_+(\Sigma)$ from $\mathrm{mod}_-(\Sigma)$ in the following sense:*

(*a*) *there are no nonzero morphisms from $\mathrm{mod}_-(\Sigma)$ to $\mathrm{mod}_{\geq}(\Sigma)$ or from $\mathrm{mod}_{\leq}(\Sigma)$ to* $\mathrm{mod}_+(\Sigma)$,

(*b*) *for each nonzero morphism $f : X \to Y$ with $X \in \mathrm{mod}_+(\Sigma)$, $Y \in \mathrm{mod}_-(\Sigma)$, and each $x \in \mathbb{X}$, there exists a factorization $f = [X \to U \to Y]$ for some $U \in \mathcal{U}_x$,*

(*c*) *each $\mathcal{U}_x$ contains a sincere Σ-module.*

Moreover, each of the subcategories $\mathrm{mod}_+(\Sigma)$, $\mathrm{mod}_0(\Sigma)$, $\mathrm{mod}_-(\Sigma)$ is closed under extensions and almost-split sequences. □

Lenzing and de la Peña proved that the preceding theorem characterizes the concealed-canonical algebras [78]. In Chapter 8 we will study the separation property for a more general class of algebras.

7.1.3 It was shown in [75, Proposition 5.7] that for a concealed-canonical algebra $\Sigma = \mathrm{End}(T)$, where T is realized on a weighted projective line $\mathbb{X}$, the representation type of Σ coincides with that of $\mathrm{coh}(\mathbb{X})$. Further, if $\mathbb{X}$ is wild, then Σ is strictly wild, more precisely admits a homological embedding $\mathrm{mod}(\Delta) \hookrightarrow \mathrm{mod}(\Sigma)$ for some wild hereditary algebra Δ. Recall that a functor $T : \mathcal{A} \to \mathcal{C}$ between abelian categories is called *homological* if it induces isomorphisms $\mathrm{Ext}^i_{\mathcal{A}}(A_1, A_2) \to \mathrm{Ext}^i_{\mathcal{C}}(A_1, A_2)$ for all $i \geq 0$, and all $A_1, A_2 \in \mathcal{A}$. A homological functor is in particular a full and exact embedding.

By a modification of the proof the result can be generalized to an almost concealed-canonical algebra.

Lemma 7.1.3 [75, 5.6] *Let E be a vector bundle on $\mathbb{X}$ of rank r. For each line bundle L of sufficiently large degree there exists an embedding $E \hookrightarrow L^r$.* □

Theorem 7.1.3 *Let $\Sigma = \mathrm{End}(T)$ be an almost concealed-canonical algebra where T is a tilting sheaf on a wild weighted projective line. Then there exists a wild hereditary algebra Δ and a homological embedding $\mathrm{mod}(\Delta) \hookrightarrow \mathrm{mod}(\Sigma)$. Consequently Σ is strictly wild.*

Proof. Assume that $T = T' \oplus T''$ where T' is a vector bundle and T'' a finite length sheaf. We denote $\Sigma' = \mathrm{End}(T')$. According to the preceding lemma we choose a line bundle L in $\mathrm{mod}_+(\Sigma')$ such that each indecomposable direct summand of T' embeds into a power of L.

Now, all indecomposable objects but a finite number of the right perpendicular category $L^\perp$, formed in $\mathrm{coh}(\mathbb{X})$, are contained in $\mathrm{mod}_{\geq}(\Sigma)$. Indeed, $\mathrm{Ext}^1_{\mathbb{X}}(L,X) = 0$ implies $\mathrm{Ext}^1_{\mathbb{X}}(T',X) = 0$ which yields $L^\perp \cap \mathrm{vect}(\mathbb{X}) = 0$. On the other hand, $L^\perp \cap \mathrm{coh}_0(\mathbb{X})$ contains only finite length sheaves from exceptional tubes $\mathcal{T}_i$ of a quasi-length less than the rank of $\mathcal{T}_i$, hence this set contains only finitely many indecomposables $\{Q_1, \ldots, Q_m\}$. Identifying $L^\perp$ with a module category $\mathrm{mod}(H)$ over a wild hereditary algebra H, the Q_i's are preinjective modules.

Let E be a quasi-simple regular exceptional H-module and $E^\perp$ the right perpendicular category to E formed in the module category $\mathrm{mod}(H)$. We consider the embedding

$$F : E^\perp \hookrightarrow \mathrm{mod}(H) \cong L^\perp \hookrightarrow \mathrm{coh}(\mathbb{X}).$$

Replacing if necessary, E by some Auslander-Reiten-translate $\tau_H^{-n} E$ we can assume that $Q_i \notin E^\perp$ for $i = 1, \ldots, m$. Therefore F yields an embedding $F' : E^\perp \hookrightarrow \mathrm{mod}(\Sigma)$. Since $E^\perp \hookrightarrow \mathrm{mod}(H)$ and $L^\perp \hookrightarrow \mathrm{coh}(\mathbb{X})$ are homological embeddings, F' is homological, too. Furthermore, by [116, Theorem B], the algebra H is wild, which finishes the proof. □

Theorem 7.1.4 [75, 5.8] *Let* $\Sigma = \mathrm{End}(T)$ *be an almost concealed-canonical algebra. Suppose that* $T = T' \oplus T''$ *where* T' *is a vector bundle and* T'' *a finite length sheaf and denote* $\Sigma' = \mathrm{End}(T')$. *Then*

(i) $\mathrm{mod}_+(\Sigma)$ *coincides with* $\mathrm{mod}_+(\Sigma')$.

(ii) $\mathrm{mod}_0(\Sigma)$ *consists of the components* $\mathcal{U}'_x = \mathcal{U}_x$, *with* $x \notin \{\lambda_1, \dots, \lambda_m\}$ *and of components* $\mathcal{U}'_y$, *obtained by "coray deletion" from the components* $\mathcal{U}_y$ *of sheaves on* $\mathbb{X}$ *concentrated at* y, *for* $y \in \{\lambda_1, \dots, \lambda_m\}$. *The latter ones are those containing projective* Σ*-modules.*

(iii) The family $(\mathcal{U}'_x)_{x \in \mathbb{X}}$ *is a separating tubular family for* $\mathrm{mod}(\Sigma)$ *(usually not sincere).*

(iv) There are only finitely many objects in $\mathrm{mod}_0^{[1]}(\Sigma)$. □

7.2 Regular components

In this section T denotes a tilting sheaf on a weighted projective line $\mathbb{X}$ of arbitrary type. Let $\Sigma = \mathrm{End}(T)$ be the attached almost concealed-canonical algebra. Here we describe the *regular components* in the Auslander-Reiten quiver of $\mathrm{mod}(\Sigma)$, i.e. the components without projective and injective modules.

Fore this we will compare the Auslander-Reiten translations in $\mathrm{coh}(\mathbb{X})$ and $\mathrm{mod}(\Sigma)$, which are denoted by $\tau_{\mathbb{X}}$ and τ_Σ respectively. Recall that $\tau_{\mathbb{X}}$ is given by a line bundle shift with the canonical element $\vec{\omega}$.

7.2.1 Similarly as in [29, 3.5] for each $F \in \mathrm{coh}(\mathbb{X})$ there is a short exact sequence

$$0 \to F_+ \to F \to F_- \to 0, \qquad \text{with} \quad F_+ \in \mathrm{coh}_{\geq}(T),\ F_- \in \mathrm{coh}_{\leq}(T).$$

In fact, F_+ is the largest subsheaf of F belonging to $\mathrm{coh}_{\geq}(T)$.

The following result is similar to a result of Hoshino [55] concerning relative Auslander-Reiten sequences for torsion pairs in module categories.

Proposition 7.2.1 *(a) For each indecomposable module* $M \in \mathrm{mod}_{\geq}(\Sigma)$ *we have* $\tau_\Sigma M = (\tau_{\mathbb{X}} M)_+$.

(b) For each indecomposable module $M \in \mathrm{mod}_{\leq}(\Sigma)$ *we have* $\tau_\Sigma^- M = (\tau_{\mathbb{X}}^- M)_-$

Proof. (a) was proved in [77, 5.1] if T is the canonical tilting sheaf, the general case follows straightforward.

(b)For $M, N \in \mathrm{mod}_{\leq}(\Sigma)$ we have

$$\mathrm{Hom}_\Sigma(\tau_\Sigma^- M, N) \cong \underline{\mathrm{Hom}}_\Sigma(\tau_\Sigma^- M, N) \cong \overline{\mathrm{Hom}}_\Sigma(M, \tau_\Sigma N),$$

where $\underline{\mathrm{Hom}}_\Sigma(X,Y)$ (resp. $\overline{\mathrm{Hom}}_\Sigma(X,Y)$) denotes the group $\mathrm{Hom}_\Sigma(X,Y)$ modulo the subgroup consisting of all Σ-homomorphisms from X to Y which factor through projective (resp. injective) modules. This follows from the facts that all projective modules are in $\mathrm{mod}_{\geq}(\Sigma)$, in particular $\tau_\Sigma M$ and N have no nonzero projective direct summand, and that there are no nonzero homomorphisms from $\mathrm{mod}_{\leq}(\Sigma)$ to $\mathrm{mod}_{\geq}(\Sigma)$.

Now, invoking the Auslander-Reiten formula and the Serre duality for $\mathrm{coh}(\mathbb{X})$ we obtain $\mathrm{Hom}_\Sigma(\tau_\Sigma^- M, N) \cong D\mathrm{Ext}^1_\Sigma(N, M) \cong D\mathrm{Ext}^1_{\mathbb{X}}(N, M) \cong \mathrm{Hom}_{\mathbb{X}}(M, N(\vec{\omega})) \cong \mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}}^- M, N)$.

Applying the functor $\mathrm{Hom}(-, N)$ to the exact sequence

$$0 \to (\tau_{\mathbb{X}} M)_+ \to \tau_{\mathbb{X}} M \to (\tau_{\mathbb{X}} M)_- \to 0$$

we see that $\mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}}^- M, N) \cong \mathrm{Hom}_{\mathbb{X}}((\tau_{\mathbb{X}}^- M)_-, N)$ since there are no nonzero homomorphisms from $\mathrm{coh}_{\geq}(T)$ to $\mathrm{coh}_{\leq}(T)$. The last term equals $\mathrm{Hom}_\Sigma((\tau_{\mathbb{X}}^- M)_-, N)$ because both are modules in $\mathrm{mod}_{\leq}(\Sigma)$. Therefore we obtain isomorphisms $\mathrm{Hom}_\Sigma(\tau_\Sigma^- M, N) \cong \mathrm{Hom}_\Sigma((\tau_{\mathbb{X}}^- M)_-, N)$, which are functorial in $N \in \mathrm{mod}_{\leq}(\Sigma)$, and consequently $\tau_\Sigma^- M \cong (\tau_{\mathbb{X}}^- M)_-$. □

Corollary 7.2.2 *(i) For each indecomposable module* $M \in \mathrm{mod}_{\leq}(\Sigma)$ *we have* $\mathrm{rk}(\tau_\Sigma^- M) \geq \mathrm{rk}(M)$.

(ii) Let M *be indecomposable in* $\mathrm{mod}_-(\Sigma)$. *Then* $\mathrm{rk}(\tau_\Sigma^- M) = \mathrm{rk}(M)$ *if and only if* $\tau_\Sigma^- M = \tau_{\mathbb{X}}^- M$.

Proof. (i) The inequality follows from Proposition 7.2.1 and the exact sequence

$$0 \to (\tau_{\mathbb{X}}^- M)_+ \to \tau_{\mathbb{X}}^- M \to \tau_\Sigma^- M \to 0$$

and the fact that the application of $\tau_{\mathbb{X}}^-$ does not change the rank. Note that $\mathrm{rk}((\tau_\Sigma^- M)_+) \leq 0$.

(ii) Suppose that M is indecomposable in $\mathrm{mod}_-(\Sigma)$ and $\mathrm{rk}(\tau_\Sigma^- M) = \mathrm{rk}(M)$. From the exact sequence above we infer that $\mathrm{rk}((\tau_{\mathbb{X}}^- M)_+) = 0$. By our assumption we have that $\tau_\Sigma^- M = F[1]$ for some $F \in \mathrm{vect}(\mathbb{X})$. Because there are no nonzero morphisms from finite length sheaves to vector bundles it follows that $(\tau_{\mathbb{X}}^- M)_+) = 0$. □

Corollary 7.2.3 *(i) For each indecomposable module* $M \in \mathrm{mod}_{\geq}(\Sigma)$ *we have* $\mathrm{rk}(\tau_\Sigma M) \leq \mathrm{rk}(M)$.

(ii) Assume in addition that T *is a tilting bundle and let* M *be indecomposable in* $\mathrm{mod}_+(\Sigma)$. *Then* $\mathrm{rk}(\tau_\Sigma M) = \mathrm{rk}(M)$ *if and only if* $\tau_\Sigma M \cong \tau_{\mathbb{X}} M$.

Proof. (i) The inequality follows from Proposition 7.2.1 and the exact sequence

$$0 \to \tau_\Sigma M \to \tau_{\mathbb{X}} M \to (\tau_{\mathbb{X}} M)_- \to 0.$$

(ii) Let M be indecomposable in $\mathrm{mod}_+(\Sigma)$ and assume that $\mathrm{rk}(\tau_\Sigma M) = \mathrm{rk} M$. Then $\mathrm{rk}((\tau_{\mathbb{X}} M)_-) = 0$. Because for a tilting bundle $\mathrm{coh}_{\leq}(T)$ does not contain sheaves of rank zero we obtain $(\tau_{\mathbb{X}} M)_- = 0$, consequently $\tau_\Sigma M \cong \tau_{\mathbb{X}} M$. □

7.2.4 We recall the notion of a cone in an Auslander-Reiten component [63]. Assume that an object Z belongs to a component $\mathcal{C}$ of $\mathrm{coh}(\mathbb{X})$ or of $\mathrm{mod}(\Sigma)$. Then the *τ-cone* $(\to Z)$ (resp. the *τ^--cone* $(Z \to)$) is the full subquiver of $\mathcal{C}$ formed by all objects which are predecessors (resp. successors) of Z.

Theorem 7.2.4 *Let Σ be an almost concealed-canonical algebra and $\mathcal{C}$ be an Auslander-Reiten component in* $\mathrm{mod}_{<}(\Sigma)$ *different from a preinjective component. Then there exists an indecomposable $Z \in \mathcal{C}$ such that the τ_{Σ}^{-}-cone $(Z \to)$ in $\mathcal{C}$ is a full subquiver of a component in* $\mathrm{vect}(\mathbb{X})[1]$.

Proof. Applying Corollary 7.2.2 (i) and the assumption that $\mathcal{C}$ is not a preinjective component, we can find an indecomposable $Z \in \mathcal{C}$ such that the τ_{Σ}-orbit of Z does not contain an injective Σ-module and $0 > \mathrm{rk}(Z) = \mathrm{rk}(\tau_{\Sigma}^{-t}Z)$ for all $t \geq 0$. Let

$$0 \to Z \xrightarrow{\alpha} Y_1 \oplus Y_2 \xrightarrow{\beta} \tau_{\mathbb{X}}^{-} Z \to 0 \qquad (*)$$

be the Auslander-Reiten sequence in $\mathrm{vect}(\mathbb{X})[1]$. Applying Corollary 7.2.2 (i) we infer that $\tau_{\mathbb{X}}^{-}Z \cong \tau_{\Sigma}^{-}Z$, in particular $\tau_{\mathbb{X}}^{-}Z \in \mathrm{mod}_{<}(\Sigma)$. Applying the functor $\mathrm{Hom}_{\mathbb{X}}(T,-)$ we see that also $Y_1 \oplus Y_2$ is in $\mathrm{mod}_{<}(\Sigma)$.

Moreover, if $f : Z \to U$ is a morphism in $\mathrm{mod}(\Sigma)$ which is not a split monomorphism, then there is a morphism $g : Y_1 \oplus Y_2 \to U$ in $\mathrm{mod}(\Sigma)$ such that $f = g \circ \alpha$. Indeed, in case $U \in \mathrm{mod}_{<}(\Sigma)$ we can use the Auslander-Reiten factorization property in $\mathrm{coh}(\mathbb{X})[1]$, and in case $U \in \mathrm{mod}_{\geq}(\Sigma)$ f is zero. Thus $(*)$ is also an Auslander-Reiten sequence in $\mathrm{mod}(\Sigma)$.

Repeating this argument, first for the Auslander-Reiten sequence

$$0 \to \tau_{\mathbb{X}}^{-}Z \to \tau_{\mathbb{X}}^{-}Y_1 \oplus \tau_{\mathbb{X}}^{-}Y_2 \to \tau_{\mathbb{X}}^{-2}Z \to 0,$$

then for the meshes adjacent to the two already studied and continuing this process we see that the whole $\tau_{\mathbb{X}}^{-}$-cone $(Z \to)$ consist of Auslander-Reiten sequences in $\mathrm{mod}(\Sigma)$. Therefore the τ^{-}-cones $(Z \to)$ in $\mathcal{C}$ and in $\mathrm{vect}(\mathbb{X})$ coincide. □

Remark. It will follow from the results in Section 7.3 that $\mathrm{mod}(\Sigma)$ has a unique preinjective component.

Corollary 7.2.5 *Let Σ be a wild almost concealed-canonical algebra and $\mathcal{C}$ a regular Auslander-Reiten component in* $\mathrm{mod}_{-}(\Sigma)$. *Then $\mathcal{C}$ is of type $\mathbb{Z}A_{\infty}$.*

Proof. Let $Z \in \mathcal{C}$ be such that the τ^{-}-cones $(Z \to)$ in $\mathcal{C}$ and $\mathrm{vect}(\mathbb{X})[1]$ coincide. The application of τ_{Σ} does not produce projective Σ-modules, thus the result follows from 2.2.9.

7.2.6 Let $T = T' \oplus T''$ be a tilting sheaf on $\mathbb{X}$ with $T' \in \mathrm{vect}(\mathbb{X})$ and $T'' \in \mathrm{coh}_0(\mathbb{X})$. Define $\Sigma' = \mathrm{End}(T')$. It follows from 2.4.2 that T' is a tilting bundle on a weighted projective line $\mathbb{X}'$ with the property that the right perpendicular category formed in $\mathrm{coh}(\mathbb{X})$ to all simple composition factors of the objects of T'' is equivalent to $\mathrm{coh}(\mathbb{X}')$. Moreover, $\mathrm{mod}_{+}(\Sigma)$ coincides with $\mathrm{mod}_{+}(\Sigma')$, by 7.1.4. Using these notations we have

Theorem 7.2.6 *Let Σ be an almost concealed-canonical algebra and $\mathcal{C}$ be an Auslander-Reiten component in* $\mathrm{mod}_{+}(\Sigma)$ *different from a preprojective component. Then there exists an indecomposable $Z \in \mathcal{C}$ such that the τ_{Σ}-cone $(\to Z)$ in $\mathcal{C}$ is a full subquiver of a component of* $\mathrm{vect}(\mathbb{X}')$.

Proof. Similarly as in the proof of Theorem 7.2.4 there is an indecomposable $Z \in \mathcal{C}$ of the property that the τ_{Σ}-orbit of Z does not contain a projective Σ-module and $0 < \mathrm{rk}(Z) = \mathrm{rk}(\tau_{\Sigma}^{t}Z)$ for all $t \geq 0$. Then for the Auslander-Reiten sequence

$$0 \to \tau_{\mathbb{X}'}Z \to Y \to Z \to 0$$

in $\mathrm{vect}(\mathbb{X}')$ we have $\mathrm{rk}(\tau_{\Sigma'}Z) = \mathrm{rk}(\tau_{\mathbb{X}'}Z) = \mathrm{rk}(Z)$. Therefore by Corollary 7.2.3 (ii), $\tau_{\mathbb{X}'}Z = \tau_{\Sigma'}Z \in \mathrm{mod}_{+}(\Sigma') = \mathrm{mod}_{+}(\Sigma)$. Now one can follow the dual of the arguments of the proof of Theorem 7.2.4.

Corollary 7.2.7 *Let Σ be a wild concealed-canonical algebra and $\mathcal{C}$ a regular component in* $\mathrm{mod}_{+}(\Sigma)$. *Then $\mathcal{C}$ is of type $\mathbb{Z}A_{\infty}$.* □

Remark. If T is a tilting sheaf on a wild weighted projective line $\mathbb{X}$, then $\mathbb{X}'$ can be wild, tubular or domestic, thus for the almost concealed-canonical algebra Σ a regular component in $\mathrm{mod}_{+}(\Sigma)$ can be of type $\mathbb{Z}A_{\infty}$, a stable tube or of type $\mathbb{Z}\Delta$ for an extended Dynkin graph Δ.

7.3 The wing decomposition of a tilting bundle

In this section we assume that $\mathbb{X}$ is wild and T is a tilting bundle on $\mathbb{X}$. The following theorem is the analogue of the result of Strauß [116, Theorem 7.5] concerning tilting modules without nonzero preinjective direct summands over connected (wild) hereditary algebras. The proof can be done along the arguments of [116] applying 2.4.3.

Theorem 7.3.1 *Let T be a tilting bundle over a wild weighted projective line $\mathbb{X}$. Then there exists a decomposition*

$$T = T_P \oplus T_I$$

which satisfies the following conditions:

(i) The perpendicular category $T_I^{\perp}$, formed in $\mathrm{coh}(\mathbb{X})$, *is equivalent to the module category of a connected wild hereditary algebra.*

(ii) T_P is $T_I^{\perp}$-preprojective.

(iii) The preprojective component of the algebra $\Sigma_P = \mathrm{End}(T_P)$ is a full component of the Auslander-Reiten quiver for Σ. Moreover, this is the only preprojective component for Σ. □

7.3.2 Let $T = T_P \oplus T_I$ be the decomposition of a tilting bundle from the theorem above. Now we apply results of [99] and [64] in order to obtain a wing decomposition for T. By 2.2.9 all components of the Auslander-Reiten quiver of $\mathrm{vect}(\mathbb{X})$ are of the form $\mathbb{Z}A_{\infty}$, thus the indecomposable direct summands of T_I determine wings in the sense of [100, 3.3]. Recall that for an indecomposable vector bundle W on $\mathbb{X}$ with quasi-length m and quasi-socle X, contained in a component $\mathcal{C}$, the *wing* $\mathcal{W}(W)$ of W is the defined to be the mesh-complete full subquiver of $\mathcal{C}$ given by the vertices ${}^{[r]}(\tau_{\mathbb{X}}^{-t}X)$ with $1 \leq r \leq m$, $0 \leq t \leq m - r$ (recall that ${}^{[r]}X$ is the indecomposable with quasi-length r and quasi-socle X).

Now, if W is an indecomposable direct summand of T of quasi-length s, then in the wing $\mathcal{W}(W)$ of W there are s indecomposable direct summands of T and they form a *branch* in the sense of [100, 4.4].

Further, for an indecomposable direct summand W of T_1 no summand of T_P is contained in $\mathcal{W}(W)$. If W_1 and W_2 are summands of T_1 such that $\mathcal{W}(W_i) \not\subseteq \mathcal{W}(W_j)$ for $i \neq j$, then $\mathcal{W}(W_1) \cap \mathcal{W}(W_2) = \emptyset$. Therefore T has a decomposition

$$T = T_P \oplus \bigoplus_{i=1}^{l} T(M_i)$$

such that $T(M_i)$ is a tilting object, hence a branch, in the wing $\mathcal{W}(M_i)$ and furthermore the wings $\mathcal{W}(M_i)$ are pairwise disjoint. Observe that M_i is a direct summand of $T(M_i)$.

Finally, we want to distinguish the branches $T(M_i)$ which do not allow nonzero morphisms to other branches. Define $T'(M_j) = T_P \oplus \bigoplus_{i \neq j} T(M_i)$. Since the quiver of T has no oriented cycles there exists an M_j such that $T'(M_j) \in T^{\perp}(M_j)$.

Let $\{W_1, ..., W_r\}$ be the set of these M_j's and $\{V_1, ..., V_s\}$ be the others. Then we have

$$T = T_P \oplus \bigoplus_{i=1}^{s} T(V_i) \oplus \bigoplus_{j=1}^{r} T(W_j).$$

We call this decomposition the *wing decomposition* of T.

Observe that for each V_i there exists a W_j and a sequence of nonzero maps $(f_l)_{1 \leq l \leq t}$

$$(*) \qquad V_i = V_{i_1} \xrightarrow{f_1} V_{i_2} \xrightarrow{f_2} \dots V_{i_t} \xrightarrow{f_t} V_{i_{t+1}} = W_j$$

It follows from 2.3.3 that each f_l is either a monomorphism or an epimorphism. In fact we have

Lemma 7.3.2 *In the sequence* $(*)$ *above every morphism is an epimorphism.*

Proof. Assume that some $f_l : V_{i_l} \to V_{i_{l+1}}$ is a monomorphism. Denote by T'' the direct sum of all branches $T(M_j)$ where $M = W$ or $M = V$ such that $M_j \neq V_{i_l}$ and there is a chain of nonzero maps

$$V_{i_l} \to M_{k_1} \to M_{k_2} \to \dots \to M_{k_u} = M_j$$

and let T' be the complement of T'' in T. Now, the perpendicular category $(T'')^{\perp}$ is equivalent to a module category $\text{mod}(H)$ over a hereditary algebra H. Observe that T' is in $(T'')^{\perp}$. Since f_l is a monomorphism we have an embedding $V_{i_l} \hookrightarrow T''$.

We claim that V_{i_l} is projective in $(T'')^{\perp}$. Indeed, if Z is an arbitrary object in $(T'')^{\perp}$, then $\text{Ext}^1_{\mathbf{X}}(T'', Z) = 0$ and therefore $\text{Ext}^1_{(T'')^{\perp}}(V_{i_l}, Z) = \text{Ext}_{\mathbf{X}}(V_{i_l}, Z) = 0$. Then V_{i_l} is preprojective in $\text{mod}(\Sigma')$ where $\Sigma' = \text{End}(T')$. Since T_P is contained in T', V_{i_l} is also preprojective in $\text{mod}(\Sigma_P)$ hence in $\text{mod}(\Sigma)$. Consequently, V_{i_l} is a direct summand from T_P by Theorem 7.3.1, a contradiction. □

7.3.3 If T is a tilting sheaf with wing decomposition

$$T = T_P \oplus \bigoplus_{i=1}^{s} T(V_i) \oplus \bigoplus_{j=1}^{r} T(W_j)$$

then we consider

$$\overline{T} = T_P \oplus \bigoplus_{i=1}^{s} \overline{V_i} \oplus \bigoplus_{j=1}^{r} \overline{W_j}$$

where $\overline{V_i}$ (resp. $\overline{W_j}$) is the direct sum of the projectives in the wing $\mathcal{W}(V_i)$ (resp. $\mathcal{W}(W_j)$). It is easily checked (compare [100, 4.4]) that $\overline{T}$ is a tilting sheaf again and

$$\overline{T} = T_P \oplus \bigoplus_{i=1}^{s} \overline{V_i} \oplus \bigoplus_{j=1}^{r} \overline{W_j}$$

is the wing decomposition of $\overline{T}$. We call $\overline{T}$ the *normalized form* or the *normalization* of T. As in [64, Lemma2.5.] we have

Lemma 7.3.3 *Let T be a tilting sheaf with wing decomposition*

$$T = T_P \oplus \bigoplus_{i=1}^{s} T(V_i) \oplus \bigoplus_{j=1}^{r} T(W_j)$$

and let $\overline{T}$ be the normalization of T.

(a) Assume that $F \in \text{coh}(\mathbf{X})$ is not contained in the wings $\mathcal{W}(\tau_{\mathbf{X}} V_i)$ and $\mathcal{W}(\tau_{\mathbf{X}} W_j)$ for all i, j. Then $F \in \text{coh}_{\geq}(T)$ if and only if $F \in \text{coh}_{\geq}(\overline{T})$.

(b) Assume that $F \in \text{coh}(\mathbf{X})$ is not contained in the wings $\mathcal{W}(V_i)$ and $\mathcal{W}(W_j)$ for all i, j. Then $F \in \text{coh}_{\leq}(T)$ if and only if $F \in \text{coh}_{\leq}(\overline{T})$. □

7.3.4 We will use the following information about wings proved in the situation of modules in [64] (see also [99]) and easily seen to be valid in our situation.

Lemma 7.3.4 *Let U be indecomposable in $\text{vect}(\mathbf{X})$ with quasi-length r and quasi-top X. Then we have*

(a) For an indecomposable vector bundle Y in $\text{vect}(\mathbf{X})$ which is not in $\text{add}(\mathcal{W}(U))$ the following conditions are equivalent:

(1) $\text{Hom}_{\mathbf{X}}(Y, U) = 0$,

(2) $\text{Hom}_{\mathbf{X}}(Y, \tau^i_{\mathbf{X}} X) = 0$ for $i = 0 \dots r-1$,

(3) $\text{Hom}_{\mathbf{X}}(Y, W) = 0$ for all $W \in \text{add}(\mathcal{W}(U))$.

(b) For an indecomposable vector bundle Z in $\text{vect}(\mathbf{X})$ which is not in $\text{add}(\mathcal{W}(U))$ the following conditions are equivalent:

(1) $\text{Hom}_{\mathbf{X}}(U, Z) = 0$,

(2) $\text{Hom}_{\mathbf{X}}(\tau^i_{\mathbf{X}} X, Z) = 0$ for $i = 0 \dots r-1$,

(3) $\text{Hom}_{\mathbf{X}}(W, Z) = 0$ for all $W \in \text{add}(\mathcal{W}(U))$. □

Lemma 7.3.5 *Let W be an indecomposable vector bundle in* $\mathrm{vect}(\mathbb{X})$ *with quasi-length m and quasi-top X. Then the following conditions are equivalent*

(a) $X, \tau_{\mathbb{X}} X, \ldots, \tau_{\mathbb{X}}^{m-1} X$ are pairwise orthogonal.

(b) If $Z, Y \in \mathrm{add}(\mathcal{W}(W))$, then $\mathrm{rad}^{\infty}(Z, Y) = 0$. □

Here rad denotes the *Jacobson radical* of the category $\mathrm{coh}(\mathbb{X})$ and the *infinite radical* rad^{∞} is the intersection of all powers rad^i, $i \geq 1$ of rad. If one of the two conditions of the lemma is satisfied we call $\mathcal{W}(W)$ a *standard wing*.

Lemma 7.3.6 *Let W be an indecomposable vector bundle on $\mathbb{X}$ with quasi-length m and let R be the indecomposable in* $\mathrm{vect}(\mathbb{X})$ *such that there is an irreducible epimorphism from R to W. Then $\mathcal{W}(R)$ is a standard wing if and only if W is exceptional.* □

7.4 Non-regular components for concealed-canonical algebras

In this section we describe the non-regular components for concealed-canonical algebras. The results are similar to those in the case of tilting modules without preinjective direct summands over wild hereditary algebras studied by Kerner.

7.4.1 The following theorem is an analogue of [64, Theorem2]. Observe that contrary to this situation we obtain more precise information by comparing the ranks and degrees of the quasi-simples in the wing decomposition.

Theorem 7.4.2 *Let T be a tilting bundle on a wild weighted projective line $\mathbb{X}$ with wing decomposition*

$$T = T_P \bigoplus_{i=1}^{s} T(V_i) \oplus \bigoplus_{j=1}^{r} T(W_j).$$

Denote by X_j the quasi-socle of W_j and let $R_j \twoheadrightarrow W_j$ be an irreducible epimorphism for $j = 1, \ldots, r$. Then we have

(a) $R_j \in \mathrm{coh}_{\geq}(T)$ for $j = 1, \ldots, r$.

(b) Let l be such that $\mathrm{rk}(X_l)$ is minimal and $\mu(X_l)$ is maximal among the X_j's with minimal rank. Then

(i) $\tau_{\mathbb{X}}^2 X_l \in \mathrm{coh}_{\geq}(T)$ and

(ii) The $\tau_{\mathbb{X}}$-cone $(\to \tau_{\mathbb{X}}^2 X_l)$ is contained in $\mathrm{coh}_{\geq}(T)$ and is a full subquiver of the non-regular component in $\mathrm{mod}(\Sigma)$ containing W_l.

Proof. By Lemma 7.3.3 we can assume that T is normalized. Therefore let

$$T = T_P \oplus \bigoplus_{i=1}^{s} \overline{V}_i \oplus \bigoplus_{j=1}^{r} \overline{W}_j$$

using the notation of 7.3.3.

(a) First, W_l is exceptional, thus by Lemma 7.3.6, $\mathcal{W}(W_l)$ is a standard wing and by Lemma 7.3.5, $\mathrm{Hom}_{\mathbb{X}}(R_j, \tau_{\mathbb{X}} W_j) = 0$. Moreover, Lemma 7.3.4 implies that $\mathrm{Hom}_{\mathbb{X}}(R_j, \tau_{\mathbb{X}} \overline{W}_j) = 0$. Now, let $T = \overline{W}_j \oplus T'(W_j)$ and consider the exact sequence

$$0 \to \tau_{\mathbb{X}} W_j \to R_j \to Z_j \to 0.$$

Then we get $\mathrm{Hom}_{\mathbb{X}}(Z_j, \tau_{\mathbb{X}} T'(W_j)) = 0$ because otherwise $0 \neq \mathrm{Hom}_{\mathbb{X}}(W_j, \tau_{\mathbb{X}} T'(W_j)) \cong D\mathrm{Ext}^1_{\mathbb{X}}(T'(W_j), W_j)$ which is impossible. Further, from the wing decomposition of T we obtain $\mathrm{Hom}_{\mathbb{X}}(W_j, T'(W_j)) = 0$, hence $\mathrm{Hom}_{\mathbb{X}}(R_j, \tau_{\mathbb{X}} T'(W_j)) = 0$. It follows that $\mathrm{Hom}_{\mathbb{X}}(R_j, \tau_{\mathbb{X}} T) = 0$, and consequently $\mathrm{Ext}^1_{\mathbb{X}}(T, R_j) = 0$.

(b) We know from (a) that $R_l \in \mathrm{coh}_{\geq}(T)$, thus by Proposition 7.2.1, $\tau_{\Sigma} R_l \cong (\tau_{\mathbb{X}} R_l)_+$ and we have an exact sequence

$$0 \to \tau_{\Sigma} R_l \to \tau_{\mathbb{X}} R_l \to (\tau_{\mathbb{X}} R_l)_- \to 0.$$

Set $Q = (\tau_{\mathbb{X}} R_l)_-$. In the same way as in [64, Lemma 2.3] we conclude that $Q \in \mathrm{add}(\tau_{\mathbb{X}}(\bigoplus_{i=1}^{s} V_i \oplus \bigoplus_{j=1}^{r} W_j))$.

We claim that $\tau_{\mathbb{X}}^2 X_l \in \mathrm{coh}_{\geq}(T)$. Assume first that Q is of the form $(\tau_{\mathbb{X}} W_l)^{\oplus m}$ for some m. Since $\mathcal{W}(R_l)$ is a standard wing, $\mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}} R_l, \tau_{\mathbb{X}} W_l) = k$ and therefore applying the functor $\mathrm{Hom}_{\mathbb{X}}(-, W_l)$ to the exact sequence (3) we obtain $m = 1$. Hence we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tau_{\Sigma} R_l & \longrightarrow & \tau_{\mathbb{X}} R_l & \longrightarrow & \tau_{\mathbb{X}} W_l & \longrightarrow & 0 \\
 & & \vdots & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \tau_{\mathbb{X}}^2 X_l & \longrightarrow & \tau_{\mathbb{X}} R_l & \longrightarrow & \tau_{\mathbb{X}} W_l & \longrightarrow & 0
\end{array}$$

The induced morphism is an isomorphism, in particular we infer that $\tau_{\mathbb{X}}^2 X_l \cong \tau_{\Sigma} R_l \in \mathrm{coh}_{\geq}(T)$.

Suppose now that Q contains an indecomposable direct summand $\tau_{\mathbb{X}} Y \in \mathrm{add}(\tau_{\mathbb{X}}(\bigoplus_{i=1}^{s} V_i \oplus \bigoplus_{j \neq l} W_j))$. Then there is an epimorphism $\tau_{\mathbb{X}} R_l \twoheadrightarrow \tau_{\mathbb{X}} Y$ and therefore an epimorphism $R_l \twoheadrightarrow Y$. Moreover, applying the functor $\mathrm{Hom}_{\mathbb{X}}(-, Y)$ to the exact sequence

$$0 \to \tau_{\mathbb{X}} X_l \to R_l \to W_l \to 0$$

we obtain a nonzero map $f : \tau_{\mathbb{X}} X_l \to Y$. Now, $\mathrm{Ext}^1_{\mathbb{X}}(Y, \tau_{\mathbb{X}} X_l) \cong D\mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}} X_l, \tau_{\mathbb{X}} Y_l) \cong D\mathrm{Hom}(X_l, Y_l) = 0$, therefore by 2.3.3, f is an epimorphism or a monomorphism. Clearly f is not an isomorphism, because $Y \in \mathrm{coh}_{\geq}(T)$ but $\tau_{\mathbb{X}} X_l \in \mathrm{coh}_{\leq}(T)$.

Assume first that f is an epimorphism. If $Y = W_j$ for some j, then $\mathrm{rk}(X_l) > \mathrm{rk}(W_j) \geq \mathrm{rk}(X_j)$, contrary to the assumption on l. If $Y = V_i$ for some i, then using Lemma 7.3.2, we can compose f with an epimorphism $V_i \to W_j$, and again $\mathrm{rk}(X_l) > \mathrm{rk}(W_j) \geq \mathrm{rk}(X_j)$ gives a contradiction.

In case f is a monomorphism we also have a monomorphism $\tau_{\mathbb{X}}^2 X_l \hookrightarrow \tau_{\mathbb{X}} Y$. Applying the functor $\mathrm{Hom}_{\mathbb{X}}(T, -)$ we see that $\mathrm{Hom}_{\mathbb{X}}(T, \tau_{\mathbb{X}}^2 X_l) = 0$. Now, applying the functor $\mathrm{Hom}_{\mathbb{X}}(T, -)$ to the exact sequence

$$0 \to \tau_{\mathbb{X}}^2 X_l \to \tau_{\mathbb{X}} R_l \to \tau_{\mathbb{X}} W_l \to 0$$

we conclude $\mathrm{Hom}_{\mathbb{X}}(T, \tau_{\mathbb{X}} R_l) = 0$, which means that $\tau_{\mathbb{X}} R_l \in \mathrm{coh}_{\leq}(T)$. Therefore $\tau_\Sigma R_l = 0$, hence R_l is projective in $\mathrm{mod}(\Sigma)$, which is impossible. This finishes the proof that $\tau_{\mathbb{X}}^2 X_l \in \mathrm{coh}_{\geq}(T)$.

Now we show by induction on n that $\tau_{\mathbb{X}}^n X_l \in \mathrm{coh}_{\geq}(T)$. From the induction hypothesis and Proposition 7.2.1 we see that $\tau_\Sigma(\tau_{\mathbb{X}}^{n-1} X_l) \cong (\tau_{\mathbb{X}}^n X_l)_+$, thus there is an exact sequence

$$0 \to \tau_\Sigma(\tau_{\mathbb{X}}^{n-1} X_l) \to \tau_{\mathbb{X}}^n X_l \to Q \to 0.$$

Again by the arguments of [64, Lemma 2.3] we have $Q \in \mathrm{add}(\tau_{\mathbb{X}}(\bigoplus_{i=1}^s V_i \oplus \bigoplus_{j=1}^r W_j))$. Assume that $Q \neq 0$. Then there is an epimorphism $f : \tau_{\mathbb{X}}^{n-1} X_l \twoheadrightarrow Y$ where Y is some W_j or some V_i. Using Lemma 7.3.2 the second case can be reduced to the first one. Now, if $f : \tau_{\mathbb{X}}^{n-1} X_l \to W_j$ is an epimorphism but not an isomorphism then $\mathrm{rk}(X_l) > \mathrm{rk}(W_j) \geq \mathrm{rk}(X_j)$, which contradicts the choice of l. On the other hand, if f is an isomorphism then $W_j = X_j$ and therefore $\mathrm{rk}(X_j) = \mathrm{rk}(X_l)$ but $\mu(X_j) = \mu(X_l((n-1)\vec{\omega})) > \mu(X_l)$, again a contradiction to the assumption on l. Hence $Q = 0$ and consequently $\tau_{\mathbb{X}}^n X_l \cong \tau_\Sigma(\tau_{\mathbb{X}}^{n-1} X_l)$, in particular $\tau_{\mathbb{X}}^n X_l \in \mathrm{coh}_{\geq}(T)$.

Finally, as in the proof of Theorem 7.2.4 one shows that all Auslander-Reiten sequences in the $\tau_{\mathbb{X}}$-cone $(\to \tau_{\mathbb{X}}^2 X_l)$ in $\mathrm{vect}(\mathbb{X})$ are also Auslander-Reiten sequences in $\mathrm{mod}(\Sigma)$. Using $\tau_\Sigma R_l \cong \tau_{\mathbb{X}}^2 X_l$ and the existence of an irreducible morphism from R_l to W_l we see that the cone $(\to \tau_{\mathbb{X}}^2 X_l)$ and W_l are in the same component in $\mathrm{mod}(\Sigma)$. □

7.4.3 In analogy to Strauß's definition [116, 7.2] of a special direct summand of a tilting module we define a special direct summand of a tilting bundle. In contrast to the situation of a tilting module, which has regular but no preinjective direct summand [116, 7.3] we can show the existence of a special summand using the rank and the degree functions. Observe that the condition is the same as in the preceding theorem. We note further that a modified version of the following theorem can also used in the induction step of [64, Theorem 1] which completes the gap there.

Definition 7.4.3 *Let T be a tilting bundle. An indecomposable direct summand $S \in \mathrm{add}(T)$ is called a special summand of T, if S is a sink summand of T and T has no $\overline{S}^\perp$-preinjective direct summands.*

Here $\overline{S}$ denotes the direct sum of all indecomposable projectives in the wing $\mathcal{W}(S)$. Observe that the definition makes sense because $\overline{S}^\perp$ is equivalent to a module category.

Theorem 7.4.3 *Let T be a normalized tilting bundle on a wild weighted projective line $\mathbb{X}$ with wing decomposition $T = T_P \oplus \bigoplus_{i=1}^s \overline{V}_i \oplus \bigoplus_{j=1}^r \overline{W}_j$ Denote by X_j the quasi-socle of W_j. Let l be such that $\mathrm{rk}(X_l)$ is minimal and $\mu(X_l)$ is maximal among the X_j's with minimal rank. Then W_l is a special summand.*

Proof. Let us consider the bundle $T' = T'(W_l) = T_P \oplus \bigoplus_{i=1}^s \overline{V}_i \oplus \bigoplus_{j \neq l} \overline{W}_j$ in $\overline{W_l}^\perp$. We define $\Sigma' = \mathrm{End}(T')$. Note that an indecomposable direct summand from T_P is preprojective in $\mathrm{mod}(\Sigma)$, thus preprojective in $\mathrm{mod}(\Sigma')$, and consequently not preinjective in $\overline{W_l}^\perp$.

Next we show that no $X_j, j \neq l$ is preinjective in $\overline{W_l}^\perp$. Fix such an X_j. By [77, Theorem 2.7] there exists an N such that $\mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}}^{-N} X_j, X_j) \neq 0$. Now, consider the chain of irreducible maps in $\mathrm{coh}(\mathbb{X})$:

$$X_j \overset{\mu_0}{\hookrightarrow} Y_j \overset{\epsilon_0}{\twoheadrightarrow} \tau_{\mathbb{X}}^- X_j \overset{\mu_1}{\hookrightarrow} \tau_{\mathbb{X}}^- Y_j \overset{\epsilon_1}{\twoheadrightarrow} \ldots \tau_{\mathbb{X}}^{-n} X_j \overset{\mu_n}{\hookrightarrow} \tau_{\mathbb{X}}^{-n} Y_j \overset{\epsilon_n}{\twoheadrightarrow} \tau_{\mathbb{X}}^{-n-1} X_j \overset{\mu_{n+1}}{\hookrightarrow} \tau_{\mathbb{X}}^{-n-1} Y_j \overset{\epsilon_{n+1}}{\twoheadrightarrow} \ldots$$
$$\ldots \overset{\mu_{N-1}}{\hookrightarrow} \tau_{\mathbb{X}}^{-N+1} Y_j \overset{\epsilon_{N-1}}{\twoheadrightarrow} \tau_{\mathbb{X}}^{-N} X_j \qquad (*)$$

where all μ_n are monomorphisms and all ϵ_n are epimorphisms.

In case all $\tau_{\mathbb{X}}^{-n} X_j$ and all $\tau_{\mathbb{X}}^{-n} Y_j$ appearing in $(*)$ belong to $\overline{W_l}^\perp$ we obtain a cycle in $\overline{W_l}^\perp$ and then X_j is regular. Thus we can assume that one $\tau_{\mathbb{X}}^{-n} Z_j$ with $Z = X$ or $Z = Y$ is not contained in $\overline{W_l}^\perp$.

We claim that $\mathrm{Hom}_{\mathbb{X}}(\overline{W}_l, \tau_{\mathbb{X}}^{-n} X_j) = 0$ for $n = 0, \ldots, N$. First, as a consequence of Theorem 7.4.2 we have $0 = \mathrm{Ext}_{\mathbb{X}}^1(T, \tau_{\mathbb{X}}^n X_l) \cong \mathrm{DHom}(\tau_{\mathbb{X}}^n X_l, \tau_{\mathbb{X}} T)$ for $n \geq 2$. Moreover, $\mathrm{Hom}_{\mathbb{X}}(W_l, X_j) = 0$ which implies by Lemma 7.3.4, $\mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}}^{-m} X_l, X_j) = 0$ for $m = 0, 1, \ldots, t$ where $t+1$ is the quasi-length of W_l. Therefore $\mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}}^{-m} X_l, \tau_{\mathbb{X}}^{-n} X_j) = 0$ for $m = 0, 1, \ldots, t$ and $n = 0, 1, \ldots, N$. Observe that $\tau_{\mathbb{X}}^{-n} X_j \notin \mathrm{add}(\mathcal{W}(W_l))$. Indeed, otherwise X_l is in the τ-orbit of X_j which implies that X_l and X_j have equal rank. Since the wings $\mathcal{W}(W_l)$ and $\mathcal{W}(W_j)$ are disjoint, this would imply $X_l = \tau_{\mathbb{X}}^{-m} X_j$ for some $m \geq 0$, hence $\mu(X_l) < \mu(X_j)$, which contradicts the assumption on l. Therefore $\tau_{\mathbb{X}}^{-n} X_j \notin \mathrm{add}(\mathcal{W}(W_l))$ and consequently $\mathrm{Hom}_{\mathbb{X}}(\overline{W}_l, \tau_{\mathbb{X}}^{-n} X_j) = 0$ by Lemma 7.3.4.

It follows that in our case some $\mathrm{Ext}_{\mathbb{X}}^1(\overline{W}_l, \tau_{\mathbb{X}}^{-i} Z_j) \neq 0$ for some $\tau_{\mathbb{X}}^{-i} Z_j$, $Z = X$ or $Z = Y$. Because the ϵ_i are epimorphisms, the first sheaf of $(*)$ which is not contained in $\overline{W_l}^\perp$ is some $\tau_{\mathbb{X}}^{-n} Y_j$. For this n we have $\mathrm{Ext}_{\mathbb{X}}^1(W_l, \tau_{\mathbb{X}}^{-n} Y_j) \neq 0$. Now, by [56] the embedding $W_l^\perp \hookrightarrow \mathrm{coh}(\mathbb{X})$ admits a left adjoint functor $l : \mathrm{coh}(\mathbb{X}) \to W_l^\perp$. Then we can proceed as in [116, Lemma 7.2]. The object $l(\tau_{\mathbb{X}}^{-n} Y_j)$ is indecomposable by [116, 2.2], using $\mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}}^{-n} Y_j, W_l) \cong \mathrm{DExt}_{\mathbb{X}}^1(W_l, \tau_{\mathbb{X}}^{-n+1} Y_j) = 0$ by the choice of n. Moreover, the map $l(\mu_n) : l(\tau_{\mathbb{X}}^{-n} X_j) \to l(\tau_{\mathbb{X}}^{-n} Y_j)$ is nonzero. Now, $W_l^\perp$ is the coproduct of $\overline{W_l}^\perp$ and the category of a finite wing by [116, Theorem 3.5] and we conclude that $l(\tau_{\mathbb{X}}^{-n} Y_j) \in \overline{W_l}^\perp$. It follows that $\mathrm{Hom}_{\mathbb{X}}(l(\tau_{\mathbb{X}}^{-n} Y_j), R_l) \cong \mathrm{Hom}_{\mathbb{X}}(l(\tau_{\mathbb{X}}^{-n} Y_j), W_l)$, where R_l is defined as in Theorem 7.4.2. By the construction of the functor l (see [30]) the last term is nonzero. Thus we have a chain of nonzero maps in $\overline{W_l}^\perp$:

$$X_j \to Y_j \to \tau_{\mathbb{X}}^- X_j \to \tau_{\mathbb{X}}^- Y_j \to \ldots \tau_{\mathbb{X}}^{-n} X_j = l(\tau_{\mathbb{X}}^{-n} X_j) \to \tau_{\mathbb{X}}^{-n} Y_j \to R_l.$$

By [116] R_l is regular in $\overline{W_l}^\perp$. Therefore X_j and consequently no direct summand of $\overline{W}_j$ is preinjective in $\overline{W_l}^\perp$, because irreducible maps between the projectives in wings remain irreducible in $\overline{W_l}^\perp$.

In order to finish the proof it remains to show that no V_i is preinjective in $\overline{W_l}^\perp$. By Lemma 7.3.2, for each V_i there is some epimorphism $f : V_i \twoheadrightarrow W_j$. If $j \neq l$ we conclude from the fact that W_j is not preinjective that V_i is not preinjective. If $j = l$, then $f : V_i \to W_l$ factors through the middle term of the Auslander-Reiten sequence ending in W_l, and since $\mathrm{Hom}_{\mathbb{X}}(V_i, \tau_{\mathbb{X}} \overline{W}_l) \cong \mathrm{DExt}^1(\overline{W}_l, V_i) = 0$ it factors through R_l. Now the fact that R_l is regular in $\overline{W_l}^\perp$ implies that V_i is not preinjective in $\overline{W_l}^\perp$. □

Theorem 7.4.4 *Let T be a tilting bundle on a wild weighted projective line $\mathbb{X}$ with wing decomposition*

$$T = T_P \oplus \bigoplus_{i=1}^{s} T(V_i) \oplus \bigoplus_{j=1}^{r} T(W_j).$$

Define $\Sigma = \mathrm{End}(T)$ and $\Sigma_P = \mathrm{End}(T_P)$. Let $\mathcal{C}$ be a component in $\mathrm{mod}_+(\Sigma)$. Then there exists an indecomposable $Z \in \mathcal{C}$ such that the τ_Σ^--cone $(Z \to)$ is a full subquiver of a component in $\mathrm{mod}(\Sigma_P)$.

Proof. The proof of this theorem is similar as the proof of [64, Theorem 1].

We can assume that $\mathcal{C}$ is not the preprojective component and furthermore that $T = T_P \oplus \bigoplus_{i=1}^s \overline{V}_i \oplus \bigoplus_{j=1}^r \overline{W}_j$ is normalized. Define $T'(W_j) = T_P \oplus \bigoplus_{i=1}^s \overline{V}_i \oplus \bigoplus_{t \neq j} \overline{W}_t$.

Choose l such that $\mathrm{rk}(X_l)$ is minimal and $\mu(X_l)$ is maximal among the X_j's with minimal rank. To simplify notations we write $W = W_l$ and $X = X_l$, where as before X_l is the quasi-socle of W_l. Let $Z \in \mathcal{C}$. We first will show that for some $N \geq 0$,

$$\mathrm{Hom}_{\mathbf{X}}(W, \tau_\Sigma^{-t} Z) = 0 \qquad \text{for } t \geq N. \tag{7.1}$$

By [77, 2.9] there is an integer M such that

$$\mathrm{Hom}_{\mathbf{X}}(\tau_{\mathbf{X}}^i X, Z) = 0 \qquad \text{for } i \geq M. \tag{7.2}$$

Let $i \geq 2$. We know from Theorem 7.4.2 that $\tau_{\mathbf{X}}^i X \in \mathrm{coh}_{\geq}(T)$ and for these objects the application of τ_Σ^- and $\tau_{\mathbf{X}}^-$ coincides. Therefore the application of τ_Σ^- gives an isomorphism

$$\overline{\mathrm{Hom}}_\Sigma(\tau_{\mathbf{X}}^{i+1} X, \tau_\Sigma^{-t} Z) \cong \underline{\mathrm{Hom}}_\Sigma(\tau_{\mathbf{X}}^i X, \tau_\Sigma^{-t-1} Z). \tag{7.3}$$

The first term of (7.3) equals $\mathrm{Hom}_\Sigma(\tau_{\mathbf{X}}^{i+1} X, \tau_\Sigma^{-t} Z)$ because $\mathrm{mod}_+(\Sigma)$ contains no nonzero injective Σ-modules and the second term equals $\mathrm{Hom}_\Sigma(\tau_{\mathbf{X}}^i X, \tau_\Sigma^{-t-1} Z)$ because for $i \geq 2$, $\mathrm{Hom}_\Sigma(\tau_{\mathbf{X}}^i X, T) = \mathrm{Hom}_{\mathbf{X}}(\tau_{\mathbf{X}}^i X, T) \cong \mathrm{DExt}^1(T, \tau_{\mathbf{X}}^{i+1}) = 0$ and therefore a nontrivial factorization through a projective module is not possible. Iterating the arguments above $t+1$ times we obtain $\mathrm{Hom}_\Sigma(\tau_{\mathbf{X}}^i X, \tau_\Sigma^{-t-1} Z) \cong \mathrm{Hom}_\Sigma(\tau_{\mathbf{X}}^{i+t+1} X, Z)$, which vanishes by (7.2) for $t \geq M - i - 1$.

Thus we have shown that there exists an $N \in \mathbf{N}$ such that

$$\mathrm{Hom}_{\mathbf{X}}(\tau_{\mathbf{X}}^i X, \tau_\Sigma^{-t} Z) = 0 \qquad \text{for } i \geq 2 \text{ and } t \geq N. \tag{7.4}$$

Now we prove that

$$\mathrm{Hom}_{\mathbf{X}}(\tau_{\mathbf{X}}^i X, \tau_\Sigma^{-t} Z) = 0 \qquad \text{for } i \geq 0 \text{ and } t \geq N+2. \tag{7.5}$$

Consider the exact sequence

$$0 \to \tau_\Sigma^{-t} Z \to \tau_{\mathbf{X}}(\tau_\Sigma^{-t-1} Z) \xrightarrow{p} Q_t \to 0$$

where $Q_t = (\tau_{\mathbf{X}}(\tau_\Sigma^{-t-1} Z))_-$. By [64, Lemma 2.3], $Q_t \in \mathrm{add}(\tau_{\mathbf{X}}(\bigoplus_{i=1}^s V_i \oplus \bigoplus_{i=1}^r W_j))$. Now, for $t \geq N$, $\mathrm{Hom}_{\mathbf{X}}(\tau_{\mathbf{X}}^2 X, \tau_\Sigma^{-t} Z) = 0$ and $\mathrm{Hom}(\tau_{\mathbf{X}}^2 X, Q_t) = 0$ because there are no nonzero morphisms from $\mathrm{coh}_{\geq}(T)$ to $\mathrm{coh}_{\leq}(T)$ and consequently $0 = \mathrm{Hom}_{\mathbf{X}}(\tau_{\mathbf{X}}^2 X, \tau_{\mathbf{X}}(\tau_\Sigma^{-t-1} Z)) \cong \mathrm{Hom}_{\mathbf{X}}(\tau_{\mathbf{X}} X, \tau_\Sigma^{-t-1} Z)$.

Assume now that there is a nonzero morphism $f \in \mathrm{Hom}_{\mathbf{X}}(\tau_{\mathbf{X}} X, \tau_{\mathbf{X}}(\tau_\Sigma^{-t-1} Z)) \cong \mathrm{Hom}_{\mathbf{X}}(X, \tau_\Sigma^{-t-1} Z)$ for $t \geq N+1$ Applying the functor $\mathrm{Hom}_{\mathbf{X}}(\tau_{\mathbf{X}} X, -)$ to the exact sequence above we see from $\mathrm{Hom}_{\mathbf{X}}(\tau_{\mathbf{X}} X, \tau_\Sigma^{-t} Z) = 0$ that the composition $p \circ f : \tau_{\mathbf{X}} X \to Q_t$ is nonzero. Since it factorizes over $\tau_{\mathbf{X}}(\tau_\Sigma^{-t-1} Z)$ it is in $\mathrm{rad}^\infty(\tau_{\mathbf{X}} X, Q_t)$.

Let $Q = E_1 \oplus E_2$ where $E_1 \in \mathrm{add}(\tau_{\mathbf{X}} W)$ and E_2 is without direct summand isomorphic to $\tau_{\mathbf{X}} W$ and decompose $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$, $p_i : \tau_{\mathbf{X}}(\tau_\Sigma^{-t-1} Z) \to E_i$. Then $p_2 \circ f = 0$ by the wing decomposition of T. It follows that $0 \neq p_1 \circ f \in \mathrm{rad}^\infty(\tau_{\mathbf{X}} X, (\tau_{\mathbf{X}} W)^m)$ which gives a contradiction to the fact that $\mathcal{W}(W)$ is a standard wing. Thus the formula (5) holds.

Let q be the quasi-length of W and denote by ${}^{[j]}X$ the indecomposable vector bundle with quasi-length j and quasi-socle X. We show by induction on u that for $1 \leq u \leq q$ there exists an $N' \in \mathbf{N}$ such that

$$\mathrm{Hom}_{\mathbf{X}}({}^{[j]}(\tau_{\mathbf{X}}^i X), \tau_\Sigma^{-t} Z) = 0 \tag{7.6}$$

for all $i \geq 0$, $j \leq u$, $t \geq N'$. The case $u = 1$ was already proved. For $u \geq 0$ we consider the Auslander-Reiten sequence

$$0 \to {}^{[u-1]}(\tau_{\mathbf{X}}^i X) \to {}^{[u]}(\tau_{\mathbf{X}}^i X) \oplus {}^{[u-2]}(\tau_{\mathbf{X}}^{i-1} X) \to {}^{[u-1]}(\tau_{\mathbf{X}}^{i-1} X) \to 0$$

(where for $u = 1$ the middle term consists only of the first summand). Applying the functor $\mathrm{Hom}_{\mathbf{X}}(-, \tau_\Sigma^{-t} Z)$ we obtain by induction that $\mathrm{Hom}_{\mathbf{X}}({}^{[u]}(\tau_{\mathbf{X}}^i X), \tau_\Sigma^{-t} Z) = 0$ for $t \geq N'$ and $i \geq 1$. In order to show that also $\mathrm{Hom}_{\mathbf{X}}({}^{[u]}X, \tau_\Sigma^{-t} Z) = 0$ assume contrary that there is a nonzero map $f' \in \mathrm{Hom}_{\mathbf{X}}({}^{[u]}X, \tau_\Sigma^{-t} Z)$. Then for the corresponding $f \in \mathrm{Hom}_{\mathbf{X}}(\tau_{\mathbf{X}}^{[u]} X, \tau_{\mathbf{X}}(\tau_\Sigma^{-t} Z))$ the composition $p \circ f$ is nonzero and is in $\mathrm{rad}^\infty(\tau_{\mathbf{X}} X(u), Q_t)$. Using again the decomposition $Q_t = E_1 \oplus E_2$, and $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ we have $p_2 = 0$ and then $0 \neq p_1 \circ f \in \mathrm{rad}^\infty({}^{[u]}(\tau_{\mathbf{X}} X), (\tau_{\mathbf{X}} W)^m)$, a contradiction because $\mathcal{W}(\tau_{\mathbf{X}} W)$ is a standard wing. It follows that $\mathrm{Hom}_{\mathbf{X}}({}^{[u]}X, \tau_\Sigma^{-t} Z) = 0$ for $t \geq N'$.

For $u = q$ we obtain $\mathrm{Hom}_{\mathbf{X}}(W, \tau_\Sigma^{-t} Z) = 0$ which proves formula (7.1). As a consequence for $Z' = \tau_\Sigma^{-N'} Z$ the τ_Σ^--cone $(Z' \to)$ consists of modules over $\mathrm{End}(T'(W))$. Now the perpendicular category $\overline{W_l}^\perp$ is equivalent to a module category over a hereditary algebra H. Under this equivalence $T'(W)$ corresponds to a tilting module in $\mathrm{mod}(H)$, which is, by Theorem 7.4.3, without a preinjective direct summand. The modules of $(Z' \to)$ are contained in the class of torsion-free modules $\mathcal{Y}_{\mathrm{mod}(H)}(T'(W))$ of the torsion pair in $\mathrm{mod}(H)$ defined by the tilting module $T'(W)$. Moreover, the Auslander-Reiten sequences of $\mathcal{C}$ which are in this cone are also Auslander-Reiten sequences in $\mathcal{Y}_{\mathrm{mod}(H)}(T'(W))$. This means that $(Z' \to)$ is part of a component in $\mathcal{Y}_{\mathrm{mod}(H)}(T'(W))$ and then our result follows from [64, Theorem 1]. □

Corollary 7.4.4 *Let Σ be a wild concealed-canonical algebra and $\mathcal{C}$ a component in $\mathrm{mod}_+(\Sigma)$ different from the preprojective component. Then the stable part of $\mathcal{C}$ is of type $\mathbf{Z}A_\infty$.* □

Corollary 7.4.5 *Let $\mathbf{X}$ be a wild weighted projective line, T a tilting bundle on $\mathbf{X}$ and $\Sigma = \mathrm{End}(T)$ the corresponding concealed-canonical algebra. Then T defines bijections between the following three sets:*

- *$\Omega(\mathbf{X})$ of components of $\mathrm{vect}(\mathbf{X})$,*
- *$\Omega_+(\Sigma)$ of components of $\mathrm{mod}_+(\Sigma)$ different from the preprojective component,*
- *$\Omega(\Sigma_P)$ of regular components of $\mathrm{mod}(\Sigma_P)$.*

Proof. Let $\mathcal{C}$ be a component of $\mathrm{mod}_+(\Sigma)$ different from the preprojective component. It follows from Theorem 7.2.4 and Theorem 7.4.4 that there exist a unique component $\mathcal{C}'$ in $\mathrm{vect}(\mathbb{X})$ and a unique regular component $\mathcal{C}''$ in $\mathrm{mod}(\Sigma_P)$ such that $\mathcal{C}$ and $\mathcal{C}'$ coincide on a τ-cone, and $\mathcal{C}$ and $\mathcal{C}''$ coincide on a τ^--cone. Thus we obtain injective maps

$$\mu_1 : \Omega_+(\Sigma) \to \Omega(\mathbb{X}) \quad \text{and} \quad \mu_2 : \Omega_+(\Sigma) \to \Omega(\Sigma_P).$$

Let $\mathcal{D}$ be a component of $\mathrm{vect}(\mathbb{X})$ and X a quasi-simple vector bundle in $\mathcal{D}$. We have $\mathrm{DExt}^1_{\mathbb{X}}(T, \tau^n_{\mathbb{X}} X) = \mathrm{Hom}_{\mathbb{X}}(\tau^n_{\mathbb{X}} X, \tau_{\mathbb{X}} T) = 0$ for all $n \geq n_0$ by [77, Theorem 2.9]. Therefore all objects in the τ-cone $(\to \tau^{n_0}_{\mathbb{X}} X)$ are in $\mathrm{mod}_+(\Sigma)$ and the Auslander-Reiten sequences of that cone are also Auslander-Reiten sequences in $\mathrm{mod}(\Sigma)$. Consequently μ_1 is surjective.

In order to show that also μ_2 is surjective we proceed as in [64, Theorem 3]. Σ is an iterated branch enlargement (compare 8.3)

$$\Sigma = C_0[Z_1, Q_1] \ldots [Z_m, Q_m],$$

where $C_0 = \Sigma_P$ and, for $j = 1, \ldots, m$, $C_j = C_0[Z_1, Q_1] \ldots [Z_j, Q_j]$ is obtained by a one point-extension of C_{j-1} by a quasi-simple C_{j-1}-module Z_j and then rooting a linear quiver $Q_j = \circ \to \circ \to \ldots \to \circ$ at the module Z_j. Now, let $\mathcal{D}$ be a regular component in $\mathrm{mod}(\Sigma_P)$ and Y a quasi-simple object in $\mathcal{D}$. Applying Theorem 7.4.3 we see that, for $j = 0, \ldots, m-1$, C_j is a tilted algebra of some wild connected hereditary algebra with tilting module without preinjective direct summand. Therefore by [64, Corollary 3.2] there are $N_j \in \mathbb{N}$, $j = 0, \ldots, m-1$, such that $\mathrm{Hom}_{C_j}(Z_{j+1}, \tau^{-l}_{C_j} Y) = 0$ for $l \geq N_j$. It follows that the Auslander-Reiten sequences of a τ^--cone $(\tau^{-n}_{\Sigma_P} Y \to)$ are also Auslander-Reiten sequences in $\mathrm{mod}(\Sigma)$ and consequently μ_2 is surjective. □

Remark. Invoking the duality $D : \mathrm{coh}_+(T) \to \mathrm{coh}_-(T)$, $\quad F \mapsto F^{\check{}}(\vec{c} + \vec{\omega})$ we have for a tilting bundle the same results for $\mathrm{mod}_-(\Sigma)$.

7.5 Non-regular components for almost concealed-canonical algebras

The results of the preceding section can be generalized to almost concealed-canonical algebras. Assume that $T = T' \oplus T''$ is a tilting sheaf on a wild weighted projective line $\mathbb{X}$ with T' a vector bundle and T'' a finite length sheaf. Because $\mathrm{mod}_+(\Sigma)$ coincides with $\mathrm{mod}_+(\Sigma')$ where $\Sigma' = \mathrm{End}(T')$, by 7.1.4, the structure of the components of $\mathrm{mod}_+(\Sigma)$ follows from the description of the previous sections (see 7.2.6 and 7.4.4).

In order to describe the left hand side of a component of $\mathrm{mod}_-(\Sigma)$ we use the dual wing decomposition. The proofs of the following results are dual to those of 7.4 and are therefore omitted.

Theorem 7.5.1 *Let T be a tilting sheaf over a wild weighted projective line $\mathbb{X}$. Then there exists a decomposition*

$$T = T_2 \oplus T_I$$

which satisfies the following conditions:

(i) The left perpendicular category ${}^\perp T_2$ is equivalent to the module category of a connected wild hereditary algebra.

(ii) T_I is ${}^\perp T_2$-preinjective.

(iii) The preinjective component of the algebra $\Sigma_I = \mathrm{End}(T_I)$ is a full component of the Auslander-Reiten quiver for Σ. Moreover, this is the only preinjective component for Σ. □

7.5.2 Dually to 7.3.2 we have a decomposition

$$T = \bigoplus_{j=1}^{a} T(W_j) \oplus \bigoplus_{i=1}^{b} T(V_i) \oplus T_I.$$

Theorem 7.5.2 *Let T be a tilting sheaf on a wild weighted projective line $\mathbb{X}$ with a decomposition*

$$T = \bigoplus_{j=1}^{a} T(W_j) \oplus \bigoplus_{i=1}^{b} T(V_i) \oplus T_I.$$

Denote by Z_j the quasi-top of W_j and let $R_j \twoheadrightarrow W_j$ be an irreducible epimorphism for $j = 1, \ldots, a$. Then we have

(a) $R_j \in \mathrm{coh}_{\leq}(T)$ for $j = 1, \ldots, a$.

(b) Let l be such that $\mathrm{rk}(Z_l)$ is maximal and $\mu(Z_l)$ is minimal among the X_j's with maximal rank. Then

(i) $\tau^-_{\mathbb{X}} Z_l \in \mathrm{coh}_{\leq}(T)$ and

(ii) The $\tau^-_{\mathbb{X}}$-cone $(\tau^-_{\mathbb{X}} Z_l \to)$ is contained in $\mathrm{coh}_{\leq}(T)$ and $(\tau^-_{\mathbb{X}} Z_l \to)[1]$ is a full subquiver of the non-regular component in $\mathrm{mod}(\Sigma)$ containing $\tau_{\mathbb{X}} W_l$. □

Observe that this is the dual situation of Theorem 7.4.2 shifted by $\tau_{\mathbb{X}}$. Moreover, because T is not contained in $\mathrm{coh}_0(\mathbb{X})$, X_l is a vector bundle, and therefore we can apply dual arguments to those as used in the proof of Theorem 7.4.2.

Theorem 7.5.3 *Let T be a tilting sheaf on a wild weighted projective line $\mathbb{X}$ with decomposition*

$$T = \bigoplus_{j=1}^{a} T(W_j) \oplus \bigoplus_{i=1}^{b} T(V_i) \oplus T_I.$$

Denote by Z_j the quasi-top of W_j. Let l be such that $\mathrm{rk}(Z_l)$ is maximal and $\mu(Z_l)$ is minimal among the Z_j's with maximal rank. Then no direct summand of T is ${}^\perp \underline{W_l}$-preprojective, where $\underline{W_l}$ denotes the direct sum of all injectives in the wing $\mathcal{W}(W_l)$. □

The theorem can be proved again by using dual arguments with simple modifications. It is essential for the induction step of the following result.

Theorem 7.5.4 *Let T be a tilting sheaf on a wild weighted projective line $\mathbb{X}$ with decomposition*

$$T = \bigoplus_{j=1}^{a} T(W_i) \oplus \bigoplus_{i=1}^{b} T(V_i) \oplus T_I.$$

Define $\Sigma = \mathrm{End}(T)$ and $\Sigma_I = \mathrm{End}(T_I)$. Let $\mathcal{C}$ be a component in $\mathrm{mod}_{\leq}(\Sigma)$. Then there exists an indecomposable $Z \in \mathcal{C}$ such that the τ_Σ-cone $(\to Z)$ is a full subquiver of a component in $\mathrm{mod}(\Sigma_I)$. □

Corollary 7.5.5 *Let T be a tilting sheaf on a wild weighted projective line $\mathbb{X}$ and $T = T' \oplus T''$ with $T' \in \text{vect}(\mathbb{X})$ and $T'' \in \text{coh}_0(\mathbb{X})$. Furthermore, let $T = T_2 \oplus T_I$ be the decomposition of Theorem 7.5.1. Then T'' is a direct summand of T_I. Furthermore all modules from $\text{mod}_0^{[1]}(\Sigma)$ are preinjective.*

Proof. Let $\mathcal{C}$ be a component in $\text{mod}_{\leq}(\Sigma)$ different from the preinjective component containing some injective Σ-module of the form $Y = \tau_{\mathbb{X}} M_j[1]$ where $M = W$ or V and $M_j \in \text{coh}_0(\mathbb{X})$. It follows from Theorem 7.2.4 and Theorem 7.5.4 that Y has infinitely many successors in the component $\mathcal{C}$. This is impossible, since the Σ-modules from $\text{mod}_0^{[1]}(\Sigma)$ have only finitely many successors in $\text{mod}(\Sigma)$. □

Corollary 7.5.6 *Let $\mathbb{X}$ be a wild weighted projective line, T a tilting sheaf on $\mathbb{X}$ and $\Sigma = \text{End}(T)$ the corresponding almost concealed-canonical algebra. Then T defines bijections between the following three sets:*

- *$\Omega(\mathbb{X})$ of components of $\text{vect}(\mathbb{X})$,*
- *$\Omega_-(\Sigma)$ of components of $\text{mod}_-(\Sigma)$ different from the preinjective component,*
- *$\Omega(\Sigma_I)$ of regular components of $\text{mod}(\Sigma_I)$.*

Proof. A component of $\text{mod}_-(\Sigma)$ coincides on a τ-cone with a component of $\text{mod}(\Sigma_I)$ and on a τ^--cone with a component of $\text{vect}(\mathbb{X})$. Now we can apply similar arguments as in the proof of Corollary 7.4.5. For the proof of the surjectivity of μ_1 we use the fact that there are no nonzero morphisms from T'' to vector bundles. □

Chapter 8

Tilting complexes

8.1 Tilting complexes and exceptional sequences

Let $\mathbb{X}$ be a weighted projective line of arbitrary type. In this section we define the notion of a tilting complex on $\mathbb{X}$ and discuss the relationships between tilting complexes and exceptional sequences.

Definition 8.1.1 *We call an object T in $\mathcal{D}^b(\text{coh}(\mathbb{X})) = \mathcal{D}$ a tilting complex if it satisfies the following two conditions:*

(i) $\text{Hom}_{\mathcal{D}}(T, T[i]) = 0$ for $i \neq 0$,

(ii) The indecomposable direct summands of T generate $\mathcal{D}^b(\text{coh}(\mathbb{X}))$ as a triangulated category.

An object T in $\mathcal{D}^b(\text{coh}(\mathbb{X}))$ which satisfies only the vanishing condition (i) is called a partial tilting complex.

The notion of a tilting complex for a module category has been introduced by Rickard in [97] and generalizes that of a tilting module (see [17], [45]). In our situation we have

Theorem 8.1.1 *If T is a tilting complex in $\mathcal{D}^b(\text{coh}(\mathbb{X}))$ with endomorphism ring $\Sigma = \text{End}_{\mathcal{D}}(T)$, then the derived functor*

$$\mathbf{R}\text{Hom}(T, -) : \mathcal{D}^b(\text{coh}(\mathbb{X})) \to \mathcal{D}^b(\text{mod}(\Sigma))$$

is an equivalence of triangulated categories. Conversely, each triangle equivalence $\mathcal{D}^b(\text{coh}(\mathbb{X})) \overset{\cong}{\to} \mathcal{D}^b(\text{mod}(\Sigma))$, for a finite dimensional k-algebra Σ, is given by a tilting complex T in $\mathcal{D}^b(\text{coh}(\mathbb{X}))$ such that $\Sigma = \text{End}_{\mathcal{D}}(T)$. □

The proof for tilting sheaves [5, Theorem 3.1.2], [29, Theorem 3.2] generalizes to the more general case of tilting complexes.

A finite dimensional algebra Σ such that $\text{mod}(\Sigma)$ is equivalent to $\text{coh}(\mathbb{X})$ arises from $\text{coh}(\mathbb{X})$ by a finite sequence of tilts/cotilts, that is there is a sequence $\Omega_1, \ldots, \Omega_n = \Sigma$, where Ω_1 is the endomorphism algebra of a tilting sheaf in $\text{coh}(\mathbb{X})$, and for $i = 2, \ldots n-1$ the algebra Ω_{i+1} is the endomorphism algebra of a tilting or cotilting module over $\text{End}(\Omega_i)$ [43].

A complex is called *multiplicity-free* if its indecomposable direct summands are non-isomorphic. Furtheron all tilting complexes are assumed to be multiplicity-free, which of course is no loss of generality.

8.1.2 The following lemma shows that the indecomposable objects of a tilting complex for coh($\mathbb{X}$) may be arranged in a complete exceptional sequence. It works also for an arbitrary abelian hereditary k-category having a tilting complex.

Lemma 8.1.2 *The indecomposable direct summands of a multiplicity-free partial tilting complex T in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ can be ordered such that they form an exceptional sequence ϵ. Moreover, if T is a tilting complex then ϵ is complete.*

Proof. The complex T decomposes into a direct sum $\bigoplus T_i[n_i]$ for some partial tilting objects T_i in coh($\mathbb{X}$) and $n_i \in \mathbb{Z}$, in particular we have $\mathrm{Ext}^1_{\mathbb{X}}(T_i, T_i) = 0$. From the vanishing condition for a tilting complex we obtain $\mathrm{Hom}_{\mathbb{X}}(T_i, T_j) = 0$ and $\mathrm{Ext}^1_{\mathbb{X}}(T_i, T_j) = 0$ for $n_i > n_j$. Let $T_i = \bigoplus_{l=1}^{s} X_l^{(i)}$ be a decomposition into indecomposable objects of coh($\mathbb{X}$).

Then, according to 2.3.3, $\mathrm{End}(X_l^{(i)}) = k$ and the $X_l^{(i)}$ can be ordered in such a way that $\mathrm{Hom}_{\mathbb{X}}(X_{l_a}^{(i)}, X_{l_b}^{(i)}) = 0$ if $l_a > l_b$. Therefore we obtain an exceptional sequence.

Further, if T is a tilting complex, then the number of indecomposable direct summands of T equals the rank of the Grothendieck, and the exceptional sequence is complete. □

We call an exceptional sequence of the form described in the preceding lemma a *tilting sequence*. Similarly an exceptional sequence corresponding to a partial tilting complex is called a *partial tilting sequence*.

8.1.3 From the lemma above and Lemma 3.1.2 we obtain

Lemma 8.1.3 *Let T be a multiplicity-free complex in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ such that* $\mathrm{Hom}_{\mathcal{D}}(T, T[i]) = 0$ *for $i \neq 0$. Then the following conditions are equivalent:*

(a) The indecomposable direct summands of T generate $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ as a triangulated category.

(b) The number of indecomposable direct summands of T coincides with the rank of the Grothendieck group $\mathrm{K}_0(\mathbb{X})$. □

8.1.4 It was shown in [86] that for a module category over a hereditary algebra of type $\mathbb{A}_n$ a complete exceptional sequence can be considered as a tilting complex. More precisely, let Q a Dynkin quiver and $A = kQ$ the path algebra. Then Q is of type $\mathbb{A}_n$ if and only if for each complete exceptional sequence $\epsilon = (X_1, X_2, \ldots, X_n)$ in $\mathcal{D}^b(\mathrm{mod}(A))$ there are integers $j_i, 1 \le i \le n$, such that $\bigoplus_{i=1}^{n} X_i[j_i]$ is a tilting complex.

For the category coh($\mathbb{P}^1$) of coherent sheaves on the projective line each exceptional sequence is of the form $(\mathcal{O}(n), \mathcal{O}(n+1))$ and can be therefore considered as a tilting sheaf, for the general case however we have the following result proved in [86].

Proposition 8.1.5 *Let $\mathbb{X}$ be a weighted projective line such that at least one weight is greater than one. Then there is a complete exceptional sequence in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ which cannot be shifted to a tilting complex.*

Proof. Forming perpendicular categories to simple exceptional sheaves of finite length one reduces to the weight sequence (2).

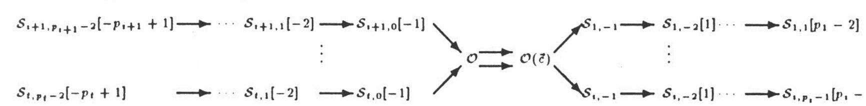
In this case for the exceptional sequence

$$\epsilon = (\mathcal{O}, \mathcal{S}_{1,1}, \mathcal{O}(\vec{x}_1)),$$

where $\mathcal{S}_{1,1}$ is the the cokernel of the exact sequence $0 \to \mathcal{O}(\vec{x}_1) \xrightarrow{X_1} \mathcal{O}(\vec{c}) \to \mathcal{S}_{1,1} \to 0$, we have $\mathrm{Hom}_{\mathbb{X}}(\mathcal{O}, \mathcal{S}_{1,1}) \neq 0$, $\mathrm{Hom}_{\mathbb{X}}(\mathcal{O}, \mathcal{O}(\vec{x}_1)) \neq 0$ and $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{S}_{1,1}, \mathcal{O}(\vec{x}_1)) \neq 0$. Therefore ϵ cannot be shifted to a tilting complex. □

8.1.6 If T is a tilting complex on a weighted projective line $\mathbb{X}$, then the indecomposable direct summands of T are taken from consecutive copies of coh($\mathbb{X}$) in the derived category. The number of these copies is called the *width* of the complex. The following example shows that the width of a tilting complex can be equal to $p_1 + p_2 - 2$, where $\mathbf{p} = (p_1, p_2, \ldots, p_t)$ is the weight type of $\mathbb{X}$.

Example. The complex



is a tilting complex on $\mathbb{X} = \mathbb{X}(\mathbf{p}, \lambda)$. We have indicated the quiver of the endomorphism algebra $\Sigma = \mathrm{End}(T)$. Note that T contains all but one simple finite length sheaves from each exceptional tube.

8.2 Tilting complexes for weighted projective lines with parameters

8.2.1 We have seen in 3.4.2 that for weighted projective lines $\mathbb{X} = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ and $\mathbb{X}' = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda}')$ of the same weight type there is a nice bijection between the exceptional objects in coh($\mathbb{X}$) and coh($\mathbb{X}'$), respectively. This bijection is not functorial but it preserves the K-theoretical data. Now, by the same arguments, there is a constructive bijection Ψ from the set of complete exceptional sequences in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ to the set of complete exceptional sequences in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}'))$; Ψ associates to a sequence of the form $\epsilon = g.\mathbf{K}$ the sequence $\epsilon = g.\mathbf{K}'$ constructed by the same element g of the group $G_n = \mathbb{Z}^n \ltimes B_n$. Recall that $\mathbf{K} = (\mathcal{O}_{\mathbb{X}}, \mathcal{O}_{\mathbb{X}}(\vec{x}_1), \ldots, \mathcal{O}_{\mathbb{X}}((p_t - 1)\vec{x}_t), \mathcal{O}_{\mathbb{X}}(\vec{c}))$ and $\mathbf{K}' = (\mathcal{O}_{\mathbb{X}'}, \mathcal{O}_{\mathbb{X}'}(\vec{x}_1), \ldots, \mathcal{O}_{\mathbb{X}'}((p_t - 1)\vec{x}_t), \mathcal{O}_{\mathbb{X}'}(\vec{c}))$ are the tilting sequences corresponding to the canonical tilting sheaves on $\mathbb{X}$ and $\mathbb{X}'$, respectively.

Proposition 8.2.1 *The function Ψ described above induces a bijection between the tilting complexes on $\mathbb{X}$ and $\mathbb{X}'$, respectively.*

Proof. We know from 8.1.2 that the indecomposable objects of tilting complexes in coh($\mathbb{X}$) and coh($\mathbb{X}'$) may be arranged in complete exceptional sequences. Further, the condition that an exceptional sequence $\epsilon = (E_1, \ldots, E_n)$ in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$, is a tilting sequence can be expressed in certain vanishing conditions for the spaces $\mathrm{Hom}_{\mathbb{X}}(F_a, F_b)$ and $\mathrm{Ext}^1_{\mathbb{X}}(F_a, F_b)$, $a < b$, where $E_i = F_i[n_i]$ for $F_i \in \mathrm{coh}(\mathbb{X})$ and $n_i \in \mathbb{Z}$.

Now, by construction, for $\epsilon = g.\mathbf{K} = (E_1, \ldots, E_n)$ and $\epsilon' = g.\mathbf{K}' = (E'_1, \ldots, E'_n)$ the classes $[E_i]$ and $[E'_i]$ are equal under the natural identification of the Grothendieck groups. Moreover, according to 2.3.4, for an exceptional pair (A, B) appearing in ϵ or ϵ' at most one of the spaces $\mathrm{Hom}(A, B)$, $\mathrm{Ext}^1(A, B)$ is nonzero. It follows that ϵ' satisfies the same vanishing conditions as ϵ. Therefore ϵ is a tilting sequence if and only if ϵ' is. □

8.2.2 As a consequence of 8.2.1 we see that if $\Sigma = \mathrm{End}(T)$ is the endomorphism algebra of a tilting complex on a weighted projective line $\mathbb{X} = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ then we have a family of such algebras by varying the parameter sequence.

Theorem 8.2.2 *The quiver of the endomorphism algebra of a tilting complex on a weighted projective line $\mathbb{X} = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ is independent of $\boldsymbol{\lambda}$.*

Proof. Assume that $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ are two parameter sequences. Let T be a tilting complex on $\mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ and $\epsilon = g.\mathbf{K}$ a tilting sequence for T. We have to show that the quiver of $\Sigma = \mathrm{End}(T)$ coincides with the quiver of the endomorphism algebra Σ' of the tilting complex for the corresponding tilting sequence $\epsilon' = g.\mathbf{K}'$ on $\mathbb{X}(\mathbf{p}, \boldsymbol{\lambda}')$. We write $\epsilon = (P_1, \ldots, P_n)$ and $\epsilon' = (P'_1, \ldots, P'_n)$. The members of these sequences are identified with the indecomposable projective modules over Σ and Σ', respectively. Let $(S_1, \ldots, S_n)$ (resp.$(S'_1, \ldots, S'_n)$) denote the corresponding sequences of simple Σ (resp. Σ') modules. Observe that $([S_1], \ldots, [S_n])$ is a basis of $\mathrm{K}_0(\mathbb{X})$, dual to the basis $([P_1], \ldots, [P_n])$, and $([S'_1], \ldots, [S'_n])$ is a basis of $\mathrm{K}_0(\mathbb{X}') = \mathrm{K}_0(\mathbb{X})$, dual to the basis $([P'_1], \ldots, [P'_n])$. Since the classes of the corresponding projective modules coincide, we conclude that $[S_i] = [S'_i]$ for all i.

Consider the map ν: {exceptional objects in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$} $\to \mathbb{Z}$ which maps each exceptional object $X = E[m]$, $E \in \mathrm{coh}(\mathbb{X})$, to its "copy number" m. In the same manner we define a copy number map for the exceptional objects in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}'))$. We have $\nu(P_i) = \nu(P'_i)$, by construction. Now, there are nonzero morphisms $P_i \to S_i$ and $P'_i \to S'_i$, hence $\nu(S_i), \nu(S'_i) \in \{\nu(P_i), \nu(P_i) + 1\}$. It follows that $\nu(S_i) = \nu(S'_i)$, because otherwise either $0 \neq \mathrm{rk}(S_i) = -\mathrm{rk}(S'_i)$ or $\mathrm{rk}(S_i) = \mathrm{rk}(S'_i) = 0$ and $\deg(S_i) = -\deg(S'_i)$, a contradiction.

We claim that $\dim_k \mathrm{Ext}^p_\Sigma(S_i, S_j) = \dim_k \mathrm{Ext}^p_{\Sigma'}(S'_i, S'_j)$ for all i, j and all p. Indeed by [39, Chapter IV, Lemma 1.11] (compare 9.6.4), there exists at most one p_0 such that $\mathrm{Ext}^{p_0}_\Sigma(S_i, S_j) \neq 0$ and at most one p'_0 such that $\mathrm{Ext}^{p'_0}_{\Sigma'}(S'_i, S'_j) \neq 0$. Therefore we have

$$\begin{aligned}\chi([S_i], [S_j]) &= \sum_p (-1)^p \dim_k \mathrm{Hom}_{\mathcal{D}}(S_i, S_j[p]) \\ &= \sum_p (-1)^p \dim_k \mathrm{Ext}_\Sigma(S_i, S_j) \\ &= (-1)^{p_0} \dim_k \mathrm{Ext}^{p_0}_\Sigma(S_i, S_j).\end{aligned}$$

If $\chi([S_i], [S_j]) = 0$ then all $\mathrm{Ext}^p_\Sigma(S_i, S_j)$ vanish and the same holds for all $\mathrm{Ext}^p_{\Sigma'}(S'_i, S'_j)$. On the other hand, if $\chi([S_i], [S_j]) \neq 0$ then $p_0 = \nu(S_i) - \nu(S_j)$ or $p_0 = \nu(S_i) - \nu(S_j) + 1$ and the sign of $\chi([S_i], [S_j])$ determines p_0 uniquely. In the same way p'_0 can be determined and the claim follows. Since the dimensions of the Ext^1-spaces between the simple modules give the numbers of arrows for the algebra, the proof is finished. □

Remark. Observe that the proof gives more; for the endomorphism algebra of a tilting complex on a weighted projective line $\mathbb{X} = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ all dimensions of the spaces $\mathrm{Ext}^p_\Sigma(S_i, S_j)$ between the simple Σ-modules are independent of $\boldsymbol{\lambda}$.

Corollary 8.2.2 *The global dimension of the endomorphism algebra of a tilting complex on a weighted projective line $\mathbb{X} = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ is independent of $\boldsymbol{\lambda}$.* □

Theorem 8.2.3 *Let E be an exceptional vector bundle on a weighted projective line $\mathbb{X} = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ and H the hereditary algebra such that the right perpendicular category $E^\perp$ is equivalent to the category of modules over H. Then H is independent of $\boldsymbol{\lambda}$.*

Proof. For a fixed choice of parameters $\boldsymbol{\lambda}$ we consider an exceptional vector bundle E on $\mathbb{X} = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ and the hereditary algebra H such that $E^\perp$ is equivalent to $\mathrm{mod}(H)$. Denote by $(S_1, \ldots, S_{n-1})$ a complete exceptional sequence in $\mathcal{D}^b(E^\perp)$ consisting of the simple H-modules, in some order. Then $\epsilon = (S_1, \ldots, S_{n-1}, E)$ is a complete exceptional sequence in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$, therefore $\epsilon = g.\mathbf{K}$ for some element $g \in G_n$.

For another choice of parameters $\boldsymbol{\lambda}'$ we have the exceptional sequence $\epsilon' = g.\mathbf{K}' = (S'_1, \ldots, S'_{n-1}, E')$ in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}'))$ and the hereditary algebra H' such that $(E')^\perp$ is equivalent to $\mathrm{mod}(H')$. Obviously the S'_i are H'-modules. Moreover, the sequence $(S_1, \ldots, S_{n-1})$ is Hom-orthogonal, thus the sequence $(S'_1, \ldots, S'_{n-1})$ has the same property. It follows from Theorem 3.1.5 that $S'_1, \ldots, S'_{n-1}$ are the simple H'-modules. Because ϵ and ϵ' are constructed using the same group element, the dimensions between the Ext^1-spaces of the corresponding simple modules coincide, which gives H and H' are isomorphic. □

Corollary 8.2.4 *Let $\epsilon = (E_1, \ldots, E_r, E_{r+1}, \ldots, E_s, E_{s+1}, \ldots, E_n)$ be a tilting sequence on a weighted projective line $\mathbb{X} = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ such that $E_1, \ldots, E_r, E_{s+1}, \ldots, E_n$ are, up to translation, finite length sheaves and E_{r+1}, E_s are, up to translation, vector bundles. Then the algebras $\mathrm{End}(\bigoplus_{i=1}^{s-1} E_i)$ and $\mathrm{End}(\bigoplus_{i=r+2}^{n} E_i)$ are independent of $\boldsymbol{\lambda}$.*

Proof. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ be two parameter sequences. We consider tilting sequences $\epsilon = g.\mathbf{K} = (E_1, \ldots, E_r, E_{r+1}, \ldots, E_s, E_{s+1}, \ldots, E_n)$ on $\mathbb{X} = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ and $\epsilon' = g.\mathbf{K}' = (E'_1, \ldots, E'_r, E'_{r+1}, \ldots, E'_s, E'_{s+1}, \ldots, E'_n)$ on $\mathbb{X}' = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda}')$. By assumption, $\mathrm{rk}(E_1) = \ldots = \mathrm{rk}(E_r) = \mathrm{rk}(E_{s+1}) = \ldots = \mathrm{rk}(E_n) = 0$ and $\mathrm{rk}(E_{r+1}) \neq 0$, $\mathrm{rk}(E_s) \neq 0$. Note that the corresponding objects in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}'))$ have the same ranks.

We denote by ν the "copy number maps" for $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ and $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}'))$ as in the proof of 8.2.2. Let $\mathcal{C}$ (resp. $\mathcal{C}'$) be the right perpendicular category formed in $\mathrm{coh}(\mathbb{X})$ (resp. $\mathrm{coh}(\mathbb{X}')$) to the sheaves $E_s[-\nu(E_s)], E_{s+1}[-\nu(E_{s+1})], \ldots, E_n[-\nu(E_n)]$, (resp. $E'_s[-\nu(E'_s)], E'_{s+1}[-\nu(E'_{s+1})], \ldots, E'_n[-\nu(E'_n)]$). We deduce from 2.4.2 and 8.2.3 that $\mathcal{C} \cong \mathcal{C}' \cong \mathrm{mod}(H)$ for some hereditary algebra H.

Furthermore, the sequences $(E_1, \ldots, E_{s-1})$ and $(E'_1, \ldots, E'_{s-1})$ are tilting sequences in $\mathcal{D}^b(\mathrm{mod}(H))$. Now, the classes $[E_i]$, $[E'_i]$ coincide and determine unique exceptional H-modules M_i such that $E_i = M_i[n_i]$ and $E'_i = M_i[n'_i]$, for $i = 1, \ldots, s-1$ (compare 3.4.1). Since ϵ and ϵ' are constructed using the same group element we also have $n_i = n'_i$. Therefore $\mathrm{End}(\bigoplus_{i=1}^{s-1} E_i)$ and $\mathrm{End}(\bigoplus_{i=1}^{s-1} E'_i)$ are isomorphic.

Forming left perpendicular categories one shows dually that $\mathrm{End}(\bigoplus_{i=r+2}^{n} E_i)$ is independent of $\boldsymbol{\lambda}$. □

Theorem 8.2.5 *Let T be a tilting complex on a weighted projective line $\mathbb{X} = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$, where $\mathbf{p} = (p_1, p_2, \ldots, p_t)$, all $p_i \geq 2$ and $t \geq 4$. Then $\Sigma = \mathrm{End}(T)$ is representation-infinite.*

Proof. Suppose that for a certain choice of parameters $\overline{\boldsymbol{\lambda}}$ there is a tilting complex $T^{\overline{\boldsymbol{\lambda}}}$ in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}(\mathbf{p}, \overline{\boldsymbol{\lambda}}))$ such that $\Sigma^{\overline{\boldsymbol{\lambda}}} = \mathrm{End}(T^{\overline{\boldsymbol{\lambda}}})$ is representation-finite. Let

$$\epsilon^{\overline{\boldsymbol{\lambda}}} = g.\overline{\mathbf{K}} = (E_1^{\overline{\boldsymbol{\lambda}}}, \ldots, E_s^{\overline{\boldsymbol{\lambda}}}, E_{s+1}^{\overline{\boldsymbol{\lambda}}}, \ldots, E_n^{\overline{\boldsymbol{\lambda}}})$$

be a tilting sequence for $T^{\overline{\boldsymbol{\lambda}}}$. We assume that, up to translation, $E_s^{\overline{\boldsymbol{\lambda}}}$ is a vector bundle and $E_{s+1}^{\overline{\boldsymbol{\lambda}}}, \ldots, E_n^{\overline{\boldsymbol{\lambda}}}$ are finite length sheaves.

By 8.2.1 there is a tilting sequence $\epsilon^{\boldsymbol{\lambda}} = g.\mathbf{K} = (E_1^{\boldsymbol{\lambda}}, \ldots, E_s^{\boldsymbol{\lambda}}, E_{s+1}^{\boldsymbol{\lambda}}, \ldots, E_n^{\boldsymbol{\lambda}})$ for each parameter sequence $\boldsymbol{\lambda}$. Note that the ranks of the $E_i^{\boldsymbol{\lambda}}$ are independent of $\boldsymbol{\lambda}$. Moreover, a tilting sequence contains at least two vector bundles, hence $s > 1$.

For $m = s-1, \ldots, n$ we define $\Sigma_m^{\boldsymbol{\lambda}} = \mathrm{End}(\oplus_{i=1}^{m} E_i^{\boldsymbol{\lambda}})$. By 8.2.4, $\Gamma = \Sigma_{s-1}^{\boldsymbol{\lambda}}$ is independent of $\boldsymbol{\lambda}$. Now the algebra $\Sigma_s^{\boldsymbol{\lambda}}$ is given as a one-point coextension $[M^{\boldsymbol{\lambda}}]\Gamma$ of Γ by a Γ-module $M^{\boldsymbol{\lambda}}$. The dimension vector of $M^{\boldsymbol{\lambda}}$ equals

$$\mathbf{v} = (\dim_k \mathrm{Hom}(E_1^{\boldsymbol{\lambda}}, E_s^{\boldsymbol{\lambda}}), \ldots, \dim_k \mathrm{Hom}(E_{s-1}^{\boldsymbol{\lambda}}, E_s^{\boldsymbol{\lambda}}))$$

and is independent of $\boldsymbol{\lambda}$. By assumption $\Sigma^{\overline{\boldsymbol{\lambda}}}$ is representation-finite, hence Γ is representation-finite as a factor algebra of $\Sigma^{\overline{\boldsymbol{\lambda}}}$. But then there are only finitely many non-isomorphic Γ-modules of dimension vector $\mathbf{v}$. Hence there are only finitely many isomorphism classes of algebras $\Sigma_s^{\boldsymbol{\lambda}}$.

Writing each algebra $\Sigma_m^{\boldsymbol{\lambda}}$ as a one-point coextension of $\Sigma_{m-1}^{\boldsymbol{\lambda}}$, $m = s+1, \ldots, n$, we conclude in the same way that there are only finitely many isomorphism classes of algebras $\Sigma_n^{\boldsymbol{\lambda}}$. Consequently, there are, up to equivalence, only finitely many categories of the form $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})))$ with fixed weight type $\mathbf{p}$. For $t \geq 4$ this gives a contradiction to [75, Theorem 2.3], [30, Proposition 9.2]. □

Note that the proof above and its dual actually show that for a tilting sequence $\epsilon = (E_1, \ldots, E_n)$ on a weighted projective line $\mathbb{X} = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$, where $\mathbf{p} = (p_1, p_2, \ldots, p_t)$, all $p_i \geq 2$ and $t \geq 4$, the algebras $\mathrm{End}(\oplus_{i=1}^{n-1} E_i)$ and $\mathrm{End}(\oplus_{i=2}^{n} E_i)$ are representation-infinite.

We remark also that for wild weighted projective lines with $t = 3$ there are tilting complexes with representation-finite endomorphism algebras.

8.3 Branch enlargements of concealed-canonical algebras

In this section we consider branch enlargements in the sense of Assem and Skowroński of concealed-canonical algebras.

In [2, Theorem 2.5] the representation-infinite algebras which are derived equivalent to tame hereditary (resp. tubular) algebras were described as domestic (resp. tubular) branch enlargements of tame concealed algebras.

In 8.4 and 8.5 we will give for the domestic and tubular case a handy criterion whether or not the endomorphism ring of a tilting complex on a weighted projective line $\mathbb{X}$ is representation-finite. Moreover, we show how in the representation-infinite case the characterization of Assem and Skowroński can be obtained in an easy way considering tilting complexes on $\mathbb{X}$.

We refer to [2, 2.1 - 2.3] for the concept of branch enlargements. We also use the terminology developed there.

Definition 8.3.1 *A tilting complex on a weighted projective line $\mathbb{X}$*

$$T = T_{-n_1}[-n_1] \oplus \ldots \oplus T_{-1}[-1] \oplus T_+ \oplus T_0 \oplus T_1[1] \oplus \ldots \oplus T_{n_2}[n_2] \qquad (*)$$

with $T_+ \in \mathrm{vect}(\mathbb{X})$ and $T_i \in \mathrm{coh}_0(\mathbb{X})$ for all $-n_1 \leq i \leq n_2$, is called to be of the form (*).

8.3.2 Let Σ_+ be a concealed-canonical algebra, realized by a tilting bundle on a weighted projective line $\mathbb{Y} = \mathbb{X}(\mathbf{p}', \boldsymbol{\lambda})$, $\mathbf{p}' = (p_1', \ldots, p_{t'}')$. Let t be an integer with $t \geq t'$. For each $i = 1, \ldots, t$ we select a sequence $\mathcal{S}_i'(j)$, $j = 1, \ldots, h_i$, of $h_i \geq 0$ pairwise non-isomorphic simple objects from $\mathrm{mod}_0(\Sigma_+) = \mathrm{coh}_0(\mathbb{Y})$, which are concentrated at λ_i. Note that we allow that $p_i' = 1$, accordingly that $\mathcal{S}_i'(j)$ is an ordinary simple object from $\mathrm{mod}_0(\Sigma_+) = \mathrm{coh}_0(\mathbb{Y})$.

Let $J'(i) \coprod J''(i)$ be a decomposition of $J(i) = \{1, \ldots, h_i\}$ into, possibly empty, disjoint subsets. Further we select for each $j \in J'(i)$ (resp. $j \in J''(i)$) an extension branch $\mathcal{B}_i'(j)$ with extension root (resp. coextension branch $\mathcal{B}_i''(j)$ with coextension root) $a_i(j)$ of length $\ell_i(j)$.

The algebra Σ obtained from Σ_+ by forming first the multi-point extension-coextension

$$[\mathrm{D}\mathcal{S}_i'(j)]_{j \in J'(i)}^{i=1\ldots t} \Sigma_+ [\mathcal{S}_i'(j)]_{j \in J''(i)}^{i=1\ldots t}$$

and then rooting each extension branch $\mathcal{B}_i'(j)$ in $\mathrm{D}\mathcal{S}_i'(j)$ (resp. each coextension branch $\mathcal{B}_i''(j)$ in $\mathcal{S}_i'(j)$) is said to be *obtained from Σ_+ by branch enlargement.*

If all $J'(i)$ are empty and all $\mathcal{B}_i''(j)$ are truncated branches in the sense of Ringel [100], then Σ is a branch coenlargement as considered in [75] and therefore an almost concealed-canonical algebra. If, for each $i = 1, \ldots, t$, one of the sets $J'(i)$ or $J''(i)$ is empty and in addition all $\mathcal{B}_i'(j)$ and $\mathcal{B}_i''(j)$ are truncated branches, then Σ is a semiregular branch coenlargement as studied by Lenzing and Skowroński in [79]. In particular, these algebras were shown to be quasitilted of canonical type (compare 9.6.1).

We put

$$p_i = p_i' + \sum_{j=1}^{h_i} \ell_i(j)$$

for $i = 1, \ldots, t$. Moreover, we let $\mathbb{X}$ be the weighted projective line which attaches the weights p_i to the points λ_i for $i = 1, \ldots, t$. Note that we are identifying the sets underlying $\mathbb{X}$ and $\mathbb{Y}$ with the projective line over k.

Theorem 8.3.2 *(i) If T is a tilting complex of the form (*), then $\Sigma = \mathrm{End}(T)$ is obtained by branch enlargement from the concealed-canonical algebra $\Sigma_+ = \mathrm{End}(T_+)$.*

(ii) If Σ is an algebra obtained from a concealed-canonical algebra Σ_+ by branch enlargement, then Σ can be realized as the endomorphism algebra of a tilting complex of the form (*) on a weighted projective line.

Proof. (ii) For each $i = 1, \dots, t$ we arrange the simple sheaves $\mathcal{S}_i'(j)$ on $\mathbf{Y}$ in such a way that $\mathcal{S}_i'(j) = \tau_{\mathbf{Y}}^{-m_i'(j)}\mathcal{S}_i'$ for some simple sheaf $\mathcal{S}_i'$ concentrated at λ_i and where

$$0 < m_i'(1) < m_i'(2) < \dots < m_i'(h_i) \le p_i',$$

We further put $m_i(j) = m_i'(j) + \sum_{r=1}^{j} \ell_i(r)$. Let $\mathcal{S}_i$ be a simple sheaf on $\mathbf{X}$ concentrated at λ_i. We can select mutually non-adjacent segments

$$\mathcal{I}_i(j) = \left\{\mathcal{S}_i(j), \tau_{\mathbf{X}}\mathcal{S}_i(j), \dots, \tau_{\mathbf{X}}^{\ell_i(j)-1}\mathcal{S}_i(j)\right\}, \quad j = 1, \dots, h_i$$

of simple sheaves on $\mathbf{X}$ concentrated at λ_i, where $\mathcal{S}_i(j) = \tau_{\mathbf{X}}^{-m_i(j)}\mathcal{S}_i$.

Now, the perpendicular category

$$^{\perp}\{\mathcal{I}_i(j),\ i = 1, \dots, t,\ j \in J'(i)\} \cap \{\mathcal{I}_i(j),\ i = 1, \dots, t,\ j \in J''(i)\}^{\perp}$$

is equivalent to $\mathrm{coh}(\mathbf{Y})$ and we will identify these two categories. Recall that Σ_+ is realized by a tilting bundle T_+ on $\mathbf{Y}$.

Note further that, for $j \in J''(j)$, the object $\mathcal{S}_i(j)^{[\ell_i(j)+1]}$ belongs to $\mathrm{coh}(\mathbf{Y})$ and agrees with the simple sheaf $\mathcal{S}_i'(j)$ from $\mathrm{coh}(\mathbf{Y})$, and dually, for $j \in J'(i)$, the object $\tau_{\mathbf{X}}^{-}\mathcal{S}_i(j)^{[\ell_i(j)+1]}$ belongs to $\mathrm{coh}(\mathbf{Y})$ and agrees with the simple sheaf $\mathcal{S}_i'(j)$ from $\mathrm{coh}(\mathbf{Y})$.

Denote by $\mathcal{C}_i(j)$ the subcategory of $\mathrm{coh}(\mathbf{X})$ generated by $\mathcal{I}_i(j)$. Then $\mathcal{C}_i(j)$ can be identified with the module category over a path algebra for a Dynkin quiver $Q_i(j)$ of type $\mathbb{A}_{\ell_i(j)}$ with linear orientation.

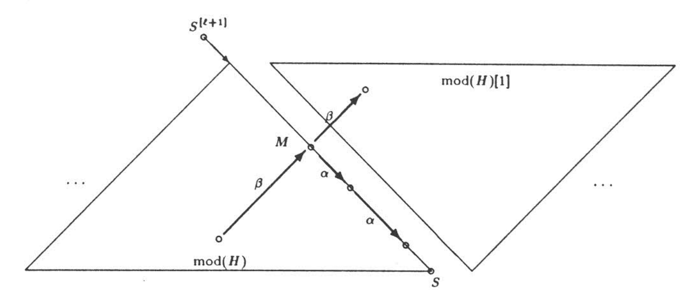
According to [1] and [44] each extension branch $\mathcal{B}_i'(j)$ (resp. coextension branch $\mathcal{B}_i''(j)$) can be realized as a tilting complex $U_i(j)$ (resp. $V_i(j)$) in $\mathcal{D}^b(\mathrm{mod}(kQ_i(j))$.

Let $j \in J''(i)$. For simplicity of notation we write $kQ_i(j) = H$, $\mathcal{B}_i''(j) = \mathcal{B}$, $\ell_i(j) = \ell$, $V_i(j) = V$ and $\mathcal{S}_i(j) = \mathcal{S}$. We can assume that V is of the form

$$V = V^{(-m)}[-m] \oplus \dots \oplus V^{(-1)}[-1] \oplus V^{(0)}[0] \oplus V^{(1)}[1] \dots V^{(m')}[m'],$$

where all $V^{(r)}$ are in $\mathrm{mod}(H)$, the coextension root a is represented by a module, say M, which is a direct summand of $V^{(0)}$, and also the vertex corresponding to α, if it exists, has this property. Note that M is an injective H-module. Further, for an indecomposable direct summand N of $V^{(0)}$ such that $\mathrm{Hom}(M, N) \neq 0$, N is injective in $\mathrm{mod}(H)$ and the vertex of N in the coextension branch corresponds to some α^n, $n \ge 0$. Conversely, each vertex of $\mathcal{B}$ of the form α^n, $n \ge 0$, is represented by an indecomposable direct summand of $V^{(0)}$ which is an injective H-module.

The situation is illustrated in the following picture



Dually, we can realize each extension branch $\mathcal{B}_i'(j)$ by a tilting complex $U = U_i(j)$ in $\mathcal{D}^b(\mathrm{mod}(kQ_i(j))$ of the form

$$U = U^{(-n)}[-n] \oplus \dots \oplus U^{(-1)}[-1] \oplus U^{(0)}[0] \oplus U^{(1)}[1] \dots U^{(n')}[n']$$

where all $U^{(r)}$ are in $\mathrm{mod}(kQ_i(j))$, the extension root a is represented by an object, say $M'[-1]$, which is an indecomposable direct summand of $U^{(-1)}[-1]$, and also the vertex corresponding to α^{-1}, if it exists, has this property.

We claim that

$$\Omega = \bigoplus_{i=1,\dots,t,\ j\in J'(i)} \mathcal{B}_i'(j) \oplus T_+ \oplus \bigoplus_{i=1,\dots,t,\ j\in J''(i)} \mathcal{B}_i''(j)$$

is a tilting complex on $\mathbf{X}$ such that $\mathrm{End}(\Omega)$ is isomorphic to Σ.

First, by the choice of the $\mathcal{S}_i(j)$ we have $\mathrm{Hom}_{\mathcal{D}}(E, E'[n]) = 0$, for $n \in \mathbb{Z}$, if E and E' are indecomposable direct summands of T of finite length belonging to different extension or coextension branches. Therefore in order to show that T satisfies the vanishing condition for a tilting complex it is enough to prove the following assertions.

(a) $\mathrm{Ext}^1_{\mathbf{X}}(E, T_+) = 0$ for each summand E of $V^{(s)}$ for all s and all $V = V_i(j)$.
(b) $\mathrm{Hom}_{\mathbf{X}}(T_+, E) = 0$ for each summand E of $V^{(s)}$ for $s \neq 0$ and all $V = V_i(j)$.
(c) $\mathrm{Hom}_{\mathbf{X}}(E, T_+) = 0$ for each summand E of $U^{(s)}$ for all s and all $U = U_i(j)$.
(d) $\mathrm{Ext}^1_{\mathbf{X}}(E, T_+) = 0$ for each summand E of $U^{(s)}$ for $s \neq -1$ and all $U = U_i(j)$.

Assertion (a) is a consequence of the fact that T_+ is in $\{\mathcal{I}_i(j),\ i = 1, \dots, t,\ j \in J''(i)\}^{\perp}$.

Now, if E is an indecomposable direct summand of some $V^{(s)}$ with $s \neq 0$, then E is not injective in $\mathrm{mod}(H)$. Furthermore, there is an exact sequences in $\mathrm{mod}(H)$ of the form

$$0 \to^{[\ell']} (\tau_{\mathbf{X}}^{b}\mathcal{S}) \to E \to \tau_{\mathbf{X}}^{a}\mathcal{S} \to 0$$

with $1 \le a \le b \le \ell - 1$ and $1 \le \ell' \le \ell$. Each $F \in \mathrm{vect}(\mathbf{Y})$ satisfies $0 = \mathrm{Ext}^1_{\mathbf{X}}(\tau_{\mathbf{X}}^{c}\mathcal{S}, F) \cong D\mathrm{Hom}_{\mathbf{X}}(F, \tau_{\mathbf{X}}^{c+1}\mathcal{S})$ for $0 \le c \le \ell - 1$, thus applying the functor $\mathrm{Hom}_{\mathbf{X}}(T_+, -)$ to the exact sequences above we obtain $\mathrm{Hom}_{\mathbf{X}}(T_+, E) = 0$, which proves (b). Dually one shows (c) and (d).

Observe that Ω consists of pairwise non-isomorphic indecomposable objects whose number agrees with the rank of the Grothendieck group $\mathrm{K}_0(\mathbf{X})$. Consequently, Ω is a tilting complex on $\mathbf{X}$.

Finally, $\mathrm{End}(\Omega) \cong \Sigma$ follows from the definition of a branch enlargement and the following facts:

(e) The natural epimorphism $\mathcal{S}^{[\ell+1]} \twoheadrightarrow \mathcal{S}^{[h]}$ induces an isomorphism $\mathrm{Hom}_{\mathbf{X}}(F, \mathcal{S}^{[\ell+1]}) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{X}}(F, \mathcal{S}^{[h]})$, for each $F \in \mathrm{vect}(\mathbf{Y})$, $1 \le h \le \ell$ and for $\mathcal{S} = \mathcal{S}_i(j)$, $j \in J''(i)$.

(f) $\mathrm{Hom}_{\mathbf{X}}(T_+, E) = 0$ for each indecomposable direct summand E of $V^{(0)} = V_i^{(0)}(j)$ which does not correspond to a vertex of the form α^n, $n \ge 0$, in the corresponding coextension branch.

(g) The natural monomorphism ${}^{[h]}(\tau_{\mathbf{X}}^{\ell-1}\mathcal{S}) \hookrightarrow \tau_{\mathbf{X}}^{-}\mathcal{S}^{[\ell+1]}$ induces an isomorphism $\mathrm{Ext}^1_{\mathbf{X}}(\tau_{\mathbf{X}}^{-}\mathcal{S}^{[\ell+1]}, F) \xrightarrow{\cong} \mathrm{Ext}^1_{\mathbf{X}}({}^{[h]}(\tau_{\mathbf{X}}^{\ell-1}\mathcal{S}), F)$, for each $F \in \mathrm{vect}(\mathbf{Y})$, $1 \le h \le \ell$ and for $\mathcal{S} = \mathcal{S}_i(j)$, $j \in J'(i)$.

(h) $\mathrm{Ext}^1_{\mathbf{X}}(E, T_+) = 0$ for each indecomposable direct summand E of $U^{(-1)} = U_i^{(-1)}(j)$ which does not correspond to a vertex of the form α^{-n}, $n \ge 0$ in the corresponding extension branch.

Assertion (e) follows by applying the functor $\mathrm{Hom}_{\mathbf{X}}(F, -)$ to the exact sequence

$$0 \to {}^{[\ell+1-h]}(\tau_{\mathbf{X}}^{\ell}\mathcal{S}) \to \mathcal{S}^{[\ell+1]} \to \mathcal{S}^{[h]} \to 0.$$

Indeed, we have $0 = \mathrm{Ext}^1_{\mathbf{X}}(\mathcal{S}^{[\ell]}, F) \cong \mathrm{DHom}(F, \tau_{\mathbf{X}}\mathcal{S}^{[\ell]})$. Therefore the monomorphism ${}^{[\ell+1-h]}(\tau^{\ell}\mathcal{S}) \hookrightarrow \tau_{\mathbf{X}}\mathcal{S}^{[\ell]}$ yields $\mathrm{Hom}_{\mathbf{X}}(F, {}^{[\ell+1-h]}\tau_{\mathbf{X}}^{\ell}\mathcal{S}) = 0$. On the other hand we have $\mathrm{Ext}^1_{\mathbf{X}}(F, {}^{[\ell+1-h]}\tau_{\mathbf{X}}^{\ell}\mathcal{S}) = \mathrm{DHom}_{\mathbf{X}}({}^{[\ell+1-h]}\tau_{\mathbf{X}}^{\ell-1}\mathcal{S}, F) = 0$, and (e) follows.

The assertion (f) follows by the same kind of arguments as in the proof of (b). Dually one proves (g) and (h) which completes the proof of assertion (ii).

(i) Assume that T is of the form

$$T = T_{-n_1}[-n_1] \oplus \ldots \oplus T_{-1}[-1] \oplus T_+ \oplus T_0 \oplus T_1[1] \oplus \ldots \oplus T_{n_2}[n_2]$$

with $T_+ \in \mathrm{vect}(\mathbf{X})$ and $T_i \in \mathrm{coh}_0(\mathbf{X})$ for all $-n_1 \le i \le n_2$. Then the perpendicular category

$$^{\perp}\{T_{-n_1}, \ldots, T_{-1}\} \cap \{T_0, T_1, \ldots, T_{n_2}\}^{\perp}$$

is equivalent to a category of coherent sheaves on a weighted projective line $\mathbf{Y}$ and T_+ can be considered as a tilting bundle on $\mathbf{Y}$.

The finite length parts are, by [1], tilting complexes $\mathcal{B}_i(j)$ in wings of the tubes of $\mathrm{coh}_0(\mathbf{X})$ in such a way that their segments of simple objects are non-adjacent. Accordingly the $\mathcal{B}_i(j)$ are branches and the statement follows in the same way as in the proof of (i) from the definition of a branch enlargement. □

Remark 8.3.3 *Let Σ be an algebra obtained from a concealed-canonical algebra Σ_+ by branch enlargement. Then the algebra Σ_+ is uniquely determined, provided the weighted projective line $\mathbf{X}$ is not tubular.*

Proof. The statement follows from the uniqueness, up to sign, of the rank function [75, Lemma 2.5]. □

8.3.4 For a tilting complex $T = \bigoplus_n T_n[n]$ such that $T_n \in \mathrm{coh}(\mathbf{X})$, with endomorphism ring $\Sigma = \mathrm{End}(T)$, an indecomposable object X of $\mathcal{D}^b(\mathrm{coh}(\mathbf{X}))$ belongs to $\mathrm{mod}(\Sigma)$ if and only if

$$\mathrm{Hom}_{\mathcal{D}}(T, X[n]) = 0 \qquad \text{for each integer} \quad n \ne 0.$$

Since each indecomposable object in $\mathcal{D}^b(\mathrm{coh}(\mathbf{X}))$ is, up to translation, a coherent sheaf, $\mathrm{mod}(\Sigma)$ is the additive closure of all

$$\mathrm{mod}^{[j]}(\Sigma) = \{X[j] \in \mathrm{coh}(\mathbf{X})[j] \mid \mathrm{Hom}_{\mathbf{X}}(T_i, X) = 0,\ i \ne j,\ \mathrm{Ext}^1_{\mathbf{X}}(T_i, X) = 0,\ i \ne j-1\}.$$

For a branch enlargement of a concealed canonical algebra we obtain the following global description of the module category.

Proposition 8.3.4 *Let $T = T_{-n_1}[-n_1] \oplus \ldots \oplus T_{-1}[-1] \oplus T_+ \oplus T_0 \oplus T_1[1] \oplus \ldots \oplus T_{n_2}[n_2]$ be a tilting complex on a weighted projective line $\mathbf{X}$ with $T_+ \in \mathrm{vect}(\mathbf{X})$ and $T_i \in \mathrm{coh}_0(\mathbf{X})$ for all $-n_1 \le i \le n_2$ and let Σ be the endomorphism algebra of T.*

Then each indecomposable Σ-module belongs to one of the following subcategories

(a) $\mathrm{mod}_0^{[j]}(\Sigma)$ consisting of all $E[j]$, where $E \in \mathrm{coh}_0(\mathbf{X})$ satisfies $\mathrm{Hom}_{\mathbf{X}}(T_+, E) = 0$, $\mathrm{Hom}_{\mathbf{X}}(T_i, E) = 0$ for $i \ne j$, $\mathrm{Ext}^1_{\mathbf{X}}(T_i, E) = 0$ for $i \ne j-1$; $j = -n_1, \ldots, -1$,

(b) $\mathrm{mod}_+(\Sigma)$ consisting of all $E \in \mathrm{vect}(\mathbf{X})$ satisfying $\mathrm{Ext}^1_{\mathbf{X}}(T_+, E) = 0$, $\mathrm{Ext}^1_{\mathbf{X}}(T_i, E) = 0$ for $i \ne -1$,

(c) $\mathrm{mod}_0(\Sigma)$ consisting of all $E \in \mathrm{coh}_0(\mathbf{X})$ satisfying $\mathrm{Hom}_{\mathbf{X}}(T_i, E) = 0$ for $i \ne 0$, $\mathrm{Ext}^1_{\mathbf{X}}(T_i, E) = 0$ for $i \ne -1$,

(d) $\mathrm{mod}_-(\Sigma)$ consisting of all $E[1]$, where $E \in \mathrm{vect}(\mathbf{X})$ satisfies $\mathrm{Hom}_{\mathbf{X}}(T_+, E) = 0$, $\mathrm{Ext}^1_{\mathbf{X}}(T_i, E) = 0$ for $i \ne 0$.

(f) $\mathrm{mod}_0^{[j]}(\Sigma)$, consisting of all $E[j]$, where $E \in \mathrm{coh}_0(\mathbf{X})$ satisfies $\mathrm{Hom}_{\mathbf{X}}(T_+, E) = 0$, $\mathrm{Hom}_{\mathbf{X}}(T_i, E) = 0$ for $i \ne j$, $\mathrm{Ext}^1_{\mathbf{X}}(T_i, E) = 0$ for $i \ne j-1$; $j = 1, \ldots, n_2+1$.

The subcategories $\mathrm{mod}_0^{[j]}(\Sigma)$, $j = -n_1, \ldots, -1, 1, \ldots, n_2+1$, have only finitely many indecomposable objects.

Moreover, the objects of the additive closure from T (resp. $\tau_{\mathbf{X}}T[1]$) are the projective (resp. injective) Σ-modules.

Further, in the ordering $\mathrm{mod}_0^{[-n_1]}(\Sigma), \ldots, \mathrm{mod}_0^{[-1]}(\Sigma), \mathrm{mod}_+(\Sigma), \mathrm{mod}_0(\Sigma), \mathrm{mod}_-(\Sigma), \mathrm{mod}_0^{[1]}(\Sigma), \ldots, \mathrm{mod}_0^{[n_2+1]}(\Sigma)$ there are no nonzero morphisms from the right to the left. □

Proof. Using the notations above, each indecomposable Σ-module belongs to one of the subcategories $\mathrm{mod}^{[j]}(\Sigma)$. Let us denote

$$\begin{aligned} \mathrm{mod}_0^{[j]}(\Sigma) &= \mathrm{coh}_0(\mathbf{X}) \cap \mathrm{mod}^{[j]}(\Sigma) \\ \mathrm{mod}_+(\Sigma) &= \mathrm{vect}(\mathbf{X}) \cap \mathrm{mod}^{[0]}(\Sigma) \\ \mathrm{mod}_-(\Sigma) &= \mathrm{vect}(\mathbf{X}) \cap \mathrm{mod}^{[1]}(\Sigma). \end{aligned}$$

For simplicity of notation we write $\mathrm{mod}_0(\Sigma)$ instead of $\mathrm{mod}_0^{[0]}(\Sigma)$.

Since there are no morphisms from finite length sheaves to vector bundles, we obtain the desired characterizations for $\mathrm{mod}_+(\Sigma)$, mod_0, $\mathrm{mod}_-(\Sigma)$ and $\mathrm{mod}_0^{[1]}$. It is also clear that $\mathrm{mod}_0^{[1]}(\Sigma)$ has only finitely many indecomposables.

We show that, for $j \neq 0, 1$, the subcategories $\mathrm{mod}_0^{[j]}(\Sigma)$ have only finitely many indecomposable objects and that they all are, up to translation, finite length sheaves.

According to 2.4, for $j = -n_1, \dots, -1$, the subcategory $\mathrm{mod}_0^{[j]}(\Sigma)$ is contained in the perpendicular category

$$\mathcal{A} = (T_+ \oplus T_0 \oplus T_1 \oplus \dots \oplus T_{n_2})^\perp,$$

which is equivalent to a module category $\mathrm{mod}(H)$ for a hereditary algebra H. We see also that there is a tilting complex

$$T' = T_{-n_1}[-n_1] \oplus \dots \oplus T_{-1}[-1]$$

in $\mathcal{D}^b(\mathcal{A})$. Now, the endomorphism algebra of T' consists of branches which implies that H is a product of algebras $H_1 \times \dots \times H_r$, where H_i is the path algebra of a Dynkin quiver of type $\mathbb{A}_{m_i}$, $i = 1, \dots, r$. Consequently, each $\mathrm{mod}^{[j]}(\Sigma)$ has only finitely many indecomposables. Furthermore, for each i, the objects in $\mathrm{mod}(H_i)$ corresponding to finite length sheaves form a successor closed subcategory having the shape of a wing $\mathcal{W}_i$. But $\mathrm{mod}(H_i) = \mathcal{W}_i$, because otherwise T' cannot be a tilting complex for H. Thus $\mathrm{mod}^{[j]}(\Sigma)$ consists only of objects of the form $E[j]$ with $E \in \mathrm{coh}_0(\mathbb{X})$. Finally, it is easy to see that they are characterized by the conditions given in the theorem.

For $j = 2, \dots, n_2 + 1$, the subcategory $\mathrm{mod}_0^{[j]}(\Sigma)$ is contained in the perpendicular category $(T_{-n_1} \oplus \dots \oplus T_{-1} \oplus T_0 \oplus T_+)^\perp$ and we can proceed analogously.

The other statements are obvious. □

8.3.5 We define

$$\Sigma_l = \mathrm{End}(T_{-n_1}[-n_1] \oplus \dots \oplus T_{-1}[-1] \oplus T_+),$$
$$\Sigma_r = \mathrm{End}(T_+ \oplus T_0 \oplus T_1[1] \oplus \dots \oplus T_{n_2}[n_2]).$$

Obviously Σ_l and Σ_r are full convex subcategories of Σ. The following proposition follows easily from the Proposition 8.3.4.

Proposition 8.3.5 *The support of an indecomposable Σ-module either belongs to Σ_l or else to Σ_r.* □

Corollary 8.3.5 *Σ is tame if and only both Σ_l and Σ_r are tame.* □

8.3.6 We show that an algebra Σ obtained from a concealed-canonical algebra by branch enlargement admits a separating family of standard (see [100]) tubes. This generalizes results of [75], [78] and [79].

Let $\mathrm{mod}^l(\Sigma)$ (resp. $\mathrm{mod}^r(\Sigma)$) denote the additive closure of the union of the subcategories $\mathrm{mod}_0^{[j]}(\Sigma)$, $j = -n_1, \dots, -1$ and $\mathrm{mod}_+(\Sigma)$ (resp. $\mathrm{mod}_-(\Sigma)$ and $\mathrm{mod}_0^{[j]}(\Sigma)$, $j = 1, \dots, n_2 + 1$). Then each indecomposable Σ-module belongs to exactly one of the subcategories $\mathrm{mod}^l(\Sigma)$, $\mathrm{mod}_0(\Sigma)$, $\mathrm{mod}^r(\Sigma)$ and in this ordering there are no nonzero morphisms from the right to the left.

Theorem 8.3.6 *Let Σ be an algebra obtained from a concealed-canonical algebra by branch enlargement.*

(i) The indecomposables from $\mathrm{mod}_0(\Sigma)$ form a family of standard tubes.

(ii) Each morphism from a module M of $\mathrm{mod}^l(\Sigma)$ to a module N of $\mathrm{mod}^r(\Sigma)$ factors through a module U from $\mathrm{mod}_0(\Sigma)$.

(iii) Each Auslander-Reiten component of $\mathrm{mod}(\Sigma)$ has support in Σ_l or Σ_r.

Proof. (i) According to Proposition 8.3.4, $\mathrm{mod}_0(\Sigma)$ is obtained from $\mathrm{coh}_0(\mathbb{X})$ by ray and coray deletion, which implies (i).

(ii) Let $f : M \to N$ be a nonzero morphism with $M \in \mathrm{mod}^l(\Sigma)$ and $N \in \mathrm{mod}^r(\Sigma)$. Clearly we can assume that M and N are indecomposable. Then M belongs to $\mathrm{mod}_+(\Sigma)$ and N to $\mathrm{mod}_-(\Sigma)$.

Now we can follow the arguments of [29, Corollary 2.7, Proposition 4.3]. Write $M = G$ and $N = F[1]$ for $F, G \in \mathrm{coh}(\mathbb{X})$. There is an exact sequence

$$0 \to F \to \overline{F} \to E \to 0$$

such that $\mathrm{Ext}^1_{\mathbb{X}}(G, \overline{F}) = 0$ and E belongs to a fixed component $\mathcal{C}$ of $\mathrm{coh}_0(\mathbb{X})$. Applying the functor $\mathrm{Hom}_{\mathbb{X}}(G, -)$ we see that $f \in \mathrm{Hom}_\Sigma(M, N) = \mathrm{Ext}^1_{\mathbb{X}}(G, F)$ can be lifted to a morphism $u \in \mathrm{Hom}(G, E)$. Obviously, if $\mathcal{C}$ is a component which does not contain direct summands of T then E belongs to $\mathrm{mod}_0(\Sigma)$.

(iii) follows from Proposition 8.3.5 and the fact that the indecomposables from $\mathrm{mod}_0(\Sigma)$ form a union of full components. □

Observe that a component of $\mathrm{mod}_0(\Sigma)$ can contain both projective and injective modules.

8.4 Tilting complexes for domestic weighted projective lines

8.4.1 The following theorem deals with algebras derived equivalent to tame hereditary algebras.

Theorem 8.4.1 *Let T be a tilting complex on a domestic weighted projective line $\mathbb{X}$ with endomorphism ring Σ. Then the following conditions are equivalent:*

(i) T is, up to translation in the derived category, of the form the (*) (see 8.3.1).

(ii) Σ is a obtained from a concealed-canonical algebra by branch enlargement.

(iii) Σ is representation-infinite.

Proof. (i) $\Leftrightarrow$ (ii) by Theorem 8.3.2.

(ii) $\Rightarrow$ (iii) By 7.1.3 the endomorphism algebra of a tilting bundle $\Sigma_+ = \mathrm{End}(T_+)$ is representation-infinite. Hence Σ is representation-infinite, because it contains Σ_+ as a convex full subcategory.

(iii) $\Rightarrow$ (i) Since Σ is of domestic weight type, the derived category $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ is equivalent to the derived category $\mathcal{D}^b(\mathrm{mod}(A))$ of modules for a tame hereditary algebra $A = kQ$. Therefore $\mathrm{ind}(\Sigma)$ is the union of all

$$\mathrm{ind}^{[n]}(\Sigma) = \{X = M[n], M \in \mathrm{ind}(A) \mid \mathrm{Hom}_{\mathcal{D}}(T, M[n+i]) \neq 0 \text{ for all } i \neq 0\}.$$

Observe that only finitely many $\mathrm{ind}^{[n]}(\Sigma)$ are nonzero.

Suppose that T has a direct summand of the form $E[m]$ for some preprojective A-module E and $m \in \mathbb{Z}$. We claim that in this case $\mathrm{ind}^{[n]}(\Sigma)$ is finite for $n \neq m$. For this it suffices to show that $\mathrm{Hom}(E, M) \neq 0$ for almost all indecomposable A-modules M. We further can assume that E is a projective A-module, corresponding to a vertex x. Then the quiver obtained from Q by removing x is a disjoint union of Dynkin quivers and the fact above follows.

Similarly, if T has a direct summand of the form $E'[m']$ for some preinjective A-module E' and some $m' \in \mathbb{Z}$, then $\mathrm{ind}^{[n]}(\Sigma)$ is finite for $n \neq m' + 1$.

Now, assume that T is not, up to translation, of the form (*). Then T has indecomposable direct summands $E[i]$ and $E'[j]$ satisfying one of the following three conditions:

(a) $j \neq i, i+1$,

(b) $j = i$ and E is preprojective and E' is preinjective.

(b) $j = i+1$ and both E, E' are preprojective or preinjective, or E is preprojective and E' is preinjective.

In each case we conclude that all $\mathrm{ind}^{[n]}(\Sigma)$ are finite, which gives a contradiction. □

8.5 Tilting complexes for tubular weighted projective lines

8.5.1 Let $T = \oplus_{i=1}^{n} T_i$ be a tilting complex on a tubular weighted projective line $\mathbb{X}$ such that all T_i are indecomposable in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$. Set $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. On the set $\overline{\mathbb{Q}}$ we have the natural order: $(q_1, n_1) \leq (q_2, n_2)$ if and only if $n_1 < n_2$ or $n_1 = n_2$ and $q_1 \leq q_2$. We consider the function associating to each T_i the slope and the copy number

$$\nu : \{1, \ldots, n\} \to \overline{\mathbb{Q}} \times \mathbb{Z}, \qquad \nu(i) = (q_i, n_i)$$

where $T_i = E_i[n_i]$ for a coherent sheaf E_i of slope q_i. Let (q_b, n_b) (resp. (q_e, n_e)) be the minimum (resp. maximum) of the image of ν.

Definition 8.5.1 *We say that the tilting complex T is in bad position if it satisfies the following two conditions:*

(a) There are i_1, i_2 such that

$$(q_b, m_1 - 1) < \nu(i_1) < (q_b, m_1) \leq (q_b, m_2 - 1) < \nu(i_2) < (q_b, m_2)$$

for some $m_1, m_2 \in \mathbb{Z}$.

(b) There are j_1, j_2 such that

$$(q_e, m_1' - 1) < \nu(j_1) < (q_e, m_1') \leq (q_e, m_2' - 1) < \nu(j_2) < (q_e, m_2')$$

for some $m_1', m_2' \in \mathbb{Z}$.

Otherwise T is called in good position.

Observe that under the assumption $q_b = \infty$ (resp. $q_e = \infty$) condition (a) (resp. condition (b)) is nothing then the statement that T contains vector bundles from different copies of $\mathrm{coh}(\mathbb{X})$ in the derived category.

Let us denote by T_b (resp. T_e) the direct sum of all summands T_i of T such that $\nu(i) = (q_b, n_b)$ (resp. $\nu(i) = (q_e, n_e)$) and consider the decomposition $T = T_b \oplus T''$ (resp. $T = T' \oplus T_e$). Finally, we write $\Sigma = \mathrm{End}(T)$, $\Sigma' = \mathrm{End}(T')$ and $\Sigma'' = \mathrm{End}(T'')$.

8.5.2 The following two theorems deal with algebras derived equivalent to tubular algebras.

Theorem 8.5.2 *Let T be a tilting complex on a tubular weighted projective line $\mathbb{X}$ with endomorphism ring Σ. Then the following conditions are equivalent:*

(i) T is in bad position.

(ii) Σ' and Σ'' are representation-finite.

(iii) Σ is representation-finite.

Proof. (i) $\Leftrightarrow$ (ii) By means of an automorphism of $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ (see Chapter 4) we can assume that $q_b = \infty$ and $n_b = 0$. In this case the indecomposable objects of T_b form tilting objects in wings which are contained in exceptional tubes of $\mathrm{coh}_0(\mathbb{X})$. Then the left perpendicular category ${}^{\perp}T_b$ is equivalent to a sheaf category $\mathrm{coh}(\mathbb{Y})$ for some weighted projective line $\mathbb{Y}$ of domestic weight type and T'' can be considered as a tilting complex in $\mathcal{D}^b(\mathrm{coh}(\mathbb{Y}))$.

By 8.4.1, T'' has indecomposable summands $E[n]$ and $E'[n']$, with $E, E' \in \mathrm{vect}(\mathbb{Y})$ and $n \neq n'$, if and only if Σ'' is representation-finite. This proves that condition (a) is equivalent to the fact that Σ'' is representation-finite.

Dually one shows that condition (b) is satisfied if and only if Σ' is representation-finite.

(ii) $\Rightarrow$ (iii) Applying if necessary, an automorphism of $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ we can again assume that $q_b = \infty$ and $n_b = 0$. Then $T = T_0 \oplus T_1[1] \oplus \ldots \oplus T_r[r]$ with $T_b = T_0 \in \mathrm{coh}_0(\mathbb{X})$ and $T_i \in \mathrm{coh}(\mathbb{X})$ for $1 \leq i \leq r$. We have $r \geq 2$, because otherwise T'' is, up to translation, a tilting sheaf and then Σ'' is an almost concealed-canonical algebra. This implies that Σ'' is representation-infinite, contrary to our assumption.

We claim that the support of each indecomposable Σ-module belongs either to Σ' or to Σ''. Indeed, each indecomposable Σ-module belongs to a subcategory $\mathrm{mod}^{[j]}(\Sigma) = \mathrm{coh}^{[j]}(T)[j]$ where $\mathrm{coh}^{[j]}(T) = \{F \in \mathrm{coh}(\mathbb{X}) \mid \mathrm{Hom}_{\mathbb{X}}(T_i, F) = 0 \text{ for } i \neq j,\ \mathrm{Ext}^1_{\mathbb{X}}(T_i, F) = 0 \text{ for } i \neq j-1\}$. Now, for $j \geq 2$ the conditions $\mathrm{Hom}_{\mathbb{X}}(T_0, F) = 0$ and $\mathrm{Ext}^1_{\mathbb{X}}(T_0, F) = 0$ imply that the support of each indecomposable $M \in \mathrm{mod}^{[j]}(\Sigma)$ belongs to Σ''. On the other hand, for $j \leq 1$, we conclude from $\mathrm{Hom}_{\mathbb{X}}(T_r, F) = 0$ and $\mathrm{Ext}^1_{\mathbb{X}}(T_r, F) = 0$ that the support of each indecomposable $M \in \mathrm{mod}^{[j]}(\Sigma)$ belongs to Σ'. Consequently Σ is representation-finite.

(iii) $\Rightarrow$ (ii) is obvious. □

Theorem 8.5.3 *Let T be a tilting complex on a tubular weighted projective line $\mathbb{X}$ with endomorphism ring Σ. Then the following conditions are equivalent:*

(i) T is in good position.

(ii) There is an automorphism $\Phi \in \mathrm{Aut}(\mathcal{D}^b(\mathrm{coh}(\mathbb{X})))$ such that $\Phi(T)$ is of the form (*) (see 8.3.1).

(iii) Σ is obtained from a concealed-canonical algebra by branch enlargement.
(iv) Σ is representation-infinite.

Proof. (i) $\Rightarrow$ (ii) Suppose first that T is a tilting complex which does not satisfy the condition (a). Let (q_b, n_b) be the minimum of the image of ν for T. Then $T = \Phi(\overline{T})$ for some tilting complex $\overline{T}$, having as minimum for the image of ν the pair $(\infty, 0)$, and an automorphism $\Phi \in \mathrm{Aut}(\mathcal{D}^b(\mathrm{coh}(\mathbb{X})))$ being the composition of the n_b-th power of the translation functor and a telescopic functor $\Phi q_b, \infty$.

Now, if the image of ν for $\overline{T}$ contains two pairs (q_1, n_1), (q_2, n_2) with $q_1 \neq \infty$, $q_2 \neq \infty$, then $n_1 = n_2$. This is a consequence of the fact that Φ preserves the order on $\overline{\mathbb{Q}}$. Therefore the objects of $\overline{T}$ corresponding to vector bundles are in the same copy of $\mathrm{coh}(\mathbb{X})$ in the derived category, and consequently $\overline{T}$ is of the form (*).

Similarly, in case T does not satisfy the condition (b) one shows that $T = \Phi(\overline{T})$ for some tilting complex $\overline{T}$ of the form (*) and an automorphism $\Phi \in \mathrm{Aut}(\mathcal{D}^b(\mathrm{coh}(\mathbb{X})))$ being the composition of the n_e-th power of the translation functor and a telescopic functor $\Phi q_e, \infty$.

(ii) $\Rightarrow$ (iii) follows from Theorem 8.3.2.

(iii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (i) follows from Theorem 8.5.2. □

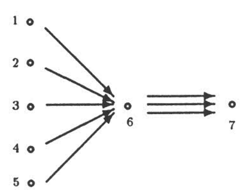
8.5.4 Using other methods algebras derived equivalent to tubular algebras were studied by Barot and de la Peña [7] and by Barot [6].

8.5.5 We have seen that for a domestic or tubular weighted projective line $\mathbb{X}$ the endomorphism ring of a tilting complex which contains vector bundles from different copies of $\mathrm{coh}(\mathbb{X})$ (in the tubular case after application of an automorphism) is representation-finite. The following example shows that this is no longer true if $\mathbb{X}$ is wild.

Example. Let $\mathbb{X}$ be a weighted projective line of weight type $(2,2,2,2,2)$. The complex

$$T = \bigoplus_{i=1}^{5} \mathcal{O}(\vec{x}_i) \oplus \mathcal{O}(\vec{c}) \oplus \mathcal{O}(-\vec{\omega})[1]$$

is a tilting complex. Observe that T is obtained from the canonical tilting sequence by mutation of $\mathcal{O}$ to the right end. The quiver of $\Sigma = \mathrm{End}(T)$ is the following



while the relations involve the parameters (note that $\dim_k \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{x}_i), \mathcal{O}(-\vec{\omega})) = 2$ for $i = 1, \ldots, 5$). Obviously Σ is wild. □

Chapter 9

Hyperelliptic weighted projective lines

In this chapter we will study tilting complexes T and their endomorphism rings Σ for weighted projective lines of type $(2, \ldots, 2)$, t entries, in detail. We are mainly interested in the hyperelliptic case, however we allow also that $t \leq 4$. First we describe properties which are independent of the representation type, later we will turn our attention to those tilting complexes T having tame endomorphism rings.

9.1 Structure of a tilting complex on a hyperelliptic weighted projective line

9.1.1 In our further investigation the Riemann-Roch theorem will play an important role. We can rephrase Proposition 2.3.5 for a partial tilting sequence of length 2 in the derived category in the following way.

Proposition 9.1.1 *Let $\mathbb{X}$ be a weighted projective line of type $(2, \ldots, 2)$ and let (X, Y) be a partial tilting pair in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$. Then*

$$\dim_k \mathrm{Hom}_{\mathcal{D}}(X, Y) = \begin{vmatrix} \mathrm{rk}(X) & \mathrm{rk}(Y) \\ \deg(X) & \deg(Y) \end{vmatrix}.$$

Proof. Up to translation in the derived category we have the following three possibilities: (a) $X, Y \in \mathrm{coh}(\mathbb{X})$, (b) $X \in \mathrm{coh}(\mathbb{X})$ and $Y \in \mathrm{coh}(\mathbb{X})[1]$, (c) $X \in \mathrm{coh}(\mathbb{X})$ and $Y \in \mathrm{coh}(\mathbb{X})[n]$, for some $n \neq 0, 1$.

In the case (a) the result is proved in Proposition 2.3.5.

To deduce the formula in the case (b), assume that $Y = Z[1]$ for some $Z \in \mathrm{coh}(\mathbb{X})$. Then $\mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathrm{Ext}^1_{\mathbb{X}}(X, Z)$. Furthermore from Proposition 2.3.5 we conclude that

$$-\dim_k \mathrm{Ext}^1_{\mathbb{X}}(X, Z) = \begin{vmatrix} \mathrm{rk}(X) & \mathrm{rk}(Z) \\ \deg(X) & \deg(Z) \end{vmatrix}.$$

On the other hand $\mathrm{rk}(Z) = -\mathrm{rk}(Y)$ and $\deg(Z) = -\deg(Y)$ which establishes the formula.

Finally, in the case (c) we assume that $Y = Z[n]$ for some $Z \in \mathrm{coh}(\mathbb{X})$ and some integer $n \neq 0, 1$. In this situation we have $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ by the heredity of $\mathrm{coh}(\mathbb{X})$. Now if $\mu(X) < \mu(Z)$, we would have $\mathrm{Hom}_{\mathbb{X}}(X, Z) \neq 0$ by Proposition 2.3.5 (ii), contrary to the fact that (X, Y) is a partial tilting pair. Furthermore, by Proposition 2.3.5 (iii), $\mu(X) > \mu(Z)$ implies $\mathrm{Ext}^1_{\mathbb{X}}(X, Z) \neq 0$, which again gives a contradiction. Therefore $\mu(X) = \mu(Z) = \mu(Y)$. We conclude from Theorem 3.5.1 that, up to sign, $\mathrm{rk}(X) = \mathrm{rk}(Y)$ and $\deg(X) = \deg(Y)$, and the right hand side of the formula vanishes, too. □

Corollary 9.1.1 *Assume that A and B are exceptional sheaves on a weighted projective line $\mathbb{X}$ of type $(2, ..., 2)$.*

(i) Let $A \oplus B$ be a partial tilting sheaf. Then $\mathrm{Hom}_{\mathbb{X}}(A, B) \neq 0$ if and only if $\mu(A) < \mu(B)$.

(ii) Let $A \oplus B[1]$ be a partial tilting complex. Then $\mu(A) \geq \mu(B)$ and equality holds if and only if $\mathrm{Ext}^1_{\mathbb{X}}(A, B) = 0$. □

9.1.2 As a consequence of the result above we show that the width of a tilting complex on $\mathbb{X}$ is bounded by 2.

Theorem 9.1.2 *Let T be a tilting complex on a weighted projective line $\mathbb{X}$ of type $(2, ..., 2)$. Then T is, up to translation, of the form $T = U \oplus V[1]$ where $U, V \in \mathrm{coh}(\mathbb{X})$.*

Proof. Since the algebra $\Sigma = \mathrm{End}(T)$ is connected, the indecomposable direct summands of T belong to consecutive copies of $\mathrm{coh}(\mathbb{X})$ in the derived category. Assume, contrary to the assumption, that T is spread over at least three copies. Then by means of a translation we have summands A, $B[1]$, $C[2]$ of T for some indecomposable sheaves $A, B, C \in \mathrm{coh}(\mathbb{X})$.

From 9.1.1 we deduce that $\mu(A) \geq \mu(B)$ and $\mu(B) \geq \mu(C)$. Moreover, applying 9.1.1 to A and C we see that $\mu(A) = \mu(C)$.

It follows that all indecomposable direct summands of T have the same slope. This implies that Σ is not connected, a contradiction. □

Remark 9.1.2 *The method of the proof carries over to connected partial tilting complexes on weighted projective lines $\mathbb{X}$ of type $(2, \ldots, 2)$.*

9.1.3 Let $T = U \oplus V[1]$ be a tilting complex on a weighted projective line $\mathbb{X}$ of type $(2, ..., 2)$ with $U, V \in \mathrm{coh}(\mathbb{X})$. Decompose T into a direct sum of indecomposable objects $T = \bigoplus_{i \in I} T_i$ and denote by $q_1, q_2, ..., q_{n_0}$ (resp. $q'_1, q'_2, ..., q'_{n_1}$) the pairwise different slopes of the indecomposable direct summands of U (resp. $V[1]$). Assume that $q_1 < q_2 < ... < q_{n_0}$ and $q'_1 < q'_2 < ... < q'_{n_1}$ and set $n = n_0 + n_1$. Consider the following function defined on the set of vertices of the quiver Q for $\Sigma = \mathrm{End}(T)$

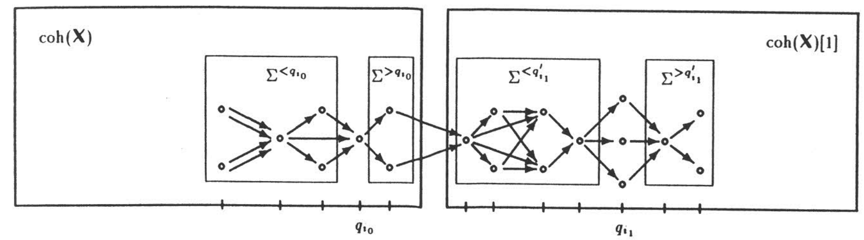
$$s : Q_0 \to \{1, ..., n\}$$

given by $s(T_i) = a$ if T_i is an indecomposable direct summand of U such that $\mu(T_i) = q_a$, and $s(T_i) = n_0 + b$ if T_i is an indecomposable direct summand of $V[1]$ such that $\mu(T_i) = q_b$. Moreover, write $q_i = \frac{d_i}{r_i}$ with d_i, r_i a coprime pair of integers and $r_i \geq 0$ and do the the same for q'_i. Denote by $i_0 \in \{1, ..., n_0\}$ [resp. $i_1 \in \{1, ..., n_1\}$] the unique index characterized by the following property:

(*) if $q_i > q_{i_0}$ then $r_i < r_{i_0}$ and if $q_i < q_{i_0}$ then $r_i \leq r_{i_0}$ (resp. if $q'_i > q'_{i_1}$ then $r'_i < r'_{i_1}$ and if $q'_i < q'_{i_1}$ then $r'_i \leq r'_{i_1}$).

Furthermore, let $T^{<q_{i_0}}$ [resp. $T^{>q_{i_0}}$] be the full subcategory of $\mathrm{coh}(\mathbb{X})$ formed by all objects of U of slope less [resp. greater] than q_{i_0} and $T^{<q'_{i_1}}$ [resp. $T^{>q'_{i_1}}$] the full subcategory formed by all objects of $V[1]$ of slope less [resp. greater] than $q_{i'_1}$. Finally, denote by $\Sigma^{<q_{i_0}} = \mathrm{End}(T^{<q_{i_0}})$, $\Sigma^{>q_{i_0}} = \mathrm{End}(T^{>q_{i_0}})$, $\Sigma^{<q'_{i_1}} = \mathrm{End}(T^{<q'_{i_1}})$ and $\Sigma^{>q'_{i_1}} = \mathrm{End}(T^{>q'_{i_1}})$ the corresponding endomorphism algebras .

The situation is illustrated in the following picture.



With the notations above we have the following properties for $\Sigma = \mathrm{End}(T) = kQ/I$.

Proposition 9.1.3 *(i) $\dim_k \mathrm{Hom}_{\mathcal{D}}(T_i, T_j)$ depends only on the slopes and the copy numbers of T_i and T_j.*

(ii) The quiver Q has no oriented cycles. More precisely, for $i \neq j$ there is a path from T_i to T_j in Q if and only if $s(T_i) < s(T_j)$.

(iii) If all pathes from T_i to T_j in Q belong to I, then $s(T_i) = 1$ and $s(T_j) = n$.

(iv) Let both T_i and T_j be indecomposable direct summands of U (resp. $V[1]$). If $\mu(T_i) < \mu(T_j) \leq q_{i_0}$ (resp. $\mu(T_i) < \mu(T_j) \leq q'_{i_1}$), then there is a monomorphism $T_i \hookrightarrow T_j$ and if $q_{i_0} \leq \mu(T_i) < \mu(T_j)$ (resp. $q'_{i_1} \leq \mu(T_i) < \mu(T_j)$), then there is an epimorphism $T_i \twoheadrightarrow T_j$.

(v) The algebras $\Sigma^{<q_{i_0}}$, $\Sigma^{>q_{i_0}}$, $\Sigma^{<q'_{i_1}}$ and $\Sigma^{>q'_{i_1}}$ are hereditary.

Proof. (i) follows from the Riemann-Roch formula and the fact that rank and degree for an exceptional sheaf are coprime, by Theorem 3.5.1.

(ii) and (iii) are consequences of 9.1.1.

(iv) By 2.3.3, each nonzero homomorphism between indecomposable direct summands of U is a monomorphism or an epimorphism. Moreover, in a chain of nonzero homomorphisms between direct summands of U an epimorphism cannot be followed by a monomorphism. Furthermore we know from Theorem 3.5.1 that all exceptional objects of a fixed slope have the same rank.

Now, assume that T_i and T_j are indecomposable direct summands of U such that $\mu(T_i) = q_1 < \mu(T_j) = q_2 \leq q_{i_0}$. If $q_2 = q_{i_0}$, then by (ii) there is a homomorphism $T_i \to T_j$ which by the choice of i_0 is a monomorphism. If $q_2 \neq q_{i_0}$ we choose an arbitrary direct summand T_l of slope q_{i_0}. Then we have a monomorphism $T_j \hookrightarrow T_l$. Further, there is a nonzero homomorphism $T_i \to T_j$, which by the argument above is a monomorphism, as well. The other statements are proved by the same method.

(v) Let us show that $\Sigma^{<q_{i_0}}$ is a hereditary algebra. Denote by $T^{q_{i_0}}$ the direct sum of all direct summands of U of slope q_{i_0}. If $q_{i_0} = \infty$ then $T^{<q_{i_0}} = 0$, by the choice of q_{i_0}, and there is nothing to show. Thus, we assume that $q_{i_0} \neq \infty$, in this case the right perpendicular category $(T^{q_{i_0}})^\perp$ is equivalent to a category of modules over a hereditary algebra H. Now $T^{<q_{i_0}} \in (T^{q_{i_0}})^\perp$, applying (ii) and condition (i) of the Definition 8.1.1.

We show that $T^{<q_{i_0}}$ is projective in $(T^{q_{i_0}})^\perp$. For this let A be an indecomposable direct summand of $T^{<q_{i_0}}$. Choose an arbitrary direct summand A' of $T^{q_{i_0}}$. By (iv) there is a monomorphism $f : A \hookrightarrow A'$. Now, for an arbitrary object $X \in (T^{q_{i_0}})^\perp$, f gives rise to an exact sequence $\mathrm{Ext}^1_{\mathbf{X}}(A', X) \to \mathrm{Ext}^1_{\mathbf{X}}(A, X) \to 0$. By assumption $\mathrm{Ext}^1_{\mathbf{X}}(A', X) = 0$, and therefore $\mathrm{Ext}^1_{\mathbf{X}}(A, X)$ vanishes, too. Hence $\mathrm{Ext}^1_{(T^{q_{i_0}})^\perp}(T^{<q_{i_0}}, -) = 0$, which shows that $\Sigma^{<q_{i_0}}$ is hereditary. The other statements are proved analogously. □

9.1.4 We denote for an exceptional pair (X, Y) in $\mathcal{D}^b(\mathrm{coh}(\mathbf{X}))$ by $h_{X,Y}$ the k-dimension of $\mathrm{Hom}_{\mathcal{D}}(X, Y)$.

In 9.6 we will need the following lemma

Lemma 9.1.4 *Let $\mathbf{X}$ be a weighted projective line of type $(2, ..., 2)$. Then there is no partial tilting sequence (A, B, C) such that $h_{A,B} = 2$, $h_{B,C} = 1$ and $h_{A,C} = 0$.*

Proof. Assume to the contrary that there is a partial tilting complex $A \oplus B \oplus C$ satisfying the conditions above. Then by 9.1.2, A and C are in consecutive copies of $\mathrm{coh}(\mathbf{X})$ in the derived category. Moreover, by 9.1.1, $\mu(A) = \mu(C)$. Since for an exceptional sheaf rank and degree are coprime, we infer that $\mathrm{rk}(C) = -\mathrm{rk}(A)$ and $\deg(C) = -\deg(A)$. Hence the Riemann-Roch formula yields

$$2 = h_{A,B} = \begin{vmatrix} \mathrm{rk}(A) & \mathrm{rk}(B) \\ \deg(A) & \deg(B) \end{vmatrix} \quad \text{and} \quad 1 = h_{B,C} = \begin{vmatrix} \mathrm{rk}(B) & -\mathrm{rk}(A) \\ \deg(B) & -\deg(A) \end{vmatrix},$$

a contradiction. □

9.2 Diophantine equations

It seems to be typical that tilting procedures in a geometrical context produce certain diophantine equations. In the case of the projective plane $\mathbf{P}^2$ the Markov equation mentioned in the introduction plays an important role, for other del surfaces there are equations of similar type [104], [89].

In this section we deduce two types of diophantine equations for (partial) tilting complexes on hyperelliptic weighted projective lines, which have essential consequences for the structure of the endomorphism algebras of these complexes.

Proposition 9.2.1 *Let (A, B, C) be a partial tilting sequence in $\mathcal{D}^b(\mathrm{coh}(\mathbf{X}))$ for a weighted projective line of type $(2, ..., 2)$. Then*

$$h_{A,C} \cdot \mathrm{rk}(B) = h_{A,B} \cdot \mathrm{rk}(C) + h_{B,C} \cdot \mathrm{rk}(A).$$

Proof. Consider the matrix

$$M = \begin{pmatrix} \mathrm{rk}(A) & \mathrm{rk}(B) & \mathrm{rk}(C) \\ \mathrm{rk}(A) & \mathrm{rk}(B) & \mathrm{rk}(C) \\ \deg(A) & \deg(B) & \deg(C) \end{pmatrix}.$$

Laplace expansion yields

$$\begin{aligned} 0 &= |M| \\ &= \mathrm{rk}(A) \begin{vmatrix} \mathrm{rk}(B) & \mathrm{rk}(C) \\ \deg(B) & \deg(C) \end{vmatrix} - \mathrm{rk}(B) \begin{vmatrix} \mathrm{rk}(A) & \mathrm{rk}(C) \\ \deg(A) & \deg(C) \end{vmatrix} + \mathrm{rk}(C) \begin{vmatrix} \mathrm{rk}(A) & \mathrm{rk}(B) \\ \deg(A) & \deg(B) \end{vmatrix}. \end{aligned}$$

Hence the proposition follows from Proposition 9.1.1 . □

Corollary 9.2.1 *Assume that A, B, C, X, Y are exceptional sheaves on a weighted projective line $\mathbf{X}$ of type $(2, ..., 2)$.*

(i) Let $A \oplus B \oplus C$ be a partial tilting sheaf on $\mathbf{X}$ such that $\mu(A) < \mu(B) < \mu(C)$. If $h_{A,B} \geq h_{A,C}$ and $h_{B,C} \geq h_{A,C}$, then there is a monomorphism $A \hookrightarrow B$ and an epimorphism $B \twoheadrightarrow C$.

(ii) Let $A \oplus B \oplus C$ be a partial tilting sheaf on $\mathbf{X}$ and assume that there is a monomorphism $A \hookrightarrow B$ and an epimorphism $B \twoheadrightarrow C$. Then $h_{A,C} \leq 2\max(h_{A,B}, h_{B,C})$.

(iii) Let $A \oplus X[1] \oplus Y[1]$ be a partial tilting complex on $\mathbf{X}$ such that there is an epimorphism $X \twoheadrightarrow Y$. Then $h_{A,X[1]} > h_{A,Y[1]}$.

(iv) Let $B \oplus A \oplus X[1]$ be a partial tilting complex on $\mathbf{X}$ such that there is a monomorphism $B \hookrightarrow A$. Then $h_{A,X[1]} > h_{B,X[1]}$.

Proof. (i) We infer from the Riemann-Roch formula that $h_{A,C} \neq 0$. Thus the proposition yields $\mathrm{rk}(B) = \frac{h_{B,C}}{h_{A,C}} \cdot \mathrm{rk}(A) + \frac{h_{A,B}}{h_{A,C}} \cdot \mathrm{rk}(C) \geq \mathrm{rk}(A) + \mathrm{rk}(C)$. This is impossible in case there are both monomorphisms (resp. both epimorphisms) from A to B and from B to C.

(ii) Suppose to the contrary that $h_{A,C} > 2\max(h_{A,B}, h_{B,C})$. Certainly we can assume that $\mathrm{rk}(B) > 0$. The assumptions imply that $\mathrm{rk}(B) \geq \mathrm{rk}(A)$ and $\mathrm{rk}(B) > \mathrm{rk}(C)$. Thus

$$\begin{aligned} h_{A,C} \cdot \mathrm{rk}(B) &> 2\max(h_{A,B}, h_{B,C}) \cdot \mathrm{rk}(B) \\ &> \max(h_{A,B}, h_{B,C}) \cdot (\mathrm{rk}(A) + \mathrm{rk}(C)) \\ &\geq h_{A,B} \cdot \mathrm{rk}(C) + h_{B,C} \cdot \mathrm{rk}(A) \end{aligned}$$

which contradicts the proposition.

(iii) Observe first that $h_{A,X[1]} \neq 0$. Indeed, otherwise $\mu(A) = \mu(X) < \mu(Y)$, which gives rise to a nonzero homomorphism from A to Y, contradicting condition (i) of Definition 8.1.1. Denote by K the kernel of an epimorphism $X \twoheadrightarrow Y$. Application of $\mathrm{Hom}_{\mathbf{X}}(A, -)$ to the exact sequence $0 \to K \to X \to Y \to 0$ yields $h_{A,X[1]} \geq h_{A,Y[1]}$. Now assume that $h_{A,X[1]} = h_{A,Y[1]}$. According to the proposition, we get $\mathrm{rk}(Y) = \mathrm{rk}(X) + \frac{h_{X,Y}}{h_{A,Y[1]}}\mathrm{rk}(A)$. Consequently $\mathrm{rk}(Y) \geq \mathrm{rk}(X)$, which contradicts the fact that there is an epimorphism $X \twoheadrightarrow Y$.

(iv) follows similarly. □

As a consequence of assertion (i) of the corollary, none of the hereditary algebras $\Sigma^{<q_{10}}$, $\Sigma^{>q_{10}}$, $\Sigma^{<q'_{11}}$ and $\Sigma^{>q'_{11}}$ can obtain as a full subcategory a path algebra given by a Dynkin quiver of type $\mathbb{A}_3$ with linear orientation. From this it follows that there exists a bound c such that for $t > c$ each tilting complex on a hyperelliptic weighted projective line with t weights is wild. In 9.4 we will determine this number c precisely.

Proposition 9.2.2 *Let (A, B, C, D) be a partial tilting sequence in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ for a weighted projective line of type $(2, ..., 2)$. Then*

$$h_{A,C} \cdot h_{B,D} = h_{A,B} \cdot h_{C,D} + h_{A,D} \cdot h_{B,C}.$$

Proof. Consider the matrix

$$M = \begin{pmatrix} \mathrm{rk}(A) & \mathrm{rk}(B) & \mathrm{rk}(C) & \mathrm{rk}(D) \\ \deg(A) & \deg(B) & \deg(C) & \deg(D) \\ \mathrm{rk}(A) & \mathrm{rk}(B) & \mathrm{rk}(C) & \mathrm{rk}(D) \\ \deg(A) & \deg(B) & \deg(C) & \deg(D) \end{pmatrix}.$$

By Laplace expansion with respect to the first two rows and application of the Riemann-Roch formula we obtain

$$\begin{aligned} 0 &= h_{A,B} \cdot h_{C,D} - h_{A,C} \cdot h_{B,D} + h_{A,D} \cdot h_{B,C} + h_{B,C} \cdot h_{A,D} - h_{B,D} \cdot h_{A,C} + h_{C,D} \cdot h_{A,B} \\ &= 2(h_{A,B} \cdot h_{C,D} - h_{A,C} \cdot h_{B,D} + h_{A,D} \cdot h_{B,C}), \end{aligned}$$

which finishes the proof. □

9.3 Layered algebras

9.3.1 Let $\Sigma = kQ/I$ be a finite dimensional k-algebra. For vertices $i, j \in Q_0$ we denote by $h_{i,j}$ the k-dimension of $\mathrm{Hom}_\Sigma(P(i), P(j))$, where $P(i)$, $P(j)$ are the corresponding indecomposable projective Σ-modules.

Definition 9.3.1 *The algebra Σ is called a layered algebra if there is a surjective map $s : Q_0 \to \{1, ..., n\}$, for some $n \in \mathbb{N}$, satisfying the following conditions:*

(i) $h_{i,j}$ depends only on $s(i)$ and $s(j)$,
(ii) $h_{i,i} = 1$,
(iii) if $h_{i,j} \neq 0$ and $i \neq j$ then $s(i) < s(j)$,
(iv) if $s(i) < s(j)$ and $h_{i,j} = 0$ then $s(i) = 1$ and $s(j) = n$,
(v) if $s(i) < s(j) < s(l) < s(m)$ then $h_{i,l}h_{j,m} = h_{i,j}h_{l,m} + h_{i,m}h_{j,l}$.

9.3.2 Applying the results of the previous sections we have the following theorem

Theorem 9.3.2 *Let Σ be an algebra derived equivalent to a canonical algebra of type $(2, ..., 2)$. Then Σ is a layered algebra.* □

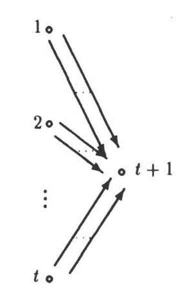
9.3.3 Identifying the category of modules over the t-subspace problem algebra Λ_0 with the right perpendicular category $L^\perp$ of a line bundle on a weighted projective line $\mathbb{X}$ of type $(2, ..., 2)$, t entries, a tilting complex in $\mathcal{D}^b(\mathrm{mod}(\Lambda_0))$ can be considered as a connected partial tilting complex in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$. Thus, again by 9.1, and 9.2, we have

Theorem 9.3.3 *Let Σ be an algebra derived equivalent to a t-subspace problem algebra. Then Σ is a layered algebra.* □

9.3.4 Let E be an exceptional vector bundle on a weighted projective line of type $(2, ..., 2)$. We know that the right perpendicular category $E^\perp$ is equivalent to a category of modules over a hereditary algebra H. Then H is also a layered algebra. Indeed, since the embedding $E^\perp \hookrightarrow \mathrm{coh}(\mathbb{X})$ is exact, the projective H-modules form a connected partial tilting complex in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ and the result follows again by 9.1, and 9.2.

In case E is a line bundle, H is known to be the t-subspace problem algebra. As an example we determine here the hereditary algebra in the "next more complicated" case of an omnipresent exceptional vector bundle of minimal rank.

Proposition 9.3.4 *Let E be an omnipresent exceptional vector bundle of rank $t - 1$ on a hyperelliptic weighted projective line with t weights. Then the right perpendicular category $E^\perp$ is equivalent to the category of modules over the path algebra for the following quiver*



where for each i, $1 \leq i \leq t$, the number of arrows from i to $t + 1$ equals $t - 3$.

Proof. Recall from Theorem 6.3.6 that, up to a line bundle shift, E is given as the middle term of an exact sequence

$$0 \to \mathcal{O}(-\vec{\omega}) \to E \to \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{c}), \mathcal{O}(-\vec{\omega})) \otimes \mathcal{O}(\vec{c}) \to 0.$$

It follows that $\mathcal{O}(\vec{x}_i)$, for $1 \leq i \leq t$, and $\mathcal{O}(\vec{c} + \vec{\omega})$ belong to $E^\perp$.

We prove that the $\mathcal{O}(\vec{x}_i)$ are projective and that $\mathcal{O}(\vec{c} + \vec{\omega})$ is injective in $E^\perp$. In order to show this suppose that $A \in E^\perp$. Now, there is a monomorphism $\mathcal{O}(\vec{x}_i) \hookrightarrow E$. Application of $\mathrm{Hom}(-, A)$ yields an exact sequence $\mathrm{Ext}^1_{\mathbb{X}}(E, A) \to \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{x}_i), A) \to 0$. From $\mathrm{Ext}^1_{\mathbb{X}}(E, A) = 0$ it follows that $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{x}_i), A) = 0$, and consequently $\mathcal{O}(\vec{x}_i)$ is projective in $E^\perp$. Further, there is an epimorphism $E \twoheadrightarrow \mathcal{O}(\vec{c})$. Application of $\mathrm{Hom}_{\mathbb{X}}(-, A)$ yields an exact sequence $0 \to \mathrm{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{c}), A) \to \mathrm{Hom}_{\mathbb{X}}(E, A) = 0$. Invoking Serre duality we get $0 = \mathrm{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{c}), A) \cong \mathrm{DExt}^1_{\mathbb{X}}(A, \mathcal{O}(\vec{c}+\vec{\omega}))$. This proves that $\mathrm{Ext}^1_{E^\perp}(-, \mathcal{O}(\vec{c}+\vec{\omega})) = 0$, and consequently $\mathcal{O}(\vec{c}+\vec{\omega})$ is injective in $E^\perp$.

Since $\mathrm{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{x}_i), \mathcal{O}(\vec{c}+\vec{\omega})) \cong \mathrm{DExt}^1_{\mathbb{X}}(\mathcal{O}(\vec{c}), \mathcal{O}(\vec{x}_i)) = 0$, it is impossible that in $E^\perp$ the module $\mathcal{O}(\vec{c}+\vec{\omega})$ is the injective envelope of the top of one of the $\mathcal{O}(\vec{x}_i)$. Hence $\mathcal{O}(\vec{c}+\vec{\omega})$ coincides with the injective indecomposable corresponding to the remaining vertex of the quiver, and therefore the remaining indecomposable projective in $E^\perp$ is of the form $\tau^-_{D^b(E^\perp)}\mathcal{O}(\vec{c}+\vec{\omega})[-1]$. Finally, $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{c}+\vec{\omega}), \mathcal{O}(\vec{x}_i)) \cong \mathrm{DHom}(\mathcal{O}(\vec{x}_i), \mathcal{O}(\vec{c}+2\vec{\omega})) = \mathrm{DHom}(\mathcal{O}, \mathcal{O}(\vec{x}_i + 2\vec{\omega}))$ and $\vec{x}_i + 2\vec{\omega} = \vec{x}_i + (t-4)\vec{c}$. This shows that the dimension of $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{c}+\vec{\omega}), \mathcal{O}(\vec{x}_i))$ equals $t-3$ which gives the number of arrows for the quiver. □

9.3.5 Let Σ be a layered algebra with layer function $s : Q_0 \to \{1, ..., n\}$. For $i = 1, ..., n$ we choose an indecomposable projective Σ-module P_i such that $s(P_i) = i$. Further we denote by v_i the number of indecomposable projectives P with $s(P) = i$.

The *layer triangle* of Σ is defined as

$$H(\Sigma) = \begin{pmatrix} & & & & h_{1,n} & & & & \\ & & & \cdot & & \cdot & & & \\ & & \cdot & & & & \cdot & & \\ & h_{1,3} & & & & & & h_{n-2,n} & \\ h_{1,2} & & h_{2,3} & & \cdots & & h_{n-2,n-1} & & h_{n-1,n} \end{pmatrix}$$

Moreover, we define the *Cartan triangle* of Σ as

$$C(\Sigma) = \begin{pmatrix} & & & & & h_{1,n} & & & & & \\ & & & & \cdot & & \cdot & & & & \\ & & & \cdot & & & & \cdot & & & \\ & & h_{1,3} & & & & & & h_{n-2,n} & & \\ & h_{1,2} & & h_{2,3} & & \cdots & & h_{n-2,n-1} & & h_{n-1,n} & \\ - & - & - & - & - & - & - & - & - & - & - \\ v_1 & & v_2 & & & \cdots & & & v_{n-1} & & v_n \end{pmatrix}$$

Note that the Cartan triangle contains the same information as the Cartan matrix of Σ.

The property of being layered is preserved when passing from Σ to its opposite algebra Σ^{op}. Moreover, if Σ' is a full subcategory of Σ then Σ' is also layered, possible with a smaller number of layers. In this case we call $H(\Sigma^{op})$ (resp. $C(\Sigma^{op})$) the opposite layer (resp. Cartan) triangle and $H(\Sigma')$ (resp. $C(\Sigma')$) a subtriangle of $H(\Sigma)$ (resp. $C(\Sigma)$).

9.3.6 We will study layered algebras which are tame. In our investigation we will frequently use a result of Dowbor and Skowroński [28], which states that a locally bounded category is tame if and only if its (finite) full subcategories are tame.

Proposition 9.3.6 *Let Σ be a tame layered algebra. Then $n \leq 5$ and the layer triangle $H(\Sigma)$ or its dual is a subtriangle of one of the following*

$$\begin{pmatrix} & & & 0 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 1 & & 1 & & 1 \end{pmatrix}, \begin{pmatrix} & & & 0 & & & \\ & & 1 & & 1 & & \\ & 2 & & 1 & & 2 & \\ 1 & & 1 & & 1 & & 1 \end{pmatrix},$$

$$\begin{pmatrix} & & 0 & & \\ & 2 & & 1 & \\ 2 & & 2 & & 1 \end{pmatrix}, \begin{pmatrix} & & 0 & & \\ & 2 & & 2 & \\ 2 & & 2 & & 2 \end{pmatrix}, \begin{pmatrix} & 1 & \\ 2 & & 2 \end{pmatrix}.$$

Proof. Since Σ is tame all entries in $H = H(\Sigma)$ are smaller than or equal to 2, because otherwise Σ contains as a full subcategory a wild algebra. Moreover, whenever $h_{i,i+1} = 1$ and $h_{i+1,i+2} = 2$ (resp. $h_{i,i+1} = 2$ and $h_{i+1,i+2} = 1$) it follows that $h_{i,i+2} \neq 2$, by the same argument. This gives in the case $n = 3$, up to duality, the following

$$\begin{pmatrix} & s & \\ 1 & & 1 \end{pmatrix}, \begin{pmatrix} & s & \\ 1 & & 2 \end{pmatrix}, \begin{pmatrix} & s & \\ 2 & & 2 \end{pmatrix}, \quad \text{for } s = 0 \text{ or } 1, \quad \begin{pmatrix} & 2 & \\ 1 & & 1 \end{pmatrix}, \begin{pmatrix} & 2 & \\ 2 & & 2 \end{pmatrix}.$$

as possible layer triangles.

Suppose now that $n = 4$. Then there are three possibilities: $h_{1,3} \cdot h_{2,4} = 1$, $h_{1,3} \cdot h_{2,4} = 4$ and $h_{1,3} \cdot h_{2,4} = 2$. Applying condition (iv) and (v) of Definition 9.3.1 we conclude that in the first and the second case the only possible triangles are

$$\begin{pmatrix} & & 0 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} & & 0 & & \\ & 2 & & 2 & \\ 2 & & 2 & & 2 \end{pmatrix}, \text{ respectively.}$$

In the third case, up to duality, H is one of the following

$$\begin{pmatrix} & & 0 & & \\ & 1 & & 2 & \\ 2 & & 1 & & 1 \end{pmatrix}, \begin{pmatrix} & & 0 & & \\ & 1 & & 2 & \\ 1 & & 2 & & 2 \end{pmatrix} \text{ or } \begin{pmatrix} & & 1 & & \\ & 1 & & 2 & \\ 1 & & 1 & & 1 \end{pmatrix} = \tilde{H}.$$

Now assume that Σ is a layered algebra with $n \geq 5$ layers. Forming layered algebras by taking full convex subcategories of Σ with 4 layers we infer that the triangles

$$H' = \begin{pmatrix} & & h_{1,4} & & \\ & h_{1,3} & & h_{2,4} & \\ h_{1,2} & & h_{2,3} & & h_{3,4} \end{pmatrix}, H'' = \begin{pmatrix} & & h_{2,5} & & \\ & h_{2,4} & & h_{3,5} & \\ h_{2,3} & & h_{3,4} & & h_{4,5} \end{pmatrix}$$

belong to the layer triangles we have classified before. Moreover, we have $h_{1,4} \neq 0$ and $h_{2,5} \neq 0$. Thus, up to duality, $H' = \tilde{H}$ and $H'' = (\tilde{H})^{op}$. From the equation $h_{1,3} \cdot h_{2,5} = h_{1,2} \cdot h_{3,5} + h_{1,5} \cdot h_{2,3}$ we obtain $h_{1,5} = 0$. Therefore $n = 5$ and $H(\Sigma)$ is the first triangle of our list. Dually the case $H'' = \tilde{H}$ gives the second one. □

Corollary 9.3.7 *Let Σ be a tame layered algebra. Then the Cartan triangle $C(\Sigma)$ or its dual is a subtriangle of one of the following*

$$\begin{pmatrix} & & & & 0 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 1 & & 1 & & 1 & \\ - & - & - & - & - & - & - & - & - \\ 2 & & 1 & & 2 & & 1 & & 2 \end{pmatrix}, \begin{pmatrix} & & & & 0 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 1 & & 1 & & 1 & \\ - & - & - & - & - & - & - & - & - \\ 1 & & 1 & & 3 & & 1 & & 1 \end{pmatrix},$$

$$\begin{pmatrix} & & & & 0 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 1 & & 1 & & 1 & \\ - & - & - & - & - & - & - & - & - \\ 3 & & 1 & & 1 & & 1 & & 3 \end{pmatrix},$$

$$\left(\begin{array}{c} 0 \\ 1 \quad 1 \\ 2 \quad 1 \quad 2 \\ 1 \quad 1 \quad 1 \quad 1 \\ - - - - - - - - - \\ 1 \quad 2 \quad 1 \quad 2 \quad 1 \end{array}\right), \left(\begin{array}{c} 0 \\ 1 \quad 1 \\ 2 \quad 1 \quad 2 \\ 1 \quad 1 \quad 1 \quad 1 \\ - - - - - - - - - \\ 1 \quad 3 \quad 1 \quad 1 \quad 1 \end{array}\right),$$

$$\left(\begin{array}{c} 0 \\ 1 \quad 2 \\ 2 \quad 1 \quad 1 \\ - - - - - - - \\ 1 \quad 1 \quad 4 \quad 1 \end{array}\right), \left(\begin{array}{c} 0 \\ 2 \quad 1 \\ 2 \quad 2 \quad 1 \\ - - - - - - - \\ 1 \quad 1 \quad 1 \quad 4 \end{array}\right), \left(\begin{array}{c} 0 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ - - - - - - - \\ 4 \quad 1 \quad 1 \quad 4 \end{array}\right),$$

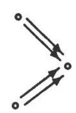
$$\left(\begin{array}{c} 0 \\ 1 \quad 1 \\ 1 \quad 1 \quad 1 \\ - - - - - - - \\ 2 \quad 2 \quad 2 \quad 2 \end{array}\right), \left(\begin{array}{c} 0 \\ 2 \quad 2 \\ 2 \quad 2 \quad 2 \\ - - - - - - - \\ 1 \quad 1 \quad 1 \quad 1 \end{array}\right), \left(\begin{array}{c} 1 \\ 2 \quad 2 \\ - - - - - \\ 1 \quad 1 \quad 1 \end{array}\right).$$

Proof. Adapting to the layer triangles obtained in the proposition the possible sequences $v = (v_1, ..., v_n)$ of numbers of objects in each layer, we obtain by tameness necessary conditions for v. It is easily checked that for all other choices of v an algebra Σ with layer triangle $C(\Sigma)$ contains a full subcategory which is wild. We illustrate this in two typical cases, for the remaining ones the arguments are similar.

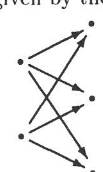
Suppose first that Σ is a tame layered algebra with

$$C(\Sigma) = \left(\begin{array}{c} 0 \\ 1 \quad 1 \\ 2 \quad 1 \quad 2 \\ 1 \quad 1 \quad 1 \quad 1 \\ - - - - - - - - - \\ a \quad b \quad c \quad d \quad e \end{array}\right)$$

Then $a = c = e = 1$, because otherwise Σ contains as a full subcategory a path algebra of the quiver



or its dual. Moreover, we infer that $b + d \leq 4$. Indeed, otherwise Σ or Σ^{op} contains as a full subcategory a hereditary algebra given by the quiver



or a full subcategory which is a one-point extension of the dual of the 4-subspace problem algebra A by an A-module M of dimension type

$$\left(\begin{array}{cc} & 1 \\ & 1 \\ 1 & \\ & 1 \\ & 1 \end{array}\right).$$

Observe that in the second case M must contain an indecomposable projective direct summand and consequently Σ is wild.

By the same method we get in the case that Σ is a tame layered algebra with

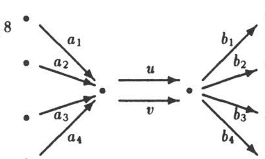
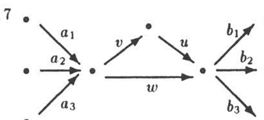
$$C(\Sigma) = \left(\begin{array}{c} 0 \\ 1 \quad 1 \\ 1 \quad 1 \quad 1 \\ - - - - - - - \\ a \quad b \quad c \quad d \end{array}\right)$$

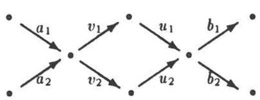
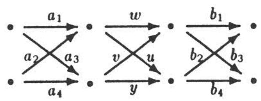
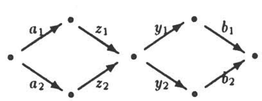
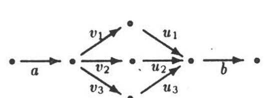
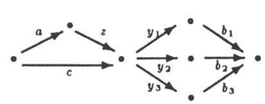
the necessary conditions $a+b \leq 4$, $a+c \leq 4$, $b+c \leq 4$, $b+d \leq 4$ and $c+d \leq 4$. It follows that $C(\Sigma)$ is a subtriangle of the second, the third or the ninth Cartan triangle listed in the corollary. □

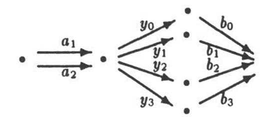
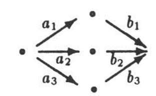
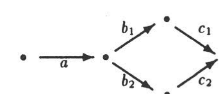
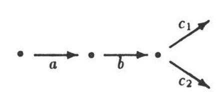
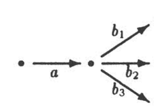
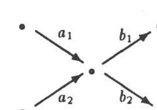
9.4 Tame algebras derived equivalent to hyperelliptic algebras

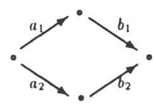
Theorem 9.4.1 *Let Σ be a finite dimensional k-algebra. Then Σ is tame and derived equivalent to a canonical algebra of type $(2, ..., 2)$ if and only if Σ or Σ^{op} belongs to the following list.*

List 9.4 Tame algebras derived equivalent to canonical algebras of type $(2, 2, ..., 2)$, t entries

algebra	t	quiver	relations	layer triangle
$\Sigma_1(r,s)$ $0 \leq r,s \leq 4$	$0, ..., 8$		$(u - \lambda'_i v)a_i = 0$ $b_j(u - \lambda''_j v) = 0$	$\left(\begin{array}{c} 0 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \end{array}\right)$
$\Sigma_2(r,s)$ $0 \leq r,s \leq 3$	$1, ..., 7$		$(w - \lambda'_i uv)a_i = 0$ $b_j(w - \lambda''_j uv) = 0$	$\left(\begin{array}{c} 0 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 1 \quad 1 \quad 1 \end{array}\right)$

$\Sigma_3(r,s)$ $0 \le r,s \le 2$	2, ..., 6		$(u_2v_2 - \lambda_i' u_1v_1)a_i = 0$ $b_j(u_2v_2 - \lambda_i'' u_1v_1) = 0$	$\begin{pmatrix} & & & 0 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 1 & & 1 & & 1 \end{pmatrix}$
$\Sigma_4(r,s)$ $0 \le r,s \le 2$	2, ..., 6		$ua_1 - ya_3 = 0$ $ua_2 - ya_4 = 0$ $va_3 - \lambda_3 wa_1 = 0$ $va_4 - \lambda_4 wa_2 = 0$ $b_1w - b_2u = 0$ $b_3w - b_4u = 0$ $b_2y - \lambda_5 b_1 v = 0$ $b_4y - \lambda_6 b_3 v = 0$	$\begin{pmatrix} & & 0 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \end{pmatrix}$
$\Sigma_5(r,s)$ $0 \le r,s \le 1$	3,4,5		$y_1(z_1a_1 - z_2a_2) = 0$ $y_2(z_1a_1 - \lambda_4 z_2a_2) = 0$ $(b_2y_2 - b_1y_1)z_1 = 0$ $(b_2y_2 - \lambda_5 b_1y_1)z_2 = 0$	$\begin{pmatrix} & & & 0 & & & \\ & & 1 & & 1 & & \\ & 2 & & 1 & & 2 & \\ 1 & & 1 & & 1 & & 1 \end{pmatrix}$
$\Sigma_6(r,s)$ $0 \le r,s \le 1$	3,4,5		$u_3v_3 - u_2v_2 + u_1v_1 = 0$ $(u_2v_2 - \lambda_4 u_1v_1)a = 0$ $b(u_2v_2 - \lambda_5 u_1v_1) = 0$	$\begin{pmatrix} & & & 0 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 1 & & 1 & & 1 \end{pmatrix}$
$\Sigma_7(r,s)$ $0 \le r,s \le 1$	3,4,5		$b_3y_3 - b_2y_2 + b_1y_1 = 0$ $(b_2y_2 - \lambda_5 b_1y_1)z = 0$ $y_1c = 0$ $y_2(za - c) = 0$ $y_3(za - \frac{\lambda_4 - 1}{\lambda_4}c) = 0$	$\begin{pmatrix} & & & 0 & & & \\ & & 1 & & 1 & & \\ & 2 & & 1 & & 2 & \\ 1 & & 1 & & 1 & & 1 \end{pmatrix}$

$\Sigma_8(r,s)$ $0 \le r,s \le 1$	3,4,5		$b_3y_3 - b_2y_2 + b_1y_1 = 0$ $b_0y_0 - b_2y_2 + \lambda_5 b_1y_1 = 0$ $y_0a_1 = 0$ $y_1a_2 = 0$ $y_2(a_2 - a_1) = 0$ $y_3(a_2 - \frac{\lambda_4}{\lambda_4 - 1}a_1) = 0$	$\begin{pmatrix} & & 0 & & \\ & 1 & & 2 & \\ 2 & & 1 & & 1 \end{pmatrix}$
Σ_9	3		$b_1a_1 - b_2a_2 = 0$ $b_1a_1 - b_3a_3 = 0$	$\begin{pmatrix} & 1 & \\ 1 & & 1 \end{pmatrix}$
Σ_{10}	3		$c_1b_1 - c_2b_2 = 0$ $c_1b_1a = 0$	$\begin{pmatrix} & & 0 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \end{pmatrix}$
Σ_{11}	3		$c_iba = 0$	$\begin{pmatrix} & & 0 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \end{pmatrix}$
Σ_{12}	3		$b_ia = 0$	$\begin{pmatrix} & 0 & \\ 1 & & 1 \end{pmatrix}$
Σ_{13}	3		$b_ia_j = 0$	$\begin{pmatrix} & 0 & \\ 1 & & 1 \end{pmatrix}$

Σ_{14}	2		$b_i a_j = 0$	$\left(\begin{smallmatrix} & 0 & \\ 1 & & 1 \end{smallmatrix}\right)$

In the list the $\Sigma_i(r, s)$ stand for families of algebras, where r and s denote the numbers of vertices in the first and the last layer, respectively. These numbers can be chosen as it is indicated. We have always drawn the quiver for which r and s are maximal, the other algebras are obtained by deleting vertices and modifying the relations in the obvious way.

In the description of the relations the parameters are assumed to be pairwise distinct. Moreover three of them can be chosen as 0, 1 and (formally) ∞, this is already realized for algebras of type Σ_5 - Σ_8. Thus for $t \geq 4$ each $\Sigma_i(r, s)$ denotes a family of algebras depending on $t - 3$ parameters. Observe that in case both r and s are nonzero the fact that the parameters are pairwise distinct implies that all pathes from the first to the last layer are zero.

The module category of an algebra with parameters as stated in the list is always derived equivalent to coh($\mathbb{X}$) for a weighted projective line with parameters ($\lambda_1 = \infty, \lambda_2 = 0, \lambda_3 = 1, \ldots \lambda_t$). Note that in the cases $\Sigma_7(r, s)$ and $\Sigma_8(r, s)$ a rational fraction in one of the parameters appears.

For the convenience of the reader we have included the corresponding layer triangles.

The theorem will be proved in 9.6. We remark that the theorem gives an alternative proof of the fact that for weighted projective lines of type $(2, ..., 2)$, t entries, $t \geq 4$, there are no tilting complexes with representation-finite endomorphism algebras (compare Theorem 8.2.5).

Corollary 9.4.2 *Let Σ be a tame algebra derived equivalent to a canonical algebra of type $(2, ..., 2)$, t-entries. Then $t \leq 8$.*

9.4.3 We recall from [42] that an algebra is called *quasitilted* if it is of the form $A = \mathrm{End}(T)$, where T is a tilting object in a locally finite hereditary abelian k-category. Equivalently these algebras are characterized by the following two conditions:

(i) gl.dim $A \leq 2$.

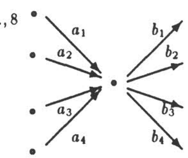
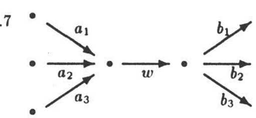
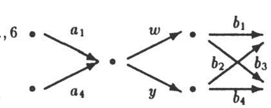
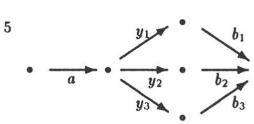
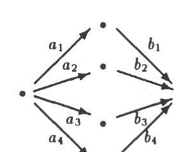
(ii) If X is a finitely generated indecomposable A-module then either $\mathrm{pd}_A X \leq 1$ or $\mathrm{id}_A X \leq 1$.

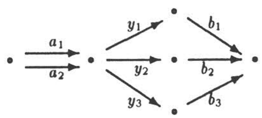
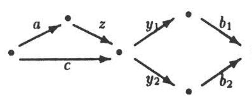
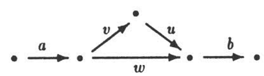
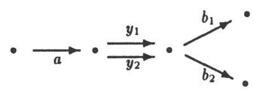
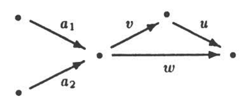
Corollary 9.4.3 *Let Σ be a tame algebra derived equivalent to a canonical algebra of type $(2, ..., 2)$. Then Σ is quasitilted.*

9.5 Tame algebras derived equivalent to subspace problem algebras

Theorem 9.5.1 *Let Σ be a finite dimensional k-algebra. Then Σ is tame and derived equivalent to a t-subspace problem algebra if and only if Σ or Σ^{op} belongs to the following list.*

List 9.5 Tame algebras derived equivalent to t-subspace problem algebras

algebra	t	quiver	relations	layer triangle
$\Delta_1(r, s)$ $0 \leq r, s \leq 4$	$0, ..., 8$		$b_j a_i = 0$	$\left(\begin{smallmatrix} & 0 & \\ 1 & & 1 \end{smallmatrix}\right)$
$\Delta_2(r, s)$ $0 \leq r, s \leq 3$	$1, ...7$		$b_j w a_i = 0$	$\left(\begin{smallmatrix} & & 0 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \end{smallmatrix}\right)$
$\Delta_3(r, s)$ $0 \leq r, s \leq 2$	$2, ..., 6$		$b_1 w - b_2 y = 0$ $b_3 w - b_4 y = 0$ $b_i w a_i = 0$ $b_i y a_i = 0$	$\left(\begin{smallmatrix} & & 0 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \end{smallmatrix}\right)$
$\Delta_4(r, s)$ $0 \leq r, s \leq 1$	$3, 4, 5$		$b_1 y_1 - b_2 y_2 = 0$ $b_1 y_1 - b_3 y_3 = 0$ $b_i y_i a = 0$	$\left(\begin{smallmatrix} & & 0 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 1 \end{smallmatrix}\right)$
$\Delta_5(r)$ $0 \leq r \leq 1$	$4, 5$		$b_1 a_1 - b_2 a_2 = 0$ $b_1 a_1 - b_3 a_3 = 0$ $b_1 a_1 - b_4 a_4 = 0$	$\left(\begin{smallmatrix} & 1 & \\ 1 & & 1 \end{smallmatrix}\right)$

$\Delta_6(r,s)$ $0 \le r,s \le 1$	3,4,5		$b_3y_3 - b_2y_2 + b_1y_1 = 0$ $b_1y_1a_2 = 0$ $b_2y_2a_1 = 0$ $y_1a_1 = 0$ $y_2a_2 = 0$ $y_3(a_2 - a_1) = 0$	$\begin{pmatrix} & & 0 & & \\ & 1 & & 2 & \\ 2 & & 1 & & 1 \end{pmatrix}$
$\Delta_7(r,s)$ $0 \le r,s \le 1$	3,4,5		$(b_2y_2 - b_1y_1)z = 0$ $y_1c = 0$ $y_2(za - c) = 0$ $b_2y_2c = 0$	$\begin{pmatrix} & & & 0 & & & \\ & & 1 & & 1 & & \\ & 2 & & 1 & & 2 & \\ 1 & & 1 & & 1 & & 1 \end{pmatrix}$
Δ_8	4		$(w - uv)a = 0$ $bw = 0$	$\begin{pmatrix} & & & 0 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 1 & & 1 & & 1 \end{pmatrix}$
Δ_9	4		$b_1y_1 = 0$ $b_2y_2 = 0$ $(y_1 - y_2)a = 0$	$\begin{pmatrix} & & 0 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{pmatrix}$
Δ_{10}	4		$wa_1 = 0$ $uva_2 = 0$	$\begin{pmatrix} & & 1 & & \\ & 1 & & 2 & \\ 1 & & 1 & & 1 \end{pmatrix}$

The theorem will be proved in 9.6.

Corollary 9.5.2 *Let Σ be a tame algebra derived equivalent to a t-subspace problem algebra. Then $t \le 8$.*

Corollary 9.5.3 *Let Σ be a tame algebra derived equivalent to a t-subspace problem algebra. Then Σ is quasitilted.*

9.5.4 Using a result of Lenzing [73] describing the types of hereditary categories of "wild hereditary type" we obtain a stronger version of the last corollary.

Corollary 9.5.4 *Let Σ be a tame algebra derived equivalent to a wild t-subspace problem algebra. Then Σ is a tilted algebra.*

9.6 Proofs of the classification results

In this section we prove the statements of 9.4 and 9.5. We first recall some results of Lenzing and Skowroński concerning quasitilted algebras of canonical type [79]. Then in 9.6.2 and 9.6.3 we will find realizations of the algebras of list 9.4 and 9.5 as endomorphism algebras of tilting complexes. Finally, applying the results of 9.3 and information about Coxeter polynomials we will show that the endomorphism algebras of tilting complexes, we are interested in, belong to our lists.

9.6.1 Let $\mathbf{X}$ be a weighted projective line of arbitrary weight type. An abelian k-category $\mathcal{H}$ is said to be *of canonical type* if it derived equivalent to a category of coherent sheaves $\mathrm{coh}(\mathbf{X})$.

Recall that the category of finite length sheaves $\mathrm{coh}_0(\mathbf{X})$ decomposes into a coproduct $\coprod_{\lambda\in\mathbf{X}}\mathcal{U}_\lambda$ where $\mathcal{U}_\lambda$ denotes the uniserial category of finite length sheaves concentrated at λ. A *cut* in $\mathrm{coh}(\mathbf{X})$ is a pair $(\mathcal{C}',\mathcal{C}'')$ of extension closed subcategories of $\mathrm{coh}(\mathbf{X})$ such that $\mathrm{Hom}(\mathcal{C}'',\mathcal{C}') = 0$, and moreover each indecomposable object in $\mathrm{coh}(\mathbf{X})$ either belongs to $\mathcal{C}'$ or to $\mathcal{C}''$. By [79, Proposition 2.2.] for each cut $(\mathcal{C}',\mathcal{C}'')$ in $\mathrm{coh}(\mathbf{X})$, the additive closure $\mathcal{H}$ of $\mathcal{C}'' \vee \mathcal{C}'[1]$ in $\mathcal{D}^b(\mathrm{coh}(\mathbf{X}))$ is a hereditary abelian k-category which is derived equivalent to $\mathrm{coh}(\mathbf{X})$ and has a tilting object.

Conversely, for a hereditary abelian k-category $\mathcal{H}$ each equivalence of triangulated categories $\mathcal{D}^b(\mathcal{H}) \stackrel{\cong}{\to} \mathcal{D}^b(\mathrm{coh}(\mathbf{X}))$ produces a cut $(\mathcal{C}',\mathcal{C}'')$ in $\mathrm{coh}(\mathbf{X})$ such that $\mathcal{H}$ is equivalent to the additive closure of $\mathcal{C}'' \vee \mathcal{C}'[1]$.

Let $\mathbf{X}'\coprod\mathbf{X}''$ be a decomposition of $\mathbf{X}$ into disjoint subsets, and write $\mathcal{C}_0' = \coprod_{\lambda\in\mathbf{X}'}\mathcal{U}_\lambda$, $\mathcal{C}_0'' = \coprod_{\lambda\in\mathbf{X}''}\mathcal{U}_\lambda$. Obviously, $(\mathrm{add}(\mathrm{vect}(\mathbf{X}) \vee \mathcal{C}_0''),\mathcal{C}_0')$ is a cut of $\mathrm{coh}(\mathbf{X})$, accordingly the additive closure $\mathcal{C}(\mathbf{X}',\mathbf{X}'')$ of $\mathcal{C}_0'[-1] \vee \mathrm{vect}(\mathbf{X}) \vee \mathcal{C}_0''$ in $\mathcal{D}^b(\mathrm{coh}(\mathbf{X}))$ is hereditary abelian with a tilting object and is derived-equivalent to $\mathrm{coh}(\mathbf{X})$. Note that $\mathcal{C}(\emptyset,\mathbf{X})$ [resp. $\mathcal{C}(\mathbf{X},\emptyset)$] agrees with $\mathrm{coh}(\mathbf{X})$ [resp. $(\mathrm{coh}(\mathbf{X}))^{op}$].

Let $\mathbf{X}$ be a weighted projective line of genus one. Recall that, for $q \in \mathbf{Q} \cup \{\infty\}$, $\mathcal{C}_q$ denotes the additive closure of indecomposable sheaves of slope q, $q \in \mathbf{Q} \cup \{\infty\}$. For an irrational number r let $\mathcal{C}'^{(r)}$ [resp. $\mathcal{C}''^{(r)}$] be the additive closure of all $\mathcal{C}_q$ with $q < r$ [resp. $r < q \le \infty$]. Then the additive closure $\mathcal{C}\langle r\rangle$ of $\mathcal{C}''^{(r)} \cup \mathcal{C}'^{(r)}[1]$ in $\mathcal{D}^b(\mathrm{coh}(\mathbf{X}))$ is hereditary abelian with a tilting object and is derived equivalent to $\mathrm{coh}(\mathbf{X})$.

It is proved in [79, Proposition 2.3.] that each hereditary k-category $\mathcal{H}$ of canonical type is equivalent to exactly one of the following

(a) $\mathrm{mod}(A)$, where A is a tame hereditary algebra,
(b) $\mathcal{C}(\mathbf{X}',\mathbf{X}'')$ for some decomposition $\mathbf{X}'\coprod\mathbf{X}''$ of $\mathbf{X}$,

(c) $\mathcal{C}\langle r\rangle$, where r is an irrational number and $\mathbf{X}$ is of tubular weighted type.

We are interested in representation-infinite quasitilted algebras of canonical type. They were characterized in [79, Theorem 3.4.] as follows

Theorem 9.6.1 *The following assertions are equivalent for a k-algebra Σ.*

(i) Σ is representation-infinite and quasitilted of canonical type.

(ii) Σ is isomorphic to the endomorphism ring of a tilting object in a category $\mathcal{C}(\mathbf{X}', \mathbf{X}'')$.

(iii) Σ is a semiregular branch enlargement of a concealed-canonical algebra.

(iv) The category mod(Σ) *admits a sincere separating family of semiregular standard tubes.* □

Recall that a tube is called semiregular if it does not contain both a projective and an injective object. Further a semiregular branch enlargement in the meaning of [79] is a branch enlargement in the sense of 8.3.2, for which all extension and coextension branches are truncated branches, and moreover, from a fixed tube either only extension branches or only coextension branches are taken.

Specifying to weighted projective lines of type $(2, ..., 2)$, t entries, we obtain the following description. Fix a decomposition $\mathbf{X}' \coprod \mathbf{X}''$ of $\mathbf{X}$ and a tilting object

$$T = T_0'[-1] \oplus T_+ \oplus T_0''$$

in $\mathcal{H} = \mathcal{C}(\mathbf{X}', \mathbf{X}'')$ with $T_+ \in \mathrm{vect}(\mathbf{X})$ and $T_0', T_0'' \in \mathrm{coh}_0(\mathbf{X})$. Then T_0' and T_0'' are direct sums of simple exceptional finite length sheaves. We assume that T_0' or T_0'' is nonzero. Since T is a tilting complex, we see that T_+ is in the perpendicular category $(T_0'' \oplus \tau_{\mathbf{X}}^- T_0')^\perp$ which, by 2.4.2, is equivalent to a category of coherent sheaves coh($\mathbf{Y}$) for a a weighted projective line $\mathbf{Y}$ of type $(2, ..., 2)$, t' entries, with $t' < t$. Thus T^+ is a tilting bundle on coh($\mathbf{Y}$) and the endomorphism algebra $\Sigma = \mathrm{End}(T)$ is a multi-point extension-coextension of the concealed-canonical algebra $\Sigma_+ = \mathrm{End}(T_+)$ (see 8.3.2). Now, under the assumptions that $\mathbf{X}$ is hyperelliptic, that Σ is tame and that both T_0' and T_0'' are nonzero, it follows from the results of 9.3.6 that $\mathbf{Y}$ is of domestic type.

Conversely, let T_+ be a tilting bundle on a weighted projective line of type $(2, ..., 2)$, t' entries. Choose ordinary points λ_i', $i = t'+1, ..., t'+r$ and λ_j'', $j = t'+r+1, ..., t'+r+s$. Let $\mathbf{X}$ be the weighted projective line obtained from $\mathbf{Y}$ which attaches the weight 2 to the points $\lambda_1', ..., \lambda_r', \lambda_1'', ..., \lambda_s''$ (note that we are identifying the sets underlying $\mathbf{X}$ and $\mathbf{Y}$ with the projective line over k). Further, let $\mathbf{X}' \coprod \mathbf{X}''$ be a decomposition of $\mathbf{X}$ such that each λ_i' belongs to $\mathbf{X}'$ and that each λ_j'' belongs to $\mathbf{X}''$. Then the complex $T_0'[-1] \oplus T_+ \oplus T_0''$, where $T_0' = \bigoplus_{i=t'+1}^{t'+r} \mathcal{S}_{i,0}$ and $T_0'' = \bigoplus_{j=t'+r+1}^{t'+r+s} \mathcal{S}_{j,1}$, is a tilting object in $\mathcal{C}(\mathbf{X}', \mathbf{X}'')$. Here $\mathcal{S}_{i,0}$, $\mathcal{S}_{i,1}$ are the simple finite length sheaves given by the exact sequences

$$0 \to \mathcal{O} \to \mathcal{O}(\vec{x}_i) \to \mathcal{S}_{i,0} \to 0,$$

$$0 \to \mathcal{O}(\vec{x}_i) \to \mathcal{O}(\vec{c}) \to \mathcal{S}_{i,1} \to 0$$

(compare 2.2.5).

For a tilting object $T_0'[-1] \oplus T_+ \oplus T_0''$ in $\mathcal{C}(\mathbf{X}', \mathbf{X}'')$ we write, as in 8.3.5,

$$\Sigma_l = \mathrm{End}(T_0'[-1] \oplus T_+) \qquad \text{and} \qquad \Sigma_r = \mathrm{End}(T_+ \oplus T_0'').$$

Then Σ_l^{op} and Σ_r are almost concealed-canonical algebras. According to 8.3.5, the support of each indecomposable Σ-module belongs to Σ_l or else to Σ_r, and therefore Σ is tame if and only if both Σ_l and Σ_r are tame. Now, applying Theorem 7.1.3, which states that an almost concealed-canonical algebra realized on a wild weighted projective line is wild again, we obtain the following handy criterion for tameness.

Proposition 9.6.1 *Let $\mathbf{X}$ be a weighted projective line of type $(2, ..., 2)$, and let $T_0'[-1] \oplus T_+ \oplus T_0''$ be a tilting object in $\mathcal{C}(\mathbf{X}', \mathbf{X}'')$. Then Σ is tame if and only if Σ_l and Σ_r have less than 7 vertices.* □

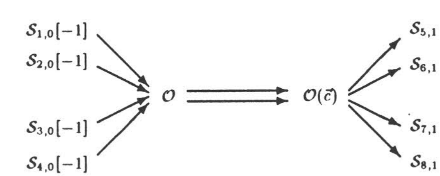
9.6.2 In this subsection we provide realizations for the algebras Σ of list 9.4 by tilting complexes on weighted projective lines of type $(2, ..., 2)$, t entries. We always give a tilting complex for the algebras $\Sigma_i(r, s)$ with maximal r and s, the modifications for the other cases are easily done.

For $t \geq 4$ all these tilting complexes are of the form $T_0'[-1] \oplus T_+ \oplus T_0''$ with $T_+ \in \mathrm{vect}(\mathbf{X})$, $T_0', T_0'' \in \mathrm{coh}_0(\mathbf{X})$, and can be therefore regarded as tilting objects in a category $\mathcal{C}(\mathbf{X}', \mathbf{X}'')$. Moreover, in this situation T_+ can be considered as a tilting bundle on a weighted projective line of type $(2,2,2)$, $(2,2)$, (2) or else on $\mathbf{P}^1$. In the first case we denote by F the uniquely determined rank 2 vector bundle of slope $\frac{1}{2}$.

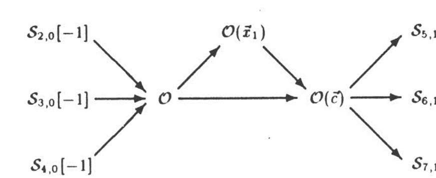
For $t < 3$ the tilting complexes of list 9.6.2 can be regarded as tilting modules in categories mod(A) for tame hereditary algebras A.

List 9.6.2 Realization of the algebras of list 9.4 as endomorphism algebras of tilting complexes

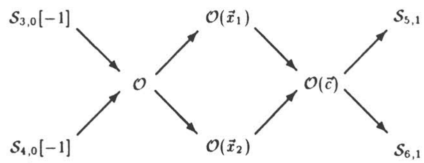
$\Sigma_1(4,4)$



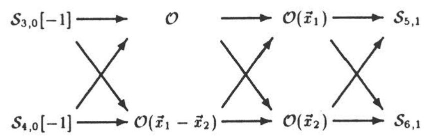
$\Sigma_2(3,3)$



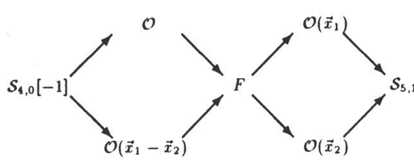
$\Sigma_3(2,2)$



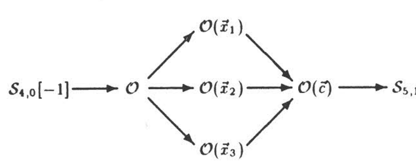
$\Sigma_4(2,2)$



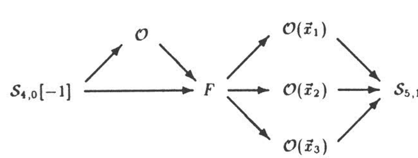
$\Sigma_5(1,1)$



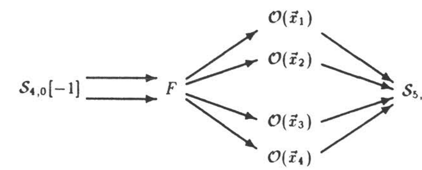
$\Sigma_6(1,1)$



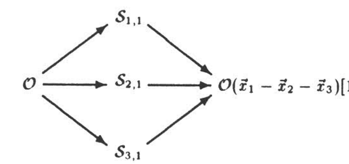
$\Sigma_7(1,1)$



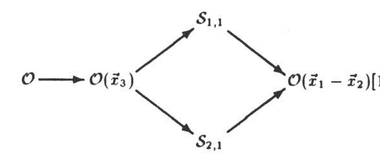
$\Sigma_8(1,1)$



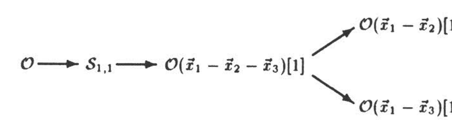
Σ_9



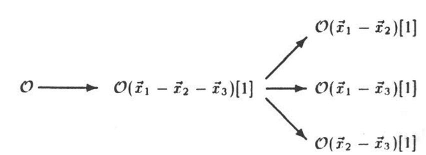
Σ_{10}



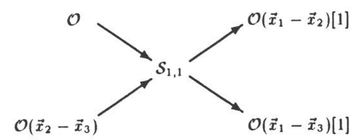
Σ_{11}



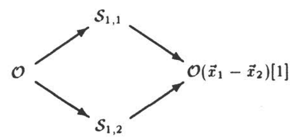
Σ_{12}



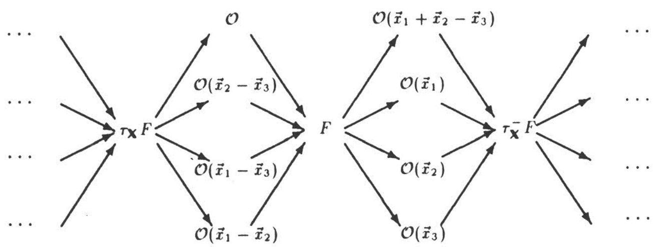
Σ_{13}



Σ_{14}



In each case one proves that the listed complexes are in fact tilting complexes having the corresponding algebras as endomorphism algebras. In fact, the given complexes have the correct numbers of indecomposable direct summands. Moreover, the fulfilment of condition (i) of Definition 8.1.1 is a consequence of the formula determining the Hom-dimension between line bundles, the Serre duality, the interaction between line bundles and finite length sheaves (see the exact sequences in 9.6.1) and the fact that the rank-2 vector bundle F has a line bundle filtration (for instance given by an Auslander-Reiten sequence). Recall that the Auslander-Reiten quiver of the sheaf category for a weighted projective line of type $(2,2,2)$ has a unique vector bundle component of the form



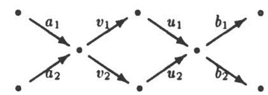
The tools mentioned above also allow to determine the quivers and relations for the given complexes. We illustrate this in detail in two typical cases, the arguments in the remaining ones are of the same type.

Also, it follows from 9.6.1 that all endomorphism algebras of the complexes listed here are quasitilted and tame.

9.6.2.1 First, as a typical example for a tilting complex whose indecomposable direct summands are only line bundles and finite length sheaves, we consider a realization for an algebra of type Σ_3. Take the tilting complex

$$T = \mathcal{S}_{3,0}[-1] \oplus \mathcal{S}_{4,0}[-1] \oplus \mathcal{O} \oplus \mathcal{O}(\vec{x}_1) \oplus \mathcal{O}(\vec{x}_2) \oplus \mathcal{O}(\vec{c}) \oplus \mathcal{S}_{5,1} \oplus \mathcal{S}_{6,1}$$

on a weighted projective line $\mathbb{X} = \mathbb{X}((2,2,2,2,2,2),(\lambda_1,\lambda_2,\dots,\lambda_6))$. We can assume that $\lambda_1 = \infty$, $\lambda_2 = 0$ and $\lambda_3 = 1$. Our aim is to show that $\Sigma = \mathrm{End}(T)$ is isomorphic to the algebra $\Sigma_3(2,2)$ as described in 9.4. Obviously the quiver Q of Σ has the shape



We will give a presentation $\epsilon : kQ/I \xrightarrow{\cong} \Sigma$ such that I coincides with the ideal described in the list. First define $\epsilon(u_1) = X_1$, $\epsilon(v_1) = X_1$, $\epsilon(u_2) = X_2$, $\epsilon(v_2) = X_2$, where the X_i's denote the obvious multiplications $\mathcal{O} \to \mathcal{O}(\vec{x}_i)$ and $\mathcal{O}(\vec{x}_i) \to \mathcal{O}(\vec{c})$, respectively.

For $i = 3,4$ there are exact sequences

$$\eta_i : \qquad 0 \longrightarrow \mathcal{O} \xrightarrow{X_i} \mathcal{O}(\vec{x}_i) \longrightarrow \mathcal{S}_{i,0} \longrightarrow 0$$

which give rise to morphisms $\alpha_i \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{S}_{i,0}[-1], \mathcal{O})$. We define $\epsilon(a_1) = \alpha_3$ and $\epsilon(a_2) = \alpha_4$. We further choose, for $j = 5,6$, nonzero morphisms $\beta_j \in \mathrm{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{c}), \mathcal{S}_{j,1})$ and define $\epsilon(b_1) = \beta_5$ and $\epsilon(b_2) = \beta_6$.

Now, for $j = 5,6$, $\mathrm{Hom}_{\mathbb{X}}(\mathcal{O}, \mathcal{S}_{j,1})$ is one-dimensional. Consider the exact sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{u} \mathcal{O}(\vec{c}) \xrightarrow{\kappa_j} \mathcal{S}_{j,1}^{[2]} \longrightarrow 0$$

where $u = X_2^2 - \lambda_j X_1^2$ and $\mathcal{S}_{j,1}^{[2]}$ is the sheaf of quasi-length 2 such that there is an irreducible epimorphism $\nu_j : \mathcal{S}_{j,1}^{[2]} \twoheadrightarrow \mathcal{S}_{j,1}$. Up to a scalar we have $\beta_j = \nu_j \kappa_j$. Hence $\beta_j u = 0$, which implies $\beta_j X_2^2 - \lambda_j \beta_j X_1^2 = 0$, and consequently $b_l(u_2 v_2 - \lambda_{l+4} u_1 v_1) \in I = \ker \epsilon$, for $l = 1,2$.

On the other hand, we have $\dim_k \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{S}_{i,0}, \mathcal{O}(\vec{c})) = 1$, for $i = 3,4$. Note that the exact sequence $u \cdot \eta_i$, obtained from η_i in the pushout diagram along the morphism $u = X_2^2 - \lambda_i X_1^2$, splits. Therefore $0 = u \cdot \alpha_i = (X_2^2 - \lambda_i X_1^2)\alpha_i$, which implies that the elements $(u_2 v_2 - \lambda_{l+2} u_1 v_1) a_i$ belong to I for $l = 1,2$.

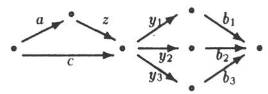
Observe, that the fact that the parameters are pairwise distinct implies that all $b_j u_l v_l a_i$ are in I.

9.6.2.2 Next we determine the endomorphism ring of the tilting complex

$$T = \mathcal{S}_{4,0}[-1] \oplus \mathcal{O} \oplus F \oplus \mathcal{O}(\vec{x}_1) \oplus \mathcal{O}(\vec{x}_2) \oplus \mathcal{O}(\vec{x}_3) \oplus \mathcal{S}_{5,1}$$

on a weighted projective line $\mathbb{X} = \mathbb{X}((2,2,2,2,2),(\lambda_1,\lambda_2,\dots,\lambda_5))$. Again we assume that $\lambda_1 = \infty$, $\lambda_2 = 0$ and $\lambda_3 = 1$.

The quiver Q of $\Sigma = \mathrm{End}(T)$ is easily calculated, it is of the form



We shall choose representatives for the arrows in a special way in order to get a presentation $\epsilon : kQ/I \xrightarrow{\cong} \Sigma$ such that I coincides with the ideal for $\Sigma_7(1,1)$ as described in 9.4. For this we first fix a nonzero map $\beta_c \in \mathrm{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{c}), \mathcal{S}_{5,1})$ and define $\epsilon(b_j)$ to be the compositions $\beta_j : \mathcal{O}(\vec{x}_j) \xrightarrow{X_j} \mathcal{O}(\vec{c}) \xrightarrow{\beta_c} \mathcal{S}_{5,1}$, $j = 1,2,3$. Further we choose a nonzero map $\zeta \in \mathrm{Hom}_{\mathbb{X}}(\mathcal{O}, F)$. Using that $\mathrm{Hom}_{\mathbb{X}}(\zeta, \mathcal{O}(\vec{x}_j)) : \mathrm{Hom}_{\mathbb{X}}(F, \mathcal{O}(\vec{x}_j)) \to \mathrm{Hom}_{\mathbb{X}}(\mathcal{O}, \mathcal{O}(\vec{x}_j))$ are isomorphisms, we see that there are $\xi_j \in \mathrm{Hom}_{\mathbb{X}}(F, \mathcal{O}(\vec{x}_j))$ such that $\xi_j\zeta = X_j$. We define $\epsilon(y_j) = \xi_j$.

We have $\dim_k \mathrm{Hom}_{\mathbb{X}}(F, \mathcal{S}_{5,1}) = 2$ and $\dim_k \mathrm{Hom}_{\mathbb{X}}(\mathcal{O}, \mathcal{S}_{5,1}) = 1$. Applying the functor $\mathrm{Hom}_{\mathbb{X}}(-, \mathcal{O}(\vec{c}))$ to the exact sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{\zeta} F \longrightarrow \mathcal{O}(\vec{x}_1 + \vec{x}_2 - \vec{x}_3) \longrightarrow 0,$$

the relation $X_3^2 - X_2^2 - X_1^2$ in $\mathrm{Hom}_{\mathbb{X}}(\mathcal{O}, \mathcal{O}(\vec{c}))$ gives a relation $X_3\xi_3 - X_2\xi_2 - X_1\xi_1$ in $\mathrm{Hom}_{\mathbb{X}}(F, \mathcal{O}(\vec{c}))$. Composing with β_c we see that $\beta_3\xi_3 - \beta_2\xi_2 + \beta_1\xi_1 = 0$. Next, we calculate an additional relation in $\mathrm{Hom}_{\mathbb{X}}(\mathcal{O}, \mathcal{S}_{5,1})$. To do this, we conclude similarly as in 9.6.2.1 that $0 = \beta_c(X_2^2 - \lambda_5 X_1^2)$. Thus $0 = \beta_2 X_2 - \lambda_5 \beta_1 X_1 = \beta_2\xi_2\zeta - \lambda_5\beta_1\xi_1\zeta$.

We complete the definition of ϵ by choices of representatives for the arrows a and c at the left hand side of the quiver. Let $\epsilon(a) = \alpha$ where α is the exact sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{X_4} \mathcal{O}(\vec{x}_4) \longrightarrow \mathcal{S}_{4,0} \longrightarrow 0.$$

Now, $\epsilon(c)$ has to be chosen as a morphism $\gamma \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{S}_{4,0}[-1], F)$ which does not factorize through $\mathcal{O}$. We choose γ as a morphism factorizing through $\mathcal{O}(\vec{x}_2 - \vec{x}_3)$ using the pushout diagram

$$\begin{array}{lccccccccc}
\gamma' : & 0 & \longrightarrow & \mathcal{O}(\vec{x}_2 - \vec{x}_3) & \longrightarrow & \mathcal{O}(\vec{x}_2 - \vec{x}_3 - \vec{x}_4) & \longrightarrow & \mathcal{S}_{4,0} & \longrightarrow & 0 \\
 & & & \downarrow \zeta_1 & & \downarrow & & \downarrow = & & \\
\gamma = \zeta_1\gamma' : & 0 & \longrightarrow & F & \longrightarrow & M & \longrightarrow & \mathcal{S}_{4,0} & \longrightarrow & 0
\end{array}$$

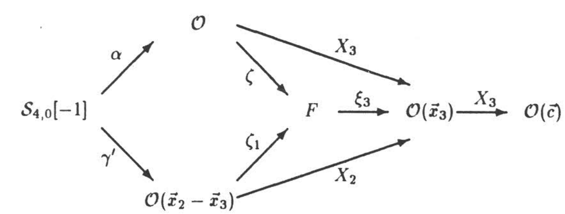
where $\gamma' \in \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{S}_{4,0}, \mathcal{O}(\vec{x}_2 - \vec{x}_3))$ and $\zeta_1 \in \mathrm{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{x}_2 - \vec{x}_3), F)$ satisfy, additionally, the conditions $\xi_2\zeta_1 = X_3$ in $\mathrm{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{x}_2 - \vec{x}_3), \mathcal{O}(\vec{x}_2))$ and $X_3\gamma' = X_2\alpha$ in $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{S}_{4,0}, \mathcal{O}(\vec{x}_2))$. Note that these choices are possible, because
$\mathrm{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{x}_2 - \vec{x}_3), \xi_2) : \mathrm{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{x}_2 - \vec{x}_3), F) \to \mathrm{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{x}_2 - \vec{x}_3), \mathcal{O}(\vec{x}_2))$ and
$\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{S}_{4,0}, X_3)) : \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{S}_{4,0}, \mathcal{O}(\vec{x}_2 - \vec{x}_3)) \to \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{S}_{4,0}, \mathcal{O}(\vec{x}_2))$ are isomorphisms.

We obtain that $y_1 c \in I$. Indeed, application of $\mathrm{Hom}_{\mathbb{X}}(\mathcal{S}_{4,0}, -)$ to the exact sequence

$$0 \longrightarrow \mathcal{O}(\vec{x}_2 - \vec{x}_3) \xrightarrow{\zeta_1} F \xrightarrow{\xi_1} \mathcal{O}(\vec{x}_1) \longrightarrow 0$$

gives $\xi_1\gamma = \xi_1\zeta_1\gamma' = 0$. Furthermore, $y_2(c - za) \in I$, because $0 = X_3\gamma' - X_2\alpha = \xi_2\zeta_1\gamma' - \xi_2\zeta\alpha = \xi_2(\gamma - \zeta\alpha)$.

In order to find a relation in $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{S}_{4,0}, \mathcal{O}(\vec{x}_3))$ consider the following (non-commutative) diagram



Similarly as in 9.6.2.1 there is a relation $X_4^2\alpha = 0$ in $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{S}_{4,0}, \mathcal{O}(\vec{c}))$, hence $X_1^2\alpha = \frac{1}{\lambda_4}X_2^2\alpha$. Invoking the relation $X_3^2 - X_2^2 + X_1^2 = 0$ we infer that $X_3^2\alpha = \frac{\lambda_4 - 1}{\lambda_4}X_2^2\alpha$. Now, $X_3\xi_3\gamma = X_3\xi_3\zeta_1\gamma' = X_3X_2\gamma' = X_2X_3\gamma' = X_2^2\alpha$, and therefore $X_3(\xi_3\zeta\alpha - \frac{\lambda_4 - 1}{\lambda_4}\xi_3\gamma) = X_3\xi_3\zeta\alpha - \frac{\lambda_4 - 1}{\lambda_4}X_3\xi_3\gamma = X_3^2\alpha - \frac{\lambda_4 - 1}{\lambda_4}X_2^2\alpha = 0$.

Since $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{S}_{4,0}, X_3)) : \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{S}_{4,0}, \mathcal{O}(\vec{x}_3)) \to \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{S}_{4,0}, \mathcal{O}(\vec{c}))$ is an isomorphism, we obtain $\xi_3(\zeta\alpha - \frac{\lambda_4 - 1}{\lambda_4}\gamma)$, and consequently $y_3(za - \frac{\lambda_4 - 1}{\lambda_4}c) \in I$.

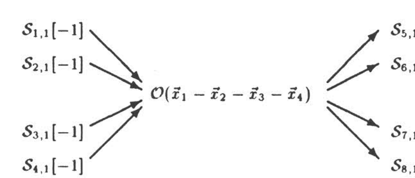
Observe again that the fact that the parameters are pairwise distinct implies that all pathes between the two extreme vertices belong to I.

9.6.3 In this subsection we will realize the algebras $\Delta_i(r,s)$, for $1 \le i \le 7$, of list 9.5 as endomorphism algebras of tilting complexes for subspace problem algebras. Note that the remaining algebras Δ_i, $8 \le i \le 10$, have 5 vertices and appear already in list 9.4. Thus they can be realized by tilting complexes in the derived category of coherent sheaves over a weighted projective line of type $(2,2,2)$, which is equivalent to the derived category of modules over the 4-subspace problem algebra.

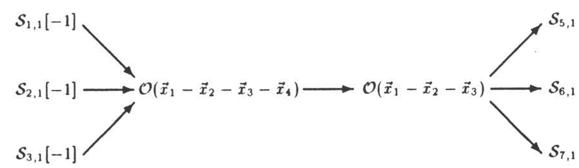
For the relevant realizations we identify the category of modules over the t-subspace problem algebra Λ_0 with the left perpendicular category ${}^{\perp}\mathcal{O}$ to the structure sheaf $\mathcal{O}$, formed in a sheaf category $\mathrm{coh}(\mathbb{X})$ for a weighted projective line of type $(2,\ldots,2)$, t entries. Under this identification a tilting complex in $\mathcal{D}^b(\mathrm{mod}(\Lambda_0))$ is the same as a partial tilting complex in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ having $t+1$ indecomposable direct summands which are contained in $\mathcal{D}^b({}^{\perp}\mathcal{O})$.

List 9.6.3 Realization of the algebras of list 9.5 as endomorphism algebras of tilting complexes

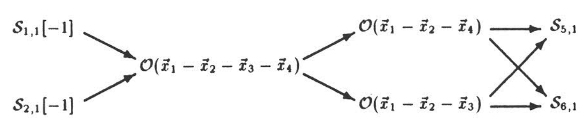
$\Delta_1(4,4)$



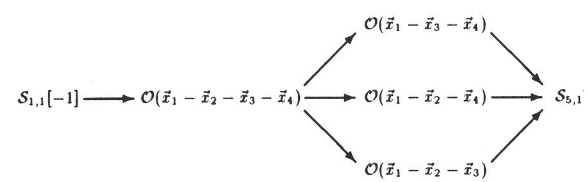
$\Delta_2(3,3)$



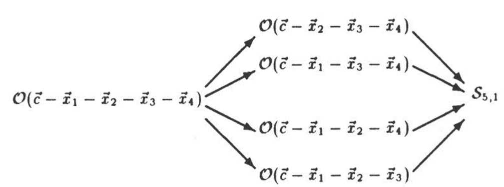
$\Delta_3(2,2)$



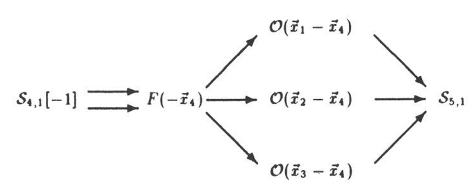
$\Delta_4(1,1)$



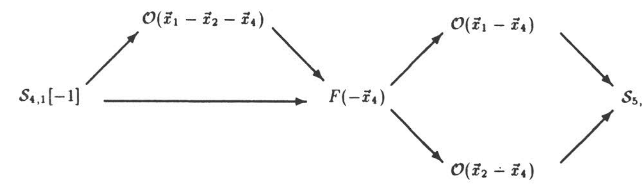
$\Delta_5(1)$



$\Delta_6(1,1)$



$\Delta_7(1,1)$



Using the same methods as in 9.6.2 it is easy to check that the given complexes in fact satisfy the vanishing condition of Definition 8.1.1 and are contained in $\mathcal{D}^b({}^{\perp}\mathcal{O})$. Moreover, the complexes have the correct numbers of indecomposable direct summands. Furthermore, changing if necessary the representatives for the arrows, their endomorphism algebras are isomorphic to the corresponding algebras Δ_i. Note that here parameters in the relations do not occur.

Observe also that each of the algebra of list 9.5, except $\Delta_5(1)$, can be considered as a full subcategory of an algebra of list 9.4. In particular, these algebras are tame, and by [42, Chapter II, Proposition 1.15] quasitilted. On the other hand $\Delta_5(1)$, as a one-point extension of a tame hereditary algebra by an indecomposable injective module, is tame and quasitilted, too.

9.6.4 It remains to show that an arbitrary algebra derived equivalent to a hyperelliptic or to a subspace problem algebra is isomorphic to an algebra of our lists. In order to determine the quivers for such algebras we will use the following result.

Lemma 9.6.4 *Let A be a finite dimensional k-algebra such that $\mathcal{D}^b(\mathrm{mod}(A))$ is triangle-equivalent to $\mathcal{D}^b(\mathcal{H})$ for a hereditary category $\mathcal{H}$. Let S, S' be simple A-modules. Then there exists at most one $i \geq 0$ such that $\mathrm{Ext}^i_A(S, S') \neq 0$.* □

The lemma was formulated in [39, Chapter IV, Lemma 1.11] in case $\mathcal{H}$ is a module category for a finite dimensional hereditary k-algebra, the proof is easily modified to the more general situation.

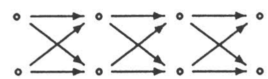
As a consequence the lemma implies that the quiver of A does not contain parallel paths v, w such that v is zero but w is nonzero in A.

If, in particular, $\Sigma = \mathrm{End}(T)$ is the endomorphism ring of a tilting complex in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ for some weighted projective line $\mathbb{X}$ of type $(2, ..., 2)$, or else in $\mathcal{D}^b(\mathrm{mod}(\Lambda_0))$ for a subspace problem algebra Λ_0, and if moreover Σ is tame, then the quiver of Σ is uniquely determined by the Cartan matrix of Σ. Indeed, we know that in this case the Cartan triangle is one of the triangles characterized in Corollary 9.3.7. Now, applying the lemma one easily proves that in each case there is only one possibility for the quiver.

Example. Let Σ be tame and the endomorphism ring of a tilting complex in $\mathcal{D}^b(\mathrm{coh}(\mathbb{X}))$ such that

$$C(\Sigma) = \left(\begin{array}{ccccccc} & & & 0 & & & \\ & & 1 & & 1 & & \\ & 1 & & 1 & & 1 & \\ \frac{}{2} & - & \frac{}{2} & - & \frac{}{2} & - & \frac{}{2} \end{array} \right)$$

Then the quiver of Σ is of the form



because no path of length 2 can be zero. □

9.6.5 In the converse of the proofs of our classification theorems an important role is played by investigating Coxeter polynomials. For a finite dimensional algebra A with Cartan matrix C_A the *Coxeter matrix* is defined as

$$\Phi_A := -{}^tC_A \cdot C_A^{-1}$$

where ${}^tC_A$ denotes the transpose of the matrix C_A. It is the matrix of the automorphism $\varphi_A : K_0(A) \to K_0(A)$ given by the formula $\varphi_A([P(S)]) = -[I(S)]$ for each simple A-module S. Here $P(S)$ and $I(S)$ denote the projective cover and the injective hull of S, respectively. The map φ_A is said to be the *Coxeter transformation* of A.

The characteristic polynomial $\Psi_A = \det(TE - \Phi_A)$ is called the *Coxeter polynomial* of A. It follows from [37] that two algebras which are derived equivalent share the same Coxeter polynomial. In [77] the Coxeter polynomials for canonical algebras and for path algebras of stars are determined. In particular, for the algebras we are interested in we have

Proposition 9.6.6 *(i) If* $\mathbb{X}$ *is a weighted projective line of type* $(2, ..., 2)$, *t entries, then*

$$\Psi_A = (T-1)^2 \cdot (T+1)^t.$$

(ii) If Λ_0 *is the t-subspace algebra, then*

$$\Psi_A = (T+1)^{t-1} \cdot (T^2 - (t-2)T + 1).$$

□

9.6.7 Now we continue the proofs of Theorem 9.4.1 and Theorem 9.5.1. Assume that A is a tame algebra which is derived equivalent to a canonical algebra of type $(2, ..., 2)$ or to a t-subspace problem algebra. We have to show that A or A^{op} is isomorphic to one of the algebras of list 9.4 or list 9.5, respectively.

Since A is tame the Cartan triangle $C(A)$ is one of the Cartan triangles described in 9.3.7. It follows from 9.1.4 and the explicit calculation of the Coxeter polynomials that the Cartan triangle of A coincides with one of the Cartan triangles of an algebra of list 9.4 or list 9.5. Moreover, according to 9.6.4, the quiver Q of A is determined uniquely by its Cartan triangle.

Finally we have to determine the relations for A. First, in case A is derived equivalent to a subspace problem algebra, it is easily checked that for some choice of the representatives for the arrows the ideal I coincides with the ideal of the corresponding algebra of list 9.5.

The same can be done if A is derived equivalent to a domestic or tubular canonical algebra of type $(2, ..., 2)$ (observe that in the later case A depends on one parameter λ).

Now, let A be derived equivalent to a hyperelliptic algebra. We know that there is an algebra Σ of list 9.4 such that the Cartan triangles and the quivers of A and Σ coincide. According to [75], the rank function $\mathrm{rk} : K_0(\mathbb{X}) \to \mathbb{Z}$ is, up to sign, uniquely determined. From this and from the realization of Σ we deduce that A is the endomorphism ring of a tilting complex of the form $T = T_0'[-1] \oplus T_+ \oplus T_0''$ with $T_0', T_0'' \in \mathrm{coh}_0(\mathbb{X})$ and $T_+ \in \mathrm{vect}(\mathbb{X})$. Then T_+ is in $(T_0'' \oplus \tau_{\mathbb{X}}^- T_0')^{\perp}$, which is equivalent to a sheaf category $\mathrm{coh}(\mathbb{Y})$, where $\mathbb{Y}$ is a weighted projective line of type $(2,2,2)$, $(2,2)$, (2) or $\mathbb{Y} = \mathbb{P}^1$. Therefore T_+ is either, up to a line bundle shift, the canonical tilting sheaf for $\mathbb{Y}$ of type $(2,2,2)$, or it is given by a complete slice in the unique component of vector bundles of $\mathrm{coh}(\mathbb{Y})$. Again, by means of a line bundle shift, we can assume that this slice is the same as for the tilting bundle in the realization for Σ. Obviously, T_0' and T_0'' are direct sums of simple finite length sheaves on $\mathbb{X}$. Hence the endomorphism ring A of T can be calculated by the method described in 9.6.2 and it turns out that A is isomorphic to an algebra $\Sigma_i(r,s)$ for suitable parameters.

Let us illustrate the method in two typical examples.

(a) Let A be a tame algebra derived equivalent to a canonical algebra of type $(2, ..., 2)$ or to a subspace problem algebra such that $H(A) = \left(\begin{array}{ccccccc} & & & 0 & & & \\ & & 1 & & 1 & & \\ & 2 & & 1 & & 2 & \\ 1 & & 1 & & 1 & & 1 \end{array} \right)$.

According to 9.3.7 we have the conditions $a = c = e = 1$ and $b+d \leq 4$ for the sequence of the numbers of the vertices $v = (a,b,c,d,e)$. Suppose that $b = d = 1$. Then the Coxeter polynomial for an algebra having this Cartan triangle equals $(T^2 - 4T + 1)(T+1)^3$, which is neither a Coxeter polynomial of a canonical algebra of type $(2, ..., 2)$, nor of a subspace problem algebra. Hence this case is impossible. For $b = 1$, $d = 2$ we obtain the algebra $\Delta_7(1,1)$ of list 9.5 and the choices $b = 1$, $c = 3$ and $b = 2$, $d = 2$ lead to the algebras of type $\Sigma_7(1,1)$ and $\Sigma_5(1,1)$ of list 9.4, respectively.

(b) Let A be a tame algebra derived equivalent to a canonical algebra of type $(2, ..., 2)$ or to a subspace problem algebra such that $H(A) = \left(\begin{array}{ccccccc} & & & 0 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 1 & & 1 & & 1 \end{array} \right)$.

Here $v = (a,1,c,1,e)$ with $a + c \leq 4$ and $c + e \leq 4$. For $c = 3$, $a = 1 = e$ we get algebras of type $\Sigma_6(1,1)$ and for $c = 2$, $c = 1$ we obtain algebras of type $\Sigma_3(r,s)$, $\Sigma_2(r,s)$, respectively.

In the same way one proceeds for the other layer triangles, which completes the proofs. □

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