Asymptotic Expansions for Second-Order Moments of Integral Functionals of Weakly Correlated Random Functions

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Abstract

In the paper asymptotic expansions for second-order moments of integral functionals of a class of random functions are considered. The random functions are assumed to be ε-correlated, i.e. the values are not correlated excluding a ε-neighbourhood of each point. The asymptotic expansions are derived for $\varepsilon \to 0$. With the help of a special weak assumption there are found easier expansions as in the case of general weakly correlated functions ([1]).

1 Introduction

In this paper asymptotic expansions for second-order moments of integral functionals of the type

$$
{}^{\varepsilon}r(\omega) = \int_{\mathcal{D}} Q(s)\, {}^{\varepsilon}f(s,\omega)\, ds \tag{1}
$$

are considered, where $Q(\cdot)$ is a deterministic function, $\mathcal{D} \subset \mathbf{R}$ and $({}^{\varepsilon}f(\cdot)), \varepsilon > 0$, is a family of real valued random functions, indexed by a parameter ε. We will suppose the validity of the following

Assumption 1.1

1. *${}^{\varepsilon}f(\cdot)$ is weakly stationary with a correlation function*

$$
\mathbf{E}({}^{\varepsilon}f(s)\, {}^{\varepsilon}f(t)) = {}^{\varepsilon}R(t-s);
$$

2. *${}^{\varepsilon}f(\cdot)$ is centered: $\mathbf{E}({}^{\varepsilon}f(s)) = 0$;*

3. *${}^{\varepsilon}f(\cdot)$ is ε-correlated, i. e. ${}^{\varepsilon}R(s) = 0$ for $|s| \geq \varepsilon$;*

4. *the correlation functions ${}^{\varepsilon}R(\cdot)$ are generated by a correlation function $R(\cdot)$ of a 1–correlated weakly stationary process:*

$$ {}^{\varepsilon}R(s) = R\left(\frac{s}{\varepsilon}\right), \; s \in \mathbf{R}, \, \varepsilon > 0; $$

5. *the correlation function $R(\cdot)$ is continuous on $\mathbf{R}$, hence the processes ${}^{\varepsilon}f(\cdot)$ are continuous in mean square on $\mathbf{R}$.*

The integral in (1) is assumed to be a mean square integral, under weak conditions it coincides a. s. with the pathwise integral.
The family $({}^{\varepsilon}f(\cdot))$, $\varepsilon > 0$, for example can be a family of so called weakly correlated random functions. In the theory of weakly correlated random functions (cf. [1], [2]) asymptotic expansions of the type

$$ \mathbf{E}({}^{\varepsilon}r_1 \cdot {}^{\varepsilon}r_2 \cdot \ldots \cdot {}^{\varepsilon}r_m) = \begin{cases} c_m^1 \varepsilon^{\frac{m}{2}} + \ldots & \text{for even } m \\ c_m^1 \varepsilon^{\frac{m+1}{2}} + \ldots & \text{for odd } m > 1 \end{cases} $$

are derived. The indices 1 to m refer to deterministic functions $Q_1(\cdot), \ldots, Q_m(\cdot)$ and domains $\mathcal{D}_1, \ldots, \mathcal{D}_m$. Here we will consider only second-order moments and propose a new method of obtaining such asymptotic expansions, which seems to be easier and clarifies in a certain sense the structure of asymptotic expansions in the case of correlation functions. The main difference to the general theory of weakly correlated random functions consists in the explicitely given generating condition 4 of assumption 1.1.
In the following treatment the concept of correlation moments of weakly stationary processes is used.

Definition 1.1 *Let be $j \in \mathbf{N}_0 = \{0, 1, 2, \ldots\}$ and $R(\cdot)$ a continuous correlation function of a weakly stationary process with $R(0) = 1$ and*

$$ \int_{-\infty}^{\infty} |s|^j |R(s)| \, ds < \infty. $$

Then

$$ \mu_j = \int_{-\infty}^{\infty} s^j R(s) \, ds = \begin{cases} 0 & j \text{ odd} \\ 2 \int\limits_0^{\infty} s^j R(s) \, ds & j \text{ even} \end{cases} $$

is called the correlation moment of j-th order of the correlation function or the random process and

$$ \nu_j = \int_{-\infty}^{\infty} |s|^j R(s) \, ds = 2 \int_0^{\infty} s^j R(s) \, ds $$

is called the absolute correlation moment of j-th order.

We remark some properties of correlation moments:

1. From the positive definiteness of the correlation function it follows
$$\mu_0 = \nu_0 \geq 0.$$

2. This is not true for higher-order correlation moments, i. e. there exist correlation functions and numbers $j \in \mathbf{N}$ with $\nu_j < 0$.

3. For ε-correlated stationary processes correlation moments of all orders exist and it holds
$$\lim_{j\to\infty} \nu_j = \lim_{j\to\infty} \mu_j = 0.$$

We also note, that for 1–correlated weakly stationary random processes the following version of the Shannon-Kotelnikov-sampling theorem is valid:

Proposition 1.1 *Let*
$$S(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(s)\exp(-i\alpha s)\,ds = \frac{1}{\pi}\int_0^1 R(s)\cos(\alpha s)\,ds$$
denote the spectral density of a 1–correlated weakly stationary random process. Then it holds the representation
$$S(\alpha) = \lim_{m\to\infty}\sum_{n=-m}^{m}\frac{\sin(\alpha - n\pi)}{\alpha - n\pi}S(n\pi),$$
for all $\alpha \in \mathbf{R}$ with $\frac{\sin 0}{0} := 1$.

In the following treatment we will also suppose that the function $Q(\cdot)$ satisfies the

Assumption 1.2 *The deterministic function $Q(\cdot)$ is piecewise $N-$ times absolutely continuous differentiable on $\mathcal{D}$, $N \in \mathbf{N}_0$ and its derivatives up to the order $(N+1)$ belong to the space $L^2(\mathcal{D})$.*

The piecewise differentiability condition on $Q(\cdot)$ means, that the set of discontinuity points of the function or the derivatives is at most countable and has no points of accumulation and further that the function and its derivatives have finite one-sided limits in this points.
In order to obtain propositions about the limit behaviour only the assumptions on $Q(\cdot)$ can be weakened.

2 Asymptotic expansions of variances

From (1) it follows that

$$\begin{aligned}\mathbf{E}({}^{\varepsilon}r^2) &= \int_{\mathcal{D}}\int_{\mathcal{D}} Q(s)Q(t)\mathbf{E}({}^{\varepsilon}f(s)\,{}^{\varepsilon}f(t))\,ds\,dt \\ &= \int_{\mathcal{D}}\int_{\mathcal{D}} Q(s)Q(t)\,{}^{\varepsilon}R(t-s)\,ds\,dt \\ &= \int_{\mathcal{D}}\int_{\mathcal{D}} Q(s)Q(t)\,R\left(\frac{t-s}{\varepsilon}\right)\,ds\,dt.\end{aligned}$$

The substitution of the variables $t = t, u = \frac{t-s}{\varepsilon}$ yields

$$\mathbf{E}({}^{\varepsilon}r^2) = \varepsilon \int\int_{\mathcal{D}'} Q(t-\varepsilon u)Q(t)R(u)\,dt\,du \tag{2}$$

with the transformed domain of integration $\mathcal{D}' \subset \mathbf{R}^2$.
Now the cases $\mathcal{D} = \mathbf{R}$, $\mathcal{D} = \mathbf{R}_+$ and $\mathcal{D} = [a, b]$, a, b finite, are considered explicitely.

2.1 The case $\mathcal{D} = \mathbf{R}$

Here we consider the random variable

$${}^{\varepsilon}r = \int_{-\infty}^{\infty} Q(s)\,{}^{\varepsilon}f(s)\,ds,$$

so that

$$\begin{aligned}\mathbf{E}({}^{\varepsilon}r^2) &= \varepsilon \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(t-\varepsilon u)Q(t)R(u)\,dt\,du \\ &= \varepsilon \int_{-1}^{1} R(u) \int_{-\infty}^{\infty} Q(t-\varepsilon u)Q(t)\,dt\,du.\end{aligned}$$

With the notation

$$\phi(u) = \int_{-\infty}^{\infty} Q(t-u)Q(t)\,dt,$$

we have

$$\mathbf{E}({}^{\varepsilon}r^2) = \varepsilon \int_{-1}^{1} \phi(\varepsilon u)R(u)\,du$$

and we can see, that the value of the variance depends on the correlation function $R(\cdot)$ and the behavior of the function $\phi(\cdot)$ in a neighbourhood of zero. Now we apply the Taylor expansion of the function $\phi(\cdot)$:
From assumption 1.2 we find for all $t \in \mathbf{R}$, $u \in [-1, 1]$

$$Q(t-\varepsilon u) = \sum_{j=0}^{N} Q^{(j)}(t)\frac{(-\varepsilon u)^j}{j!} + \tilde{\rho}_{N+1}(t, u, \varepsilon)$$

with

$$\tilde{\rho}_{N+1}(t,u,\varepsilon) \;=\; \frac{1}{N!}\int_t^{t-\varepsilon u} Q^{(N+1)}(v)(t-\varepsilon u - v)^N\,dv$$

and it follows

$$\begin{aligned}\mathbf{E}({}^{\varepsilon}r^2) \;&=\; \sum_{j=0}^{N}\frac{(-1)^j\varepsilon^{j+1}}{j!}\cdot\int_{-\infty}^{\infty}Q(t)Q^{(j)}(t)\,dt\cdot\int_{-1}^{1}u^jR(u)\,du + \\ &+\frac{\varepsilon}{N!}\int_{-1}^{1}\int_{-\infty}^{\infty}\int_t^{t-\varepsilon u}Q^{(N+1)}(v)(t-\varepsilon u-v)^N dv Q(t)dtR(u)du.\end{aligned} \tag{3}$$

For the quantities

$$q_j = \int_{-\infty}^{\infty} Q(t)Q^{(j)}(t)\,dt,\;\; j=0,\dots,N, \tag{4}$$

which do not depend on the random process, partial integration yields if $Q(\cdot)$ is $N-$times continuous differentiable on $\mathbf{R}$

$$\begin{aligned}\int_{-\infty}^{\infty} Q(t)Q^{(j)}(t)\,dt \;&=\; \Big[Q(t)Q^{(j-1)}(t)\Big]_{t\to-\infty}^{t\to\infty} - \int_{-\infty}^{\infty}Q'(t)Q^{(j-1)}(t)\,dt \\ &=\; -\int_{-\infty}^{\infty}Q'(t)Q^{(j-1)}(t)\,dt = \dots.\end{aligned}$$

Performing this procedure we find for $k=0,1,\dots$

$$\begin{aligned}q_{2k+1} \;&=\; (-1)^k\int_{-\infty}^{\infty}Q^{(k)}(t)Q^{(k+1)}(t)dt \\ &=\; \frac{(-1)^k}{2}\Big[[Q^{(k)}(t)]^2\Big]_{t\to-\infty}^{t\to\infty} = 0 \\ q_{2k} \;&=\; (-1)^k\int_{-\infty}^{\infty}Q^{(k)}(t)Q^{(k)}(t)\,dt = (-1)^k\int_{-\infty}^{\infty}[Q^{(k)}(t)]^2\,dt.\end{aligned}$$

If $Q(\cdot)$ is only piecewise $N-$times continuous differentiable additional terms may occur.

Theorem 2.1 *Let $({}^{\varepsilon}f(s);\, s\in\mathbf{R}), \varepsilon>0$, be a family of random processes satisfying the assumption 1.1 and $Q(\cdot)$ a function, satisfying the assumption 1.2 with $\mathcal{D}=\mathbf{R}$. Then*

$$\mathbf{E}({}^{\varepsilon}r^2) \;=\; \sum_{j=0, j\,\mathrm{even}}^{N}\frac{\varepsilon^{j+1}}{j!}q_j\mu_j + \rho_{N+1}(\varepsilon),$$

where μ_j denotes the correlation moment of $j-th$ order of the correlation function, q_j is given in (4) and $\rho_{N+1}(\varepsilon)$ is the last term in (3).

Example 1 For the function $Q(t)=\exp(-\frac{t^2}{2})$ it holds

$$\mathbf{E}({}^{\varepsilon}r^2) = \sqrt{\pi}\left(\mu_0\varepsilon - \frac{1}{4}\mu_2\varepsilon^3 + \frac{1}{32}\mu_4\varepsilon^5 - \frac{1}{384}\mu_6\varepsilon^7 + \frac{1}{6144}\mu_6\varepsilon^9 + \dots\right).$$

2.2 The case $\mathcal{D} = \mathbf{R}_+$

Now integral functionals of the type

$$ {}^{\varepsilon}r = \int_0^\infty Q(s)\, {}^{\varepsilon}f(s)\, ds $$

are considered. In this case we derive from (2)

$$\begin{aligned}
\mathbf{E}({}^{\varepsilon}r^2) &= \varepsilon \int_{-\infty}^{0} R(u) \int_0^\infty Q(t-\varepsilon u)Q(t)\, dt\, du \\
&\quad + \varepsilon \int_0^\infty R(u) \int_{\varepsilon u}^\infty Q(t-\varepsilon u)Q(t)\, dt\, du \\
&= \varepsilon \int_{-\infty}^{0} R(u) \int_0^\infty Q(t-\varepsilon u)Q(t)\, dt\, du \\
&\quad + \varepsilon \int_0^\infty R(u) \int_0^\infty Q(t)Q(t+\varepsilon u)\, dt\, du \\
&= \varepsilon \int_{-\infty}^{\infty} R(u) \int_0^\infty Q(t+\varepsilon |u|)Q(t)\, dt\, du \\
&= \varepsilon \int_{-1}^{1} R(u) \int_0^\infty Q(t+\varepsilon |u|)Q(t)\, dt\, du
\end{aligned}$$

and in an analogous manner as before we find

Theorem 2.2 *Let $({}^{\varepsilon}f(s);\ s \in \mathbf{R}_+), \varepsilon > 0$, be a family of random processes satisfying the assumption 1.1 and $Q(\cdot)$ a function, satisfying the assumption 1.2 with $\mathcal{D} = \mathbf{R}_+$. Then*

$$ \mathbf{E}({}^{\varepsilon}r^2) = \sum_{j=0}^{N} \frac{\varepsilon^{j+1}}{j!} q_j \nu_j + \rho_{N+1}(\varepsilon), $$

where ν_j denotes the absolute correlation moment of $j-$th order of the correlation function and the q_j and $\rho_{N+1}(\varepsilon)$ are as follows:

$$\begin{aligned}
\rho_{N+1}(\varepsilon) &= \frac{\varepsilon}{N!} \int_{-1}^{1} R(u) \int_0^\infty Q(t) \int_t^{t+\varepsilon|u|} Q^{(N+1)}(v)(t+\varepsilon|u|-v)^N dv dt du; \\
q_j &= \int_0^\infty Q^{(j)}(t)Q(t)\, dt,\ j = 0, \dots, N,
\end{aligned}$$

and explicitely for $k = 0, 1, \dots$ if $Q(\cdot)$ is $N-$times continuous differentiable on $\mathbf{R}_+$

$$\begin{aligned}
q_{2k+1} &= \sum_{l=0}^{k-1} (-1)^{l+1} Q^{(l)}(0) Q^{(2k-l)}(0) + \frac{(-1)^{k+1}}{2} [Q^{(k)}(0)]^2 \\
q_{2k} &= \sum_{l=0}^{k-1} (-1)^{l+1} Q^{(l)}(0) Q^{(2k-1-l)}(0) + (-1)^k \int_0^\infty [Q^{(k)}(t)]^2\, dt.
\end{aligned}$$

In contradiction to the previous case now absolute correlation moments of even and odd order occur in the asymptotic expansion. This motivates the definition of the two different types of correlation moments.

Example 2 For $Q(t) = \exp(-\gamma t)$, $\gamma > 0$, $t \in \mathbf{R}_+$ we find for all $\varepsilon > 0$

$$\mathbf{E}({}^{\varepsilon}r^2) = \sum_{j=0}^{\infty} \frac{(-1)^j \gamma^{j-1}}{2j!} \varepsilon^{j+1} \nu_j.$$

2.3 The case $\mathcal{D} = [a, b]$

Now let the considered functional be

$${}^{\varepsilon}r = \int_a^b Q(s)\, {}^{\varepsilon}f(s)\, ds$$

with finite values $a, b \in \mathbf{R}$, $a < b, b - a \geq \varepsilon$.
Analogous arguments like in the previous case yield for the variance

$$\mathbf{E}({}^{\varepsilon}r^2) = \varepsilon \int_{-1}^{1} R(u) \int_a^{b-\varepsilon|u|} Q(t + \varepsilon|u|) Q(t)\, dt\, du.$$

Denoting for $0 \leq u \leq b - a$

$$\phi(u) = \int_a^{b-u} Q(t+u) Q(t)\, dt,$$

it follows

Theorem 2.3 *Let $({}^{\varepsilon}f(s);\ s \in [a, b]), \varepsilon > 0$, be a family of random processes satisfying the assumption 1.1 and $Q(\cdot)$ a function, satisfying the assumption 1.2 with $\mathcal{D} = [a, b]$. Then*

$$\mathbf{E}({}^{\varepsilon}r^2) \quad = \quad \sum_{j=0}^{N} \frac{\varepsilon^{j+1}}{j!} q_j \nu_j + \rho_{N+1}(\varepsilon),$$

where ν_j denote the absolute correlation moments of $j-$th order of the correlation function, $q_j = \phi^{(j)}(0)$ and $\rho_{N+1}(\varepsilon)$ is as follows:

$$\rho_{N+1}(\varepsilon) = \frac{2\varepsilon}{N!} \int_0^1 \int_0^u \phi^{(N+1)}(v)(u-v)^N\, dv\, R(u)\, du.$$

If $Q(\cdot)$ is $N-$times continuous differentiable on $[a, b]$ we can calculate for the values of $q_j = \phi^{(j)}(0)$

$$\begin{aligned}
q_{2k+1} \quad &= \quad -\sum_{l=0}^{k-1} (-1)^l \{Q^{(l)}(b) Q^{(2k-l)}(b) + Q^{(2k-l)}(a) Q^{(l)}(a)\} + \\
&\quad + \frac{(-1)^{k+1}}{2} \{[Q^{(k)}(b)]^2 + [Q^{(k)}(a)]^2\} \\
q_{2k} \quad &= \quad \sum_{l=0}^{k-1} (-1)^l \{Q^{(2k-1-l)}(b) Q^{(l)}(b) - Q^{(2k-1-l)}(a) Q^{(l)}(a)\} \\
&\quad + (-1)^k \int_a^b [Q^{(k)}(t)]^2\, dt.
\end{aligned}$$

3 Expansion of correlation functions

Now the mean square continuous weakly stationary process

$$^{\varepsilon}g(t) = \int_{-\infty}^{t} Q(t-s)\,^{\varepsilon}f(s)\,ds = \int_{0}^{\infty} Q(u)\,^{\varepsilon}f(t-u)\,du,$$

is examined, the deterministic function $Q(\cdot)$ is assumed to satisfy the assumption 1.2 on $\mathbf{R}_+$. In this case the correlation function can be written as (with $\tau = t_2 - t_1$)

$$\begin{aligned} ^{\varepsilon}R_{gg}(\tau) &= \mathbf{E}(\,^{\varepsilon}g(t_1)\,^{\varepsilon}g(t_2)) \\ &= \int_{-\infty}^{t_1}\int_{-\infty}^{t_2} Q(t_1-s_1)Q(t_2-s_2)\mathbf{E}(\,^{\varepsilon}f(s_1)\,^{\varepsilon}f(s_2))\,ds_2\,ds_1 \\ &= \int_{0}^{\infty}\int_{0}^{\infty} Q(u_1)Q(u_2)\,^{\varepsilon}R_{ff}(\tau+u_1-u_2)\,du_2du_1 \end{aligned}$$

and in the same way as before we derive

$$^{\varepsilon}R_{gg}(\tau) = \varepsilon\int_{-1}^{1} R(v)\int_{0}^{\infty} Q(u+|\varepsilon v-\tau|)Q(u)\,du\,dv.$$

Applying the Taylor-expansion of the function $Q(u+|\varepsilon v-\tau|)$ in a neighbourhood of the point $v_0 = |\tau|$ we can write

$$Q(u+|\varepsilon v-\tau|) = \sum_{j=0}^{N}\frac{1}{j!}Q^{(j)}(u+|\tau|)(|\varepsilon v-\tau|-|\tau|)^j + \tilde{\rho}_{N+1}(u,v,\varepsilon,\tau).$$

A calculation of the integral terms containing the correlation function $R(\cdot)$ yields

$$\begin{aligned} &\int_{-1}^{1} R(v)(|\varepsilon v-\tau|-|\tau|)^j dv = \\ &= \varepsilon^j\mu_j + \mathbf{1}_{\{|\tau|<\varepsilon\}}(\tau)\int_{\frac{|\tau|}{\varepsilon}}^{1} R(v)[(\varepsilon v-2|\tau|)^j-(-\varepsilon v)^j]\,dv \end{aligned}$$

and finally we have

$$\begin{aligned} ^{\varepsilon}R_{gg}(\tau) &= \sum_{j=0,\,j\,\mathrm{even}}^{N}\frac{\varepsilon^{j+1}}{j!}q_j(\tau)\mu_j + \rho_{N+1}(\varepsilon,\tau) + \\ &\quad +\mathbf{1}_{\{|\tau|<\varepsilon\}}(\tau)\sum_{j=1}^{N}\frac{\varepsilon}{j!}q_j(\tau)\int_{\frac{|\tau|}{\varepsilon}}^{1} R(v)[(\varepsilon v-2|\tau|)^j-(-\varepsilon v)^j]dv. \end{aligned} \tag{5}$$

The quantities

$$q_j(\tau) = \int_{0}^{\infty} Q^{(j)}(u+|\tau|)Q(u)\,du,\ \ j=0,\ldots,N,$$

are independent of the random process. So for fixed values of τ and $\varepsilon \to 0$ the following expansions are valid:

$$
\begin{aligned}
{}^{\varepsilon}R_{gg}(\tau) &= \sum_{j=0,\, j\,\mathrm{even}}^{N} \frac{\varepsilon^{j+1}}{j!} q_j(\tau)\mu_j + o(\varepsilon^{N+1}), \quad \text{if } \tau \neq 0, \\
{}^{\varepsilon}R_{gg}(0) &= \sum_{j=0}^{N} \frac{\varepsilon^{j+1}}{j!} q_j(0)\nu_j + o(\varepsilon^{N+1}).
\end{aligned}
$$

We can see, that if $q_j(0)$ and ν_j for for odd values j do not vanish a discontinuity in the expansions at the point $\tau = 0$ arises.
Examining asymptotic expansions of ${}^{\varepsilon}R_{gg}(\tau)$ as a function of τ it is necessary to consider not only the first terms in (5) but also the correction terms for $|\tau| < \varepsilon$.

4 Conclusion

Asymptotic expansions of variances or correlation functions of integral functionals involving ε-correlated weakly stationary random functions for $\varepsilon \to 0$ have been developed. In these expansions the influence of the deterministic kernel function and of the random function can be separated by using the concept of correlation moments and certain characteristics of the kernel function. In the case of random variables the expansions have the form of a power series in ε, in the case of a correlation function additionally correction terms for $|\tau| < \varepsilon$ arise. For given kernel functions and a generating correlation function it is easy to compute (at least numerically) the terms of the expansion. An estimate of the remainder term is also possible.
With respect to applications it is worth to note, that the asymptotic expansions can over- or underestimate the true value. The statement of the overestimation of the true value in [2], p.49, results from the special type of the correlation functions considered there.
An extension of these results on complex- or vector-valued processes or random fields and certain classes of nonstationary processes is also possible and will be considered in a subsequent paper.

References

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[2] J. vom Scheidt, B. Fellenberg, and U. Wöhrl. *Analyse und Simulation stochastischer Schwingungssysteme.* B. G. Teubner, Stuttgart, 1994.